

INRIA

# Optimal Transportation and Equilibria on Wireless Networks

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# Optimal Transportation

# Mass transportation

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The theory of mass transportation goes back to the original works by [Monge](#) [1781] and later by [Kantorovich](#) [1942].

The problem can be interpreted as the question:

“How do you best move given piles of sand to fill up given holes of the same total volume?”

# Mass transportation

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Example.- In  $\mathbb{N}$

$$\mu = \frac{\delta_1 + \delta_2}{2} \quad \text{and} \quad \nu = \frac{\delta_2 + \delta_3}{2}.$$

Possible costs:

- $c_1(x, y) = |x - y|$ ,
- $c_2(x, y) = |x - y|^2$ ,
- $c_3(x, y) = \sqrt{|x - y|}$ .

Possible Transports:

- We leave the mass in 2 and transport the mass in 1 into 3.  
The costs are 1, 2, and  $1/\sqrt{2}$ , respectively.
- We transport  $T(x) = x + 1$ . Then for all three cost functions the cost is one.

There are several transport applications!

# Monge Problem

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Monge's problem consists to search for the one that minimizes the cost, *i.e.*

$$(M) \quad \inf_{T_{\#}\mu=\nu} \int_X c(x, T(x)) d\mu(x).$$

Sometimes there are **not** transport applications.

**Example.-** There is no application sending  $\mu = \delta_0$  into  $\nu = \frac{\delta_0 + \delta_1}{2}$ .

We would have to **split** the mass, leave half of it in 0 and send half of it into 1.

# Monge-Kantorovich Problem

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We consider a more general strategy in which the mass in  $x$  will be distributed over the all space, following a distribution  $d\lambda_x$ .

Then  $d\pi(x, y) = d\lambda_x(y)d\mu(x)$  is the transported mass from  $x$  to  $y$ .

The cost of transporting the mass in  $x$  will be

$$\int_Y c(x, y) d\lambda_x(y).$$

and the total transport cost will be

$$\int_{X \times Y} c(x, y) d\lambda_x(y)d\mu(x) = \int_{X \times Y} c(x, y) d\pi(x, y).$$

# Monge-Kantorovich Problem

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Kantorovich's problem consists to search the optimal transport plan

$$(K) \quad \inf_{\pi \in \Pi(\mu, \nu)} \int_{X \times Y} c(x, y) d\pi(x, y)$$

Theorem.-  $(K)$  has a solution.

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# Application



# Wireless network with fixed base stations

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Consider a bounded set  $D \subseteq \mathbb{R}^2$  which is the geographic reference for a network and a nonnegative function  $f$  with unit integral representing the **density of demand** of the nodes in the network.

We fix  $k$  points  $x_1, x_2, \dots, x_k$  at which the antennas are located.

## Assumptions

- Being  $p \geq 1$  suppose that the time spent to cover a distance between  $x$  and  $x_i$  is given by  $|x - x_i|^p$
- The total amount of **time to be served** by antenna  $j$  to be  $h_j(c_j)$  where  $c_j$  is the amount of nodes asking for information to antenna  $j$ .

In order to minimize the total cost of the network

$$\inf_{(A_i) \text{ partition of } D} \sum_{i=1}^k \int_{A_i} \left[ |x - x_i|^p + h_i \left( \int_{A_i} f(x) dx \right) \right] f(x) dx.$$

# Results

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- If  $h_i$  are continuous functions, then **there exists an optimum**.
- If  $h_i$  are continuous functions and  $\eta_i(s) := sh_i(s)$  are strictly convex, then **there exists a unique optimum**.

Defining

$$(*) \begin{cases} A_i = \{x \in D : \\ |x - x_i|^p + h_i(c_i) + c_i h'_i(c_i) < |x - x_j|^p + h_j(c_j) + c_j h'_j(c_j) \quad \forall j \neq i\} \\ c_i = \int_{A_i} f(x) dx \end{cases}$$

- If  $h_i$  are differentiable and continuous in 0, then  $(*)$  is a **necessary optimality condition**.
- If  $h_i$  are differentiable in  $]0, 1]$  and continuous in 0 and  $\eta_i$  are convex, then  $(*)$  is a **necessary and sufficient optimality condition**.

# Nash Equilibrium

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A user living in  $x \in D$  and accessing to antenna  $x_i$  will be satisfied if

$$|x - x_i|^p + h_i(c_i) = \min_{j=1, \dots, k} |x - x_j|^p + h_j(c_j).$$

**Definition.-** A partition  $(A_i)_{i=1, \dots, k}$  of  $D$  is an equilibrium if for every  $i = 1, \dots, k$  the following condition holds:

$$\begin{cases} A_i &= \{x \in D : |x - x_i|^p + h_i(c_i) < |x - x_j|^p + h_j(c_j) \text{ for every } j \neq i\} \\ c_i &= \int_{A_i} f(x) dx. \end{cases}$$

# Nash Equilibrium

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Theorem.-

- Assume that the functions  $h_i$  are continuous.  
Then there exists an equilibrium !
- If in addition the functions  $h_i$  are non-decreasing.  
Then the equilibrium is unique !

# Comparison between optimum and equilibrium

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## Example 1.-

As an illustration example, suppose that on  $D = [0, 1]$  there are two antennas at coordinates  $x_1 = 1/4$  and  $x_2 = 3/4$ .

Assume that users are uniformly distributed, and take  $p = 2$ .

Suppose the first antenna can handle more demand than the second, so

$$h_1(t) = t \quad \text{and} \quad h_2(t) = (1 + \varepsilon)t$$

Then the **optimum**  $(A_1, A_2)$  is given by

$$A_1 = [0, \lambda_\varepsilon^{\text{opt}}[, \quad A_2 = ]\lambda_\varepsilon^{\text{opt}}, 1] \quad \text{with} \quad \lambda_\varepsilon^{\text{opt}} = \frac{1}{2} + \frac{\varepsilon}{5 + 2\varepsilon},$$

whereas the **equilibrium**  $(B_1, B_2)$  is

$$B_1 = [0, \lambda_\varepsilon^{\text{eq}}[, \quad A_2 = ]\lambda_\varepsilon^{\text{eq}}, 1] \quad \text{with} \quad \lambda_\varepsilon^{\text{eq}} = \frac{1}{2} + \frac{\varepsilon}{6 + 2\varepsilon} \leq \lambda_\varepsilon^{\text{opt}}.$$

# Comparison between optimum and equilibrium

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## Example 2.-

Users are uniformly distributed on  $D = [0, 1]$ .

There are two antennas at coordinates  $x_1 = 0$  and  $x_2 = 1$ . Consider  $p = 1$  and

$$h_1(s) = 100 \quad \text{and} \quad h_2(s) = \begin{cases} 0 & \text{for } 0 \leq s \leq 0.999 \\ 1 & \text{for } 0.999 \leq s \leq 1. \end{cases}$$

Then the **equilibrium**  $(B_1, B_2)$  is given by

$$B_1 = \emptyset \quad \text{and} \quad B_2 = [0, 1]$$

and the **optimum**  $(A_1, A_2)$  is

$$A_1 = [0, 0.001[ \quad \text{and} \quad A_2 = ]0.001, 1].$$

The optimum is very unfair for users living in  $A_1$ , who pay  $x + 100$ , whereas the others just pay the distance from 1.

# Main Reference

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G. Crippa, Ch. Jimenez, A. Pratelli, “Optimum and equilibrium in a transportation problem with queue penalization effect”

# Random Walk Model

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Random Walk Model is also referred to as the **Brownian Motion Model**.

The nodes change their speed and direction at each time interval:

- For every new interval  $t$ , each node randomly and uniformly chooses its new direction  $\theta(t)$  from  $(0, 2\pi]$ .
- In similar way, the new speed  $v(t)$  follows a uniform distribution or a Gaussian distribution from  $[0, V_{\max}]$ .

If the node moves according to the above rules and reaches the boundary of simulation field, the leaving node is bounced back to the simulation field with the angle of  $\theta(t)$  or  $(\pi - \theta(t))$ .

## Disadvantage

- The Random Walk model is a memoryless mobility process, however we observe that is not the case of mobile nodes in many real life applications.



# Brownian Mobility Model

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$$d\rho^+(t) = \mathbf{v}^+(\mathbf{x}, t) dt + \sigma_+(\mathbf{x}, t) dW^+(t)$$

and/or

$$d\rho^-(t) = \mathbf{v}^-(\mathbf{x}, t) dt + \sigma_-(\mathbf{x}, t) dW^-(t).$$

where  $W^+(t)$  and  $W^-(t)$  are two independent brownian motion with values in  $X \times Y$ .

Assuming as in the previous case that we know the initial distribution of the sources then by using Ito's lemma the distribution of the sources evolves in time by the Kolmogorov Forward Equation

$$\frac{\partial}{\partial s} p(x, s) = -\frac{\partial}{\partial x} [\mu(x, s) p(x, s)] + \frac{1}{2} \frac{\partial^2}{\partial x^2} [\sigma_+^2(x, s) p(x, s)].$$

for  $s \geq 0$ , with initial condition  $p(x, 0) = \rho^+(x)$

Equivalently, the initial distribution of the destinations evolves in time by the Kolmogorov Forward Equation

$$\frac{\partial}{\partial s} p(x, s) = -\frac{\partial}{\partial x} [\mu(x, s) p(x, s)] + \frac{1}{2} \frac{\partial^2}{\partial x^2} [\sigma_-^2(x, s) p(x, s)].$$

for  $s \geq 0$ , with initial condition  $p(x, 0) = \rho^-(x)$ .