MIS in the Presence of Path Correlation

1 Problem Statement

We would like to compute the measurement equation, which is given by:

$$I = \langle W, L_e \rangle + \langle W, TL_e \rangle + \left\langle W, T^2 L_e \right\rangle + \dots \quad (1)$$

where $W(x \leftarrow \omega_i)$ is the importance function, $L_e(x \rightarrow \omega_o)$ is the emittance and T is the transport operator:

$$(T\phi)(x \rightarrow \omega_o) = \int_\Omega \rho(\omega_i \rightarrow x \rightarrow \omega_o) \phi\left(h(x, \omega_i) \rightarrow \text{-}\omega_i\right) \langle N_x, \omega_i \rangle d\omega_i$$

1.1 Monte Carlo Solution

The surface area form of $(l+1)$-th term in Equation (1) is:

$$I_l = \left\langle W, T^l L_e \right\rangle = \int_{\mathcal{M}^l} f(\bar{x})\, d\mu(\bar{x}) \quad (2)$$

$$f(\bar{x}) = L_e(x_0 \rightarrow x_1)\, \Pi_l(\bar{x})\, GV(x_{l-1} \leftrightarrow x_l) W(x_{l-1} \rightarrow x_l) \quad (3)$$

$$\Pi_l(\bar{x}) = \prod_{i=0}^{l-2} GV(x_i \leftrightarrow x_{i+1}) \rho(x_i \rightarrow x_{i+1} \rightarrow x_{i+2}) \quad (4)$$

where $\mathcal{M}$ is the surface of the scene.

We can estimate the value of each I_l using Monte Carlo integration in an unbiased way, by generating N samples of dimension l (i.e. paths of length l) $\{\bar{X}_j\}, j \in \{1 \dots N\}$, from some distribution $p(\bar{x})$:

$$\frac{1}{N} \sum_{j=1}^{N} \frac{f(\bar{X}_j)}{p(\bar{X}_j)} \approx I_l \quad (5)$$

1.2 Multiple Importance Sampling

The common way to generate a l dimensional path is through local path sampling: One needs to first generate a path starting from the camera with t vertices and then a path starting from the light source with s vertices, where $s + t = l$. However there are

$l + 1$ different ways (or *strategies*) to generate a sample in this way: $(s = 0, t = l)$, $(s = 1, t = l - 1), \ldots (s = l, t = 0)$. And the different strategies have different PDFs.

Our goal is to reduce the variance by optimally combining samples coming from the different strategies. To this end, we assume that strategy i (i.e. $t = i$) has generated n_i samples and assigns weight $w_i(\bar{X}_{ij})$ and probability $p_i(\bar{X}_{ij})$ to the j-th sample $\bar{X}_{ij}$ from this strategy. It is easy to see, that if the weights fulfill $\sum_i w_i(\bar{x}) = 1$, then the bellow estimator is unbiased:

$$E\left[\sum_{i=0}^{l} \frac{1}{n_i} \sum_{j=0}^{n_i - 1} \frac{w_i(\bar{X}_{ij}) f(\bar{X}_{ij})}{p_i(\bar{X}_{ij})}\right] = I_{l-2} \qquad (6)$$

What needs to be done at this point is to find the weights. Veach shows how to do that if the samples are uncorrelated [Vea98, p.288]. However, re-using the eye path segment introduces correlation. Thus, the MIS weights have to be modified accordingly, and this is what we do in section 2.2. Notice that this correlation affects other recent GI methods, including Combinatorial BdPT [PBPP11], and Vertex Connection Merging [GKDS12].

2 MIS with Correlations

In this section we will derive a new formulation for the MIS weights in the presence of correlation.

2.1 Assumptions

To make the problem more manageable, we will make several assumptions specific to the illumination.

We first assume that to estimate I_l the illumination algorithm generates l independent eye paths: one with length $t = 1$, one with $t = 2$, ... and one with $t = l$. Each eye path will sample a different strategy. It then generates lM independent light paths: M with length $s = 1$, M with length $s = 2$, ... and M with length $s = l$. And finally it connects each pair of eye/light paths where $s + t = l$. In practice, illumination algorithms (e.g. BPT) do not generate independent eye paths. However, this correlation can be ignored with more eye paths per pixel (see [Vea98, p.307]).

With the above assumption, it follows that paths generated using different strategies are independent. Also, the paths in strategy $t = 0$ are independent, $n_i = lM$ for $0 < i < l$, and finally $n_0 = 1$ and $n_l = M$.

2.2 Derivation

We would like to minimize the variance of the estimator in Eq. (6):

$$V[F] = V\left[\sum_i \frac{1}{n_i}\sum_j F_{ij}\right] \tag{7}$$

$$= \sum_{i_1}\sum_{i_2}\sum_{j_1}\sum_{j_2} \frac{1}{n_{i_1}n_{i_2}}\,\mathrm{COV}[F_{i_1j_1}, F_{i_2j_2}] \tag{8}$$

where $F_{ij} = w_i(\bar{X}_{ij})f(\bar{X}_{ij})/p_i(\bar{X}_{ij})$

Since paths from different strategies are independent (Sect. 2.1), it follows that $\mathrm{COV}[F_{i_1j_1}, F_{i_2j_2}] = 0$ when $i_1 \neq i_2$. Furthermore

$$\mathrm{COV}[F_{ij_1}, F_{ij_2}] = E[F_{ij_1}F_{ij_2}] - E[F_{ij_1}]E[F_{ij_2}] \qquad , j_1 \neq j_2 \tag{9}$$

$$\mathrm{COV}[F_{ij}, F_{ij}] = E[F_{ij}^2] - E[F_{ij}]^2 \qquad \text{, otherwise} \tag{10}$$

Plugging this back into Eq. (7) and letting $\mu_i = E[F_{ij}]$ we obtain:

$$V[F] = \sum_i \frac{1}{n_i}E[F_{i0}^2] + \sum_i \frac{n_i - 1}{n_i}E[F_{i0}\ F_{i1}] - \sum_i \mu_i^2 \tag{11}$$

The summands in the first and the third term expand to:

$$E[F_{i0}^2] = \int_{M^l} \frac{w_i^2(\bar{x})f^2(\bar{x})}{p_i^2(\bar{x})}p_i(\bar{x})d\mu(\bar{x}) \tag{12}$$

$$\mu_i^2 = \left[\int_{M^l} w_i(\bar{x})f(\bar{x})d\mu(\bar{x})\right]^2 \tag{13}$$

To expand the second term, we first observe that any two paths $\bar{x_1}$ and $\bar{x_2}$ from the same strategy $i \in (0, l)$ share a common eye path. That is $\bar{x}_1 = \bar{z}\bar{y}_1$ and $\bar{x}_2 = \bar{z}\bar{y}_2$ where $\bar{z}$ is the common part. Thus, the joint probability of generating the two paths is given by $p_i(\bar{x}_1, \bar{x}_2) = p(\bar{z})p(\bar{y}_1)p(\bar{y}_2)$. Thus

$$E[F_{i0}F_{i1}] = \int\int \frac{f(\bar{x}_1)f(\bar{x}_2)w_i(\bar{x}_1)w_i(\bar{x}_2)}{p_i(\bar{x}_1)p_i(\bar{x}_2)}p_i(\bar{x}_1, \bar{x}_2)d\bar{\mu}(x_1)d\mu(\bar{x}_2) \tag{14}$$

$$= \int_{\mathcal{M}^i}\left[\int_{\mathcal{M}^{l-i}} f(\bar{z}\bar{y})w_i(\bar{z}\bar{y})d\mu(\bar{y})\right]^2 \frac{1}{p(\bar{z})}d\mu(\bar{z}) \tag{15}$$

2.3 Minimizing the Variance

Next, we need to minimize $V[F]$ subject to $\sum_i w_i(\bar{x}) = 1$. Similar to [Vea98], we only look at the first two terms of $V[F]$. We can use the third term to prove that no other weighting scheme is considerably better than the one we derive. Note that in contrast to [Vea98] we can take into account the third term as well, though this would make the final solution even more complex. Also, our solution will no longer be identical to the balance heuristic in this case if no correlation is present.

In essence, we need to find $l+1$ functions $w_i \in \mathcal{L}^2(\mathcal{M}, \mu)$ that minimize the functional

$$
\begin{aligned}
&J(w_0, \ldots, w_l) = \\
&= \sum_i \int_{\mathcal{M}^l} \frac{w_i^2(\bar{x}) f^2(\bar{x})}{n_i p_i(\bar{x})} d\mu(\bar{x}) + \\
&+ \sum_i \int_{\mathcal{M}^i} \frac{n_i - 1}{n_i\, p(\bar{z})} \left[\int_{\mathcal{M}^{l-i}} f(\bar{z}\bar{y}) w_i(\bar{z}\bar{y})\, d\mu(\bar{y}) \right]^2 d\mu(\bar{z})
\end{aligned} \tag{16}
$$

and also fulfill

$$
\underbrace{\sum_i \omega_i(x) - 1}_{G(w_0, \ldots, w_l)} = 0
$$

2.4 Lagrangian Multiplier

To solve the optimization problem, we can use Lagrange multipliers for Banach spaces. We first need to introduce a multiplier Λ, which in this case is a functional in the dual space of $\mathcal{L}^2(\mathcal{M}, \mu)$:

$$
\Lambda(g) = \int_{\mathcal{M}^l} \lambda(\bar{x}) g(\bar{x}) d\mu(\bar{x}) \tag{17}
$$

Next, if $w_0', \ldots, w_i'$ is a minimum that satisfies the constraints, then the Fréchet derivatives $\mathrm{D}J_i$ w.r.t. each w_i must satisfy

$$
\mathrm{D}J_i(w_i') = \Lambda \circ \mathrm{D}G_i(w_i') \tag{18}
$$

To find $\mathrm{D}J_i$ we compute the Gâteaux derivatives:

$$
\begin{aligned}
&dJ_i(w_i; h_i) = \frac{d}{d\tau} J(w_0, \ldots, w_i + \tau h_i, \ldots w_l) \Big|_{\tau=0} = \\
&= \frac{d}{d\tau} \left(\int_{\mathcal{M}^l} \frac{(w_i(\bar{x}) + \tau h_i(\bar{x}))^2 f^2(\bar{x})}{n_i p_i(\bar{x})} \right) \Bigg|_{\tau=0} + \\
&\quad + \frac{d}{d\tau} \left(\int_{\mathcal{M}^i} \frac{n_i - 1}{n_i\, p(\bar{z})} \left[\int_{\mathcal{M}^{l-i}} f(\bar{z}\bar{y}) \left(w_i(\bar{z}\bar{y}) + \tau h_i(\bar{z}\bar{y}) \right) d\mu(\bar{y}) \right]^2 d\mu(\bar{z}) \right) \Bigg|_{\tau=0} \\
&= \int_{\mathcal{M}^l} h_i(\bar{x}) \frac{2 f(\bar{x})}{n_i} \left[\frac{w_i(\bar{x}) f(\bar{x})}{p_i(\bar{x})} + \frac{n_i - 1}{p(\bar{z})} \int_{\mathcal{M}^{l-i}} f(\bar{z}\bar{y}') w_i(\bar{z}\bar{y}')\, d\mu(\bar{y'}) \right] d\mu(\bar{x})
\end{aligned} \tag{19}
$$

where $\bar{x} = \bar{z}\bar{y}$. If the above Gâteaux derivative exists for any function h_i^0, then it is easy to see that it also exists and is continuous in an open vicinity of h_i^0. Thus it is also a Fréchet derivative there.

Similarly, the partial Gâteaux derivatives of G w.r.t. w_i are given by $dG_i(w_i; h_i) = h_i(x)$, the partial Fréchet derivatives exist and are equal to the Gâteaux ones, and

$$
\Lambda \circ \mathrm{D}G_i = \int_{\mathcal{M}^l} \lambda(\bar{x}) h_i(\bar{x}) d\mu(\bar{x}) \tag{20}
$$

2.5 Solving the Necessary Conditions

We plug Eq. (19) and Eq. (20) into Eq. (18) and note that the resulting equality must hold for any choice of h_i. Thus, from the fundamental lemma of calculus of variations, we obtain the following system of equations

$$\frac{2w_i(\bar{x})f^2(\bar{x})}{n_i p_i(\bar{x})} + \frac{2(n_i-1)f(\bar{x})}{n_i\, p(\bar{z})}\int_{\mathcal{M}^{l-i}} f(\bar{z}\bar{y}')w_i(\bar{z}\bar{y}')d\mu(\bar{y'}) = \lambda(\bar{x}) \tag{21}$$

$$\sum_i w_i(\bar{x}) = 1 \tag{22}$$

Using the fact that $p_i(\bar{x}) = p(\bar{y})p(\bar{z})$, Eq. (21) becomes:

$$w_i(\bar{x}) = \frac{n_i\lambda(\bar{x})p_i(\bar{x})}{2f^2(\bar{x})} - \frac{(n_i-1)p(\bar{y})}{f(\bar{x})}\int_{\mathcal{M}^{l-i}} f(\bar{z}\bar{y}')w_i(\bar{z}\bar{y}')d\mu(\bar{y'}) \tag{23}$$

This is an integral equation with a separable kernel. If we fix the first i components of $\bar{x}$ (i.e. $\bar{z}$), we can apply Lemma A.1 and solve the integral equation, leading to

$$\begin{aligned} w_i(x) =& \frac{n_i\lambda(\bar{x})p_i(\bar{x})}{2f^2(\bar{x})} - \frac{(n_i-1)p(\bar{y})}{f(\bar{x})}\frac{\int_{\mathcal{M}^{l-i}} \frac{n_i\lambda(\bar{z}\bar{y}')p_i(\bar{z}\bar{y}')}{2f(\bar{z}\bar{y}')}d\mu(\bar{y}')}{1+\int_{\mathcal{M}^{l-i}}(n_i-1)p(\bar{y}')d\mu(\bar{y}')} && (24)\\ =& \frac{n_i\lambda(\bar{x})p_i(\bar{x})}{2f^2(\bar{x})} - \frac{(n_i-1)p_i(\bar{x})}{2f(\bar{x})}\int_{\mathcal{M}^{l-i}}\frac{\lambda(\bar{z}\bar{y}')p(\bar{y}')}{f(\bar{z}\bar{y}')}d\mu(\bar{y}') && (25)\end{aligned}$$

where we have used the fact that $\int_{\mathcal{M}^{l-i}} p(\bar{y})d\mu(\bar{y}) = 1$ and $p_i(\bar{z}\bar{y}') = p(\bar{z})p(\bar{y}')$

2.6 Computing the Lagrange Multiplier

Next, we need to find $\lambda(\bar{x})$. The weights must sum up to 1, thus we obtain

$$1 = \lambda(x)\sum_i \frac{n_i p_i(\bar{x})}{2f^2(\bar{x})} - \sum_i\left[\frac{(n_i-1)p_i(\bar{x})}{2f(\bar{x})}\int_{\mathcal{M}^{l-i}}\frac{\lambda(\bar{z}\bar{y}')p(\bar{y}')}{f(\bar{z}\bar{y}')}d\mu(\bar{y}')\right] \tag{26}$$

$$\lambda(x) = \frac{2f^2(\bar{x})}{\sum_i n_i p_i(\bar{x})} + \sum_i\left[\frac{(n_i-1)f(\bar{x})p_i(\bar{x})}{\sum_i n_i p_i(\bar{x})}\int_{\mathcal{M}^{l-i}}\frac{\lambda(\bar{z}\bar{y}')p(\bar{y}')}{f(\bar{z}\bar{y}')}d\mu(\bar{y}')\right] \tag{27}$$

To solve the above equation, we introduce

$$a_0(\bar{x}) = \frac{2f^2(\bar{x})}{\sum_j n_j p_j(\bar{x})} \tag{28}$$

$$a_{l-i}(\bar{x}) = \frac{(n_i-1)f(\bar{x})p_i(\bar{x})}{\sum_j n_j p_j(\bar{x})} \qquad , i \in \{1,\dots,l-1\} \tag{29}$$

$$b_{l-i}(\bar{x}) = \frac{p(\overline{x_{l-i}\dots x_{l-1}})}{f(\bar{x})} \qquad , i \in \{1,\dots,l-1\} \tag{30}$$

Next, we introduce the functions

$$c_{l-i}(\overline{x_0 \dots x_{i-1}}) = \int_{\mathcal{M}^i} \lambda(\overline{x_0 \dots x_{i-1} y_i \dots y_{l-1}}) b_{l-i}(\overline{x_0 \dots x_{i-1} y_i \dots y_{l-1}}) dy_i \dots dy_{l-1} \tag{31}$$

And the integral equation for $\lambda(\bar{x})$ can be re-written as

$$\lambda(\bar{x}) = a_0(\bar{x}) + a_1(\bar{x}) c_1(\overline{x_0 \dots x_{l-2}}) + a_2(\bar{x}) c_2(\overline{x_0 \dots x_{l-3}}) + \dots \tag{32}$$

If we fix $x_0 \dots x_{l-2}$, than all c_i will become constants. Thus, we can use the same logic as in Lemma A.1 to find the value of c_1 as a function of the a, b and $c_2, \dots c_l$ functions:

$$\begin{aligned} c_1(\overline{x_0 \dots x_{l-2}}) &= \frac{\int_{\mathcal{M}} a_0(\overline{x_0 \dots x_{l-2} y_{l-1}}) b_1(\overline{x_0 \dots x_{l-2} y_{l-1}}) dy_{l-1}}{1 - \int_{\mathcal{M}} a_1(\overline{x_0 \dots x_{l-2} y_{l-1}}) b_1(\overline{x_0 \dots x_{l-2} y_{l-1}}) dy_{l-1}} \\ &+ \sum_{i=2}^{l} c_i(\overline{x_0 \dots x_i}) \frac{\int_{\mathcal{M}} a_i(\overline{x_0 \dots x_{l-2} y_{l-1}}) b_1(\overline{x_0 \dots x_{l-2} y_{l-1}}) dy_{l-1}}{1 - \int_{\mathcal{M}} a_1(\overline{x_0 \dots x_{l-2} y_{l-1}}) b_1(\overline{x_0 \dots x_{l-2} y_{l-1}}) dy_{l-1}} \end{aligned} \tag{33}$$

or omitting the functions' arguments

$$c_1 = \frac{\int_{\mathcal{M}} a_0 b_1 + \sum_{i=2}^{l} c_i \int_{\mathcal{M}} a_i b_1}{1 - \int_{\mathcal{M}} a_1 b_1} \tag{34}$$

We can then plug c_1 back into the integral equation for λ obtaining

$$\lambda = a_0 + \frac{\int_{\mathcal{M}} a_0 b_1}{1 - \int_{\mathcal{M}} a_1 b_1} + c_2 \left(a_2 + \frac{\int_{\mathcal{M}} a_2 b_1}{1 - \int_{\mathcal{M}} a_1 b_1} \right) + \dots \tag{35}$$

We can now find c_2 as a function of $c_3, \dots c_l$ and the a and b functions. Thus, the solution for λ is given by the following recursive formulas:

$$a_{i,1} = a_i \tag{36}$$

$$a_{i,j+1} = a_{i,j} + \frac{\int_{\mathcal{M}^j} a_{i,j} b_i}{1 - \int_{\mathcal{M}^j} a_{j,j} b_j} \tag{37}$$

$$c_j = \frac{\int_{\mathcal{M}^j} a_{0,j} b_j + \sum_{i=j+1}^{l} c_i \int_{\mathcal{M}^j} a_{i,j} b_j}{1 - \int_{\mathcal{M}^j} a_{j,j} b_j} \tag{38}$$

2.7 Simple Case Example

In the simplest case, we have correlation only for eye paths of certain length s. In this case the weights are given by

$$w_{i \neq s}(\bar{x}) = \frac{n_i p_i(\bar{x})}{\sum_j n_j p_j(\bar{x})} \left(1 + \frac{p_s(\bar{x})}{f(\bar{x})} C(\bar{z}_s) \right) \tag{39}$$

$$w_s(\bar{x}) = \frac{n_s p_s(\bar{x})}{\sum_j n_j p_j(\bar{x})} \left(1 + \frac{p_s(\bar{x})}{f(\bar{x})} C(\bar{z}_s) \right) - \frac{p_s(\bar{x})}{f(\bar{x})} C(\bar{z}_s) \tag{40}$$

where $\bar{z}_s$ is the path formed by taking the first s vertices of the path, starting from the eye. The constant $C(\bar{z}_s)$ is given by

$$C(\bar{z}_s) = \frac{(n_s - 1)\int_{\mathcal{M}^{l-s}} \frac{f(\bar{z}_s\bar{y})p(\bar{y})}{\sum_j n_j p_j(\bar{z}_s\bar{y})} d\mu(\bar{y})}{1 - (n_s - 1)\int_{\mathcal{M}^{l-s}} \frac{p_s(\bar{z}_s\bar{y})p(\bar{y})}{\sum_j n_j p_j(\bar{z}_s\bar{y})} d\mu(\bar{y})} \tag{41}$$

A Solving Integral Equations

Lemma A.1 *Let a, b, g, and ϕ be four functions over some domain Ω. Then, the solution of the integral equation*

$$\phi(x) = g(x) + a(x)\int_\Omega b(t)\phi(t)d\mu(t) \tag{42}$$

is given by

$$\phi(x) = g(x) + a(x)\frac{\int_\Omega b(t)g(t)d\mu(t)}{1 - \int_\Omega a(t)b(t)d\mu(t)} \tag{43}$$

Proof The value of $\int_\Omega b(t)\phi(t)d\mu(t)$ is an unknown constant c which depends on ϕ. Thus

$$\phi(x) = g(x) + a(x)c \tag{44}$$

Plugging this into Eq. (42) results in

$$g(x) + a(x)c = g(x) + a(x)\left[\int_\Omega b(t)g(t)d\mu(t) + c\int_\Omega b(t)a(t)d\mu(t)\right] \tag{45}$$

$$c = \frac{\int_\Omega b(t)g(t)d\mu(t)}{1 - \int_\Omega b(t)a(t)d\mu(t)} \tag{46}$$

Combining Eq. (44) and Eq. (46), we obtain Eq. (43) ∎

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