Mathematical Methods - Lecture 11

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Outline

Some references

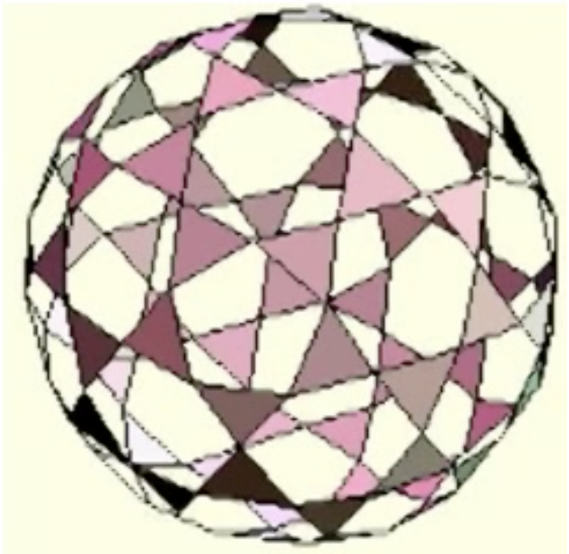
- H. Samet, Foundations of multidimensional and metric data structures, 2006.
- Gunnar Carlsson on the Shape of Data, Youtube lectures.

What is topology?

- Topology is a mathematical formalism for doing 2 things: measuring and representing shape
- A shape is the form of an object or its external boundary, outline, or external surface, as opposed to other properties such as color, texture or material
- We have to invent ways to measure shapes, to represent them in a compressed and discretized form

What is topology?

- Metric space = set for which distances between all members of the set are defined.
  - If you think about metric space, shape = infinite list of points, and infinite list of distances you'd have to define
- To find more compressed ways: triangulating space, by putting nodes and links between nodes: that captures a structure of a space



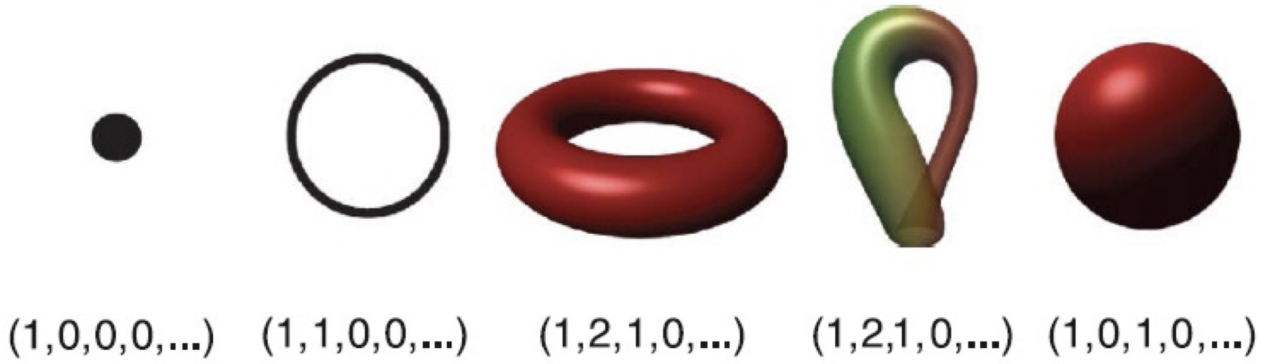
Simplicial complex

What is topology?

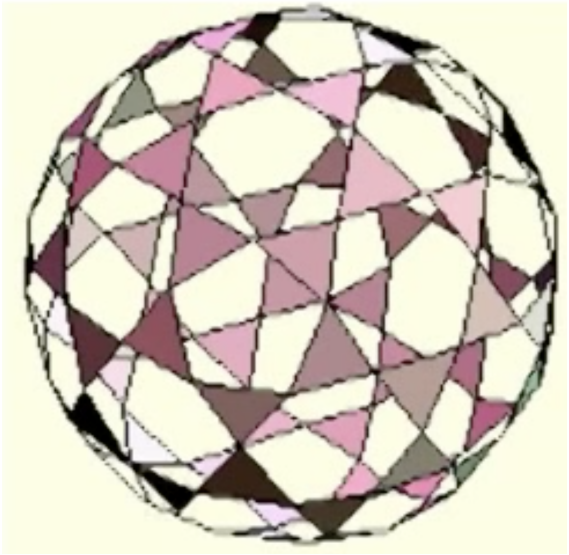
- Formalism for measuring and representing shape
- Pure mathematics since 1700's
- Last ten years ported into the point cloud world
  - Remember Euler method

Betti numbers

- The simplest measures of shape: compute number of holes
- Betti numbers: the number of k^{th} order holes

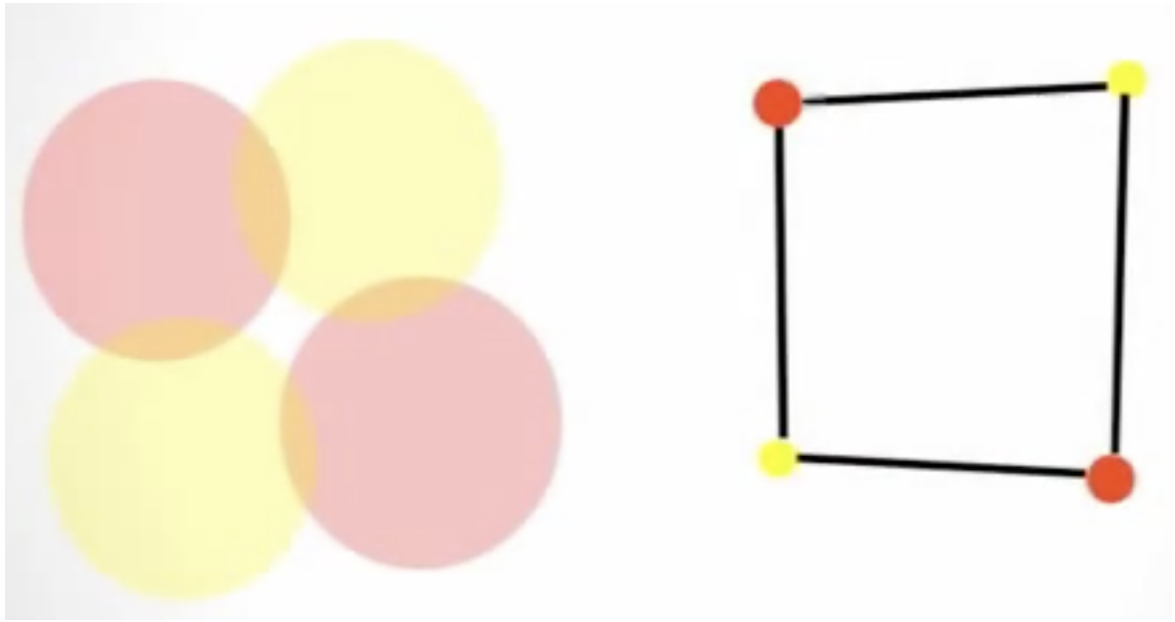


Simplicial complex



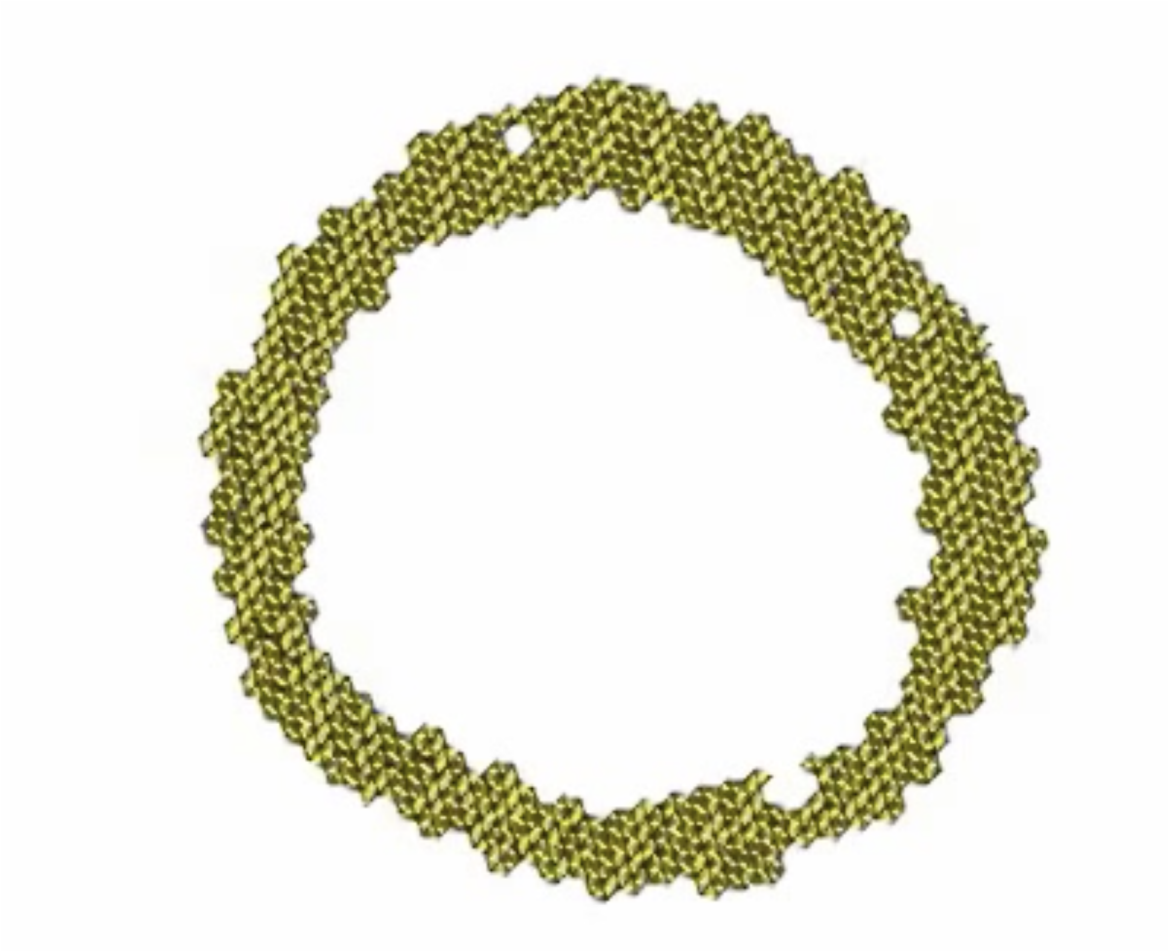
- Combinatorial representation of continuous object
- Huge compression
- Lossy compression, but retains many important features

Representing shape



- Simplicial complex on the right

Measuring the shape of data



- Features which persist across the changes of scale are important

Definition of a topological space

Let $X \neq \emptyset$. A topology on X is a collection of "open subsets" of X which satisfy the following.

(1) X, $\emptyset$ are open.

(2) the union of any family of open sets is open.

(3) the finite intersection of any collection of open sets is open.

$T = \{\text{all open subsets of } X\}$

topological space (X, T)

Definition of a topological space

Example 1:

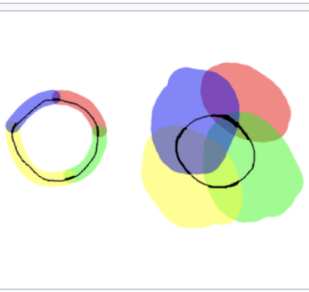
- $X = \{a, b, c\}$
- $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$
- Is τ a topology on X?

Example 2:

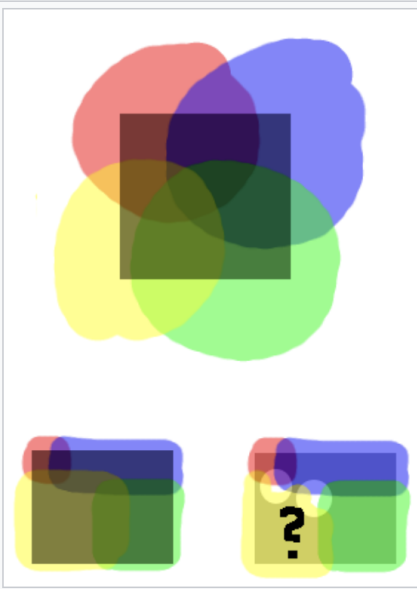
- $X = \{a, b, c\}$
- $\tau = \{\emptyset, \{a, b\}, \{a, c\}, X\}$
- Is τ a topology on X?

Topological dimension of a topological space

- Open cover of a topological space X is a family of open sets whose union contains X
- The order of a cover is the smallest number n (if it exists) such that each point of the space belongs to at most n sets in the cover

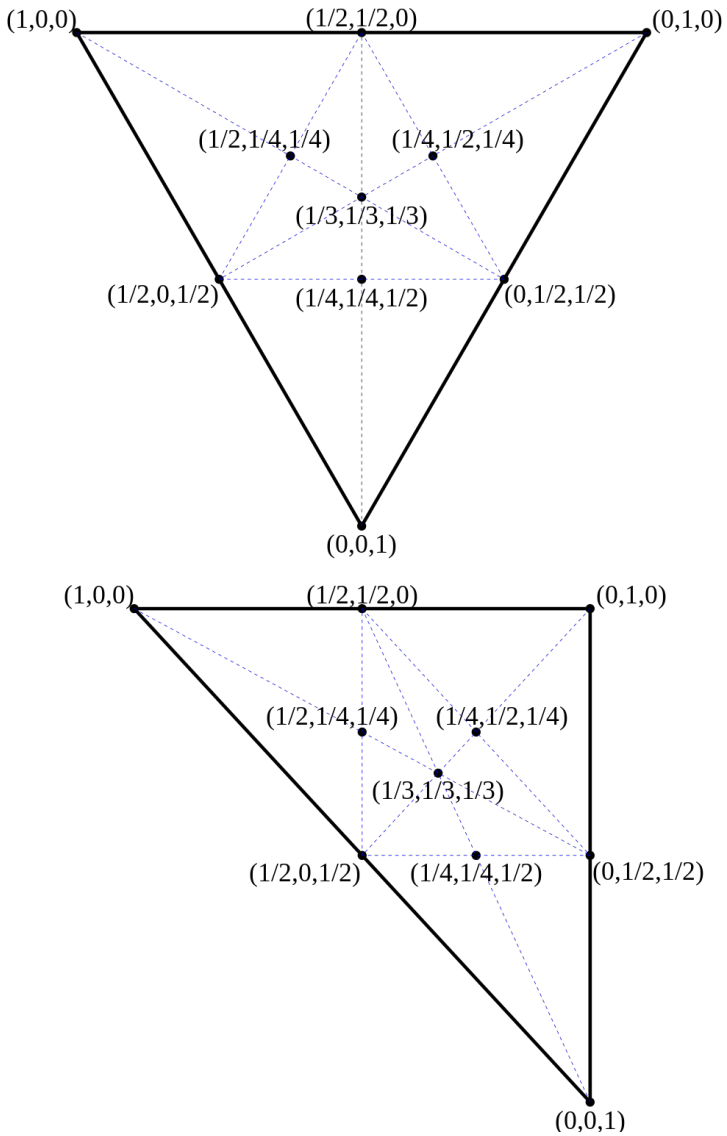


On the left is a refinement of a cover (on the right) of a circular line (black). Notice how in the refinement no point on the line is contained in more than two sets. Note also how the sets link to each other to form a "chain".



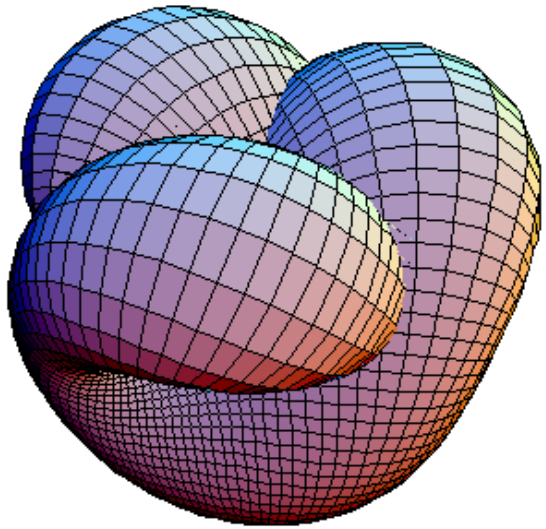
Below left is a refinement of a cover (above) of a planar shape (dark) so that all points in the shape are contained in at most three sets. Below right is an attempt to refine the cover so that no point would be contained in more than two sets. This fails in the intersection of set borders. Thus, a planar shape isn't "webby" or cannot be covered with "chains", but is in a sense thicker; i.e., its topological dimension must be higher than one.

Barycentric coordinates



What is manifold?

- Manifold is a topological space that locally resembles Euclidean space near each point
- More precisely, each point of an n-dimensional manifold has a neighbourhood that is homeomorphic to the Euclidean space of dimension n. In this more precise terminology, a manifold is referred to as an n-manifold
  - Homeomorphism = a continuous function between topological spaces that has a continuous inverse function



Homework

- What is tangent space?
- What is polytope?
- What is the relation between a triangle and a circle inside the triangle with all three sides of the triangle touching the circle once each?