Mathematical Methods - Lecture 9

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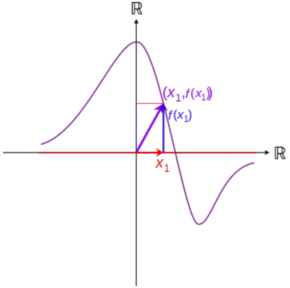
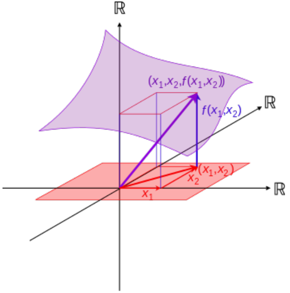
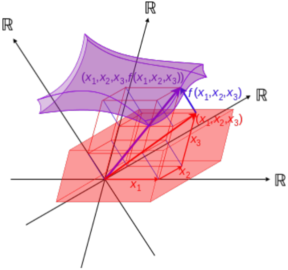
Outline

Functions of several variables

- $f(x)$ is a function of one variable x
- We may consider functions that depend on more than one variable
- **Example:** $f(x,y) = x^2 + 3xy$ depends on 2 variables x and y
  - For any pair of values x, y, $f(x,y)$ has a well-defined value
- Function $f(x_1, x_2, \ldots, x_n)$ depends on the variables $x_1, x_2, \ldots, x_n$







Partial derivatives

- A function $f(x, y)$ will have a gradient in ALL directions in the xy-plane
- **Gradient** $=$ rate of change of a function

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- A function $f(x, y)$ will have a gradient in ALL directions in the xy-plane
- **Gradient** = rate of change of a function
- The rates of change of $f(x, y)$ in the positive x- and y- directions are called **partial derivatives** wrt x and y, respectively
- Partial derivatives are extremely important in a wide range of applications!

Partial derivatives

- **Partial derivative** of $f(x, y)$ with respect to x is denoted by $\partial f / \partial x$:
  - to signify that a derivative is wrt x, but
  - to recognize that a derivative wrt y also exists

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x},$$

provided that the limit exists

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- The partial derivative of f with respect to y:

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y},$$

provided that the limit exists

Partial derivatives - Notations

- Partial derivative of $f(x, y)$ with respect to x:

$$\frac{\partial f}{\partial x} = \left(\frac{\partial f}{\partial x}\right)_y = f_x$$

- Partial derivative of f with respect to y:

$$\frac{\partial f}{\partial y} = \left(\frac{\partial f}{\partial y}\right)_x = f_y$$

- It is important when using partial derivatives to remember which variables are being held constant!

Partial derivatives

- The extension to the general n-variable case is straightforward:

$$\frac{\partial f(x_1, \ldots, x_n)}{\partial x_i} = \lim_{\Delta x_i \to 0} \frac{f(x_1, \ldots, x_i + \Delta x_i, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n)}{\Delta x_i},$$

provided that the limit exists

Partial derivatives

- Second (and higher) partial derivatives may be defined in a similar way:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{xy}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}$$

Properties of partial derivatives

- Provided that the second partial derivatives are continuous at the point in question:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$$

Partial derivatives - Example

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$$\frac{\partial^2 f}{\partial x^2} = 12xy^2, \quad \frac{\partial^2 f}{\partial y^2} = 4x^3 + 6y,$$

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$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 12x^2y$$

The chain rule

- The total derivative of $f(x, y)$ with respect to x:

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

- The total differential of the function $f(x, y)$:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

- **The chain rule:**

$$\frac{df}{du} = \frac{\partial f}{\partial x}\frac{dx}{du} + \frac{\partial f}{\partial y}\frac{dy}{du}$$

- Particularly useful when an equation is expressed in a parametric form

The chain rule

Exercise: Given that $x(u) = 1 + au$ and $y(u) = bu^3$, find the rate of change of $f(x, y) = xe^{-y}$ with respect to u

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- Using the chain rule:

$$\frac{df}{du} = (e^{-y})a + (-xe^{-y})3bu^2$$

- Substituting for x and y:

$$\frac{df}{du} = e^{-bu^3}(a - 3bu^2 - 3bau^3)$$

What is a partial differential equation?

- **Partial differential equation** (PDE) is a differential equation that contains unknown multivariable functions and their partial derivatives
- PDEs are used to describe a wide variety of phenomena: sound, heat, electrodynamics, quantum mechanics...
- Examples:
  1. transport: $u_x + u_y = 0$
  2. shock wave: $u_x + uu_y = 0$
  3. vibrating bar: $u_{tt} + u_{xxxx} = 0$

Partial differential equation

- The most general second-order PDE in two independent variables is:

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0$$

- A **solution** of a PDE is a function $y(x, y, \ldots)$ that satisfies the equation identically, at least in some region of the $x, y, \ldots$ variables
- Most of the important PDEs of physics are second-order and linear
  - For linear homogeneous PDEs, the superposition principle applies

Partial differential equation - Example

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$$u_x(x, y) = f(y),$$

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  where $f(y)$ is arbitrary

- We integrate again to get $u(x, y) = f(y)x + g(y)$. This is the general solution.
- Note that there are *two arbitrary functions* in the solution

Partial differential equation - Example 2

Exercise: Solve the PDE $u_{xx} + u = 0$

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- **Moral:** A PDE has arbitrary functions in its solution

Important PDEs

- Transport equation: $u_t + cu_x = 0$
- Wave equation: $u_{tt} = c^2 u_{xx}$
- Diffusion equation: $u_t = ku_{xx}$
- Laplace equation: $u_{xx} + u_{yy} = 0$.
  It is often written as: $\nabla^2 u = 0$ or $\Delta u = 0$, where $\Delta = \nabla^2$ is the Laplace operator

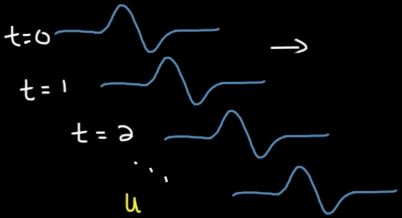
Numerical methods to solve PDEs

- The three most widely used numerical methods:
  1. Finite difference method
     - The simplest method to learn and use
     - Functions are represented by their values at certain grid points, and derivatives are approximated through differences in these values
  2. Finite element method
  3. Finite volume method
- Other methods exist: method of lines, spectral methods, multigrid methods, ...

Transport equation: intuition and derivation

Transport equation $u_t + c u_x = 0$ $\longrightarrow$ wave moving with speed c

c: constant



1. directional derivative of u in direction of $x - ct = x_0$ is zero.

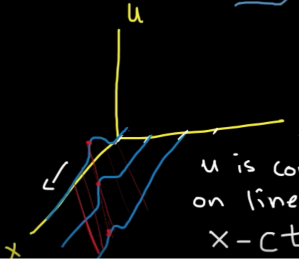
$$(c, 1)\cdot(u_x, u_t) = 0$$

$$\Rightarrow \quad c u_x + u_t = 0.$$

2. $$\frac{d}{dt} u(ct + x_0, t) = 0.$$

$$\Rightarrow \quad u_x \cdot \frac{d(ct + x_0)}{dt} + u_t \cdot \frac{d(t)}{dt}$$

$$= u_x c + u_t = 0.$$

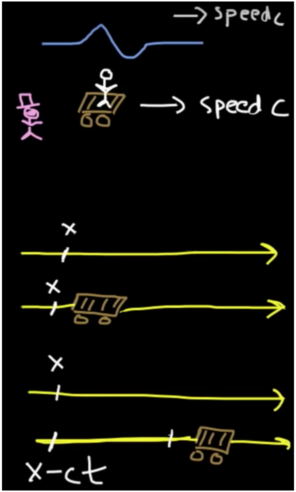


characteristic lines

u is constant on lines $x - ct = \text{const}.$

Transport equation: general solution

- Solve the transport equation $u_t + cu_x = 0$



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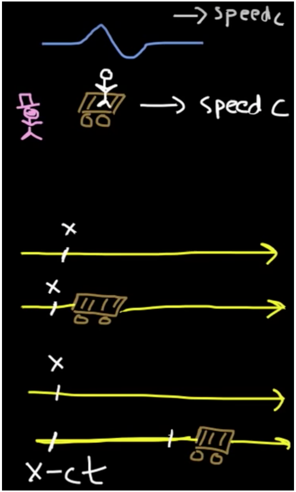
- You see a wave with speed c: $u_t + cu_x = 0$



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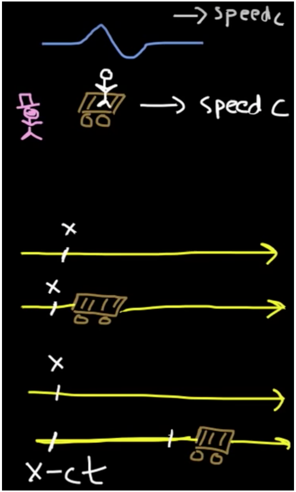

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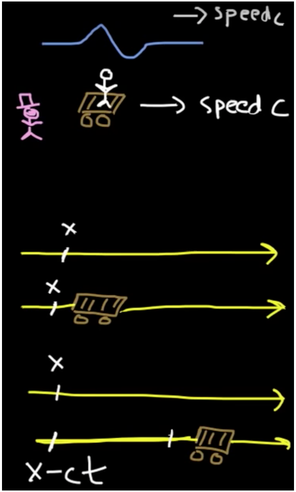
- "moving" coordinate $\xi = x - ct$



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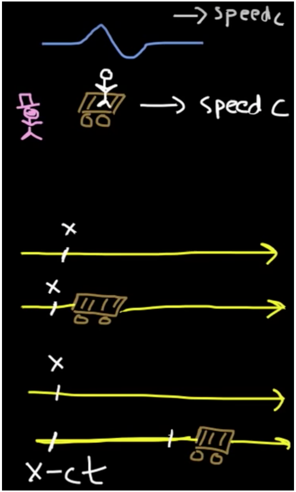
$$u(x,t) = v(\xi, t)$$



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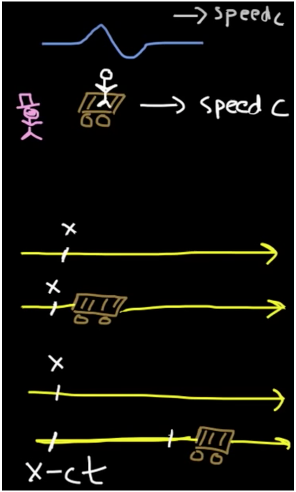
- Derivatives using the chain rule:

$$u_t = \frac{\partial v}{\partial \xi} \cdot \frac{\partial \xi}{\partial t} + \frac{\partial v}{\partial t} \cdot \frac{\partial t}{\partial t} = -c\frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial t}$$



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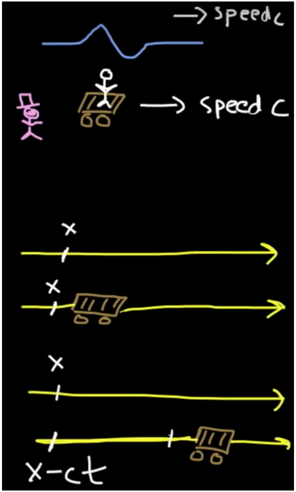

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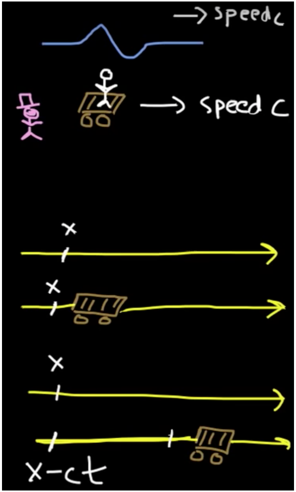
- $u_t + cu_x = -cv_\xi + v_t + cv_\xi = v_t = 0$



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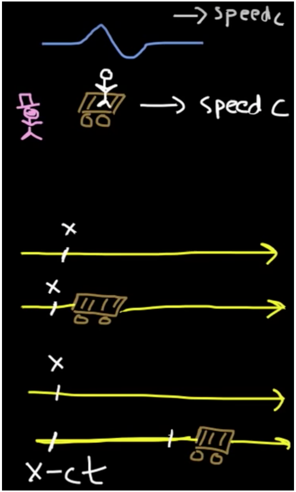

Transport equation: general solution

- **Usual coordinates** (x, t): $u(x, t)$ $\quad u_t + cu_x = 0$
- **Moving coordinates** $(\xi, t) : v(\xi, t)$ $\quad v_t = 0$ $\qquad \xi = x - ct$

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- $v_t = 0 \quad \Rightarrow \quad v = F(\xi)$

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- $v(\xi, t) = u(x, t) \quad \Rightarrow$

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- $v(\xi, t) = u(x, t) \quad \Rightarrow \quad u(x, t) = F(\xi) = F(x - ct)$
- **Conclusion:** General solution of $u_t + cu_x = 0$ is $F(x - ct)$, where F is a "once differentiable" function

Initial and boundary conditions

- Because PDEs typically have so many solutions, we single out one solution by imposing auxiliary conditions
- These conditions are motivated by the considered problem (ex.: physics)
- They come in two varieties: initial conditions and boundary conditions

Initial and boundary conditions

- An **initial condition** specifies the physical state at a particular time t_0.
- We can have one or several initial conditions:

$$u(\mathbf{x}, t_0) = \phi(\mathbf{x}) = \phi(x, y, z)$$

$$u(\mathbf{x}, t_0) = \phi(\mathbf{x}) \qquad \text{and} \qquad \frac{\partial u}{\partial t}(\mathbf{x}, t_0) = \psi(\mathbf{x}),$$

Examples: $\phi(\mathbf{x})$ is the initial position, or the initial temperature, ...

Initial and boundary conditions

- In each physical domain there is a **domain** D in which the PDE is valid.
  - *Example*: For the vibrating string, D is the interval $0 < x < l$, where l is the length of the string. The boundary of D consists only of the two points $x = 0$ and $x = l$
- To determine the solution, it is necessary to specify some **boundary conditions**
- Three most important kinds of boundary conditions:
  1. Dirichlet condition: u is specified
  2. Neumann condition: the normal derivative $\partial u/\partial n$ is specified
     - *Example*: $\partial u/\partial n = g(\mathbf{x}, t)$
  3. Robin condition: $\partial u/\partial n + au$ is specified, $a = a(x, y, z, t)$

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- In one-dimensional problems the boundary consists of two endpoints $\Rightarrow$ boundary conditions take the simple form

(D) $u(0, t) = g(t)$ and $u(l, t) = h(t)$

(N) $\frac{\partial u}{\partial x}(0, t) = g(t)$ and $\frac{\partial u}{\partial x}(l, t) = h(t)$

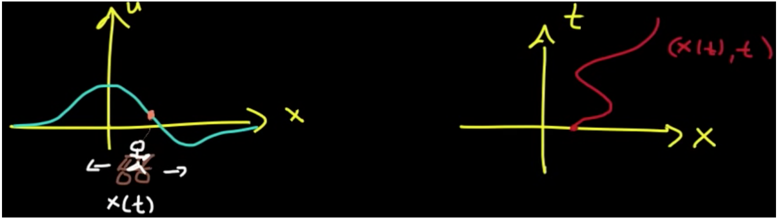
Well-posed problems

- Well-posed problems consist of:
  - a PDE in a domain
  - a set of initial and/or boundary conditions
  - or other auxiliary conditions

  that enjoy the following fundamental properties

  1. *Existence:* There exists at least one solution $u(x, t)$ satisfying all conditions
  2. *Uniqueness:* There is at most one solution
  3. *Stability:* If the data are changed a little, the solution changes only a little

Transport equation: method of characteristics



- **Technique for solving PDE by reducing to ODE**

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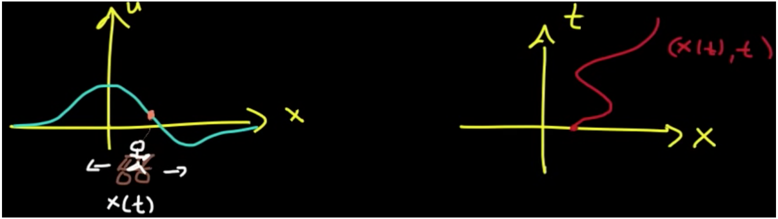

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- $x(t)$ = position of the moving observer

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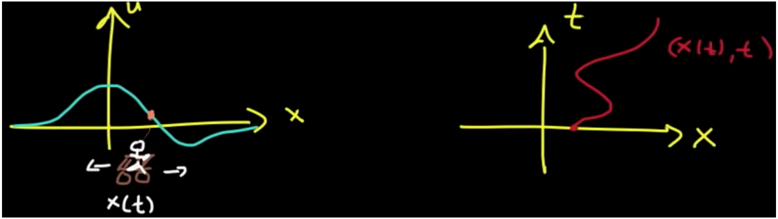

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- How does $u(x, t)$ change from observer's perspective?

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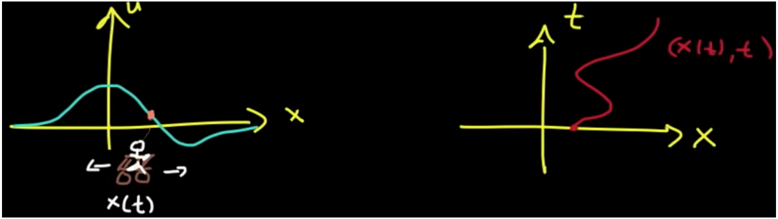

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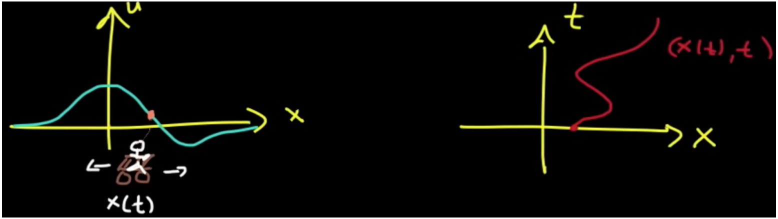
- How does $u(x,t)$ change from observer's perspective?

$$\frac{d}{dt}u(x(t),t) = \frac{du}{dt} = u_x\frac{dx}{dt} + u_t$$

Transport equation: method of characteristics



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- $x(t)$ = position of the moving observer
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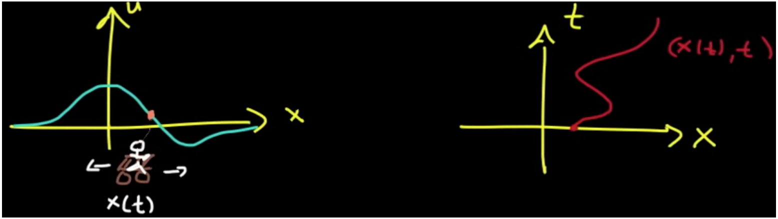

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$$\frac{d}{dt}u(x(t),t) = \frac{du}{dt} = u_x\frac{dx}{dt} + u_t \quad \Rightarrow \quad \begin{cases} dx/dt = c & \text{- moving with speed } c \\ du/dt = 0 & \text{- u not changing} \end{cases}$$

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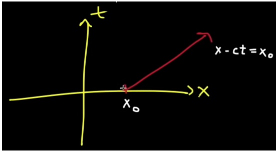
Transport equation: method of characteristics

$$u_t + cu_x = 0$$

$$\updownarrow$$

$$\frac{du}{dt} = 0 \text{ along curves given by } \frac{dx}{dt} = c$$

$$\frac{dx}{dt} = c \quad \Rightarrow \quad x = ct + x_0$$



- Need initial condition $u(x, 0) = f(x)$
- Along $x - ct = x_0$, $u(x, t) = f(x_0) = f(x - ct)$

Solution: $u(x, t) = f(x - ct)$

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Solve the initial value problem (PDE + initial condition):

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- **Solution:** $u(x,t) = t + \sin(x - ct)$

Solving PDE using Laplace transforms

- Laplace transforms are well suited to solve linear PDE with constant coefficients
- If we have a function of two variables $w(x,t)$:

$$\mathcal{L}\{w(x,t)\} = W(x,s) = \int_0^\infty e^{-st} w(x,t) dt$$

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- Transform of partial derivatives:

$$\mathcal{L}\{w_t(x,t)\} = s\mathcal{L}\{w(x,t)\} - w(x,0) = sW(x,s) - w(x,0)$$
$$\mathcal{L}\{w_{tt}(x,t)\} = s^2\mathcal{L}\{w(x,t)\} - sw(x,0) - w_t(x,0)$$
$$\mathcal{L}\{w_x(x,t)\} = \frac{\partial}{\partial x}\mathcal{L}\{w(x,t)\}$$
$$\mathcal{L}\{w_{xx}(x,t)\} = \frac{\partial^2}{\partial x^2}\mathcal{L}\{w(x,t)\}$$

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Exercise: Solve, via Laplace transforms:

$$w_x + 2xw_t = 0, \quad w(x,0) = 0, \quad w(0,t) = t, \quad t \geq 0$$

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- Second shifting theorem: $\mathcal{L}^{-1}\{e^{-cs}F(s)\} = u_c(t)f(t-c)$, where the Heaviside (unit step) function:

$$u_c(t) = \begin{cases} 0, & t < c; \\ 1, & t \geq c \end{cases}$$

$$F(s) = \frac{1}{s^2} \quad \Rightarrow \quad f(t) = t$$

$$\boxed{w(x,t)} = \mathcal{L}^{-1}\left\{\frac{1}{s^2} e^{-sx^2}\right\} = \boxed{u_{x^2}(t)[t - x^2]}$$

Partial differential equations

Other analytical methods exist:

- Separation of variables
- Change of variables
- Fundamental solution

You can read more here:

- W. A. Strauss, Partial Differential Equations: An Introduction, John Wiley & Sons Ltd, 2008