Mathematical Methods - Lecture 6

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Outline

What is integral?

- What is $\int$?

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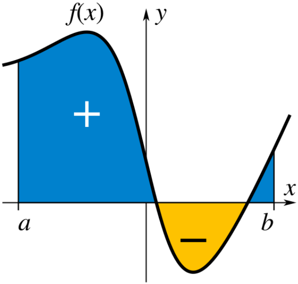
- **Integral** is a function of which a given function is the derivative

What is integral?

- Given a function f of a real variable x and an interval $[a, b]$, the **definite integral**

$$\int_a^b f(x)dx$$

is defined as the signed area of the region in the xy-plane that is bounded by the graph of f, the x-axis and the vertical lines $x = a$ and $x = b$



What is Fourier Series?

- A **Fourier series** is a series of periodic functions (typically sinusoids) that represents another periodic function
- A **Fourier series** is an expansion of a periodic function $f(x)$ in terms of infinite sum of sines and cosines:

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L} \right)$$

where a_0, a_m, b_m are constants called *Fourier coefficients*

Periodicity

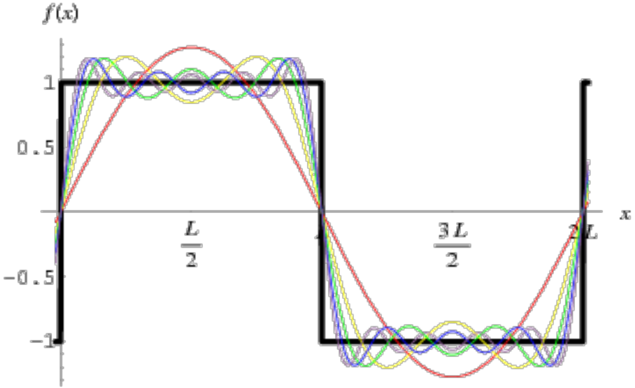
- A function $f(x)$ is said to be **periodic** when

$$f(x+T) = f(x)$$

  for some value T

- The smallest value T can be is called the **fundamental period**

What is Fourier Series?



Computing Fourier coefficients (Euler-Fourier formulas)

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L} \right)$$

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x) dx$$

$$a_m = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{m\pi x}{L} dx$$

$$b_m = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{m\pi x}{L} dx$$

Fourier convergence theorem

- If $f(x)$ is a periodic function that is piece-wise continuous over a period $2L$ and
- $f'(x)$ is piece-wise continuous over a period $2L$,
- then there exists a Fourier series that converges to f for all continuous points and $[f(x+)+f(x-)]/2$ for all discontinuous points

Integration by parts

$$\int u(x)v'(x)dx = u(x)v(x) - \int v(x)u'(x)dx$$

Example

Find the Fourier series of $f(x) = \begin{cases} x, & -\pi \leq x < 0, \\ 0, & 0 \leq x < \pi; \end{cases} \qquad f(x+2\pi) = f(x)$

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$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{0} xdx = -\frac{\pi}{2}$$

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$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos\frac{m\pi x}{\pi}dx = \frac{1}{\pi}\int_{-\pi}^{0} x\cos mxdx =$$

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$$\frac{1}{\pi}\left[\frac{x\sin mx}{m} + \frac{\cos mx}{m^2}\right]_{-\pi}^{0} = \frac{1-(-1)^m}{\pi m^2}$$

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Find the Fourier series of $f(x) = \begin{cases} x, & -\pi \leq x < 0, \\ 0, & 0 \leq x < \pi; \end{cases}$ $\qquad f(x + 2\pi) = f(x)$

- $f(x)$ is periodic with period $2\pi \Rightarrow L = \pi$

$$a_0 = -\frac{\pi}{2}, \quad a_m = \frac{1 - (-1)^m}{\pi m^2}$$

$$b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin \frac{m\pi x}{\pi} dx = \frac{1}{\pi} \int_{-\pi}^{0} x \sin mx dx =$$

$$\frac{1}{\pi} \left[\frac{-x \cos mx}{m} + \frac{\sin mx}{m^2} \right]_{-\pi}^{0} = -\frac{(-1)^m}{m}$$

Example

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- $f(x)$ is periodic with period $2\pi \Rightarrow L = \pi$

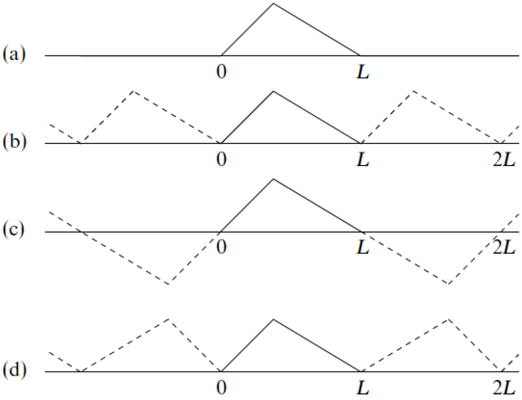
$$a_0 = -\frac{\pi}{2}, \quad a_m = \frac{1-(-1)^m}{\pi m^2}, \quad b_m = -\frac{(-1)^m}{m}$$

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L} \right) =$$

$$-\frac{\pi}{4} + \sum_{m=1}^{\infty} \left(\frac{1-(-1)^m}{\pi m^2} \cos mx - \frac{(-1)^m}{m} \sin mx \right)$$

Non-periodic functions

- To find the Fourier series of a non-periodic function only within a fixed range:
- We may continue the function outside the range so as to make it periodic



Complex Fourier series

- Fourier series expansion may be written more compactly using a complex exponential expansion
  - $\exp(irx) = \cos rx + i \sin rx$

- The complex Fourier expansion is written:

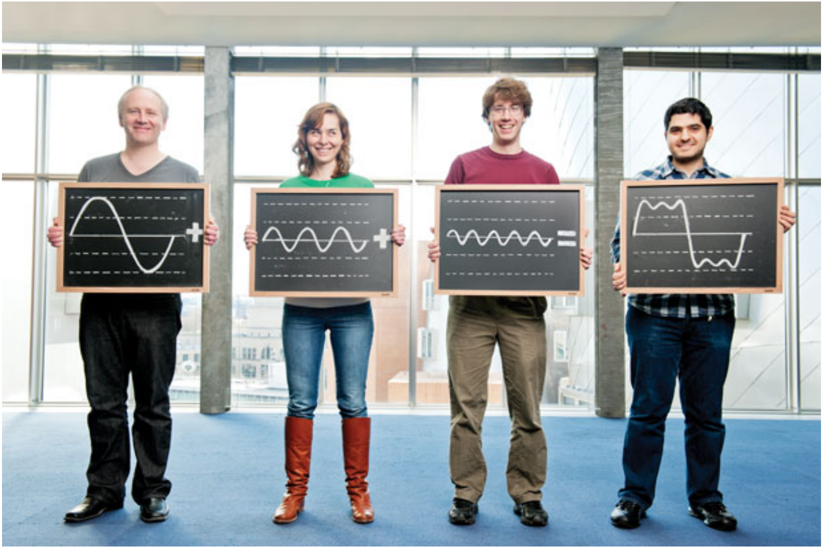
$$f(x) = \sum_{r=-\infty}^{\infty} c_r \exp\left(\frac{2\pi irx}{L}\right)$$

where the Fourier coefficients are given by

$$c_r = \frac{1}{L} \int_{-L/2}^{L/2} f(x) \exp\left(-\frac{2\pi irx}{L}\right) dx$$

- $c_r = \frac{1}{2}(a_r - ib_r), \quad c_{-r} = \frac{1}{2}(a_r + ib_r)$

Fourier transforms



Fourier transform

- **Recall:** A function of period T may be represented as

$$f(t) = \sum_{r=-\infty}^{\infty} c_r e^{2\pi irt/T} = \sum_{r=-\infty}^{\infty} c_r e^{i\omega_r t},$$

where $\omega_r = 2\pi r/T$

- As the period T tends to $\infty \Rightarrow$
  - 'frequency quantum' $\Delta\omega = 2\pi/T$ becomes vanishingly small
  - the spectrum of allowed frequencies w_r becomes a continuum
- The infinite sum of terms in the Fourier series becomes an integral
- The coefficients c_r become functions of the continuous variable ω

Fourier transform

- We define the **Fourier transform** of $f(t)$ by:

$$F(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt,$$

and its inverse by

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega$$

- Its utility is greater than the Fourier series in most practical problems
- **Important property:** A function expressed by Fourier transform can be reconstructed completely via an inverse process

Fourier transform - Example

Find the Fourier transform of the exponential decay function $f(t) = 0$ for $t < 0$ and $f(t) = Ae^{-\lambda t}$ for $t \geq 0$ $(\lambda > 0)$

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- Using the definition:

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} (0) e^{-i\omega t} dt + \frac{A}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\lambda t} e^{-i\omega t} dt =$$

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$$0 + \frac{A}{\sqrt{2\pi}} \left[-\frac{e^{-(\lambda + i\omega)t}}{\lambda + i\omega} \right]_0^\infty =$$

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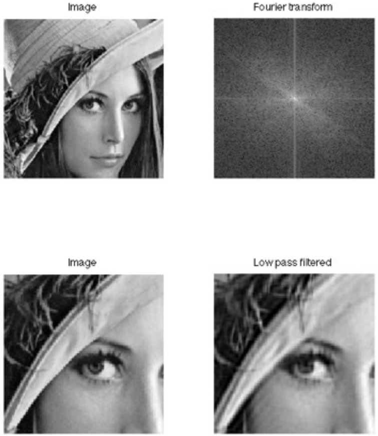
$$\frac{A}{\sqrt{2\pi}(\lambda + i\omega)}$$

Fourier transform - Example



- Most of the FT is concentrated in the center (low frequencies)
- Deleting the FT away from the center saves a lot of data (low-pass filter)

Properties of the Fourier transform

- Linear operations in one domain have corresponding operations in the other domain
  - Sometimes easier to perform
- Operation of differentiation in the time domain corresponds to multiplication by the frequency
- Convolution in the time domain corresponds to ordinary multiplication in the frequency domain

Properties of the Fourier transform (FT)

We denote the FT of $f(t)$ by $F[f(t)]$ or $\tilde{f}(\omega)$

- Linearity:

$$F[ax(t) + by(t)] = aF[x(t)] + bF[y(t)]$$

- Differentiation:

$$F[f'(t)] = i\omega\tilde{f}(\omega)$$

$$F[f''(t)] = i\omega F[f'(t)] = -\omega^2\tilde{f}(\omega)$$

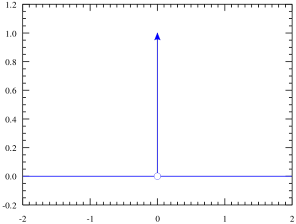
- Integration:

$$F\left[\int^t f(s)ds\right] = \frac{1}{i\omega}\tilde{f}(\omega) + 2\pi c\delta(\omega)$$

where $\delta(\cdot)$ is the Dirac δ-function

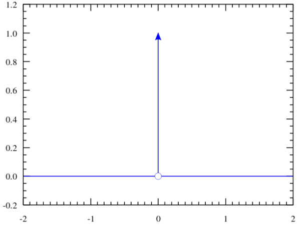
Dirac δ-function

- $\delta(t) = 0$ for $t \neq 0$



Dirac δ-function

- $\delta(t) = 0$ for $t \neq 0$
- $\int f(t)\delta(t-a)dt = f(a)$
  provided the range of integration includes $t = a$



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- $\int_{-a}^{b} \delta(t)dt = 1$ for all $a, b > 0$
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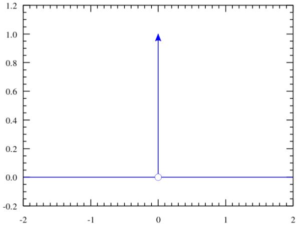

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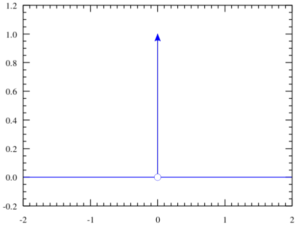
- $\int \delta(t-a)dt = 1$
  provided the range of integration includes $t = a$
- $\tilde{\delta}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(t) e^{-i\omega t} dt = \frac{1}{\sqrt{2\pi}}$



Properties of the Fourier transform (FT)

- Scaling:

$$F[f(at)] = \frac{1}{a}\tilde{f}\left(\frac{\omega}{a}\right)$$

- Translation (time shift):

$$F[f(t+a)] = e^{ia\omega}\tilde{f}(\omega)$$

- Exponential multiplication:

$$F[e^{\alpha t}f(t)] = \tilde{f}(\omega + i\alpha)$$

where α may be real, imaginary or complex

Convolution theorem

- A **convolution** is defined as the integral of the product of the two functions f and g after one is reversed and shifted:

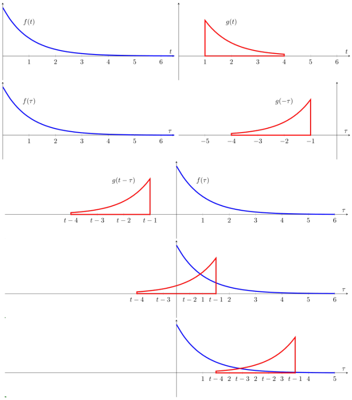
$$f * g = h(t) = \int_{-\infty}^{\infty} f(x) g(t-x) dx$$

- **Convolution theorem:**

$$F[f * g] = k \cdot F[f] \cdot F[g]$$

where k is a constant that depends on the specific normalization of the Fourier transform

  - According to our definition, $k = \sqrt{2\pi}$



Applying convolution theorem - Deconvolution

- **Deconvolution** allows us to find a true distribution $f(x)$ given an observed distribution $h(z)$ and a resolution function $g(y)$

$$\tilde{h}(k) = \sqrt{2\pi}\tilde{f}(k)\tilde{g}(k)$$

$$\Downarrow$$

$$f(x) = \frac{1}{2\pi}F^{-1}\left[\frac{\tilde{h}(k)}{\tilde{g}(k)}\right]$$

  - We divide the FT of the observed distribution by that of the resolution function for each value of k
  - Then take the inverse FT of the function so generated

Laplace transform

- We are often interested in functions $f(t)$ for which the FT does not exist
  - because $f \not\to 0$ as $t \to \infty$
- We might be interested in a given function only for $t > 0$
- This leads us to consider the **Laplace transform**, $\bar{f}(s)$ or $\mathcal{L}[f(t)]$ of $f(t)$, defined by

$$\bar{f}(s) = \int_0^\infty f(t)e^{-st}dt$$

provided that the integral exists, where $s = \sigma + i\omega$

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provided that the integral exists, where $s = \sigma + i\omega$

- We assume here that s is real
    - complex values would have to be considered in a more detailed study
- For a given function $f(t)$, there will be some real number s_0 such that
    - the integral above exists for $s > s_0$,
    - but diverges for $s \leq s_0$

Laplace transform

- The **Laplace transform**, $\bar{f}(s)$ or $\mathcal{L}[f(t)]$ of $f(t)$:

$$\bar{f}(s) = \int_0^\infty f(t)e^{-st}dt \tag{1}$$

- Through (1) we define a linear transformation
  - that converts functions of the variable t to functions of a variable s

$$\mathcal{L}[af_1(t) + bf_2(t)] = a\mathcal{L}[f_1(t)] + b\mathcal{L}[f_2(t)] = a\bar{f}_1(s) + b\bar{f}_2(s)$$

Laplace transform (LT) - Examples

Find LT of $f(t) = 1$

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$$\mathcal{L}[1] = \int_0^\infty e^{-st} dt =$$

$$\left[\frac{-1}{s} e^{-st}\right]_0^\infty = \frac{1}{s}, \quad \text{if } s > 0$$

where $s > 0$ is required for the integral to exist

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$$\bar{f}(s) = \int_0^\infty e^{at} e^{-st} dt = \int_0^\infty e^{(a-s)t} dt =$$

$$\left[\frac{e^{(a-s)t}}{a-s}\right]_0^\infty = \frac{1}{s-a}, \quad \text{if } \; s > a$$

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- Integrating by parts, we find:

$$\bar{f}_n(s) = \left[\frac{-t^n e^{-st}}{s}\right]_0^{\infty} + \frac{n}{s}\int_0^{\infty} t^{n-1} e^{-st} dt = 0 + \frac{n}{s}\bar{f}_{n-1}(s), \quad \text{if} \;\; s > 0$$

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  - $t^0 = 1 \Rightarrow \bar{f}_0 = \frac{1}{s}, \;\; \text{if} \;\; s > 0$
  - $\bar{f}_1(s) = \frac{1}{s^2}, \;\; \bar{f}_2(s) = \frac{2!}{s^3}, \;\; \ldots, \;\; \bar{f}_n(s) = \frac{n!}{s^{n+1}}, \;\; \text{if} \;\; s > 0$

Inverse Laplace transform

- Unlike for FT, the inversion of the Laplace transform is not easy to perform
  - makes use of complex variable theory
- We can use a ‘dictionary’ of the LTs of common functions

Standard Laplace transforms

$f(t)$	$\bar{f}(s)$	s_0
c	c/s	0
ct^n	$cn!/s^{n+1}$	0
$\sin bt$	$b/(s^2+b^2)$	0
$\cos bt$	$s/(s^2+b^2)$	0
e^{at}	$1/(s-a)$	a
$t^n e^{at}$	$n!/(s-a)^{n+1}$	a
$\sinh at$	$a/(s^2-a^2)$	$\|a\|$
$\cosh at$	$s/(s^2-a^2)$	$\|a\|$
$e^{at}\sin bt$	$b/[(s-a)^2+b^2]$	a
$e^{at}\cos bt$	$(s-a)/[(s-a)^2+b^2]$	a
$t^{1/2}$	$\frac{1}{2}(\pi/s^3)^{1/2}$	0
$t^{-1/2}$	$(\pi/s)^{1/2}$	0
$\delta(t-t_0)$	e^{-st_0}	0
$H(t-t_0)=\begin{cases}1 & \text{for } t\geq t_0\\ 0 & \text{for } t<t_0\end{cases}$	e^{-st_0}/s	0

Inverse Laplace transform

- For all practical purposes the inverse LT is unique and linear:

$$\mathcal{L}^{-1}[a\bar{f}_1(s) + b\bar{f}_2(s)] = af_1(t) + bf_2(t)$$

- The method of partial fractions can be useful
  - $\bar{f}_s = \frac{s+3}{s(s+1)} = \frac{3}{s} - \frac{2}{s+1}$

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- The method of partial fractions can be useful
  - $\bar{f}_s = \frac{s+3}{s(s+1)} = \frac{3}{s} - \frac{2}{s+1}$
  - $f(t) = 3 - 2e^{-t}, \quad \text{if} \;\; s > 0$

Laplace transform of derivatives

- One of the main uses of LT: **solving differential equations**
- The Laplace transform of the first derivative:

$$\mathcal{L}\left[\frac{df}{dt}\right] = \int_0^\infty \frac{df}{dt} e^{-st} dt =$$

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$$[f(t)e^{-st}]_0^\infty + s\int_0^\infty f(t)e^{-st} dt =$$

$$-f(0) + s\bar{f}(s), \quad \text{for} \;\; s > 0$$

Laplace transform of derivatives

- The Laplace transform of the first derivative:

$$\mathcal{L}\left[\frac{df}{dt}\right] = -f(0) + s\bar{f}(s), \quad \text{for} \;\; s > 0$$

- In a similar manner:

$$\mathcal{L}\left[\frac{d^2 f}{dt^2}\right] = s^2\bar{f}(s) - sf(0) - \frac{df}{dt}(0), \quad \text{for} \;\; s > 0$$

$$\mathcal{L}\left[\frac{d^n f}{dt^n}\right] = s^n\bar{f} - s^{n-1}f(0) - s^{n-2}\frac{df}{dt}(0) - \cdots - \frac{d^{n-1}f}{dt^{n-1}}(0), \quad \text{for} \;\; s > 0$$

Laplace transforms of integrals

$$\mathcal{L}\left[\int_0^t f(u)du\right] = \frac{1}{s}\mathcal{L}[f(t)]$$

Other properties of Laplace transforms

1. Multiplying $f(t)$ by e^{at} moves the origin of s by an amount of a

$$\mathcal{L}[e^{at}f(t)] = \bar{f}(s-a)$$

2.

$$\mathcal{L}[f(at)] = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$$

3.

$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n \bar{f}(s)}{ds^n}, \quad \text{for} \;\; n = 1, 2, 3, \ldots,$$

4.

$$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^{\infty} \bar{f}(u)du$$

provided $\lim_{t\to 0}[f(t)/t]$ exists

Convolution theorem for Laplace transforms

$$\mathcal{L}\left[\int_0^t f(u)g(t-u)du\right] = \bar{f}(s)\bar{g}(s)$$