Mathematical Methods - Lecture 3

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INVENTORS FOR THE DIGITAL WORLD

Outline

What is matrix?

- **Definition:** An $r \times k$ **matrix** $M = (m^i_j)$ for $i = 1, \ldots, r; j = 1, \ldots, k$ is a rectangular array of real (or complex) numbers:

$$M = \begin{pmatrix} m^1_1 & m^1_2 & \cdots & m^1_k \\ m^2_1 & m^2_2 & \cdots & m^2_k \\ \vdots & \vdots & & \vdots \\ m^r_1 & m^r_2 & \cdots & m^r_k \end{pmatrix}$$

- The numbers m^i_j are called **entries**

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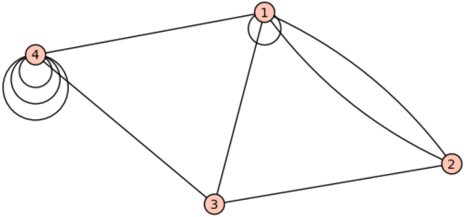
- The numbers m^i_j are called **entries**
- An $r \times 1$ matrix $v = (v^r_1) = (v^r)$ is called a **column vector**
- An $1 \times k$ matrix $v = (v^1_k) = (v_k)$ is called a **row vector**

Matrices are useful in many applications

- Adjacency matrix example



- Adjacency matrix indicates how many edges attach one vertex to another:

$$M = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 3 \end{pmatrix}$$

Properties of matrices

- Space of $r \times k$ matrices M_k^r is a vector space with the addition and scalar multiplication:

$$M + N = (m_j^i) + (n_j^i) = (m_j^i + n_j^i)$$

$$rM = r(m_j^i) = (rm_j^i)$$

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$$rM = r(m^i_j) = (rm^i_j)$$

- Multiplying an $r \times k$ matrix M by a $k \times s$ matrix N: $MN = L$:

$$L = (l^i_j),$$

$$l^i_j = \sum_{p=1}^{k} m^i_p n^p_j$$

Matrix terminology

- The entries m_i^i are called **diagonal**
- The set $\{m_1^1, m_2^2, \ldots\}$ is called **the diagonal of the matrix**
- Any $r \times r$ matrix is called a **square matrix**
- A square matrix that is zero for all non-diagonal entries is called a **diagonal matrix**

Matrix terminology

- The $r \times r$ diagonal matrix with all diagonal entries equal to 1 is called the **identity matrix**:

$$I_r = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

- $I_r M = M I_k = M$ for all M of size $r \times k$

Matrix terminology

- The **transpose** of an $r \times k$ matrix $M = (m^i_j)$ is the $k \times r$ matrix with entries

$$M^T = (\bar{m}^i_j)$$

with $\bar{m}^i_j = m^j_i$

- A matrix M is **symmetric** if $M = M^T$

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- Example:
$$\begin{pmatrix} 2 & 5 & 6 \\ 1 & 3 & 4 \end{pmatrix}^T = \begin{pmatrix} 2 & 1 \\ 5 & 3 \\ 6 & 4 \end{pmatrix}$$
- **Theorem:** Let M, N be matrices such that MN makes sense. Then $(MN)^T = N^T M^T$

Block matrices

- It is often convenient to partition a matrix M into smaller matrices called **blocks**:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 1 \\ \hline 0 & 1 & 2 & 0 \end{array}\right) = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

- Matrix operations on block matrices can be carried out by treating the blocks as matrix entries

Block matrices - Example

$$M = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 1 \\ \hline 0 & 1 & 2 & 0 \end{array}\right) = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

- $M^2 = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \left(\begin{array}{c|c} A^2 + BC & AB + BD \\ \hline CA + DC & CB + D^2 \end{array}\right)$
- $A^2 + BC = \begin{pmatrix} 30 & 37 & 44 \\ 66 & 81 & 96 \\ 102 & 127 & 152 \end{pmatrix}$, $AB + BD = \begin{pmatrix} 4 \\ 10 \\ 16 \end{pmatrix}$
- $CA + DC = \begin{pmatrix} 18 \\ 21 \\ 24 \end{pmatrix}^T$, $CB + D^2 = (2)$

Block matrices - Example

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- $M^2 = \left(\begin{array}{ccc|c} 30 & 37 & 44 & 4 \\ 66 & 81 & 96 & 10 \\ 102 & 127 & 152 & 16 \\ \hline 18 & 21 & 24 & 2 \end{array}\right)$

Matrix multiplication does not commute

- For generic $n \times n$ square matrices M and n, then $MN \neq NM$
- Example:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

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- Conclusion: For two linear transformations K and L taking $\mathbb{R}^n \to \mathbb{R}^n$ and $v \in \mathbb{R}^n$, then in general

$$K(L(v)) \neq L(K(v))$$

Trace

- **Definition:** The *trace* of a square matrix $M = (M^i_j)$ is the sum of its diagonal entries:

$$\mathrm{tr} M = \sum_{i=1}^{n} m^i_i$$

- Example:

$$\mathrm{tr} \begin{pmatrix} 2 & 7 & 6 \\ 9 & 5 & 1 \\ 4 & 3 & 8 \end{pmatrix} = 2 + 5 + 8 = 15$$

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- **Theorem:** $\mathrm{tr}(MN) = \mathrm{tr}(NM)$ for any square matrices M and N
- Another useful property: $\mathrm{tr} M = \mathrm{tr} M^T$

Linear systems redux

- We can view a linear system as a matrix equation

$$MX = V$$

  with M an $r \times k$ matrix of coefficients, X a $k \times 1$ matrix of unknowns, and V an $r \times 1$ matrix of const

- If $r = k$, we have hope to find a single solution
- An extremely useful function would be $f(M) = \frac{1}{M}$, where $M\frac{1}{M} = I$
- Then: $X = \frac{1}{M}V$
- If the system has one solution, $\frac{1}{M} = M^{-1}$ exists and is called the *inverse* of M

Inverse matrix

- **Definition:** A square matrix M is *invertible* (or *nonsingular*) if there exists M^{-1} such that

$$M^{-1}M = I = MM^{-1}$$

Inverse of a 2×2 matrix

- Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $N = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$
- Multiplying these matrices gives:

Inverse of a 2×2 matrix

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$$MN = \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix} = (ad - bc)I$$

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- Multiplying these matrices gives:

$$MN = \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix} = (ad - bc)I$$

- Then: $M^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, so long as $ad - bc \neq 0$

Three properties of the inverse

1. $(A^{-1})^{-1} = A$

2. $(AB)^{-1} = B^{-1}A^{-1}$

3. $(A^T)^{-1} = (A^{-1})^T$

Finding inverses

- Suppose $MX = V$ is a linear system with unique solution X_0. Then:

$$(M|V) \sim (I|M^{-1}V)$$

- To compute M^{-1}:

$$(M|I) \sim (I|M^{-1}I) = (I|M^{-1})$$

Finding inverses - Example

- Find $\begin{pmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{pmatrix}^{-1}$

Finding inverses - Example

- Find $\begin{pmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{pmatrix}^{-1}$
- $\left(\begin{array}{ccc|ccc} -1 & 2 & -3 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 4 & -2 & 5 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & -2 & 3 & -1 & 0 & 0 \\ 0 & 5 & -6 & 2 & 1 & 0 \\ 0 & 6 & -7 & 4 & 0 & 1 \end{array}\right) \sim$

Finding inverses - Example

- Find $\begin{pmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{pmatrix}^{-1}$

- $\left(\begin{array}{ccc|ccc} -1 & 2 & -3 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 4 & -2 & 5 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & -2 & 3 & -1 & 0 & 0 \\ 0 & 5 & -6 & 2 & 1 & 0 \\ 0 & 6 & -7 & 4 & 0 & 1 \end{array}\right) \sim$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 3/5 & -1/5 & 2/5 & 0 \\ 0 & 1 & -6/5 & 2/5 & 1/5 & 0 \\ 0 & 0 & 1/5 & 8/5 & -6/5 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & 4 & -3 \\ 0 & 1 & 0 & 10 & -7 & 6 \\ 0 & 0 & 1 & 8 & -6 & 5 \end{array}\right)$$

Finding inverses - Example

- To check our answer, we apply:

$$MM^{-1} = \begin{pmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{pmatrix} \begin{pmatrix} -5 & 4 & -3 \\ 10 & -7 & 6 \\ 8 & -6 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Linear systems and inverses

- If M^{-1} exists and is known, we can immediately solve linear systems associated to M:

$$MX = V \Rightarrow X = M^{-1}V$$

- **Example:** Solve the linear system

$$\begin{aligned} -x + 2y - 3z &= 1 \\ 2x + y &= 2 \\ 4x - 2y + 5z &= 0 \end{aligned}$$

Linear systems and inverses

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- The associated matrix equation is

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Linear systems and inverses

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- $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5 & 4 & -3 \\ 10 & -7 & 6 \\ 8 & -6 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ -4 \end{pmatrix}$

Homogeneous systems

- **Theorem:** A square matrix M is invertible if and only if the homogeneous system

$$MX = 0$$

has no non-zero solutions

- Proof: Suppose that M^{-1} exists. Then $MX = 0 \Rightarrow X = M^{-1}0 = 0$

- On the other hand, $MX = 0$ always has the solution $X = 0$

LU decomposition

- For computational reasons, a useful trick is to write a square matrix as the product of two simpler matrices:

$$M = LU$$

  - L is *lower triangular*: $L = (l^i_j), l^i_j = 0$ for all $j > i$

$$L = \begin{pmatrix} l^1_1 & 0 & 0 & \cdots \\ l^2_1 & l^2_2 & 0 & \cdots \\ l^3_1 & l^3_2 & l^3_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

  - U is *upper triangular*: $U = (u^i_j), u^i_j = 0$ for all $j < i$

$$U = \begin{pmatrix} u^1_1 & u^1_2 & u^1_3 & \cdots \\ 0 & u^2_2 & u^2_3 & \cdots \\ 0 & 0 & u^3_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

LU decomposition

- For computational reasons, a useful trick is to write a square matrix as the product of two simpler matrices:

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- $M = LU$ is called an LU decomposition of M
- Useful to compute inverse $\rightarrow$ thus to solve linear systems

LU decomposition - Example

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & d \\ a & 1 & 0 & e \\ b & c & 1 & f \end{array}\right)$$

- $x =$

LU decomposition - Example

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & d \\ a & 1 & 0 & e \\ b & c & 1 & f \end{array}\right)$$

- $x = d$

LU decomposition - Example

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & d \\ a & 1 & 0 & e \\ b & c & 1 & f \end{array}\right)$$

- $x = d$
- $y = e - ad$

LU decomposition - Example

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & d \\ a & 1 & 0 & e \\ b & c & 1 & f \end{array}\right)$$

- $x = d$
- $y = e - ad$
- $z = f - bd - c(e - ad)$

Using LU decomposition to solve linear systems

Suppose $M = LU$ and we want to solve

$$MX = LUX = V$$

Using *LU* decomposition to solve linear systems

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- Step 1: Set $W = \begin{pmatrix} u \\ v \\ w \end{pmatrix} = UX$

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- Step 1: Set $W = \begin{pmatrix} u \\ v \\ w \end{pmatrix} = UX$
- Step 2: Solve the system $LW = V$
  - Simple by forward substitution
  - Suppose the solution to $LW = V$ is W_0

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- Step 2: Solve the system $LW = V$
  - Simple by forward substitution
  - Suppose the solution to $LW = V$ is W_0
- Step 3: Solve the system $UX = W_0$
  - Easy by backward substitution
  - Solution to this system = solution to the original system

Solving linear systems - Example

Solve the system:

$$6x + 18y + 3z = 3$$
$$2x + 12y + z = 19$$
$$4x + 15y + 3z = 0$$

where $\begin{pmatrix} 6 & 18 & 3 \\ 2 & 12 & 1 \\ 4 & 15 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 6 & 0 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 6 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

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- $\begin{pmatrix} 3 & 0 & 0 \\ 1 & 6 & 0 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 3 \\ 19 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 2 & 6 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -11 \end{pmatrix}$

Solving linear systems - Example

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- $z = -11, y = 3, x = -3$

Finding an LU decomposition

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Finding an *LU* decomposition

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- There is a unique decomposition, where L has ones on the diagonal
  - $\Rightarrow$ L is called a *lower unit triangular matrix*

Finding an LU decomposition

- For a given matrix, there are many different LU decompositions
- There is a unique decomposition, where L has ones on the diagonal
  - $\Rightarrow L$ is called a *lower unit triangular matrix*
- To find LU decomposition:
  - We set $L_0 = I$ and $U_0 = M \Rightarrow L_0 U_0 = M$
  - We'll find sequences $L_1 U_1, L_2 U_2, \ldots$, such that:
    - each L_i is lower triangular
    - only last U_i is upper triangular

Finding an *LU* decomposition - Example

$$L_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad U_0 = \begin{pmatrix} 6 & 18 & 3 \\ 2 & 12 & 1 \\ 4 & 15 & 3 \end{pmatrix}$$

Finding an LU decomposition - Example

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- Zeroing the 1st column of U_0

$$U_1 = \begin{pmatrix} 6 & 18 & 3 \\ 0 & 6 & 0 \\ 0 & 3 & 1 \end{pmatrix}$$

Finding an LU decomposition - Example

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Finding an *LU* decomposition - Example

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- Zeroing the 2nd column of U_0

$$L_2 = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{2}{3} & \frac{1}{2} & 1 \end{pmatrix} \qquad U_2 = \begin{pmatrix} 6 & 18 & 3 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Finding an *LU* decomposition

- Multiplying a column of L by a constant λ and dividing the corresponding row of U by the same const does not change the product:

$$LU = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{2}{3} & \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 6 & 18 & 3 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 3 & 0 & 0 \\ 1 & 6 & 0 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 6 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- The resulting matrix looks nicer
  - but it is not in standard form

LU decomposition for non-square matrices

- For a non-square $m \times n$ matrix M, $M = LU$ with:
  - L a square lower unit triangular $m \times m$ matrix
  - U a rectangular $m \times n$ matrix
- Example:

$$\begin{pmatrix} -2 & 1 & 3 \\ -4 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} -2 & 1 & 3 \\ 0 & 2 & -5 \end{pmatrix}$$

Block *LDU* decomposition

- Let

$$M = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$$

with square blocks X, Y, Z, W, such that X^{-1} exists

- M can be decomposed as a block *LDU* decomposition:

$$M = \begin{pmatrix} I & 0 \\ ZX^{-1} & I \end{pmatrix} \begin{pmatrix} X & 0 \\ 0 & W - ZX^{-1}Y \end{pmatrix} \begin{pmatrix} I & X^{-1}Y \\ 0 & I \end{pmatrix}$$

where D is block diagonal