A simple testbed for stability analysis of quantum dissipative systems

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Abstract

We study a two-state quantum system with a nonlinearity intended to describe interactions with a complex environment, arising through a nonlocal coupling term. We study the stability of particular solutions, obtained as constrained extrema of the energy functional of the system. The simplicity of the model allows us to justify a complete stability analysis. This is the opportunity to review in details the techniques to investigate the stability issue. We also bring out the limitations of perturbative approaches based on simpler asymptotic models.

Keywords. Open quantum systems. Particles interacting with a vibrational field. Orbital stability.

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1 Introduction

In this work, we consider a simple quantum system characterized by a single degree of freedom which can take only two values, hereafter referred to as 0 and 1. The quantum system interacts with its environment, the description of which is embodied into a vibrational field, oscillating in some abstract direction $z \in \mathbb{R}^n$. Therefore the evolution of the system is governed by the ODE system

$$\begin{aligned} i\frac{\mathrm{d}}{\mathrm{d}t}u_0(t) &= u_0(t) - u_1(t) + u_0(t)\int_{\mathbb{R}^n} \sigma(z)\psi_0(t,z)\,\mathrm{d}z, \\ i\frac{\mathrm{d}}{\mathrm{d}t}u_1(t) &= u_1(t) - u_0(t) + u_1(t)\int_{\mathbb{R}^n} \sigma(z)\psi_1(t,z)\,\mathrm{d}z, \end{aligned} \tag{1}$$

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coupled to the wave equations

$$\begin{aligned}\left(\frac{1}{c^2}\partial_t^2 - \Delta\right)\psi_0(t,z) &= -\sigma(z)|u_0(t)|^2,\\ \left(\frac{1}{c^2}\partial_t^2 - \Delta\right)\psi_1(t,z) &= -\sigma(z)|u_1(t)|^2.\end{aligned} \tag{2}$$

These equations are completed by initial data

$$(u_0, u_1, \psi_0, \partial_t\psi_0, \psi_1, \partial_t\psi_1)\big|_{t=0} = (u_{0,\mathrm{init}}, u_{1,\mathrm{init}}, \psi_{0,\mathrm{init}}, \varpi_{0,\mathrm{init}}, \psi_{1,\mathrm{init}}, \varpi_{1,\mathrm{init}}). \tag{3}$$

Throughout the paper, we assume the coupling function $z \in \mathbb{R}^n \mapsto \sigma(z)$ to be nonnegative, smooth, with fast enough decay (say compactly supported to fix ideas). The free problem ($\sigma = 0$) reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}u_0\\u_1\end{pmatrix} = \frac{1}{i}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\begin{pmatrix}u_0\\u_1\end{pmatrix}. \tag{4}$$

We infer that the system oscillates with frequency 2 around a constant state: the solutions of (4) read

$$\begin{pmatrix}u_0(t)\\u_1(t)\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}(u_{0,\mathrm{init}} + u_{1,\mathrm{init}})\\ (u_{0,\mathrm{init}} - u_{1,\mathrm{init}})e^{-2it}\end{pmatrix}.$$

Hence, we are wondering how the coupling ($\sigma \neq 0$) impacts this simple dynamics. It is also worth considering the large speed regime $c \to \infty$ which leads to the following nonlinear ODE system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}u_0\\u_1\end{pmatrix} = \frac{1}{i}\begin{pmatrix}1-\kappa|u_0|^2 & -1\\ -1 & 1-\kappa|u_1|^2\end{pmatrix}\begin{pmatrix}u_0\\u_1\end{pmatrix} \tag{5}$$

where

$$\kappa = \int_{\mathbb{R}^n} \sigma(z)(-\Delta)^{-1}\sigma(z)\,\mathrm{d}z = \int_{\mathbb{R}^n} \frac{|\widehat{\sigma}(\xi)|^2}{|\xi|^2}\frac{\mathrm{d}\xi}{(2\pi)^n} > 0. \tag{6}$$

It will be interesting to compare the behavior of the asymptotic model (5) with (1)-(2); in particular we are going to point out the limitations of a perturbative approach that would try to deduce properties of (1)-(2) from the analysis of (5).

According to the ideas of quantum mechanics, the $|u_j|^2$'s represent the probability of being in the state labelled by j; in turn, the total probability should be one: we always have

$$|u_0(t)|^2 + |u_1(t)|^2 = 1. \tag{7}$$

If this property holds initially, we check that it holds forever. Moreover, the equations describe the energy exchanges between the quantum system and the environment which translates into an additional conservation property, namely, we have (detailed computations can be found in Appendix A, but the result also directly follows from the symplectic form of the problem, exhibited below, combined with Noether's theorem)

$$\text{for (1)-(2):}\qquad \frac{\mathrm{d}}{\mathrm{d}t}\Bigg(\frac{|u_0-u_1|^2}{2} + \frac{1}{4}\int_{\mathbb{R}^n}\Big(\frac{1}{c^2}(|\partial_t\psi_0|^2 + |\partial_t\psi_1|^2) + |\nabla\psi_0|^2 + |\nabla\psi_1|^2\Big)\,\mathrm{d}z + \frac{1}{2}\int_{\mathbb{R}^n}\sigma(\psi_0|u_0|^2 + \psi_1|u_1|^2)\,\mathrm{d}z\Bigg) = 0 \tag{8}$$

which becomes

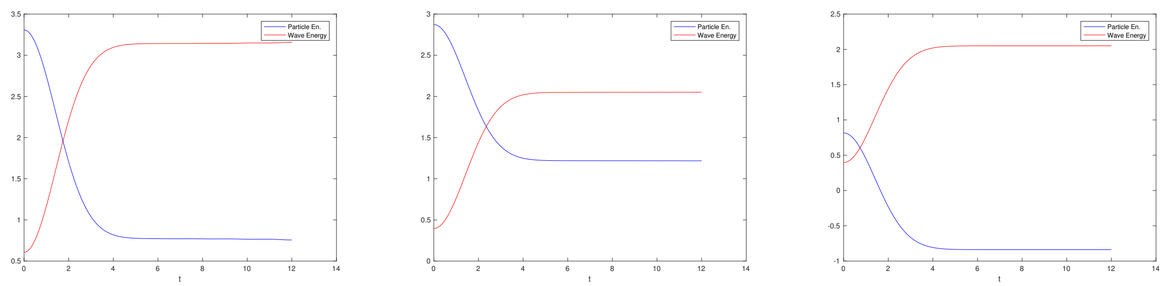
$$\text{for (5):} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{|u_0 - u_1|^2}{2} - \frac{\kappa}{4}(|u_0|^4 + |u_1|^4)\right) = 0 \tag{9}$$

for the asymptotic model (5). These conservation properties play a central role in the analysis of the equations.

The question we address comes from the modeling of *quantum open systems*. The motivation, inspired from the seminal work of Caldeira and Leggett [6], is to understand how the interactions with the environment induce some kind of dissipative effects. The intuition is that the quantum system exchanges energy with the vibrational field, and the energy is eventually evacuated "at infinity" in the z-direction; this mechanism can be interpreted as a sort of friction acting on the quantum system. For the sake of concreteness, the energy transfer mechanisms at work between the quantum system and the environment with the model (1)-(2) are illustrated in Figure 1 which show typical evolutions of the different contributions, wave and particle, to the total energy: albeit these curves are suggestive, in fact, they correspond to very different behaviors of the system, as we shall discuss below. Such an issue has been studied in details for the case of a single classical particle in [5], where the dissipation mechanisms are explicitly exhibited: the particle comes to a state at rest for large times. This situation has been further investigated in [1, 10, 11, 33, 26]; we also refer the reader to [24] or [25] for different, but related, viewpoints on the dynamic of a classical particle coupled to a complex environment. Dealing with many classical particles leads to considering Vlasov-like equations [9, 17], and the Landau damping effect exhibited in [19] can be interpreted as the result of the dissipation effects induced by the coupling, even if such a result remains weaker than the frictional behavior neatly identified in [5] for a single particle. In any case, the dissipation mechanisms are intimately related to the dispersion properties of the wave equation that need to be strong enough, an effect driven by the condition $n \geqslant 3$ on the z-direction, that will be assumed throughout the paper. In particular, it can be noticed that it guarantees the quantity defined by (6) to be finite. We refer the reader to [19] for detailed comments about this assumption. Coming back to quantum particles, one is led to systems coupling the Schrödinger equation with a wave equation: the model

$$\begin{aligned}
&i\partial_t u + \frac{\Delta_x}{2} u = \Phi u,\\
&\Phi(t,x) = \int \sigma_1(x-y)\sigma(z)\psi(t,y,z)\,\mathrm{d}z\,\mathrm{d}y,\\
&\Big(\frac{1}{c^2}\partial_t^2\psi - \Delta_z\psi\Big)(t,x,z) = -\sigma(z)\int \sigma_1(x-y)|u(t,y)|^2\,\mathrm{d}y,
\end{aligned} \tag{10}$$

is the quantum analog of the equation introduced in [5] (other quantum frameworks are discussed for instance in [2, 12, 23]). The equation is analysed when the variable x lies in $\mathbb{R}^d$ in [20] and ground states can be identified by variational approaches. However, the stability analysis of the ground states is delicate because of the nonlocal definition of the self-consistent potential, and the arguments developed for NLS ($\Phi = -|u|^2$ in the first equation of (10)) or Schrödinger-Newton ($\Phi = \frac{1}{|\cdot|} \star |u|^2$ with $d = 3$) do not adapt directly (note at least that here the coupling has a more dynamical nature). The attempt in this direction presented in [20], completed by the numerical investigation in [18], relies on a perturbative approach, inspired from [27]. However, it induces some restrictions which are not completely satisfying. In order to understand this difficulty, we have studied the simpler framework of plane waves (x lies in the torus $\mathbb{T}^d$) in [16], where the Hamiltonian structure is further exploited, in the spirit of the pioneering work [21], see also the recent overview [3]. It allows us to identify fundamental differences between (10) and its asymptotic counterpart as $c \to \infty$; in particular, the coupling with the wave equation induces spectral difficulties which make perturbative arguments inoperative. We wish to explore in further details these issues by considering the simpler systems (1)-(2)and (5).



(a) (b) (c)

Figure 1: Evolution of the "Wave contribution" $\frac{1}{4}\sum_{j=0}^{1}\int_{\mathbb{R}^n}\left(\frac{|\partial_t\psi_j|^2}{c^2}+|\nabla\psi_j|^2\right)\mathrm{d}z$ and the "Particle contribution" $\frac{|u_0-u_1|^2}{2}+\frac{1}{2}\sum_{j=0}^{1}\int_{\mathbb{R}^n}\sigma\psi_j|u_j|^2\,\mathrm{d}z$ to the Total Energy (8) associated to (1)-(2). The simulations correspond to various cases that will be discussed in details below: (a) $\tau=+1$ and large κ, (b) $\tau=-1$, (c): $\tau=+1$ and small κ

In what follows, we pay attention to solutions of (1)-(2) or (5), where the quantum particles distribution has the specific form $e^{i\omega t}(U_{*0},U_{*1})$, with U_{*0},U_{*1}, fixed complex numbers. These solutions can be classified in terms of extrema of the energy. The question we address is about the stability of these specific solutions. At first sight, the problem under consideration can be seen as a discrete version of the nonlinear Schrödinger equation: we roughly interpret u_0-u_1 and u_1-u_0 as the discrete laplacian $(\Delta^d u)_0=\frac{-u_{-1}+2u_0-u_1}{2}$, $(\Delta^d u)_1=\frac{-u_0+2u_1-u_2}{2}$ endowed with periodic conditions $u_{-1}=u_1$, $u_0=u_2$! Stability analysis relies on the properties of the energy functional which can be used as a Lyapounov functional, and establishing coercivity properties is key for proving the orbital stability of the ground state, see [37, 38] and the recent review [36]. A quite general framework has been set up in [21, 22], see also [3, 4], intended to cover the analysis of a wide class of Hamiltonian systems. However, the coupling with a vibrational environment lead to difficulties of a different nature, which are not covered by the abstract framework of [21, 22] since the nonlinearity has the form Φu, where the potential Φ is nonlocal both "in space" (here it means that it mixes the two states 0 and 1) and in time, with some kind of memory effects, so that the arguments of [21, 22] do not apply.

The interest of the two-level model is to be both simple enough to allow us to perform many explicit computations, and rich enough to exhibit interesting phenomena; in turn

- we are able to provide a complete stability analysis for the models (1)-(2) and (5);
- we review in full details the techniques for investigating such systems, and explain how they can be adapted to handle the nonlocal coupling;

- it allows us to clarify where are the main difficulties and it provides valuable hints to study more complex models. We expect this work to provide useful ideas to go back to the more challenging problem (10).

The paper is organized as follows. In Section 2, we discuss the Hamiltonian structure of the problem and make the connection appear between extrema of the energy functional and specific solutions with the form $u(t) = e^{i\omega t}(U_{*0}, U_{*1})$. Section 3 is devoted to the analysis of the asymptotic system (5), which is a mere ODE system. In Section 4, we discuss the system (1)-(2). Throughout the paper, numerical simulations illustrate the obtained statements.

2 Hamiltonian formulation, extrema of the energy and traveling-wave-like solutions

Throughout the paper, we split a complex number $u = q + ip$, where q, p are real valued. Coming back to the unknown describing the quantum state, it makes the following correspondance appear

$$U = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in \mathbb{C}^2 \longleftrightarrow X = \begin{pmatrix} q_0 \\ p_0 \\ q_1 \\ p_1 \end{pmatrix} \in \mathbb{R}^4. \tag{11}$$

2.1 Analysis of the asymptotic model

We start with the simpler system (5). Let us introduce the function

$$\mathscr{H} : (u_0, u_1) \in \mathbb{C}^2 \longmapsto \frac{|u_0 - u_1|^2}{2} - \frac{\kappa}{4}(|u_0|^4 + |u_1|^4).$$

We have observed that $t \mapsto \mathscr{H}(u_0(t), u_1(t))$ is conserved by the differential system (5). This property can be interpreted as a consequence of the following reformulation of the problem, in terms of the real valued quantities defined by (11). The conserved quantity becomes

$$\mathscr{H}(X) = \frac{|q_0 - q_1|^2}{2} + \frac{|p_0 - p_1|^2}{2} - \frac{\kappa}{4}(|q_0|^2 + |p_0|^2)^2 - \frac{\kappa}{4}(|q_1|^2 + |p_1|^2)^2, \tag{12}$$

and (5) can be cast in the *symplectic (i.e. Hamiltonian) form*

$$\frac{\mathrm{d}}{\mathrm{d}t} X = \mathscr{J} \nabla_X \mathscr{H}(X), \tag{13}$$

with $\mathscr{J}$ the skew-symmetric matrix

$$\mathscr{J} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

We are interested in specific solutions of (5) having the special form $(e^{i\omega t}U_{*0}, e^{i\omega t}U_{*1})$ where $U_{*0} = Q_{*0} + iP_{*0}$, and $U_{*1} = Q_{*1} + iP_{*1}$ are fixed complex numbers. We are led to the relation

$$-\omega X_* = \nabla_X \mathscr{H}(X_*) = \begin{pmatrix} Q_{*0} - Q_{*1} - \kappa(Q_{*0}^2 + P_{*0}^2)Q_{*0} \\ P_{*0} - P_{*1} - \kappa(Q_{*0}^2 + P_{*0}^2)P_{*0} \\ Q_{*1} - Q_{*0} - \kappa(Q_{*1}^2 + P_{*1}^2)Q_{*1} \\ P_{*1} - P_{*0} - \kappa(Q_{*1}^2 + P_{*1}^2)P_{*1} \end{pmatrix}, \tag{14}$$

which also arises when searching for the extrema of $\mathscr{H}$ under the constraint of fixed L^2 norm $|U_{*0}|^2 + |U_{*1}|^2 = 1$ with ω being interpreted as the associated Lagrange multiplier. We thus focus on this optimization viewpoint.

We write $u_j = r_j e^{i\theta_j} = q_j + ip_j$, with $r_j \geqslant 0$ and $\theta_j \in [0, 2\pi)$, and we realize that the only term depending on the angles θ_j in the expression of $\mathscr{H}(u_0, u_1)$ is

$$|u_0 - u_1|^2 = r_0^2 + r_1^2 - 2r_0 r_1 \cos(\theta_1 - \theta_0)$$

so that

$$(r_0 - r_1)^2 \leqslant |u_0 - u_1|^2 \leqslant (r_0 + r_1)^2$$

holds. The inequalities are saturated when $\theta_1 = \theta_0 \mod(2\pi)$ (left) or $\theta_1 = \theta_0 \mod(\pi)$ (right). If (u_0, u_1) minimizes $\mathscr{H}$ over the unit sphere of $\mathbb{C}^2$, we deduce from $\mathscr{H}(r_0, r_1) \leqslant \mathscr{H}(u_0, u_1)$ and $r_0^2 + r_1^2 = 1$, that $(r_0, r_1) \in [0,1] \times [0,1]$ is a minimizer too. Furthermore, if (q_0, q_1) minimizes $\mathscr{H}$ over the unit sphere of $\mathbb{R}^2$, then, for any $u_j = r_j e^{i\theta_j}$, with $r_0^2 + r_1^2 = 1$, we get $\mathscr{H}(q_0, q_1) \leqslant \mathscr{H}(r_0, r_1) \leqslant \mathscr{H}(u_0, u_1)$ so that (q_0, q_1) minimizes $\mathscr{H}$ over the unit sphere of $\mathbb{C}^2$. A similar equivalence holds for maximizing $\mathscr{H}$.

Therefore, all extrema can be described by restricting first to the case $p_0 = p_1 = 0$, and then, from the obtained (real valued) optima (q_0, q_1), by setting $u_0 = e^{i\theta_0} q_0$, $u_1 = e^{i\theta_0} q_1$, $\theta_0 \in [0, 2\pi)$. Moreover, we should also bear in mind the conservation of the L^2 norm, so that we are actually interested in extrema over the sphere $\{(q_0, q_1) \in \mathbb{R}^2,\ |q_0|^2 + |q_1|^2 = 1\}$. Accordingly, we can reinterpret the problem as a single-variable optimization problem for

$$\theta \in [0, 2\pi) \longmapsto \mathscr{H}_1(\theta) = \frac{(\cos(\theta) - \sin(\theta))^2}{2} - \frac{\kappa}{4}(\cos^4(\theta) + \sin^4(\theta)).$$

For the reader's convenience, graphs of $\theta \mapsto \mathscr{H}_1(\theta)$ are plotted for several values of κ in Fig. 2. We have

$$\mathscr{H}_1'(\theta) = -(\cos^2(\theta) - \sin^2(\theta)) + \kappa \cos(\theta)\sin(\theta)(\cos^2(\theta) - \sin^2(\theta)) = \frac{\kappa}{2}\cos(2\theta)\left(\sin(2\theta) - \frac{2}{\kappa}\right).$$

It vanishes when $\theta = \frac{\pi}{4}$ which yields the solution $q_0 = 1/\sqrt{2}$, $q_1 = 1/\sqrt{2}$, or $\theta = \frac{3\pi}{4}$, which yields the solution $q_0 = 1/\sqrt{2}$, $q_1 = -1/\sqrt{2}$. If the smallness condition $0 < \kappa < 2$ holds, this completely describes the extrema of the function $\mathscr{H}_1$. When $\kappa > 2$, we can find other solutions by setting $\theta = \frac{\arcsin(2/\kappa)}{2} \in (0, \pi/4)$ and $\theta = \frac{\pi}{2} - \frac{\arcsin(2/\kappa)}{2} \in (\pi/4/, \pi/2)$. We have

$$\mathscr{H}_1''(\theta) = \kappa\left(\cos^2(2\theta) - \sin(2\theta)\left(\sin(2\theta) - \frac{2}{\kappa}\right)\right).$$

Therefore, we distinguish the following cases:

- if $0 < \kappa < 2$, $\theta = \pi/4$ minimizes the energy ($\mathscr{H}_1''(\pi/4) = \frac{\kappa}{2}(\frac{2}{\kappa} - 1) > 0$) and $\theta = 3\pi/4$ maximizes the energy ($\mathscr{H}_1''(3\pi/4) = -\frac{\kappa}{2}(\frac{2}{\kappa} + 1) < 0$): we have $\mathscr{H}_1(\pi/4) = -\frac{\kappa}{8} \leqslant \mathscr{H}_1(\theta) \leqslant \mathscr{H}_1(3\pi/4) = 1 - \frac{\kappa}{8}$;

- if $\kappa > 2$, $\theta_\kappa^+ = \frac{\arcsin(2/\kappa)}{2}$, $\theta_\kappa^- = \frac{\pi}{2} - \frac{\arcsin(2/\kappa)}{2}$ minimize the energy ($\mathscr{H}_1''(\theta_\kappa^\pm) = \kappa\cos^2(2\theta\kappa^\pm) > 0$), $\theta = \pi/4$ is a local maximum of the energy ($\mathscr{H}_1''(\pi/4) = \frac{\kappa}{2}(\frac{2}{\kappa} - 1) < 0$) and $\theta = 3\pi/4$ maximizes the energy ($\mathscr{H}_1''(3\pi/4) = -\frac{\kappa}{2}(\frac{2}{\kappa} + 1) < 0$); we have $\mathscr{H}_1(\theta_\kappa^\pm) = \frac{1}{2}(1 - \kappa/2 - 1/\kappa) \leqslant \mathscr{H}_1(\theta) \leqslant \mathscr{H}_1(3\pi/4) = 1 - \frac{\kappa}{8}$ and $\mathscr{H}_1(\pi/4) = -\frac{\kappa}{8} \in (\mathscr{H}_1(\theta_\kappa^\pm), \mathscr{H}_1(3\pi/4))$.

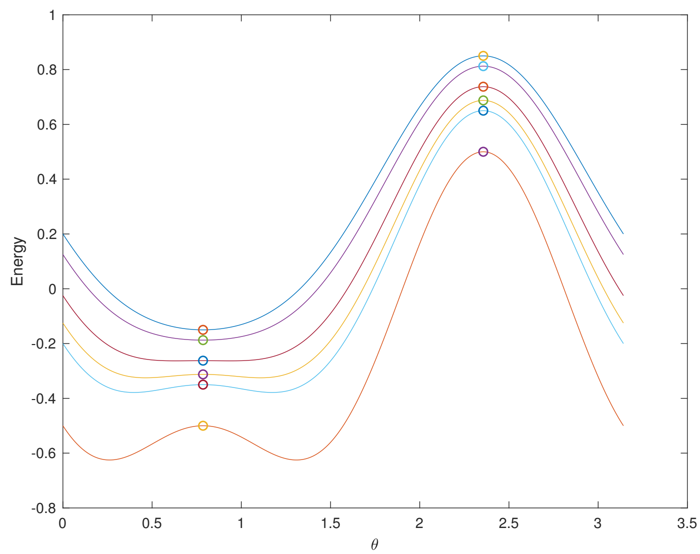


Figure 2: Graphs of $\theta \mapsto \mathscr{H}_1(\theta)$ for several values of κ ($\kappa \in \{1.2, 1.5, 2.1, 2.5, 2.8, 4\}$). The circles correspond to $(\pi/4, \mathscr{H}(\pi/4))$ and $(3\pi/4, \mathscr{H}(3\pi/4))$: $\kappa = 2$ is the threshold at which the convexity at $\pi/4$ changes.

Assuming the smallness condition

$$0 < \kappa < 2, \tag{15}$$

we thus denote

$$e^{i\omega t}U_* = \frac{e^{i\omega t}}{\sqrt{2}}\begin{pmatrix}1\\ \tau\end{pmatrix}, \qquad \tau = \pm 1 \tag{16}$$

the obtained solution of (5), with $\tau = 1$ corresponding to the state of minimal energy, and $\tau = -1$ corresponding to the state of maximal energy. Equivalently, we can consider

$$X_* = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\ \tau\\0\end{pmatrix} \tag{17}$$

so that, given the extended rotation matrix

$$R(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 & 0 \\ \sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & \cos(\theta) & -\sin(\theta) \\ 0 & 0 & \sin(\theta) & \cos(\theta) \end{pmatrix},$$

$R(\omega t)X_*$ defines a solution to (13).

When $\kappa > 2$, (16) are still solutions of (5), but for $\tau = 1$ the solution does not achieve the minimal energy. Moreover, in this situation we find two additional solutions

$$e^{i\omega t}U_{*\kappa,\pm} \text{ with } U_{*\kappa,\pm} = \begin{pmatrix} \sin(\theta_\kappa^\pm) \\ \cos(\theta_\kappa^\pm) \end{pmatrix} \text{ and } \theta_\kappa^+ = \frac{1}{2}\arcsin\left(\frac{2}{\kappa}\right),\ \theta_\kappa^- = \frac{\pi}{2} - \frac{1}{2}\arcsin\left(\frac{2}{\kappa}\right). \tag{18}$$

Since $\frac{2}{\kappa} > 0$, both $\sin(\theta_\kappa^\pm)$ and $\cos(\theta_\kappa^\pm)$ are positive. Using the elementary relation $\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}$, we can write

$$\sin(\theta_\kappa^+) = \sqrt{\frac{1-\cos(\arcsin(2/\kappa))}{2}} = \sqrt{\frac{1-\sqrt{1-(2/\kappa)^2}}{2}} = \frac{1}{2}\Big(\sqrt{1+2/\kappa} - \sqrt{1-2/\kappa}\Big),$$

and

$$\cos(\theta_\kappa^+) = \sqrt{1-\sin^2(\theta_\kappa)} = \frac{1}{2}\Big(\sqrt{1+2/\kappa} + \sqrt{1-2/\kappa}\Big),$$

. Using $\sin(\pi/2-\theta) = \cos(\theta)$ and $\cos(\pi/2-\theta) = \sin(\theta)$, we can rewrite the solution $U_{*\kappa,\pm}$ as follows

$$U_{*\kappa,+} = \frac{1}{2\sqrt{\kappa}}\begin{pmatrix} \sqrt{\kappa+2} - \sqrt{\kappa-2} \\ \sqrt{\kappa+2} + \sqrt{\kappa-2)} \end{pmatrix}, \qquad U_{*\kappa,-} = \frac{1}{2\sqrt{\kappa}}\begin{pmatrix} \sqrt{\kappa+2} + \sqrt{\kappa-2} \\ \sqrt{\kappa+2} - \sqrt{\kappa-2)} \end{pmatrix}.$$

The corresponding solution for (13) reads

$$R(\omega t)X_*, \qquad X_* = \begin{pmatrix} \sin(\theta_\kappa^\pm) \\ 0 \\ \cos(\theta_\kappa^\pm) \\ 0 \end{pmatrix} = \frac{1}{2\sqrt{\kappa}}\begin{pmatrix} \sqrt{\kappa+2} - \tau\sqrt{\kappa-2} \\ 0 \\ \sqrt{\kappa+2} + \tau\sqrt{\kappa-2} \\ 0 \end{pmatrix}, \qquad \tau = \pm 1. \tag{19}$$

Going back to (14), we find the Lagrange multiplier ω associated to all these solutions. Namely we get

$$-\omega U_{*0} = U_{*0} - U_{*1} - \kappa|U_{*0}|^2U_{*0}, \quad -\omega U_{*1} = U_{*1} - U_{*0} - \kappa|U_{*1}|^2U_{*1}.$$

Adding these relations and using $|U_{*0}|^2 + |U_{*1}|^2 = 1$, we are led to

$$2(\omega+1) - \kappa = \frac{U_{*1}}{U_{*0}} + \frac{U_{*0}}{U_{*1}} = \frac{1}{U_{*1}U_{*0}}.$$

Hence, we conclude that

$$\text{for (16)} \quad \omega = \frac{\kappa}{2} + \tau - 1 = \begin{cases} \kappa/2, & \text{if } \tau = +1, \\ -2+\kappa/2, & \text{if } \tau = -1, \end{cases} \tag{20}$$

$$\text{for (18)} \quad \omega = \kappa - 1. \tag{21}$$

2.2 Analysis of the coupled model

Writing $u_j = q_j + ip_j$ and $\varpi_j = \frac{\partial_t \psi_j}{2c^2}$, the energy functional (8) casts as

$$\mathscr{H}(X) = \frac{|q_0 - q_1|^2 + |p_0 - p_1|^2}{2} + \int_{\mathbb{R}^n} \left(c^2 \left(|\varpi_0|^2 + |\varpi_1|^2 \right) + \frac{1}{4} \left(|\nabla \psi_0|^2 + |\nabla \psi_1|^2 \right) \right) \mathrm{d}z$$
$$+ \frac{1}{2} \int_{\mathbb{R}^n} \sigma(\psi_0(|q_0|^2 + |p_0|^2) + \psi_1(|q_1|^2 + |p_1|^2)) \, \mathrm{d}z, \tag{22}$$

where X is the shorthand notation for $(q_0, p_0, q_1, p_1, \psi_0, \varpi_0, \psi_1, \varpi_1)$. Repeating the arguments used for the asymptotic model, we realize that extrema of $\mathscr{H}$ can be found by considering only the case $p_0 = p_1 = 0$ and, taking into account the constraint of normalized norm, $|u_0|^2 + |u_1|^2 = 1$, we are led to investigate the extrema of

$$\mathscr{H}_1(\theta, \psi_0, \varpi_0, \psi_1, \varpi_1) = \frac{|\cos(\theta) - \sin(\theta)|^2}{2} + \int_{\mathbb{R}^n} \left(c^2 \left(|\varpi_0|^2 + |\varpi_1|^2 \right) + \frac{1}{4} \left(|\nabla \psi_0|^2 + |\nabla \psi_1|^2 \right) \right) \mathrm{d}z$$
$$+ \frac{1}{2} \int_{\mathbb{R}^n} \sigma(\psi_0 \cos^2(\theta) + \psi_1 \sin^2(\theta)) \, \mathrm{d}z,$$

where θ lies in $[0, 2\pi)$. At the extrema, by computing the derivatives $\partial_{\varpi_j} \mathscr{H}_1$ and $\partial_{\psi_j} \mathscr{H}_1$, we infer that

$$\varpi_0 = \varpi_1 = 0$$

together with

$$-\Delta \psi_0 = -\sigma \cos^2(\theta), \qquad -\Delta \psi_1 = -\sigma \sin^2(\theta).$$

The latter relation leads to $\int_{\mathbb{R}^n} \sigma \psi_0 \, \mathrm{d}z = -\int_{\mathbb{R}^n} \sigma(-\Delta)^{-1} \sigma \, \mathrm{d}z \cos^2(\theta) = -\kappa \cos^2(\theta)$, and similarly $\int_{\mathbb{R}^n} \sigma \psi_1 \, \mathrm{d}z = -\kappa \sin^2(\theta)$. Eventually, computing $\partial_\theta \mathscr{H}_1$ yields

$$-(\cos^2(\theta) - \sin^2(\theta)) - \frac{1}{2} \int_{\mathbb{R}^n} \sigma(\psi_0 - \psi_1) \sin(2\theta) \, \mathrm{d}z.$$

Therefore, at the extrema we obtain

$$\frac{\kappa}{2} \cos(2\theta) \left(\frac{2}{\kappa} - \sin(2\theta) \right) = 0.$$

Hence, we find the same extrema as for the asymptotic model.

In particular, we set $Q_{*0} = \frac{1}{\sqrt{2}}$, $P_{*0} = 0$, $Q_{*1} = \frac{\tau}{\sqrt{2}}$, $P_{*1} = 0$, $\Psi_{*0} = \Psi_{*1} = -\frac{(-\Delta)^{-1}\sigma}{2}$, $\varpi_{*0} = \varpi_{*1} = 0$, and the energy is made minimal (resp. maximal) when $\tau = +1$ with $0 < \kappa < 2$ (resp. $\tau = -1$ without condition on κ). This analysis provides specific solutions of (1)-(2), having the special form $(e^{i\omega t} U_{*0}, e^{i\omega t} U_{*1}, \Psi_{*0}, \Psi_{*1})$ where U_{*0}, U_{*1} are fixed complex numbers and Ψ_{*0}, Ψ_{*1} are fixed functions in $L^2(\mathbb{R}^n)$. This leads to the relations

$$-\omega U_{*0} = U_{*0} - U_{*1} + U_{*0} \int_{\mathbb{R}^n} \sigma \Psi_{*0} \, \mathrm{d}z, \quad -\omega U_{*1} = U_{*1} - U_{*0} + U_{*1} \int_{\mathbb{R}^n} \sigma \Psi_{*1} \, \mathrm{d}z,$$
$$-\Delta \Psi_{*0} = -\sigma |U_{*0}|^2, \qquad -\Delta \Psi_{*1} = -\sigma |U_{*1}|^2.$$

Let Γ denote the solution of $-\Delta \Gamma = \sigma$, which can be alternatively defined by means of Fourier transform

$$\Gamma = \mathcal{F}^{-1}_{\xi \to z} \left(\frac{\widehat{\sigma}(\xi)}{|\xi|^2} \right).$$

Hence, we get

$$\Psi_{*0}(z) = -|U_{*0}|^2\Gamma(z), \qquad \Psi_{*1}(z) = -|U_{*1}|^2\Gamma(z),$$

so that U_{*0}, U_{*1} are required to satisfy

$$(\omega+1)U_{*0} - U_{*1} - \kappa|U_{*0}|^2U_{*0} = 0 = (\omega+1)U_{*1} - U_{*0} - \kappa|U_{*1}|^2U_{*1},$$

together with the physical normalisation

$$|U_{*0}|^2 + |U_{*1}|^2 = 1.$$

With the extrema discussed above, we have $|U_{*0}| = |U_{*1}| = \frac{1}{\sqrt{2}}$ and for the system

$$\begin{pmatrix} \omega+1-\kappa/2 & -1 \\ -1 & \omega+1-\kappa/2 \end{pmatrix} \begin{pmatrix} U_{*0} \\ U_{*1} \end{pmatrix} = 0$$

to admit nontrivial solutions, the dispersion relation (20) should be fulfilled. Given this condition, we conclude that

$$u_0(t) = \frac{e^{i\omega t}}{\sqrt{2}}, \quad u_1(t) = \tau\frac{e^{i\omega t}}{\sqrt{2}}, \quad \psi_0(t,z) = -\frac{\Gamma(z)}{2}, \quad \psi_1(t,z) = -\frac{\Gamma(z)}{2} \tag{23}$$

satisfies (1)-(2).

If $\kappa > 2$, we find two extra solutions which minimize the energy

$$\begin{aligned} &Q_{*0} = \sin(\theta_\kappa^\pm), \quad Q_{*1} = \cos(\theta_\kappa^\pm), \quad P_{*0} = P_{*1} = 0, \\ &U_* = \begin{pmatrix} Q_{*0} \\ Q_{*1} \end{pmatrix}, \Psi_{*0} = -|Q_{*0}|^2\Gamma, \quad \Psi_{*1} = -|Q_{*1}|^2\Gamma. \end{aligned} \tag{24}$$

With ω still given by (21), we conclude that

$$u_0(t) = e^{i\omega t}Q_{*0}, \quad u_1(t) = e^{i\omega t}Q_{*1}, \quad \psi_0(t,z) = \Psi_{*0}, \quad \psi_1(t,z) = \Psi_{*1} \tag{25}$$

satisfies (1)-(2).

Finally, we observe that the system can be expressed in the Hamiltonian formulation

$$\partial_t X = \begin{pmatrix} \mathscr{J} & 0 \\ 0 & \mathscr{J} \end{pmatrix} \nabla_X \mathscr{H}(X), \qquad \mathscr{J} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

We shall see later on a more adapted formulation that is more convenient for the stability analysis. For the time being, this formulation makes a parallel between the structure of the two models and brings out the role of energy minimization in classifying the traveling-wave-like solutions described above.

2.3 Statement of the results

Let us collect here the main statements that will be obtained (definitions of the notions of stability will be made precise later on).

Theorem 2.1 (Stability analysis for (5)**)** *Let us assume one of the following cases:*

i) $\tau = -1$,

ii) $\tau = +1$ *with* $0 < \kappa < 2$,

iii) $\kappa > 2$.

We consider the reference solution of (5) *given by* (16) *for i) and ii) or by* (18) *for iii). Then, the reference solution is spectrally and orbitally stable.*

Theorem 2.2 (Instability result for (5)**)** *Let* $\kappa > 2$. *Then, the state* $e^{i\omega t}(1/\sqrt{2}, 1/\sqrt{2})$ *is a spectrally and orbitally unstable solution of* (5).

Theorem 2.3 (Stability analysis for (1)-(2)**)** *Let* $\tau = +1$ *with* $0 < \kappa < 2$. *Then, the reference solution* (23) *of* (1)-(2) *is spectrally and orbitally stable. Let* $\kappa > 2$. *Then, the reference solution* (24)-(25) *is spectrally and orbitally stable.*

Theorem 2.4 (Instability result for (1)-(2)**)** *Let* $\tau = 1$ *with* $\kappa > 2$ *or* $\tau = -1$. *Then the reference solution* (23) *is spectrally and orbitally unstable.*

These results are in line with the analysis performed in [16] for plane waves solutions for the PDE system (10) and its asymptotic Hartree-like counterpart. It confirms that the asymptotic model has more stable solutions than the original model, and that the dynamic coupling (2) induces intricate and rich selection mechanisms. We expect this study will provide fruitful ideas to come back to (10) set for $x \in \mathbb{R}^d$, and will allow us to fill a gap in the understanding of open quantum systems.

3 Stability analysis of the asymptotic model (5)

3.1 Spectral and linearized stability

We start by linearizing (5) about the solutions (16). We search for solutions of (5) on the form

$$u_j = e^{i\omega t}(U_{*j} + v_j).$$

Using $|u+h|^2 = |u|^2 + 2\mathrm{Re}(\overline{u}h) + |h|^2$, the dispersion relation (20), and neglecting the nonlinear terms, one is led to the following linearized system

$$i\frac{\mathrm{d}}{\mathrm{d}t}v_0 = \tau v_0 - v_1 - \kappa \mathrm{Re}(v_0), \qquad i\frac{\mathrm{d}}{\mathrm{d}t}v_1 = \tau v_1 - v_0 - \kappa \mathrm{Re}(v_1). \tag{26}$$

We write $v_j = q_j + ip_j$, with q_j, p_j real-valued. The unknown is now represented by the vector $X = (q_0, p_0, q_1, p_1)$; we get

$$\frac{\mathrm{d}}{\mathrm{d}t}X = \mathbb{L}X, \qquad \mathbb{L} = \begin{pmatrix} 0 & \tau & 0 & -1 \\ \kappa-\tau & 0 & 1 & 0 \\ 0 & -1 & 0 & \tau \\ 1 & 0 & \kappa-\tau & 0 \end{pmatrix}.$$

The stability of this ODE system is related to the spectral analysis of the matrix $\mathbb{L}$: spectral stability means that the real part of the eigenvalues of $\mathbb{L}$ are all nonpositive; linearized stability means that any solution of this linear system remains uniformly bounded for any $t \geqslant 0$.

Proposition 3.1 *If $\tau = -1$, the system* (26) *is spectrally stable; if $\tau = +1$, the system* (26) *is spectrally stable under the condition* (15). *Moreover, in these situations, if* $\mathrm{Re}(v_0 + \tau v_1)\big|_{t=0} = 0$, *then the solution of* (26) *remains uniformly bounded for any $t \geqslant 0$. If $\tau = +1$ with $\kappa > 2$, the system is spectrally unstable.*

Proof. We observe that 0 is an eigenvalue of $\mathbb{L}$. Indeed, $\mathbb{L}X = 0$ leads to the independent relations

$$\begin{cases} \tau p_0 = p_1, \\ \tau p_1 = p_0 \end{cases} \text{and} \begin{cases} (\kappa - \tau) q_0 = -q_1, \\ (\kappa - \tau) q_1 = -q_0. \end{cases}$$

Since $\tau^2 = 1$, the former yields a nontrivial solution, while the latter in general $((\kappa-\tau)^2-1 = \kappa(\kappa-2\tau) \neq 0)$ has only the solution $q_0 = q_1 = 0$. Hence we find the eigenspace $\mathrm{Ker}(\mathbb{L}) = \mathrm{Span}\{(0,1,0,\tau)\}$. Note however that $\mathbb{L}$ has a Jordan block associated to the eigenvalue 0, since the kernel of

$$\mathbb{L}^2 = \begin{pmatrix} \kappa\tau - 2 & 0 & 2\tau - \kappa & 0 \\ 0 & \kappa\tau - 2 & 0 & 2\tau - \kappa \\ 2\tau - \kappa & 0 & \tau\kappa - 2 & 0 \\ 0 & 2\tau - \kappa & 0 & \tau\kappa - 2 \end{pmatrix}$$

is spanned by $\{(0,1,0,\tau),(1,0,\tau,0)\}$. This leads to solutions of (26) with norms that can grow linearly. Next, let $\lambda \neq 0$, $X \neq 0$ satisfy $\mathbb{L}X = \lambda X$. Since $\tau^2 = 1$, we observe that $\tau q_0 = -q_1$. Therefore, we obtain $\lambda p_0 = q_1 - \tau q_0 + \kappa q_0 = (-2\tau + \kappa) q_0$, together with $\lambda p_1 = q_0 + (-\tau + \kappa) q_1 = (2 - \tau\kappa) q_0$. It yields $\lambda q_0 = \tau p_0 - p_1 = -\big(\tau \frac{2\tau-\kappa}{\lambda} + \frac{2-\tau\kappa}{\lambda}\big) q_0$. A nontrivial solution q_0 exists provided λ satisfies

$$\lambda^2 = -4 + 2\tau\kappa.$$

If $\tau = -1$, we find $\lambda = \pm 2i\sqrt{1 + \kappa/2}$. If $\tau = 1$, we find $\lambda = \pm 2i\sqrt{1 - \kappa/2}$, assuming the smallness condition (15); otherwise, $\lambda = \pm 2\sqrt{\kappa/2 - 1}$ and the system admits a positive eigenvalue.

In fact, the problem (26) can be easily solved by hand. On the one hand, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(q_0 + \tau q_1) = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}t}(p_0 + \tau p_1) = \kappa(q_0 + \tau q_1)$$

so that

$$(q_0 + \tau q_1)(t) = C_1, \qquad (p_0 + \tau p_1)(t) = C_2 + C_1 \kappa t.$$

On the other hand, the pair $(q_0 - \tau q_1)$ and $(p_0 - \tau p_1)$ solves a linear system associated to the matrix

$$\begin{pmatrix} 0 & 2\tau \\ \kappa - 2\tau & 0 \end{pmatrix}$$

which is diagonalizable with eigenvalues satisfying $\lambda^2 = -4(1 - \tau\kappa/2) < 0$. The analysis of the linearized system is therefore complete. ■

Similar computations can be performed with the solutions (18). The linearized system now reads

$$i\frac{\mathrm{d}}{\mathrm{d}t} v_0 = (1 + \omega - \kappa\alpha^2) v_0 - v_1 - 2\kappa\alpha^2 \mathrm{Re}(v_0), \qquad i\frac{\mathrm{d}}{\mathrm{d}t} v_1 = (1 + \omega - \kappa\beta^2) v_1 - v_0 - 2\kappa\beta^2 \mathrm{Re}(v_1), \quad (27)$$

with

$$U_{*\kappa,\pm} = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}, \qquad \alpha = \frac{\sqrt{\kappa+2} - \tau\sqrt{\kappa-2}}{2\sqrt{\kappa}} = \sin(\theta_\kappa^\tau), \qquad \beta = \frac{\sqrt{\kappa+2} + \tau\sqrt{\kappa-2}}{2\sqrt{\kappa}} = \cos(\theta_\kappa^\tau).$$

Let us set

$$A = 1 + \omega - \kappa\alpha^2, \qquad B = 1 + \omega - \kappa\beta^2.$$

Elementary manipulations lead to

$$A = \frac{\kappa}{2} + \tau\frac{\sqrt{\kappa^2-4}}{2}, \quad B = \frac{\kappa}{2} - \tau\frac{\sqrt{\kappa^2-4}}{2}, \quad AB = 1, \quad \kappa\alpha^2 = B, \quad \kappa\beta^2 = A. \tag{28}$$

The matrix associated to the linearized system thus reads

$$\mathbb{L} = \begin{pmatrix} 0 & A & 0 & -1 \\ -A+2B & 0 & 1 & 0 \\ 0 & -1 & 0 & B \\ 1 & 0 & -B+2A & 0 \end{pmatrix}.$$

In turn, it can be checked that

$$\mathrm{Ker}(\mathbb{L}) = \mathrm{Span}\{(0,1,0,A)\}.$$

Next, let (λ, X) be an eigenpair of $\mathbb{L}$, with $\lambda \neq 0$. We observe that $\lambda A q_1 = A(Bp_1 - p_0) = -\lambda q_0$, which implies $Aq_1 + q_0 = 0$. It follows that

$$\lambda p_0 = (-A+2B)q_0 + q_1 = (-A+2B)(-Aq_1) + q_1 = A(A-B)q_1$$

and

$$\lambda p_1 = (-B+2A)q_1 + q_0 = (-B+2A)q_1 - Aq_1 = (A-B)q_1,$$

which lead to

$$\lambda q_1 = Bp_1 - p_0 = B\frac{A-B}{\lambda}q_1 - \frac{A(A-B)}{\lambda}q_1 = -\frac{q_1}{\lambda}(A-B)^2.$$

Therefore, we obtain

$$\lambda^2 = -(A-B)^2 = -(\kappa^2-4) = -\kappa^2 + 4 < 0.$$

We deduce that $\lambda \in i\mathbb{R}$.

Proposition 3.2 *The system* (27) *is spectrally stable. Moreover, if* $\mathrm{Re}(v_0 + Av_1)\big|_{t=0} = 0$, *then the solution of* (27) *remains uniformly bounded for any* $t \geqslant 0$.

Proof. The spectral stability has just been established above, all eigenvalues of $\mathbb{L}$ being with a nonpositive real part. Next, we introduce the vectors

$$\Psi = (1, 0, A, 0), \qquad \Psi_1 = \Big(0, \frac{-\tau}{2\sqrt{\kappa^2-4}}, 0, \frac{\kappa\tau + \sqrt{\kappa^2-4}}{4\sqrt{\kappa^2-4}}\Big).$$

They satisfy

$$\mathbb{L}^\intercal\Psi = 0, \qquad \mathbb{L}^\intercal\Psi_1 = \Psi.$$

Let X satisfy $\frac{\mathrm{d}}{\mathrm{d}t}X = \mathbb{L}X$. We observe that $\frac{\mathrm{d}}{\mathrm{d}t}X \cdot \Psi = \frac{\mathrm{d}}{\mathrm{d}t}(q_0 + Aq_1) = X \cdot \mathbb{L}^\intercal \Psi = 0$, and $\frac{\mathrm{d}}{\mathrm{d}t}X \cdot \Psi_1 = X \cdot \mathbb{L}^\intercal \Psi_1 = X \cdot \Psi$. Hence $X(t) \cdot \Psi = X_{\text{init}} \cdot \Psi$ is conserved and $X(t) \cdot \Psi_1 = X_{\text{init}} \cdot \Psi_1 + tX_{\text{init}} \cdot \Psi$ grows at most linearly. Assuming $X_{\text{init}} \cdot \Psi = 0$ prevents the linear growth. Finally, the pair $(Ap_0 - p_1, Aq_0 + q_1)$ satisfies the 2×2 system governed by the matrix

$$\begin{pmatrix} 0 & B-A \\ A-B & 0 \end{pmatrix}$$

whose eigenvalues are clearly purely imaginary. These observations completely characterize the solution of the linear system (27). ■

Propositions 3.1 and 3.2 are illustrated in Fig. 3 where we perform simulations of the different scenario: the stable case ((a)-(b)) requires a condition on both the coefficients (τ, κ) and the data; when the orthogonality condition of Proposition 3.1 is violated, one observes a linear growth of the L^2 norm ((c)-(d)); when the condition on the data is not fulfilled, one observes an exponential blow up ((e)-(f)).

System (5) is a mere finite dimensional differential system. As far as one is concerned with the stability of equilibrium solution of differential systems in finite dimension, spectral stability implies nonlinear stability, see e. g. [35, Prop. 1.41], [34, Th. 1.1 & 1.2]. Here, we are dealing with the notion of *orbital stability*, and the reference solutions remains time-dependent which induces some subtleties. We shall detail approaches which do not use properties specific to the finite dimensional framework, having in mind more complicated couplings.

3.2 Orbital stability

Let us set $F(X) = \frac{|X|^2}{2} = \frac{Q_0^2+Q_1^2+P_0^2+P_1^2}{2}$ and introduce the functional

$$\mathscr{E}(X) = \mathscr{H}(X) + \omega F(X)$$

with $\mathscr{H}$ defined by (12). This quantity is thus conserved by the dynamical system (13), being the sum of two conserved quantities. We observe that (14) can be reformulated as

$$\nabla \mathscr{E}(X_*) = 0 \tag{29}$$

and $\mathscr{L}$ corresponds to the Hessian of $\mathscr{E}$ evaluated at X_*. Inspired by the strategy described in [3], we introduce the level set of the solutions of (13) associated to X_*,

$$\mathscr{S} = \{X \in \mathbb{R}^4,\ F(X) = F(X_*) = 1/2\}.$$

We wish to establish a coercivity estimate, on a certain subspace, for the quadratic form $X \mapsto \mathscr{L}X \cdot X$. This is a crucial property for establishing the orbital stability, an idea that dates back to [37, 38] for Schrödinger equations, see [3, 21, 36].

With X_* given by (17), the tangent set to the level set is given by

$$T\mathscr{S} = \{X \in \mathbb{R}^4,\ \nabla F(X_*) \cdot X = 0\} = \{(q_0, p_0, q_1, p_1) \in \mathbb{R}^4,\ q_0 + \tau q_1 = 0\}.$$

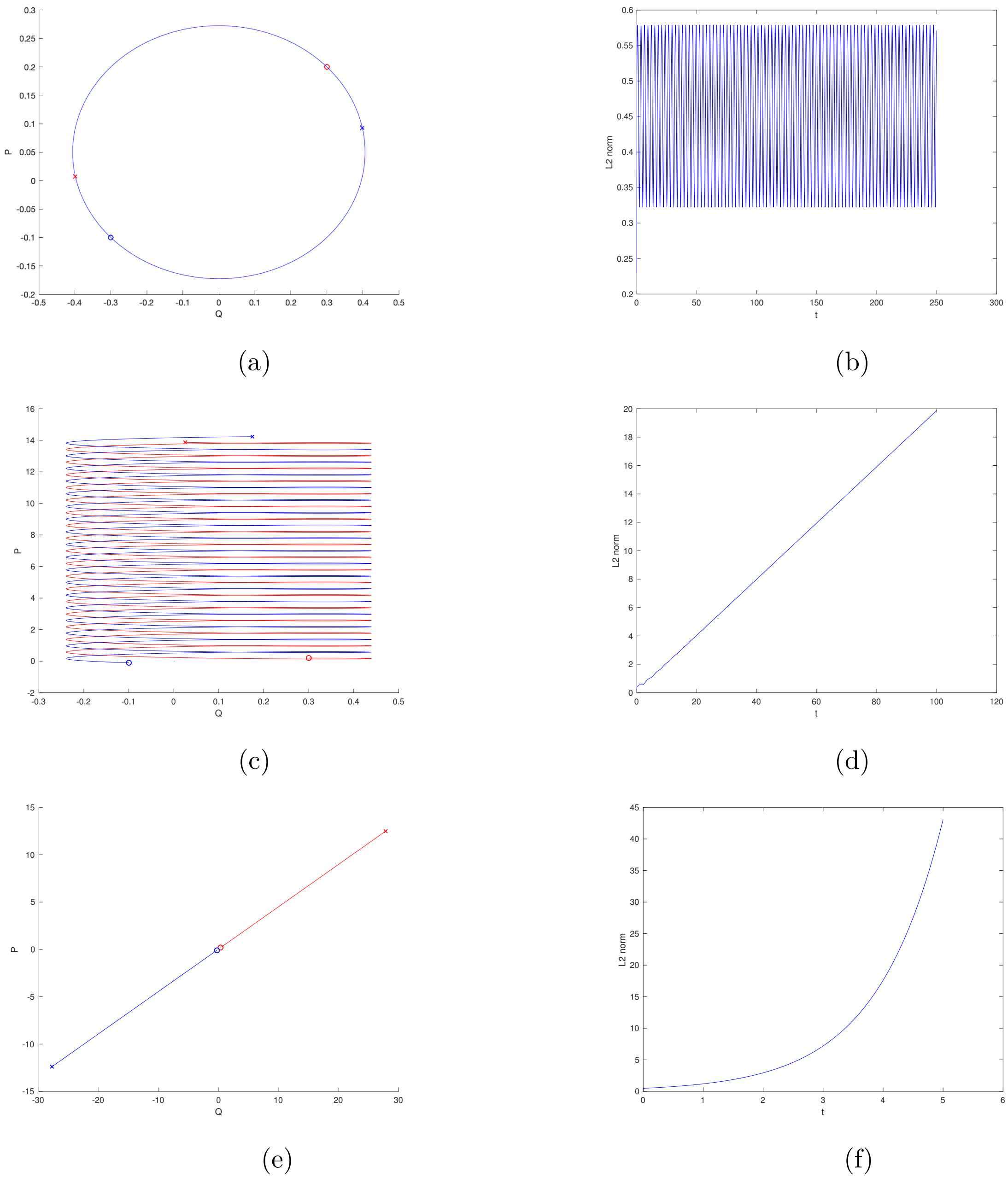


Figure 3: Simulation of the linearized asymptotic model (26). The circled points indicate the initial state, the cross indicate the final state. (a)-(b): stable case $\kappa = 1.4$ and $\tau = +1$; phase portrait at $T = 250$ (a) and evolution of the L^2 norm (b) for a well-prepared data. The solution remains in a bounded domain. Similar results can be obtained when $\tau = -1$ or, with $\kappa > 2$, for the linearized problem (27). (c)-(d): $\kappa = 1.4$ and $\tau = +1$ with ill prepared data; phase portrait at $T = 100$ (c) and evolution of the L^2 norm (d); the L^2 norm of the solution grows linearly. (e)-(f): instable case $\kappa = 2.4$ and $\tau = +1$; phase portrait at $T = 50$ (e) and evolution of the L^2 norm (f).

The orbit associated to X_* is given by

$$\mathscr{O} = \left\{ \frac{1}{\sqrt{2}}(\cos(\theta), \sin(\theta), \tau\cos(\theta), \tau\sin(\theta)),\ \theta \in \mathbb{R} \right\}$$

and we get

$$(T\mathscr{O})^\perp = \{(q_0, p_0, q_1, p_1) \in \mathbb{R}^4,\ p_0 + \tau p_1 = 0\}.$$

The reference solution associated to X_* is said to be orbitally stable if, for any $\epsilon > 0$, there exists $\delta > 0$, such that, for any solution $t \mapsto Y(t)$ of (13), $|Y(0) - X_*| \leqslant \delta$ implies that $\mathrm{dist}(Y(t) - \mathscr{O}) \leqslant \epsilon$ holds for any $t \geqslant 0$.

Remark 3.3 *Bearing in mind the transformation* (11)*, multiplying the components of* $U \in \mathbb{C}^2$ *by* $e^{i\theta}$ *is equivalent to applying the (extended) rotation* $R(\theta)$ *to* $X \in \mathbb{R}^4$*, which leaves the energy* $\mathscr{H}(X)$*, as well as* $\mathscr{E}(X)$*, invariant. The identity* $\mathscr{H}(R(\theta)X) = \mathscr{H}(X)$ *yields* $R(\theta)^\intercal \nabla\mathscr{H}(R(\theta)X) = \nabla\mathscr{H}(X)$ *and we observe that* $R(\theta)^{-1}R'(\theta) = -\mathscr{J}$*. These observations allow us to derive directly the linearized system: with* $\frac{\mathrm{d}}{\mathrm{d}t}X = \mathscr{J}\nabla\mathscr{H}(X)$ *and* $X(t) = R(\omega t)(X_* + \tilde{X}(t))$*, we get*

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{X} = \omega\mathscr{J}(X_* + \tilde{X}) + \mathscr{J}\nabla\mathscr{H}(X_* + \tilde{X}).$$

Assuming the perturbation to be small, at leading order the right hand side reads

$$\mathscr{J}(\omega X_* + \nabla\mathscr{H}(X_*)) + \mathscr{J}(\omega\tilde{X} + D^2\mathscr{H}(X_*)\tilde{X}) = 0 + \mathscr{J}\mathscr{L}\tilde{X} = \mathbb{L}\tilde{X}.$$

In order to investigate the orbital stability of the system, we recast the linearized system by using the symplectic form

$$\mathbb{L} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}}_{=\mathscr{J}} \underbrace{\begin{pmatrix} \tau-\kappa & 0 & -1 & 0 \\ 0 & \tau & 0 & -1 \\ -1 & 0 & \tau-\kappa & 0 \\ 0 & -1 & 0 & \tau \end{pmatrix}}_{=\mathscr{L}},$$

with $\mathscr{L} = D^2\mathscr{E}(X_*)$ symmetric.

Lemma 3.4 *The spectrum of the matrix* $\mathscr{L}$ *is* $\sigma(\mathscr{L}) = \{0, -\kappa, 2\tau, 2\tau-\kappa\}$ *with eigenspaces spanned respectively by*

$$\begin{aligned} X_0 &= (0,1,0,\tau), & X_{-\kappa} &= (1,0,\tau,0), \\ X_{2\tau-\kappa} &= (1,0,-\tau,0), & X_{2\tau} &= (0,1,0,-\tau). \end{aligned}$$

Hence, we get

$$\mathscr{L}X \cdot X = (\tau-\kappa)(q_0^2 + q_1^2) - 2q_1q_0 + \tau(p_0^2 + p_1^2) - 2p_1p_0.$$

As a matter of fact, when $\tau = 1$, it recasts as

$$\mathscr{L}X \cdot X = |p_0 - p_1|^2 + |q_0 - q_1|^2 - \kappa(q_0^2 + q_1^2).$$

Restricting to the subspace $T\mathscr{S} \cap (T\mathscr{O})^\perp$, we have $q_0 = -\tau q_1$ and $p_0 = -\tau p_1$, so that, still for $\tau = 1$, we get

$$\mathscr{L}X \cdot X = 4|p_0|^2 + 2(2-\kappa)|q_0|^2 \geqslant (2-\kappa)|X|^2.$$

This coercivity estimate is key in establishing the orbital stability [21, 22, 3, 4]. Surprisingly, the case $\tau = -1$ is simpler. We now work with

$$\mathscr{E}(X) = -\mathscr{H}(X) - \omega F(X).$$

We still have $\nabla\mathscr{E}(X_*) = 0$ and $D^2\mathscr{E}(X_*) = -\mathscr{L}$. The spectral decomposition of $\mathscr{L}$ implies that $-\mathscr{L}$ is coercive on $(\mathrm{Ker}(\mathscr{L}))^\perp = (T\mathscr{O})^\perp$. This allows us to justify the orbital stability.

We turn to the case where $\kappa > 2$ and $X_* = (\alpha, 0, \beta, 0)$ is given by (19). Now, we look at

$$\mathscr{L} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \mathbb{L} = \begin{pmatrix} A-2B & 0 & -1 & 0 \\ 0 & A & 0 & -1 \\ -1 & 0 & B-2A & 0 \\ 0 & -1 & 0 & B \end{pmatrix}.$$

The equations for the eigenpairs uncouple since we get

$$\begin{array}{ll} (A-\lambda)p_0 = p_1, & (B-\lambda)p_1 = p_0, \\ (A-2B-\lambda)q_0 = q_1, & (B-2A-\lambda)q_1 = q_0. \end{array}$$

The former leads to

$$\lambda(\lambda - (A+B)) = \lambda(\lambda - \kappa) = 0,$$

and the latter gives

$$(B-2A-\lambda)(A-2B-\lambda) - 1 = \lambda^2 + \lambda(A+B) + (A-2B)(B-2A) - 1 = \lambda^2 + \lambda\kappa - 2(\kappa^2 - 4) = 0.$$

This gives the eigenelements of $\mathscr{L}$.

Lemma 3.5 *We have*

$$\sigma(\mathscr{L}) = \Big\{0, \kappa, \frac{-\kappa + \sqrt{9\kappa^2 - 32}}{2}, \frac{-\kappa - \sqrt{9\kappa^2 - 32}}{2}\Big\},$$

where only the last value is negative, with eigenspaces spanned respectively by

$$X_0 = \Big(0, 1, 0, \frac{\kappa}{2} + \tau\frac{\sqrt{\kappa^2-4}}{2}\Big), \qquad X_\kappa = \Big(0, 1, 0, -\frac{\kappa}{2} + \tau\frac{\sqrt{\kappa^2-4}}{2}\Big),$$

$$X_+ = \Big(1, 0, \tau\frac{3}{2}\sqrt{\kappa^2-4} - \frac{1}{2}\sqrt{9\kappa^2-32}, 0\Big), \quad X_- = \Big(1, 0, \tau\frac{3}{2}\sqrt{\kappa^2-4} + \frac{1}{2}\sqrt{9\kappa^2-32}, 0\Big).$$

Establishing the orbital stability amounts to check the coercivity of $\mathscr{L}$ on $T\mathscr{S} \cap (T\mathscr{O})^\perp$, where, now,

$$T\mathscr{S} = \big\{X = (q_0, p_0, q_1, p_1) \in \mathbb{R}^4,\ X \cdot X_* = \alpha q_0 + \beta q_1 = 0\big\},$$

and

$$(T\mathscr{O})^\perp = \big\{X = (q_0, p_0, q_1, p_1) \in \mathbb{R}^4,\ \alpha p_0 + \beta p_1 = 0\big\}.$$

We have

$$\mathscr{L}X \cdot X = (A - 2B)q_0^2 - 2q_0q_1 + (B - 2A)q_1^2 + Ap_0^2 - 2p_0p_1 + Bp_1^2.$$

Since $AB = 1$ and $\frac{\alpha}{\beta} = B$, on $T\mathscr{S} \cap (T\mathscr{O})^\perp$, it reduces to

$$\mathscr{L}X \cdot X\big|_{T\mathscr{S}\cap(T\mathscr{O})^\perp} = \big(A + (B - 2A)B^2\big)q_0^2 + \big(A + B^3 + 2B\big)p_0^2.$$

A tedious, but elementary, computation yields

$$\mathscr{L}X \cdot X\big|_{T\mathscr{S}\cap(T\mathscr{O})^\perp} = \frac{\kappa - \tau\sqrt{\kappa^2 - 4}}{2}((\kappa^2 - 4)q_0^2 + \kappa^2 p_0^2),$$

hence the desired coercivity estimate holds.

3.3 Symplectic formulation and further comments about spectral stability

Let us focus on the spectral stability issue. For the problem (5), the spectrum of $\mathbb{L} = \mathscr{J}\mathscr{L}$ is completely determined, as seen above, and we have directly a full understanding of the linearized problem. However, for more intricate system, like (1)-(2), we do not have a direct access to the spectrum of $\mathbb{L}$. The strategy is to deduce information about stable/instable modes from the study of $\mathscr{L}$ which could be easier (in particular because $\mathscr{L}$ is symmetric). To this end, according to [7, 28], we introduce the auxilliary operators

$$\mathscr{M} = -\mathscr{J}\mathscr{L}\mathscr{J}, \qquad \mathbb{A} = \mathscr{P}\mathscr{M}\mathscr{P},$$

where $\mathscr{P}$ is the orthogonal projection on $(\mathrm{Ker}(\mathscr{L}))^\perp$. We also introduce

$$\mathbb{K} = \mathscr{P}\mathscr{L}^{-1}\mathscr{P}.$$

The counting of the eigenvalues of $\mathbb{L}$ is based on the following considerations. We are interested in the coupled system

$$\mathscr{M}X = -\lambda\tilde{X}, \qquad \mathscr{L}\tilde{X} = \lambda X. \tag{30}$$

It turns out that this problem (30) admits nontrivial solutions iff $\pm\lambda$ are eigenvalues of $\mathbb{L}$. Next, (30) admits nontrivial solutions with $\lambda \neq 0$ iff the generalized eigenvalue problem

$$\mathbb{A}X = \mu\mathbb{K}X \tag{31}$$

(which recasts as $\mathscr{M}X = \mu\tilde{X}$, $\mathscr{L}\tilde{X} = X$, with $X \in (\mathrm{Ker}(\mathscr{L}))^\perp$) admits nontrivial solutions with $\mu = -\lambda^2$. The spectral stability means that the spectrum of $\mathbb{L}$ is contained in $i\mathbb{R}$. This can be reformulated as saying that all the eigenvalues of the generalized eigenproblem (31) are real and positive. In order to count the eigenvalues μ of the generalized eigenvalue problem, we define the following quantities:

- N_n^-, the number of negative eigenvalues,
- N_n^0, the number of zero eigenvalues,
- N_n^+, the number of positive eigenvalues,

counted with their algebraic multiplicity, the eigenvectors of which are associated to nonpositive values of the the quadratic form $X \mapsto (\mathbb{K}X|X) = (\mathscr{L}^{-1}\mathscr{P}X|\mathscr{P}X)$. Moreover, let N_{C^+} be the number of generalized eigenvalues $\mu \in \mathbb{C}$ of (31) with $\mathrm{Im}(\mu) > 0$. As said above, the eigenvalues counted by N_n^- and N_{C^+} correspond to cases of instabilities for the linearized problem. We now use the counting argument of [7, Theorem 1] (see also the review [28]) which asserts that

$$N_n^- + N_n^0 + N_n^+ + N_{C^+} = n(\mathscr{L}),$$

the number of negative eigenvalues of $\mathscr{L}$. Let us check how this counting machinery works for (5).

Let us begin with the case where X_* is given by (17). We use the notation of Lemma 3.4. For further purposes, we remark that

$$\mathscr{J}X_{-\kappa} = -X_0, \qquad \mathscr{J}X_{2\tau-\kappa} = -X_{2\tau}.$$

In particular, for $\tau = -1$, $\mathscr{L}$ has three negative eigenvalues; for $\tau = +1$ and assuming (15), there are two positive eigenvalues and one negative eigenvalue but if $\tau = +1$ and (15) is violated, there are one positive eigenvalue and two negative eigenvalues. Note that

- e1) the eigenvectors $X_0, X_{-\kappa}, X_{2\tau-\kappa}, X_{2\tau}$ form a orthogonal basis of $\mathbb{R}^4$;
- e2) with $X_* = \frac{1}{\sqrt{2}}(1, 0, \tau, 0) = \frac{X_{-\kappa}}{\sqrt{2}}$ the reference solution, we have

$$X_* \cdot X_0 = X_* \cdot X_{2\tau-\kappa} = X_* \cdot X_{2\tau} = 0;$$

- e3) and $X_* \cdot X_{-\kappa} = \sqrt{2} > 0$.

We start by showing $N_n^0 = 1$. We have seen that $\mathrm{Ker}(\mathscr{L})$ is spanned by $X_0 = (0, 1, 0, \tau)$. Hence, we have to solve $\mathscr{L}\tilde{X}_0 = Y_0$ with $Y_0 = -\mathscr{J}X_0 = (-1, 0, -\tau, 0)$ and $\tilde{X}_0 \in (\mathrm{Ker}(\mathscr{L}))^\perp$. This leads to $\tilde{X}_0 = \frac{1}{\kappa}(1, 0, \tau, 0)$ which yields $\mathbb{K}Y_0 \cdot Y_0 = \mathscr{L}^{-1}Y_0 \cdot Y_0 = \tilde{X}_0 \cdot Y_0 = -\frac{2}{\kappa} < 0$ and thus $N_n^0 = 1$.

Next, solving the generalized eigenvalue problem amounts to solving

$$\begin{aligned}
-\tilde{q}_1 + \tau\tilde{q}_0 - \kappa\tilde{q}_0 &= q_0, & \tau q_0 - q_1 &= \mu\tilde{q}_0,\\
-\tilde{q}_0 + \tau\tilde{q}_1 - \kappa\tilde{q}_1 &= q_1, & \tau q_1 - q_0 &= \mu\tilde{q}_1,\\
\tau p_0 - \kappa p_0 - p_1 &= \mu\tilde{p}_0, & \tau\tilde{p}_0 - \tilde{p}_1 &= p_0,\\
-p_0 + \tau p_1 - \kappa p_1 &= \mu\tilde{p}_1, & \tau\tilde{p}_1 - \tilde{p}_0 &= p_1,
\end{aligned}$$

with $X = (q_0, p_0, q_1, p_1), \tilde{X} = (\tilde{q}_0, \tilde{p}_0, \tilde{q}_1, \tilde{p}_1) \in (\mathrm{Ker}(\mathscr{L}))^\perp$. We set

$$M_\kappa = \begin{pmatrix} \tau - \kappa & -1 \\ -1 & \tau - \kappa \end{pmatrix}. \tag{32}$$

The q and p equations decouple and we have, on the one hand

$$M_\kappa \begin{pmatrix} \tilde{q}_0 \\ \tilde{q}_1 \end{pmatrix} = \begin{pmatrix} q_0 \\ q_1 \end{pmatrix}, \qquad M_0 \begin{pmatrix} q_0 \\ q_1 \end{pmatrix} = \mu \begin{pmatrix} \tilde{q}_0 \\ \tilde{q}_1 \end{pmatrix},$$

and, on the other hand

$$M_\kappa \begin{pmatrix} p_0 \\ p_1 \end{pmatrix} = \mu \begin{pmatrix} \tilde{p}_0 \\ \tilde{p}_1 \end{pmatrix}, \qquad M_0 \begin{pmatrix} \tilde{p}_0 \\ \tilde{p}_1 \end{pmatrix} = \begin{pmatrix} p_0 \\ p_1 \end{pmatrix}.$$

It amounts to say that $(\tilde{q}_0, \tilde{q}_1)$ and $(\tilde{p}_0, \tilde{p}_1)$ are eigenvectors for μ of $M_0 M_\kappa$ and $M_\kappa M_0$, respectively. Here, we get

$$M_\kappa M_0 = \begin{pmatrix} 2-\tau\kappa & \kappa-2\tau \\ \kappa-2\tau & 2-\tau\kappa \end{pmatrix} = M_0 M_\kappa,$$

the eigenvalues of which being 0 and $4(1-\tau\kappa/2)$. We thus obtain the solutions $\tilde{X}_1 = (1, 0, -\tau, 0)$ and $\tilde{X}_2 = (0, 1, 0, -\tau)$, associated to $X_1 = \mathscr{L}\tilde{X}_1 = (2\tau-\kappa, 0, \kappa\tau-2, 0)$, $X_2 = \mathscr{L}\tilde{X}_2 = (0, 2\tau, 0, -2)$ which both belong to $(\mathrm{Ker}(\mathscr{L}))^\perp$. We compute $\mathscr{L}^{-1}X_1 \cdot X_1 = \tilde{X}_1 \cdot X_1 = 2(2\tau-\kappa)$, which is negative when $\tau = -1$ and has the sign of $2-\kappa$ when $\tau = +1$, and $\mathscr{L}^{-1}X_2 \cdot X_2 = \tilde{X}_2 \cdot X_2 = 4\tau$. Therefore, we can verify the counting formula in the following three cases

- $\tau = -1$: $n(\mathscr{L}) = 3$ and $N_n^0 = 1$, $N_n^+ = 2$, $N_n^- = 0$, which yields $N_{C^+} = 0$ and indeed we found that $\mathbb{L}$ has two purely imaginary eigenvalues, there is no exponentially unstable solution to the linearized system;
- $\tau = 1$ and $\kappa > 2$: $n(\mathscr{L}) = 2$ and $N_n^0 = 1$, $N_n^+ = 0$, $N_n^- = 1$, which yields $N_{C^+} = 0$ and indeed we found that $\mathbb{L}$ has two real eigenvalues, we can find exponentially unstable solutions to the linearized system;
- $\tau = 1$ and $0 < \kappa < 2$: $n(\mathscr{L}) = 1$ and $N_n^0 = 1$, $N_n^+ = 0$, $N_n^- = 0$, which yields $N_{C^+} = 0$ and indeed we found that $\mathbb{L}$ has two purely imaginary eigenvalues, there is no exponentially unstable solution to the linearized system.

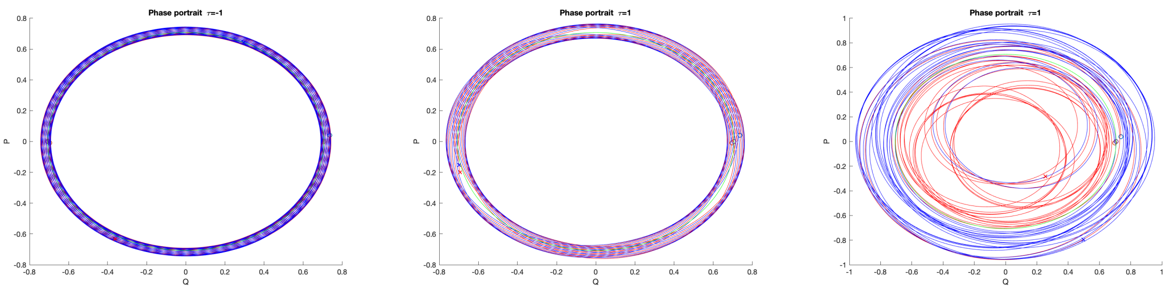
We can perform similar computations for the solution (19). We now use the notation of Lemma 3.5. We have seen that $\mathrm{Ker}(\mathscr{L})$ is spanned by $X_0 = (0, 1, 0, A)$. We start by solving $\mathscr{L}\tilde{X}_0 = Y_0$ with $Y_0 = -\mathscr{J}X_0 = (-1, 0, -A, 0)$ so that $\tilde{X}_0 = \frac{1}{2(A-B)}(-1, 0, A, 0)$ which yields $\tilde{X}_0 \cdot Y_0 = \mathbb{K}Y_0 \cdot Y_0 = -\frac{A}{2} < 0$, and thus $N_n^0 = 1$. Solving the generalized eigenvalue problem amounts to solve

$$M\begin{pmatrix}\tilde{q}_0\\ \tilde{q}_1\end{pmatrix} = \mu\begin{pmatrix}\tilde{q}_0\\ \tilde{q}_1\end{pmatrix}, \qquad M^\intercal\begin{pmatrix}\tilde{p}_0\\ \tilde{p}_1\end{pmatrix} = \mu\begin{pmatrix}\tilde{p}_0\\ \tilde{p}_1\end{pmatrix},$$
$$M = \begin{pmatrix} A & -1 \\ -1 & B \end{pmatrix}\begin{pmatrix} A-2B & -1 \\ -1 & B-2A \end{pmatrix} = \begin{pmatrix} A^2-1 & A-B \\ B-A & B^2-1 \end{pmatrix},$$

the eigenvalues of M being 0 and $\kappa^2 - 4 > 0$ (thus $N_n^- = 0$). We thus obtain the solutions $\tilde{X}_1 = (1, 0, -B, 0)$ and $\tilde{X}_2 = (0, 1, 0, B)$. Accordingly, we get $X_1 = \mathscr{L}\tilde{X}_1 = (A-B, 0, 1-B^2, 0)$, and $X_2 = \mathscr{L}\tilde{X}_2 = (0, A-B, 0, B^2-1)$, so that $\tilde{X}_1 \cdot X_1 = \tilde{X}_2 \cdot X_2 = A - 2B + B^3 = \frac{(B^2-1)^2}{B} > 0$ and $N_n^+ = 0$. Since we found $n(\mathscr{L}) = 1$, we conclude that $N_{C^+} = 0$: and there is no exponentially unstable solution to the linearized system (which is indeed consistent with the fact that $\mathbb{L}$ has two purely imaginary eigenvalues).

3.4 Instability

For $\tau = +1$, the status of the solution X_* given by (17) changes as κ overtakes the threshold 2: being a minimizer of the energy when $0 < \kappa < 2$, it becomes a local maximum when $\kappa > 2$. We have also seen that the Morse index of $\mathscr{L}$ switches from 1 to 2. In this case, we can adapt the arguments presented in [21, 29] to justify the instability of the reference solution when $\kappa > 2$ (see Figure 4). To prove this statement, we need a series of preparation lemmas, which exploit the algebraic properties of $\mathscr{L}$ and its spectral decomposition.



(a) (b) (c)

Figure 4: Simulation of the nonlinear asymptotic model: phase portrait at $T = 100$, with $\kappa = 1.4$ and $\tau = -1$ (a), with $\kappa = 1.4$ and $\tau = 1$ (b), with $\kappa = 2.4$ and $\tau = 1$ (c). The circled points indicate the initial states, the cross indicate the final states.

Lemma 3.6 *We can find a constant $c > 0$ such that for any $X \in \mathbb{R}^4$ verifying $X \cdot X_* = X \cdot X_{2-\kappa} = X \cdot X_0 = 0$, we have $\mathscr{L}X \cdot X \geqslant c|X|^2$.*

Proof. Since $(X_0, X_{-\kappa}, X_{2-\kappa}, X_2)$ forms an orthogonal basis of $\mathbb{R}^4$ and $X_* = X_{-\kappa}/\sqrt{2}$, the vector we are considering is in fact proportional to X_2: from $X = aX_2$, we deduce that

$$\mathscr{L}X \cdot X = a^2 \mathscr{L}X_2 \cdot X_2 = 2a^2|X_2|^2 = 2|X|^2.$$

■

It is convenient to split $X_* = (X_{*0}, X_{*1})$, with $X_{*0} = X_{*1} = \frac{1}{\sqrt{2}}(1, 0)$ and to consider the rotation matrix in the plane

$$R(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

We shall use the same notation for $V = (V_0, V_1) \in \mathbb{R}^2 \times \mathbb{R}^2$, $R(\theta)V = (R(\theta)V_0, R(\theta)V_1)$.

Lemma 3.7 *Let $\epsilon > 0$ and set*

$$\mathscr{U}_\epsilon = \Big\{ V = (V_0, V_1) \in \mathbb{R}^4,\ \inf_\theta |R(\theta)V - X_*|^2 \leqslant \epsilon \Big\}.$$

For any $V \in \mathscr{U}_\epsilon$, there exists $\theta_*(V) \in [0, 2\pi)$ such that

$$\inf_\theta |R(\theta)V - X_*|^2 = |R(\theta_*(V))V - X_*|^2.$$

Moreover, the following relations hold

$$(i) \qquad \forall \theta' \theta_*(R(\theta')V) = \theta_*(V) - \theta', \qquad\qquad (ii) \qquad \nabla_{V_j}\theta_*(V) = \frac{R'(\theta_*(V))^\intercal X_{*j}}{R(\theta_*(V))^\intercal X_{*j} \cdot V_j}.$$

Proof. The standard argument [21, 29] relies on an application of the implicit function theorem. Here the construction can be made fully explicit. Indeed, given $V \in \mathbb{R}^4$, the 2π-periodic function

$$\theta \longmapsto F(\theta) = |R(\theta)V - X_*|^2 = |R(\theta)V_0 - X_{*0}|^2 + |R(\theta)V_1 - X_{*1}|^2$$

admits a minimizer on $[0, 2\pi]$, characterized by

$$F'(\theta) = 2(R(\theta)V_0 - X_{*0}) \cdot R'(\theta)V_0 + 2(R(\theta)V_1 - X_{*1}) \cdot R'(\theta)V_1 = 0,$$

where

$$R'(\theta) = \begin{pmatrix} -\sin(\theta) & -\cos(\theta) \\ \cos(\theta) & -\sin(\theta) \end{pmatrix}.$$

Since

$$(R'(\theta))^\intercal R(\theta) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

the relation becomes

$$F'(\theta) = -2X_{*0} \cdot R'(\theta)V_0 - 2X_{*1} \cdot R'(\theta)V_1 = 0. \tag{33}$$

Let $V_j = (Q_j, P_j)$. Using the specific expression of X_{*j}, we obtain

$$\sin(\theta)(Q_0 + Q_1) + \cos(\theta)(P_0 + P_1) = 0,$$

which eventually determines the minimizer by

$$\tan(\theta_*(V)) = -\frac{P_0 + P_1}{Q_0 + Q_1}.$$

Differentiating (33) with respect to V_j and using $R''(\theta) = -R(\theta)$ yield

$$(R'(\theta_*(V)))^\intercal X_{*j} - (R(\theta_*(V)))^\intercal X_{*j} \cdot V_j \nabla_{V_j}\theta_*(V) = 0$$

and thus

$$\nabla_{V_j}\theta_*(V) = \frac{(R'(\theta_*(V)))^\intercal X_{*j}}{(R(\theta_*(V)))^\intercal X_{*j} \cdot V_j}.$$

Finally, from $R(\theta + \theta') = R(\theta)R(\theta')$, we infer, for any θ, θ',

$$|R(\theta_*(V) - \theta')R(\theta')V_j - X_{*j}| = |R(\theta_*(V))V_j - X_{*j}| \leqslant |R(\theta + \theta')V_j - X_{*j}| = |R(\theta)R(\theta')V_j - X_{*j}|$$

which means $\theta_*(V) - \theta' = \theta_*(R(\theta')V)$. ∎

We observe that we can move from X_* in a specific direction so that the energy decreases.

Lemma 3.8 *Let $\kappa > 2$ and set $V_s : s \in (-1/\sqrt{2}, 1/\sqrt{2}) \mapsto V_s = \sqrt{1 - 2s^2}X_* + sX_{2-\kappa}$. Then, there exists $0 < s_* < 1/\sqrt{2}$ such that for any $s \in [-s_*, s_*]$, we have $|V_s| = 1$ and $\mathscr{E}(V_s) < \mathscr{E}(X_*)$.*

Proof. We compute

$$|V_s|^2 = (1 - 2s^2)|X_*|^2 + s^2|X_{2-\kappa}|^2 + s\sqrt{1 - 2s^2}X_* \cdot X_{2-\kappa} = 1 - 2s^2 + 2s^2 + 0 = 1.$$

Next, owing to (29), we get the following Taylor expansion

$$\mathscr{E}(V_s) = \mathscr{E}\big(X_* + sX_{2-\kappa} + (\sqrt{1 - 2s^2} - 1)X_*\big) = \mathscr{E}(X_*) + \frac{s^2}{2}\mathscr{L}X_{2-\kappa} \cdot X_{2-\kappa} + s^2\epsilon(s),$$

where $\lim_{s\to 0}\epsilon(s) = 0$. The conclusion follows from the fact that

$$\mathscr{L} X_{2-\kappa} \cdot X_{2-\kappa} = (2-\kappa)|X_{2-\kappa}|^2 < 0.$$

■

We are going to use the specific directions identified in Lemma 3.8 to construct unstable solutions. The instability will be characterized by working on a suitable functional framework which is adapted to the structure of the dynamical system. Let us now consider the functional

$$A : V \in \mathscr{U}_\epsilon \longmapsto -X_2 \cdot R(\theta_*(V))V = (V_1 - V_0) \cdot \begin{pmatrix} \sin(\theta_*(V)) \\ \cos(\theta_*(V)) \end{pmatrix},$$

bearing in mind $X_2 = -\mathscr{J} X_{2-\kappa}$. By using Lemma 3.7-(ii), we get $R(\theta_*(R(\theta)V))R(\theta)V = R(\theta_*(V)-\theta)R(\theta)V = R(\theta_*(V))V$ so that $A(R(\theta)V) = A(V)$. Next, we get

$$\nabla_V A(V) = -R(\theta_*(V))^\intercal X_2 - (X_2 \cdot R'(\theta_*(V))V)\nabla_V \theta_*(V).$$

For $V = X_*$, we have $\theta_*(X_*) = 0$ and thus $X_2 \cdot R'(\theta_*(X_*))X_* = \frac{X_2 \cdot X_0}{\sqrt{2}} = 0$ and

$$\nabla_V A(X_*) = -X_2, \qquad \mathscr{J}\nabla_V A(X_*) = -\mathscr{J} X_2 = -X_{2-\kappa}. \tag{34}$$

Eventually, since $\begin{pmatrix} R(\theta) & 0 \\ 0 & R(\theta) \end{pmatrix} \mathscr{J} = -\begin{pmatrix} R'(\theta) & 0 \\ 0 & R'(\theta) \end{pmatrix}$ and $\mathscr{J}^2 = -\mathbb{I}$, we observe that

$$\begin{aligned}
\nabla_V A(V) \cdot \mathscr{J} V &= -R(\theta_*(V))^\intercal X_2 \cdot \mathscr{J} V - (X_2 \cdot R'(\theta_*(V))V)(\nabla_V \theta_*(V) \cdot \mathscr{J} V) \\
&= -R(\theta_*(V))^\intercal X_2 \cdot \mathscr{J} V + (X_2 \cdot R'(\theta_*(V))V)\frac{-R'(\theta_*(V))^\intercal X_* \cdot \mathscr{J} V}{R(\theta_*(V))^\intercal X_* \cdot V} \\
&= X_2 \cdot R'(\theta_*(V))V + (X_2 \cdot R'(\theta_*(V))V)\frac{-X_* \cdot R(\theta_*(V))V}{X_* \cdot R(\theta_*(V))V} = 0.
\end{aligned}$$

The estimate of Lemma 3.8 can be strengthened as follows.

Lemma 3.9 *Let $\kappa > 2$, set*

$$\mathscr{P}(V) = \nabla_V A(V) \cdot \mathscr{J}\nabla_V \mathscr{E}(V)$$

and let V_s be defined as in Lemma 3.8. Then, there exists $0 < s_* < 1/\sqrt{2}$ such that for any $s \in [-s_*, s_*]$, we have

$$0 < \mathscr{E}(X_*) - \mathscr{E}(V_s) < -s\mathscr{P}(V_s).$$

Proof. The proof is again based on Taylor expansion. In what follows, we denote by $\varrho(s)$ the reminder, whose expression might change from line to line, but such that $\lim_{s\to 0}\varrho(s) = 0$. Since V_s looks like $X_* + sX_{2-\kappa}$, we get, by virtue of (29) and (34),

$$\begin{aligned}
\mathscr{P}(V_s) &= s(\nabla_V A(X_*) + sD_V^2 A(X_*)X_{2-\kappa}) \cdot \mathscr{J} D_V^2 \mathscr{E}(X_*)X_{2-\kappa}) + s\varrho(s) \\
&= -sX_2 \cdot \mathscr{J}\mathscr{L} X_{2-\kappa} + s\varrho(s) = s\mathscr{L} X_{2-\kappa} \cdot X_{2-\kappa} + s\varrho(s) = s(2-\kappa)|X_{2-\kappa}|^2 + s\varrho(s).
\end{aligned}$$

Accordingly, we obtain

$$\mathscr{E}(X_*) - \mathscr{E}(V_s) + s\mathscr{P}(s) = \frac{s^2}{2}\big((2-\kappa) + \varrho(s)\big)$$

which thus remains negative for s small enough. ■

Note that $\mathscr{P}(V) = \nabla_V A(V) \cdot \mathscr{J} \nabla_V \mathscr{H}(V)$ since $\nabla_V F(V) = V$ and $\nabla_V A(V) \cdot \mathscr{J} V = 0$ for all $V \in \mathbb{R}^4$. The motivation for introducing the functional A and $\mathscr{P}$ comes from the fact that, for X solution of (13), we have

$$\frac{\mathrm{d}}{\mathrm{d}t} A(X(t)) = \nabla_U A(X(t)) \cdot \frac{\mathrm{d}}{\mathrm{d}t} X(t) = \nabla_U A(X(t)) \cdot \mathscr{J} \nabla \mathscr{H}(X(t)) = \mathscr{P}(X(t)). \tag{35}$$

Lemma 3.10 *Let $\kappa > 2$ and $\epsilon > 0$ be sufficiently small. Let $V \in \mathscr{U}_\epsilon$ be such that $|V| = |X_*|$ and $\mathscr{E}(X_*) - \mathscr{E}(V) > 0$. Then, we actually have*

$$\mathscr{E}(X_*) - \mathscr{E}(V) < -\Lambda(V)\mathscr{P}(V)$$

where

$$\Lambda(V) = \frac{R(\theta_*(V))V \cdot X_{2-\kappa}}{|X_{2-\kappa}|^2}. \tag{36}$$

Proof. For $V \in \mathbb{R}^4$, set

$$M(V) = R(\theta_*(V))V - X_* - \Lambda(V) X_{2-\kappa}, \tag{37}$$

so that $M(V) \cdot X_{2-\kappa} = 0$. Moreover, we have

$$M(V) \cdot X_0 = R(\theta_*(V))V \cdot X_0 = \sqrt{2}(-\mathscr{J} R'(\theta_*(V)V) \cdot (-\mathscr{J} X_*)) = \sqrt{2} R'(\theta_*(V))V \cdot X_* = 0,$$

by definition of $\theta_*(V)$, see (33). As a consequence, $M(V)$ lies in the orthogonal space of $\mathrm{Span}(X_0, X_{2-\kappa})$ and it can be written

$$M(V) = a(V)X_* + \tilde{M}(V), \text{ where } \tilde{M}(V) \in \mathrm{Span}(X_2). \tag{38}$$

Lemma 3.6 tells us that $\mathscr{L}\tilde{M}(V) \cdot \tilde{M}(V) \geqslant c|\tilde{M}(V)|^2$.

We start by proving

$$\mathscr{P}(V) = \mathscr{P}(R(\theta_*(V))V). \tag{39}$$

Derivating $\mathscr{H}(R(\theta)V) = \mathscr{H}(V)$ and using Lemma 3.7-(i), we get

$$R(\theta)^\intercal \nabla \mathscr{H}(R(\theta)V) = \nabla \mathscr{H}(V), \qquad R(\theta)^\intercal \nabla \theta_*(R(\theta)V) = \nabla \theta_*(V),$$

while

$$R(\theta)^\intercal = R(-\theta) = R(\theta)^{-1}, \qquad \begin{pmatrix} R'(\theta) & 0 \\ 0 & R'(\theta) \end{pmatrix} = -\mathscr{J} \begin{pmatrix} R(\theta) & 0 \\ 0 & R(\theta) \end{pmatrix}.$$

Therefore, we obtain

$$\mathscr{P}(R(\theta)V) = \nabla_U A(R(\theta)V) \cdot \mathscr{J} \nabla \mathscr{H}(R(\theta)V) = \nabla_U A(R(\theta)V) \cdot \mathscr{J} R(\theta) \nabla \mathscr{H}(V).$$

where

$$\begin{aligned}
\nabla A(R(\theta)V) &= -R(\theta_*(R(\theta)V))^\intercal X_2 - (X_2 \cdot R'(\theta_*(R(\theta)V))R(\theta)V)\ \nabla\theta_*(R(\theta)V) \\
&= -R(\theta_*(V) - \theta)^\intercal X_2 - (X_2 \cdot R'(\theta_*(V) - \theta)R(\theta)V)\ R(\theta)\nabla\theta_*(V) \\
&= -R(\theta)R(\theta_*(V))^\intercal X_2 + (X_2 \cdot \mathscr{J} R(\theta_*(V))R(-\theta)R(\theta)V)\ R(\theta)\nabla\theta_*(V) \\
&= R(\theta)\big[-R(\theta_*(V))^\intercal X_2 - (X_2 \cdot R'(\theta_*(V))V)\ \nabla\theta_*(V)\big] = R(\theta)\nabla A(V).
\end{aligned}$$

Hence, (39) holds.

Let $V \in \mathscr{U}_\epsilon$. The definition of $\Lambda(V)$ in (36), $M(V)$ in (37) and $a(V), \tilde{M}(V)$ in (38) leads to the estimates

$$\begin{aligned}
&|\Lambda(V)|^2 = \frac{|R(\theta_*(V))V \cdot X_{2-\kappa}|^2}{|X_{2-\kappa}|^4} = \frac{|(R(\theta_*(V))V - X_*) \cdot X_{2-\kappa}|^2}{|X_{2-\kappa}|^4} \leqslant \frac{|R(\theta_*(V))V - X_*|^2}{|X_{2-\kappa}|^2} \leqslant \frac{\epsilon^2}{4},\\
&|M(V)| \leqslant |R(\theta_*(V))V - X_*| + |\Lambda(V)X_{2-\kappa}| \leqslant 2\epsilon,\\
&|a(V)| \leqslant |M(V)| \leqslant 2\epsilon,\\
&|\tilde{M}(V)| \leqslant |M(V)| + |a(V)| \leqslant 4\epsilon.
\end{aligned}$$

Now, we perform a Taylor expansion on

$$\mathscr{P}(V) = \mathscr{P}(R(\theta_*(V))V) = \mathscr{P}(X_* + \Lambda(V)X_{2-\kappa} + a(V)X_* + \tilde{M}(V)),$$

based on the fact that $\varrho(V) = \Lambda(V)X_{2-\kappa} + a(V)X_* + \tilde{M}(V)$ is of the order of ϵ. Hence, we get

$$\begin{aligned}
\mathscr{P}(V) &= \nabla A(X_* + \varrho(V)) \cdot \mathscr{J}\nabla\mathscr{H}(X_* + \varrho(V))\\
&= (\nabla A(X_*) + D^2A(X_*)\varrho(V)) \cdot \mathscr{J}D^2\mathscr{H}(X_*)\varrho(V) + \mathscr{O}(\epsilon^2)\\
&= \nabla A(X_*) \cdot \mathscr{J}\mathscr{L}\varrho(V) + \mathscr{O}(\epsilon^2) = -\mathscr{L}\mathscr{J}\nabla A(X_*) \cdot \varrho(V) + \mathscr{O}(\epsilon^2)\\
&= \mathscr{L}X_{2-\kappa} \cdot (\Lambda(V)X_{2-\kappa} + a(V)X_* + \tilde{M}(V)) + \mathscr{O}(\epsilon^2)\\
&= (2-\kappa)\Lambda(V)|X_{2-\kappa}|^2 + \mathscr{O}(\epsilon^2).
\end{aligned}$$

Accordingly, we have

$$-\Lambda(V)\mathscr{P}(V) = -(2-\kappa)\Lambda(V)^2|X_{2-\kappa}|^2 + \mathscr{O}(\epsilon^3). \tag{40}$$

Similarly, we go back to the difference of energies

$$\begin{aligned}
0 < \mathscr{E}(X_*) - \mathscr{E}(V) &= \mathscr{E}(X_*) - \mathscr{E}(X_* + \varrho(V)) = -\frac{1}{2}\mathscr{L}\varrho(V) \cdot \varrho(V) + \mathscr{O}(\epsilon^3)\\
&= -\frac{1}{2}\mathscr{L}(\Lambda(V)X_{2-\kappa} + a(V)X_* + \tilde{M}(V)) \cdot (\Lambda(V)X_{2-\kappa} + a(V)X_* + \tilde{M}(V)) + \mathscr{O}(\epsilon^3)\\
&= -\frac{1}{2}\big((2-\kappa)\Lambda(V)X_{2-\kappa} - \kappa a(V)X_* + \mathscr{L}\tilde{M}(V)\big) \cdot (\Lambda(V)X_{2-\kappa} + a(V)X_* + \tilde{M}(V)) + \mathscr{O}(\epsilon^3)\\
&= -\frac{2-\kappa}{2}\Lambda(V)^2|X_{2-\kappa}|^2 + \frac{\kappa}{2}|a(V)|^2 - \frac{1}{2}\mathscr{L}\tilde{M}(V) \cdot \tilde{M}(V) + \mathscr{O}(\epsilon^3).
\end{aligned}$$

We now need to refine the estimate on $a(V) = M(V) \cdot X_* = (R(\theta_*(V))V - X_*) \cdot X_*$. To this end, we use the elementary relation

$$\begin{aligned}
0 = |V|^2 - |X_*|^2 &= |R(\theta_*(V))V|^2 - |X_*|^2 = |(R(\theta_*(V))V - X_*) + X_*|^2 - |X_*|^2\\
&= |R(\theta_*(V))V - X_*|^2 + 2(R(\theta_*(V))V - X_*) \cdot X_*\\
&= |R(\theta_*(V))V - X_*|^2 + 2a(V),
\end{aligned}$$

which yields $|a(V)| \leqslant \frac{\epsilon^2}{2}$. We are thus led to

$$\begin{aligned}
0 < \mathscr{E}(X_*) - \mathscr{E}(V) &= -\frac{1}{2}(2-\kappa)\Lambda(V)^2|X_{2-\kappa}|^2 - \frac{1}{2}\mathscr{L}\tilde{M}(V) \cdot \tilde{M}(V) + \mathscr{O}(\epsilon^3)\\
&\leqslant -\frac{(2-\kappa)}{2}\Lambda(V)^2|X_{2-\kappa}|^2 + \mathscr{O}(\epsilon^3)
\end{aligned}$$

since $\mathscr{L}\tilde{M}(V)\cdot\tilde{M}(V) \geqslant 0$. In particular, this implies that $\Lambda(V)$ does not vanish. We conclude by going back to (40). ■

We argue by contradiction to establish Theorem 2.2. We assume that X_* given by (17) is an orbitally stable solution of (13), meaning that for any $\epsilon > 0$, we can find δ such that $X_{\text{init}} \in \mathscr{U}_\delta$ implies $X(t) \in \mathscr{U}_\epsilon$ for any $t \geqslant 0$. Then, as an initial data we pick $X_{\text{init}} = V_s$ as defined in Lemma 3.8 with $s < 0$ small enough (see Lemma 3.9) so that $|V_s| = |X_*|$, $\mathscr{E}(X_*) - \mathscr{E}(V_s) = \epsilon_* > 0$ and $\mathscr{P}(V_s) > 0$. Let $t \mapsto X(t)$ be the associated solution. By using the conservation properties of the equation, we obtain

$$0 < \epsilon_* = \mathscr{E}(X_*) - \mathscr{E}(X(t)) < -\Lambda(X(t))\mathscr{P}(X(t)).$$

Since $\mathscr{P}(V_s) > 0$ and $|\Lambda(X(t))| \leqslant \frac{\epsilon}{2}$, we get $\mathscr{P}(X(t)) \geqslant C\epsilon_*$ for a certain $C > 0$. We now use (35). Consequently, there holds

$$C\epsilon_* t \leqslant \int_0^t \mathscr{P}(r)\,\mathrm{d}r = \int_0^t \frac{\mathrm{d}}{\mathrm{d}t}A(X(r))\,\mathrm{d}r = A(X(t)) - A(V_s).$$

This contradicts the stability assumption $\{X(t),\ t \geqslant 0\} \subset \mathscr{U}_\epsilon$ which implies that $A(X(t))$ remains bounded. Indeed, $|A(X(t))| \leqslant |X_2|\ |R(\theta_*(X(t)))X(t)| \leqslant |X_2|(|R(\theta_*(X(t)))X(t) - X_*| + |X_*|) \leqslant |X_2|(\epsilon + |X_*|)$.

4 Stability analysis for the coupled system (1)-(2)

4.1 Linearized equations

4.1.1 Linearization about the solution (23)

We search for solutions of (1)-(2) on the form of a perturbation of (23):

$$u_j = e^{i\omega t}(U_{*j} + v_j), \qquad \psi_j = \Psi_{*j} + \phi_j, \qquad \Psi_{*j} = -|U_{*j}|^2(-\Delta)^{-1}\sigma.$$

Using $|u+h|^2 = |u|^2 + 2\mathrm{Re}(\overline{u}h) + |h|^2$ and the dispersion relation (20), we arrive at the following linearized system

$$\begin{aligned}
i\frac{\mathrm{d}}{\mathrm{d}t}v_0 &= \tau v_0 - v_1 + \frac{1}{\sqrt{2}}\int_{\mathbb{R}^n}\sigma\phi_0\,\mathrm{d}z,\\
i\frac{\mathrm{d}}{\mathrm{d}t}v_1 &= \tau v_1 - v_0 + \frac{\tau}{\sqrt{2}}\int_{\mathbb{R}^n}\sigma\phi_1\,\mathrm{d}z,\\
\Big(\frac{1}{c^2}\partial_t^2 - \Delta\Big)\phi_0 &= -\sqrt{2}\sigma\mathrm{Re}(v_0),\\
\Big(\frac{1}{c^2}\partial_t^2 - \Delta\Big)\phi_1 &= -\tau\sqrt{2}\sigma\mathrm{Re}(v_1).
\end{aligned}$$

It is convenient to introduce new unknowns. On the one hand, we expand the complex unknown and consider its real and imaginary parts $u_j = q_j + ip_j$; on the other hand, for the wave equation, we set

$$\varphi_j = (-\Delta)^{1/2}\phi_j, \qquad \varpi_j = \frac{\partial_t\phi_j}{c}.$$

We use a block decomposition of the unknown:

$$X = \begin{pmatrix} S \\ W \end{pmatrix}, \quad W = \begin{pmatrix} W_0 \\ W_1 \end{pmatrix}, \quad S = \begin{pmatrix} S_0 \\ S_1 \end{pmatrix}, \quad W = \begin{pmatrix} W_0 \\ W_1 \end{pmatrix}, \quad S_j = \begin{pmatrix} q_j \\ p_j \end{pmatrix}, \quad W_j = \begin{pmatrix} \varphi_j \\ \varpi_j \end{pmatrix}. \tag{41}$$

Therefore, X has 8 components $(q_0, p_0, q_1, p_1, \phi_0, \varpi_0, \phi_1, \varpi_1)$ and is valued in $\mathbb{R}^4 \times (L^2(\mathbb{R}^n))^4$. With these notations, the problem casts as

$$\partial_t X = \mathbb{L}X,$$

where

$$\mathbb{L}X = \begin{pmatrix} \tau p_0 - p_1 \\ -\tau q_0 + q_1 - \dfrac{1}{\sqrt{2}} \displaystyle\int_{\mathbb{R}^n} \sigma(-\Delta)^{-1/2} \varphi_0 \, \mathrm{d}z \\ -p_0 + \tau p_1 \\ q_0 - \tau q_1 - \dfrac{\tau}{\sqrt{2}} \displaystyle\int_{\mathbb{R}^n} \sigma(-\Delta)^{-1/2} \varphi_1 \, \mathrm{d}z \\ c(-\Delta)^{1/2} \varpi_0 \\ -c(-\Delta)^{1/2} \varphi_0 - c\sqrt{2}\sigma q_0 \\ c(-\Delta)^{1/2} \varpi_1 \\ -c(-\Delta)^{1/2} \varphi_1 - c\sqrt{2}\tau\sigma q_1 \end{pmatrix}.$$

The following statements bring out the basic spectral properties of $\mathbb{L}$ and makes the symplectic structure appear. In terms of stability analysis, it implies that the linearized system is stable provided $\sigma(\mathbb{L}) \subset i\mathbb{R}$. However, the identification of the eigenvalues of $\mathbb{L}$ is now not as direct as the asymptotic problem. The symplectic structure will be crucial to decide whether or not the equation is spectrally stable.

Proposition 4.1 *Let us denote by $\check{X}$ the vector constructed from X by changing the components p_j and ϖ_j into $-p_j$ and $-\varpi_j$. Let (λ, X) be an eigenpair of $\mathbb{L}$. Then, $(-\lambda, \check{X})$, $(\overline{\lambda}, \overline{X})$ and $(-\overline{\lambda}, \overline{\check{X}})$ are as well eigenpairs of $\mathbb{L}$.*

Moreover, we can write $\mathbb{L} = \mathscr{J}\mathscr{L}$ with $\mathscr{J}$ a skew-symmetric operator and $\mathscr{L}$ a self-adjoint operator.

Proof. The first part of the claim follows by direct inspection and using the fact that $\mathbb{L}$ has real coefficients. Next, we introduce the following block-wise operator $\mathscr{J}$ and its formal inverse $\tilde{\mathscr{J}}$

$$\mathscr{J} = \begin{pmatrix} \mathscr{J}_S & 0 & 0 & 0 \\ 0 & \mathscr{J}_S & 0 & 0 \\ 0 & 0 & \mathscr{J}_W & 0 \\ 0 & 0 & 0 & \mathscr{J}_W \end{pmatrix}, \qquad \tilde{\mathscr{J}} = \begin{pmatrix} \tilde{\mathscr{J}}_{\mathscr{S}} & 0 & 0 & 0 \\ 0 & \tilde{\mathscr{J}}_S & 0 & 0 \\ 0 & 0 & \tilde{\mathscr{J}}_W & 0 \\ 0 & 0 & 0 & \tilde{\mathscr{J}}_W \end{pmatrix} \tag{42}$$

where

$$\mathscr{J}_S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \tilde{\mathscr{J}}_S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$$\mathscr{J}_W = 2c \begin{pmatrix} 0 & (-\Delta)^{1/2} \\ -(-\Delta)^{1/2} & 0 \end{pmatrix}, \quad \tilde{\mathscr{J}}_W = \frac{1}{2c} \begin{pmatrix} 0 & -(-\Delta)^{-1/2} \\ (-\Delta)^{-1/2} & 0 \end{pmatrix}.$$

We obtain

$$\mathscr{L}X = \tilde{\mathscr{J}}\mathbb{L}X = \begin{pmatrix} -q_1 + \tau q_0 + \dfrac{1}{\sqrt{2}}\displaystyle\int_{\mathbb{R}^n} \sigma(-\Delta)^{-1/2}\varphi_0 \, \mathrm{d}z \\ \tau p_0 - p_1 \\ -q_0 + \tau q_1 + \dfrac{\tau}{\sqrt{2}}\displaystyle\int_{\mathbb{R}^n} \sigma(-\Delta)^{-1/2}\varphi_1 \, \mathrm{d}z \\ -p_0 + \tau p_1 \\ \dfrac{1}{2}\varphi_0 + \dfrac{1}{\sqrt{2}}(-\Delta)^{-1/2}\sigma q_0 \\ \dfrac{1}{2}\varpi_0 \\ \dfrac{1}{2}\varphi_1 + \dfrac{\tau}{\sqrt{2}}(-\Delta)^{-1/2}\sigma q_1 \\ \dfrac{1}{2}\varpi_1 \end{pmatrix}. \tag{43}$$

We can readily check that $(\mathscr{L}X|X') = (X|\mathscr{L}X')$ holds for the inner product $(X|X') = \sum_{j=0}^{1} q_j q_j' + p_j p_j' + \int_{\mathbb{R}^n}(\varphi_j\varphi_j' + \varpi_j\varpi_j')\,\mathrm{d}z$.

The change of unknowns ensures that $\mathscr{L}$ is self-adjoint and, moreover, that the product $(\mathbb{L}X|X)$ does not involve derivatives of φ_j or ϖ_j, a property that will be useful later on (see Section 4.4). ■

A natural attempt to locate the eigenvalues of $\mathbb{L}$ would rely on a asymptotic argument from the simplified problem (26). However, this program faces severe difficulties. We have seen that the eigenvalues of the asymptotic problem lie in $i\mathbb{R}$; we would like to decide whether the eigenvalues of the coupled problem with finite wave speed c stay on the imaginary axis or split into branches with nonzero real parts. The coupling with the wave equation induces obstructions to asymptotic arguments (as for instance in [13]) that can be described as follows. Let us introduce the function

$$\epsilon \geqslant 0 \longmapsto \kappa_\epsilon = \int_{\mathbb{R}^n} \frac{|\widehat{\sigma}(\xi)|^2}{\epsilon + |\xi|^2}\frac{\mathrm{d}\xi}{(2\pi)^n}.$$

We have $0 < \kappa_\epsilon \leqslant \kappa$ and, by applying the Lebesgue theorem, we can check the continuity of $\epsilon \mapsto \kappa_\epsilon$. However, it fails to be differentiable in general since $\frac{\mathrm{d}}{\mathrm{d}\epsilon}\frac{|\widehat{\sigma}(\xi)|^2}{\epsilon+|\xi|^2} = -\frac{|\widehat{\sigma}(\xi)|^2}{(\epsilon+|\xi|^2)^2}$ is not integrable when $\epsilon = 0$ without introducing further restriction on the dimension n (as $\xi \to 0$ it behaves like $\frac{(\int \sigma(x)\,\mathrm{d}x)^2}{|\xi|^4}$). This explains why the expansion of the eigenvalues as power series of $1/c$ is misleading. Let us go back to the function

$$\lambda = a + ib \in \mathbb{C} \longmapsto \kappa_\lambda = \int_{\mathbb{R}^n} \frac{|\widehat{\sigma}(\xi)|^2}{\lambda^2 + |\xi|^2}\frac{\mathrm{d}\xi}{(2\pi)^n} = \int_{\mathbb{R}^n} \frac{|\widehat{\sigma}(\xi)|^2}{2iab + a^2 - b^2 + |\xi|^2}\frac{\mathrm{d}\xi}{(2\pi)^n}.$$

that we now define on the complex plane. The definition makes sense, except on the imaginary axis $a = 0$. Let us set $A = a^2 - b^2$ and $B = 2ab$. Since σ is radially symmetric, we are led to consider the function

$$P(A, B) = \int_0^\infty \frac{\Sigma(r)}{iB + A + r^2}\,\mathrm{d}r,$$

with $\Sigma(r) = |\widehat{\sigma}(r)|^2 r^{n-1}$. It is well-defined for $B = 0$, and $A \geqslant 0$, and for any $B \neq 0$, $A \in \mathbb{R}$; the difficulty is to deal with $B = 0$ and $A = -\mu < 0$. The lack of continuity near the imaginary axis is illustrated by the following Plemelj-like formula: for $A < 0$ fixed, the limits $B \to 0^{\pm}$ do not coincide. It reflects the jump discontinuity in the resolvent function of $-\Delta$ at the spectrum.

Lemma 4.2 *Let $\mu > 0$. Then, we have*

$$\lim_{B\to 0^{\pm}} P(-\mu, B) = \text{P.V.} \int_0^{+\infty} \frac{\Sigma(r)}{(r-\sqrt{\mu})(r+\sqrt{\mu})}\, \mathrm{d}r \mp i \frac{\pi \Sigma(\sqrt{\mu})}{2\sqrt{\mu}}.$$

For the sake of completeness, the detailed proof is provided in Appendix B. The statement can be expressed by means of the limited absorption principle for the wave equation. This difficulty we are facing can indeed be explained by coming back to the the wave equation, which has an essential spectrum lying all along the imaginary axis. As we shall detail below, we need to discuss Helmholtz-type equation $(\lambda - \Delta)u = f$. The equation perfectly makes sense provided $\lambda \in \mathbb{C}\backslash(-\infty, 0]$. For negative λ, in dimension $n = 3$, this leads to consider $u_{\pm}(x) = \int \frac{e^{\pm i\sqrt{-\lambda}|x-y|}}{|x-y|} f(y)\,\mathrm{d}y$ which both define solutions of the Helmholtz equation, with a different behavior at infinity. These solutions can be obtained as the limits of $(\lambda \pm i\epsilon - \Delta)^{-1} f$ as $\epsilon \to 0$. Hence the resolvent operator is not well-defined, and the functional integrals that one would like to apply as in [13] are misleading.

Let us further illustrate how the difficulty shows up. Searching for eigenvalues of $\mathbb{L}$, we are led to the following nonlinear equation for $\lambda \in \mathbb{C}$ (see the detailed computations in (56) below)

$$\lambda^2 + 4 - 2\tau\kappa_{\lambda^2/c^2} = 0. \tag{44}$$

We wonder whether or not there exists a solution $\lambda = a + ib$ with positive real and imaginary parts. Hence we set $A = a^2 - b^2$ and $B = 2ab$. The latter is supposed to be $\neq 0$ and we are thus led to investigate the zeros of the function

$$F : (A, B) \in \mathbb{R}^2 \longmapsto \begin{pmatrix} A + 4 - 2\tau c^2 \displaystyle\int_{\mathbb{R}^n} \frac{(A + c^2|\xi|^2)|\widehat{\sigma}(\xi)|^2}{(A + c^2|\xi|^2)^2 + B^2} \frac{\mathrm{d}\xi}{(2\pi)^n} \\ 1 + 2\tau \displaystyle\int_{\mathbb{R}^n} \frac{|\widehat{\sigma}(\xi)|^2}{(A + c^2|\xi|^2)^2 + B^2} \frac{\mathrm{d}\xi}{(2\pi)^n} \end{pmatrix}.$$

We do not find explicit solutions for the relation $F(A, B) = 0$, but the problem can be investigated numerically, based on the Newton algorithm. Note however that the Jacobian matrix $\nabla F(A, B)$ becomes singular as B tends to 0, making the problem stiffer as the solution λ is getting close to the imaginary axis. Fig. 5 displays the zeros of F in the (A, B)-plane, for several values of the wave speed c. As c becomes large, we see that the zeros tend to the eigenvalue of the asymptotic problem, which lies on the horizontal axis. It confirms the intuition that the eigenvalues of $\mathbb{L}$ for the coupled problem do have a real part, thus leading to instability, and they should converge as $c \to \infty$ to the purely imaginary eigenvalues of the

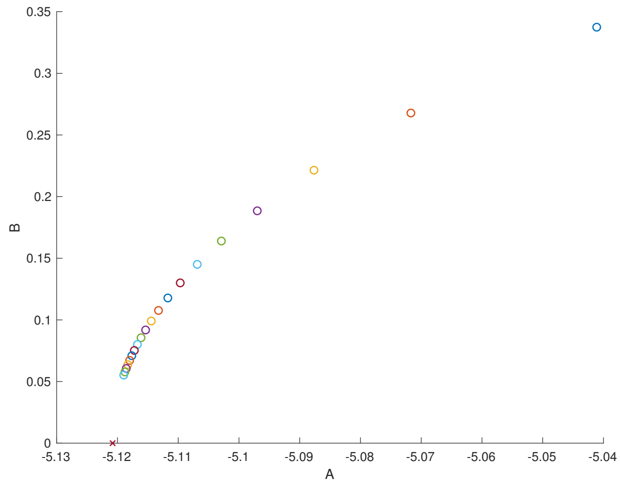


Figure 5: Numerical identification of the zeros of F for several values of the wave speed c ($\kappa = 0.5604$ and $\tau = -1$). The cross on the horizontal axis indicates the coordinates corresponding to the eigenvalue of the asymptotic problem.

asymptotic problem.

For these reasons, we are going to deduce spectral properties on $\mathbb{L}$ from the spectral analysis of $\mathscr{L}$, as proposed in [7]. Indeed, the spectral analysis of the operator $\mathscr{L}$ is easier; at least we know that the spectrum embeds into $\mathbb{R}$ due to the self-adjointness character of $\mathscr{L}$. The spectral properties of the operator $\mathscr{L}$ are summarized in the following statement. Note that, due to the coupling with the wave equation on the whole $\mathbb{R}^n$, there is a non-empty essential spectrum. From now on, we denote by $\mathbf{0} = (0, 0, 0, 0)$.

Theorem 4.3 *Let $\mathscr{L}$ be the operator defined by (43). Then, the following assertions hold:*

1. $\mathrm{Ker}(\mathscr{L}) = \mathrm{Span}(X_0)$, *with* $X_0 = (S_0, \mathbf{0})$, $S_0 = (0, 1, 0, \tau)$*;*
2. $\sigma_{\mathrm{ess}}(\mathscr{L}) = \{1/2\}$*;*
3. *If $\tau = +1$ and $0 < \kappa < 2$, $\mathscr{L}$ has one negative eigenvalue, associated to a one-dimensional eigenspace; if $\tau = +1$ and $\kappa > 2$, $\mathscr{L}$ has two negative eigenvalues, associated to one-dimensional eigenspaces; if $\tau = -1$ $\mathscr{L}$ has three negative eigenvalues associated to one-dimensional eigenspaces;*
4. *Given Y_0 a solution of $\mathscr{L} Y_0 = -\mathscr{J} X_0$, we have $(-\mathscr{J} X_0 | Y_0) < 0$.*

Proof. The operator $\mathscr{L}$ being self-adjoint, its spectrum lies in $\mathbb{R}$. Let us study the solutions of $\mathscr{L} X = \lambda X$.

In particular, we have $\lambda\varpi_j = \varpi_j/2$. Hence, when $\lambda = 1/2$, any $X = (\mathbf{0}, W)$, with $W = (0, \pi, 0, 0)$ or $(0, 0, 0, \pi)$, $\pi \in L^2(\mathbb{R}^n)$, lies in $\mathrm{Ker}(\mathscr{L} - 1/2)$. Furthermore, we also have

$$\Big(\lambda - \frac{1}{2}\Big)\varphi_0 = \frac{1}{\sqrt{2}}(-\Delta)^{1/2}\sigma q_0,$$
$$\Big(\lambda - \frac{1}{2}\Big)\varphi_1 = \frac{\tau}{\sqrt{2}}(-\Delta)^{1/2}\sigma q_1.$$

Next, we can write

$$\int_{\mathbb{R}^n} \sigma(-\Delta)^{-1/2}\varphi_j \,\mathrm{d}z = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \frac{\widehat{\sigma}\overline{\widehat{\varphi_j}}}{|\xi|}\,\mathrm{d}\xi = \int_{\mathbb{R}^n} (-\Delta)^{-1/2}\sigma\varphi_j \,\mathrm{d}z.$$

Hence, when $\lambda = 1/2$, any $X = (\mathbf{0}, W)$, with $W = (\varphi, 0, 0, 0)$ or $(0, 0, \varphi, 0)$, $\varphi \in L^2(\mathbb{R}^n)$ orthogonal to $(-\Delta)^{-1/2}\sigma$, lies in $\mathrm{Ker}(\mathscr{L} - 1/2)$. Therefore, for $\lambda = 1/2$, we have fully identified the eigenspace which is infinite-dimensional. Reasoning by a contradiction argument, based on Weyl's criterion, we can show that there is no other values in the essential spectrum of $\mathscr{L}$, see [16].

From now on, we suppose $\lambda \neq 1/2$. It allows us to infer $\varpi_0 = \varpi_1 = 0$ and

$$\varphi_0 = \frac{(-\Delta)^{-1/2}\sigma q_0}{\sqrt{2}(\lambda - 1/2)}, \qquad \varphi_1 = \tau\frac{(-\Delta)^{-1/2}\sigma q_1}{\sqrt{2}(\lambda - 1/2)}.$$

Consequently, bearing in mind $\int \sigma(-\Delta)^{-1}\sigma \,\mathrm{d}z = \kappa$, we obtain the following 4×4 system for $S = (q_0, p_0, q_1, p_1)$,

$$\lambda S = \begin{pmatrix} \tau + \dfrac{\kappa/2}{\lambda - 1/2} & 0 & -1 & 0 \\ 0 & \tau & 0 & -1 \\ -1 & 0 & \tau + \dfrac{\kappa/2}{\lambda - 1/2} & 0 \\ 0 & -1 & 0 & \tau \end{pmatrix} S.$$

We remark that the relations for (q_0, q_1) and (p_0, p_1) are uncoupled. We start by observing that $\lambda p_0 = \tau p_0 - p_1$ and $\lambda p_1 = -p_0 + \tau p_1$ which admit nontrivial solutions provided

$$(\lambda - \tau)^2 - 1 = \lambda(\lambda - 2\tau) = 0.$$

Hence, 0 and 2τ are eigenvalues for $\mathscr{L}$ with $\mathrm{Span}(0, 1, 0, \tau, \mathbf{0}) \subset \mathrm{Ker}(\mathscr{L})$, and $\mathrm{Span}(0, 1, 0, -\tau, \mathbf{0}) \subset \mathrm{Ker}(\mathscr{L} - 2\tau)$, respectively. We turn to the equations for (q_0, q_1) which admit nontrivial solutions provided

$$\Big(\lambda - \tau - \frac{\kappa/2}{\lambda - 1/2}\Big)^2 - 1 = \Big(\lambda - \tau - \frac{\kappa/2}{\lambda - 1/2} - 1\Big)\Big(\lambda - \tau - \frac{\kappa/2}{\lambda - 1/2} + 1\Big) = 0.$$

This holds iff $(\lambda - 1/2)(\lambda - \tau - 1) - \kappa/2 = 0$ or $(\lambda - 1/2)(\lambda - \tau + 1) - \kappa/2 = 0$. We distinguish the two cases:

- If $\tau = +1$, we get $(\lambda - 1/2)(\lambda - 2) - \kappa/2 = 0$ or $(\lambda - 1/2)\lambda - \kappa/2 = 0$;
- If $\tau = -1$, we get $(\lambda - 1/2)\lambda - \kappa/2 = 0$ or $(\lambda - 1/2)(\lambda + 2) - \kappa/2 = 0$.

In both cases, with the second order equation $(\lambda - 1/2)\lambda - \kappa/2 = \lambda^2 - \lambda/2 - \kappa/2 = 0$, we find the following eigenvalues of opposite signs

$$\lambda = \frac{1/2 \pm \sqrt{1/4 + 2\kappa}}{2} \in \sigma(\mathscr{L}).$$

Moreover, from $(\lambda - 1/2)(\lambda - 2\tau) - \kappa/2 = \lambda^2 - (1/2 + 2\tau)\lambda + \tau - \kappa/2 = 0$, we find

$$\lambda = \frac{1/2 + 2\tau \pm \sqrt{(1/2 - 2\tau)^2 + 2\kappa}}{2} \in \sigma(\mathscr{L}).$$

Hence, when $\tau = +1$ with $0 < \kappa < 2$, this gives two positive eigenvalues; when $\tau = -1$ or $\tau = +1$ with $\kappa > 2$ we obtain two eigenvalues of opposite signs.

Finally, $-\mathscr{J} X_0$ reads $(-1, 0, -\tau, 0, \mathbf{0})$. It is orthogonal to $\mathrm{Ker}(\mathscr{L}) = \mathrm{Span}(X_0)$ and it makes sense to consider the equation $\mathscr{L} Y_0 = -\mathscr{J} X_0$. Imposing $Y_0 \in (\mathrm{Ker}(\mathscr{L}))^\perp$, we find

$$Y_0 = \frac{1}{\kappa}\left(1, 0, \tau, 0, -\sqrt{2}(-\Delta)^{-1/2}\sigma, 0, -\sqrt{2}(-\Delta)^{-1/2}\sigma, 0\right).$$

Accordingly, we get $(-\mathscr{J} X_0 | Y_0) = -\frac{2}{\kappa} < 0$. (This product is left unchanged by adding to Y_0 any element of $\mathrm{Ker}(\mathscr{L})$.) ∎

4.1.2 Linearization about the extra solutions when $\kappa > 2$

Let us now assume $\kappa > 2$. We use the same notation as in (28). Considering a perturbation of the solution given by (24)-(25), the linearized equations read

$$\begin{aligned}
i\partial_t v_0 &= A v_0 - v_1 + \alpha \int_{\mathbb{R}^n} \sigma \phi_0 \, \mathrm{d}z, \\
i\partial_t v_1 &= B v_1 - v_0 + \beta \int_{\mathbb{R}^n} \sigma \phi_1 \, \mathrm{d}z, \\
\frac{1}{c^2}\partial_t^2 \phi_0 - \Delta \phi_0 &= -\alpha 2\sigma \mathrm{Re}(v_0), \\
\frac{1}{c^2}\partial_t^2 \phi_1 - \Delta \phi_1 &= -\beta 2\sigma \mathrm{Re}(v_1).
\end{aligned}$$

With the change of variables

$$(v_j, = q_j + i p_j, \phi_j) \to \left(q_j, p_j, \varphi_j = (-\Delta)^{1/2}\phi_j, \varpi_j = \frac{\partial_t \phi_j}{c}\right),$$

we get

$$\partial_t X = \mathbb{L} X$$

with

$$\mathbb{L}X = \begin{pmatrix} Ap_0 - p_1 \\ -Aq_0 + q_1 - \alpha \displaystyle\int_{\mathbb{R}^n} (-\Delta)^{-1/2} \sigma \varphi_0 \,\mathrm{d}z \\ -p_0 + Bp_1 \\ q_0 - Bq_1 - \beta \displaystyle\int_{\mathbb{R}^n} (-\Delta)^{-1/2} \sigma \varphi_1 \,\mathrm{d}z \\ c(-\Delta)^{1/2} \varpi_0 \\ -c(-\Delta)^{1/2} \varphi_0 - 2c\alpha\sigma q_0 \\ c(-\Delta)^{1/2} \varpi_1 \\ -c(-\Delta)^{1/2} \varphi_1 - 2c\beta\sigma q_1 \end{pmatrix}.$$

We set $\mathbb{L} = \mathscr{J}\mathscr{L}$, with $\mathscr{J}$ defined by (42) and

$$\mathscr{L}X = \begin{pmatrix} Aq_0 - q_1 + \alpha \displaystyle\int_{\mathbb{R}^n} (-\Delta)^{-1/2} \sigma \varphi_0 \,\mathrm{d}z \\ Ap_0 - p_1 \\ Bq_1 - q_0 + \beta \displaystyle\int_{\mathbb{R}^n} (-\Delta)^{-1/2} \sigma \varphi_1 \,\mathrm{d}z \\ -p_0 + Bp_1 \\ \dfrac{\varphi_0}{2} + \alpha(-\Delta)^{-1/2} \sigma q_0 \\ \dfrac{\varpi_0}{2} \\ \dfrac{\varphi_1}{2} + \beta(-\Delta)^{-1/2} \sigma q_1 \\ \dfrac{\varpi_1}{2} \end{pmatrix}. \tag{45}$$

We readily obtain the following analog to Proposition 4.1.

Proposition 4.4 *Let us denote by $\check{X}$ the vector constructed from X by changing the components p_j and ϖ_j into $-p_j$ and $-\varpi_j$. Let (λ, X) be an eigenpair of $\mathbb{L}$. Then, $(-\lambda, \check{X})$, $(\overline{\lambda}, \overline{X})$ and $(-\overline{\lambda}, \overline{\check{X}})$ are equally eigenpairs of $\mathbb{L}$.*

Moreover, we can write $\mathbb{L} = \mathscr{J}\mathscr{L}$ with $\mathscr{J}$ a skew-symmetric operator and $\mathscr{L}$ a self-adjoint operator.

The next step consists in studying the spectrum of the self-adjoint operator.

Theorem 4.5 *Let $\mathscr{L}$ be the operator defined by (45). Then, the following assertions hold:*

1. $\mathrm{Ker}(\mathscr{L}) = \mathrm{Span}(X_0)$, *with* $X_0 = (S_0, \mathbf{0})$, $S_0 = (0, 1, 0, A)$;
2. $\sigma_{\mathrm{ess}}(\mathscr{L}) = \{1/2\}$;
3. $\mathscr{L}$ *has one negative eigenvalue, associated to a one-dimensional eigenspace* ($n(\mathscr{L}) = 1$);

4. Given Y_0 a solution of $\mathscr{L}Y_0 = -\mathscr{J}X_0$, we have $(-\mathscr{J}X_0|Y_0) < 0$.

Proof. The proof of the second item follows exactly the same lines as in Theorem 4.3. We also readily check that $\mathrm{Ker}(\mathscr{L}) = \mathrm{Span}\{(0, 1, 0, A, \mathbf{0})\}$. We have $-\mathscr{J}X_0 = (-1, 0, -A, 0)$ and solving $\mathscr{L}Y_0 = -\mathscr{J}X_0$ with $Y_0 \in (\mathrm{Ker}(\mathscr{L}))^\perp$ yields

$$Y_0 = \frac{1}{2(A-B)}(-1, 0, A, 0, 2\alpha(-\Delta)^{-1/2}\sigma, 0, -2A\beta(-\Delta)^{-1/2}\sigma, 0), \tag{46}$$

and thus we get $(Y_0| - \mathscr{J}X_0) = -\frac{A}{2} < 0$.

We now study the eigenvalues $\lambda \notin \{0, 1/2\}$ of $\mathscr{L}$. We arrive at the matrix system

$$\begin{pmatrix} A + \dfrac{B}{\lambda - 1/2} & 0 & -1 & 0 \\ 0 & A & 0 & -1 \\ -1 & 0 & B + \dfrac{A}{\lambda - 1/2} & 0 \\ 0 & -1 & 0 & B \end{pmatrix} \begin{pmatrix} q_0 \\ p_0 \\ q_1 \\ p_1 \end{pmatrix} = \lambda \begin{pmatrix} q_0 \\ p_0 \\ q_1 \\ p_1 \end{pmatrix}.$$

The equations for (p_0, p_1) and (q_0, q_1) uncouple. The former leads to the relation

$$\lambda(\lambda - A - B) = 0$$

which gives the eigenvalues 0 and $A + B = \kappa$. The latter leads to the relation

$$\begin{aligned} 0 &= \left(A + \frac{B}{\lambda - 1/2} - \lambda\right)\left(B + \frac{A}{\lambda - 1/2} - \lambda\right) - 1 \\ &= \frac{1}{(2\lambda - 1)^2}\left(4\lambda^4 - 4(A + B + 1)\lambda^3 + \lambda^2 + (4A^2 + 4B^2 + A + B)\lambda - 2(A - B)^2\right) \\ &= \frac{1}{(2\lambda - 1)^2}\left(4\lambda^4 - 4(\kappa + 1)\lambda^3 + \lambda^2 + (4A^2 + 4B^2 + \kappa)\lambda - 2(A - B)^2\right) = \frac{1}{(2\lambda - 1)^2}P(\lambda) \end{aligned}$$

with P a fourth order polynomial. Descartes' rule of sign then tells us that P has exactly one negative root, see Fig. 6. We have thus proved the third item in Theorem 4.5. ■

4.2 Spectral and linearized stability

We start with the study of the spectral stability of the solution (23) of (1)-(2). Let $\mathscr{L}$ be defined by (43). According to [7], we introduce the operator

$$\mathscr{M} = -\mathscr{J}\mathscr{L}\mathscr{J}, \qquad \mathbb{A} = \mathscr{P}\mathscr{M}\mathscr{P},$$

where $\mathscr{P}$ is the orthogonal projection on $(\mathrm{Ker}(\mathscr{L}))^\perp$, and we set

$$\mathbb{K} = \mathscr{P}\mathscr{L}^{-1}\mathscr{P}.$$

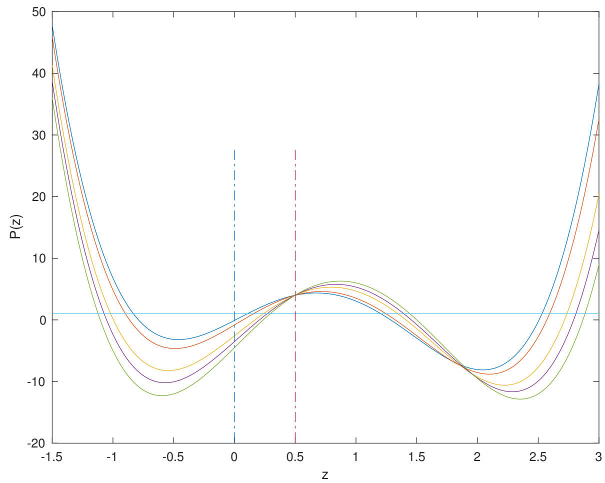


Figure 6: Graph of the polynomial function $z \mapsto P(z)$ for several values of κ ($\kappa \in \{2.01, 2.1, 2.3, 2.4, 2.5\}$).

We are interested in the generalized eigenvalue problem

$$\mathscr{M}X = \mu\tilde{X}, \qquad \mathscr{L}\tilde{X} = X.$$

Recall that X has to belong to $(\mathrm{Ker}(\mathscr{L}))^\perp$ and we need to compute the product $(\mathbb{K}X|X) = (\tilde{X}|X)$, which is thus left unchanged by adding to $\tilde{X}$ an element in $\mathrm{Ker}(\mathscr{L})$. Hence, $\tilde{X}$ can be chosen in $(\mathrm{Ker}(\mathscr{L}))^\perp$. Solving the generalized eigenvalue problem amounts to solving

$$\begin{aligned}
&\tau q_0 - q_1 = \mu\tilde{q}_0, && \tau p_0 - p_1 + c\sqrt{2}\int_{\mathbb{R}^n}\sigma\varpi_0\,\mathrm{d}z = \mu\tilde{p}_0,\\
&-q_0 + \tau q_1 = \mu\tilde{q}_1, && -p_0 + \tau p_1 + \tau c\sqrt{2}\int_{\mathbb{R}^n}\sigma\varpi_1\,\mathrm{d}z = \mu\tilde{p}_1,\\
&2c^2(-\Delta\varphi_0) = \mu\tilde{\varphi}_0, && 2c^2(-\Delta\varpi_0) + c\sqrt{2}\sigma p_0 = \mu\tilde{\varpi}_0,\\
&2c^2(-\Delta\varphi_1) = \mu\tilde{\varphi}_1, && 2c^2(-\Delta\varpi_1) + \tau c\sqrt{2}\sigma p_1 = \mu\tilde{\varpi}_1,
\end{aligned}$$

coupled to

$$\begin{aligned}
&\tau\tilde{q}_0 - \tilde{q}_1 + \frac{1}{\sqrt{2}}\int_{\mathbb{R}^n}\sigma(-\Delta)^{-1/2}\tilde{\varphi}_0\,\mathrm{d}z = q_0, && \tau\tilde{p}_0 - \tilde{p}_1 = p_0,\\
&-\tilde{q}_0 + \tau\tilde{q}_1 + \frac{\tau}{\sqrt{2}}\int_{\mathbb{R}^n}\sigma(-\Delta)^{-1/2}\tilde{\varphi}_1\,\mathrm{d}z = q_1, && -p_0 + \tau\tilde{p}_1 = p_1,\\
&\frac{\tilde{\varphi}_0}{2} + \frac{1}{\sqrt{2}}(-\Delta)^{-1/2}\sigma\tilde{q}_0 = \varphi_0, && \frac{\tilde{\varpi}_0}{2} = \varpi_0,\\
&\frac{\tilde{\varphi}_1}{2} + \frac{\tau}{\sqrt{2}}(-\Delta)^{-1/2}\sigma\tilde{q}_1 = \varphi_1, && \frac{\tilde{\varpi}_1}{2} = \varpi_1.
\end{aligned}$$

This leads to the following relations

$$\Big(-\frac{\mu}{c^2}-\Delta\Big)\varpi_0=-\frac{1}{c\sqrt{2}}\sigma p_0,\qquad\qquad \Big(-\frac{\mu}{c^2}-\Delta\Big)\varpi_1=-\tau\frac{1}{c\sqrt{2}}\sigma p_1,$$

and

$$\Big(-\frac{\mu}{c^2}-\Delta\Big)\tilde\varphi_0=-\sqrt{2}(-\Delta)^{1/2}\sigma\tilde q_0,\qquad \Big(-\frac{\mu}{c^2}-\Delta\Big)\tilde\varphi_1=-\tau\sqrt{2}(-\Delta)^{1/2}\sigma\tilde q_1.$$

When $\mu<0$, these equations can be solved by means of the Fourier transform and we get

$$\widehat{\varpi}_0(\xi)=-\frac{1}{c\sqrt{2}}\frac{\widehat{\sigma}(\xi)}{|\mu|/c^2+|\xi|^2}p_0,\qquad\qquad \widehat{\varpi}_1(\xi)=-\frac{\tau}{c\sqrt{2}}\frac{\widehat{\sigma}(\xi)}{|\mu|/c^2+|\xi|^2}p_1,$$
$$\widehat{\tilde\varphi_0}=-\frac{\sqrt{2}|\xi|\widehat{\sigma}(\xi)}{|\mu|/c^2+|\xi|^2}\tilde q_0,\qquad\qquad \widehat{\tilde\varphi_1}=-\frac{\tau\sqrt{2}|\xi|\widehat{\sigma}(\xi)}{|\mu|/c^2+|\xi|^2}\tilde q_1.$$

It follows that

$$\frac{1}{\sqrt{2}}\int_{\mathbb{R}^n}\sigma(-\Delta)^{-1/2}\tilde\varphi_0\,\mathrm{d}z=-\tilde q_0\underbrace{\int_{\mathbb{R}^n}\frac{|\widehat{\sigma}(\xi)|^2}{|\mu|/c^2+|\xi|^2}\frac{\mathrm{d}\xi}{(2\pi)^n}}_{=\kappa_{|\mu|/c^2}},\quad \frac{\tau}{\sqrt{2}}\int_{\mathbb{R}^n}\sigma(-\Delta)^{-1/2}\tilde\phi_1\,\mathrm{d}z=-\tilde q_1\kappa_{|\mu|/c^2},$$

$$c\sqrt{2}\int_{\mathbb{R}^n}\sigma\varpi_0\,\mathrm{d}z=-p_0\kappa_{|\mu|/c^2}\qquad\qquad \tau c\sqrt{2}\int_{\mathbb{R}^n}\sigma\varpi_1\,\mathrm{d}z=-p_1\kappa_{|\mu|/c^2}.$$

With the matrices defined in (32), we are thus led to

$$M_0\begin{pmatrix}\tilde p_0\\ \tilde p_1\end{pmatrix}=\begin{pmatrix}p_0\\ p_1\end{pmatrix},\qquad M_{\kappa_{|\mu|/c^2}}\begin{pmatrix}p_0\\ p_1\end{pmatrix}=\mu\begin{pmatrix}\tilde p_0\\ \tilde p_1\end{pmatrix},$$

together with

$$M_{\kappa_{|\mu|/c^2}}\begin{pmatrix}\tilde q_0\\ \tilde q_1\end{pmatrix}=\begin{pmatrix}q_0\\ q_1\end{pmatrix},\qquad M_0\begin{pmatrix}q_0\\ q_1\end{pmatrix}=\mu\begin{pmatrix}\tilde q_0\\ \tilde q_1\end{pmatrix}.$$

Since $M_{\kappa_{|\mu|/c^2}}M_0=M_0M_{\kappa_{|\mu|/c^2}}$, we deduce, like for the asymptotic model, that $\mu<0$ should be such that $\det(M_0M_{\kappa_{|\mu|/c^2}}-\mu\mathbb{I})=0$. This condition leads to

$$\begin{aligned}0&=(2-\tau\kappa_\gamma+\gamma c^2)^2-(\kappa_\gamma-2\tau)^2=(2-\tau\kappa_\gamma+\gamma c^2-\kappa_\gamma+2\tau)(2-\tau\kappa_\gamma+\gamma c^2+\kappa_\gamma-2\tau)\\&=\gamma c^2(2(2-\tau\kappa_\gamma)+\gamma c^2)\end{aligned}$$

where we set $\gamma=-\frac{\mu}{c^2}=\frac{|\mu|}{c^2}$. When $\tau=-1$ or $\tau=+1$ with $0<\kappa<2$, we have $2(2-\tau\kappa_\gamma)+\gamma c^2>0$ for any positive γ, hence there is no solution to this equation: in these cases we have $N_n^-=0$. If $\tau=+1$ and $\kappa>2$, it is thus required to make the nonlinear quantity

$$F(\gamma)=\gamma-\frac{2}{c^2}(\kappa_\gamma-2)$$

vanishes. The function F is continuous, increasing from $(0,\infty)$ to $(-\frac{2}{c^2}(\kappa-2),+\infty)$; hence there exists a unique $\gamma_c = -\frac{\mu_c}{c^2} > 0$ such that $F(\gamma_c) = 0$. Finally, we have to compute $(\mathbb{K}X, X)$. Since $X \in (\mathrm{Ker}(\mathscr{L}))^\perp$, $\mathcal{P}X = X$ and $(\mathbb{K}X, X) = (\tilde{X}, X)$. Using the equations for $(\tilde{q}_0, \tilde{q}_1)$ and $(\tilde{p}_0, \tilde{p}_1)$ together with $\gamma_c c^2 = 2(\kappa_{\gamma_c} - 2) > 0$, we deduce that the eigenvectors associated to μ_c are such that $\tilde{q}_1 = -\tilde{q}_0$ and $\tilde{p}_1 = -\tilde{p}_0$. On the one hand, choosing $\tilde{q}_0 = 1$ and $\tilde{p}_0 = 0$, we have

$$\tilde{X} = \left(1, 0, -1, 0, \mathcal{F}^{-1}\left(-\frac{\sqrt{2}|\xi|\widehat{\sigma}(\xi)}{\gamma_c + |\xi|^2}\right), 0, \mathcal{F}^{-1}\left(\frac{\sqrt{2}|\xi|\widehat{\sigma}(\xi)}{\gamma_c + |\xi|^2}\right), 0\right)$$

$$X = \left(2 - \kappa_{\gamma_c}, 0, \kappa_{\gamma_c} - 2, 0, \frac{1}{\sqrt{2}}\mathcal{F}^{-1}\left(-\frac{|\xi|\widehat{\sigma}(\xi)}{\gamma_c + |\xi|^2} + \frac{\widehat{\sigma}(\xi)}{|\xi|}\right), 0, \frac{1}{\sqrt{2}}\mathcal{F}^{-1}\left(\frac{|\xi|\widehat{\sigma}(\xi)}{\gamma_c + |\xi|^2} - \frac{\widehat{\sigma}(\xi)}{|\xi|}\right), 0\right)$$

so that

$$\begin{aligned}(\tilde{X}, X) &= -2(\kappa_{\gamma_c} - 2) + 2\int_{\mathbb{R}^n}\left(\frac{|\xi|^2|\widehat{\sigma}(\xi)|^2}{(\gamma_c + |\xi|^2)^2} - \frac{|\widehat{\sigma}(\xi)|^2}{\gamma_c + |\xi|^2}\right)\frac{\mathrm{d}\xi}{(2\pi)^n} \\ &= -\gamma_c c^2 - 2\gamma_c\int_{\mathbb{R}^n}\frac{|\widehat{\sigma}(\xi)|^2}{(\gamma_c + |\xi|^2)^2}\frac{\mathrm{d}\xi}{(2\pi)^n} < 0.\end{aligned}$$

On the other hand, choosing $\tilde{q}_0 = 0$ and $\tilde{p}_0 = 1$, we have

$$\tilde{X} = \left(0, 1, 0, -1, 0, -\frac{2\sqrt{2}}{c}\mathcal{F}^{-1}\left(\frac{\widehat{\sigma}(\xi)}{\gamma_c + |\xi|^2}\right), 0, \frac{2\sqrt{2}}{c}\mathcal{F}^{-1}\left(\frac{\widehat{\sigma}(\xi)}{\gamma_c + |\xi|^2}\right)\right)$$

$$X = \left(0, 2, 0, -2, 0, -\frac{\sqrt{2}}{c}\mathcal{F}^{-1}\left(\frac{\widehat{\sigma}(\xi)}{\gamma_c + |\xi|^2}\right), 0, \frac{\sqrt{2}}{c}\mathcal{F}^{-1}\left(\frac{\widehat{\sigma}(\xi)}{\gamma_c + |\xi|^2}\right)\right)$$

so that

$$(\tilde{X}, X) = 4 + \frac{8}{c^2}\int_{\mathbb{R}^n}\frac{|\widehat{\sigma}(\xi)|^2}{(\gamma_c + |\xi|^2)^2}\frac{\mathrm{d}\xi}{(2\pi)^n} > 0.$$

We can conclude $N_n^- = 1$.

When $\mu > 0$, the symbol $\frac{\widehat{\sigma}(\xi)}{|\xi|^2 - \mu/c^2}$ has a singularity which is not square integrable; this forces to set $p_0 = p_1 = 0$, and $\tilde{q}_0 = \tilde{q}_1 = 0$, so that $\varpi_0 = \varpi_1 = 0$, and $\tilde{\phi}_0 = \tilde{\phi}_1 = 0$. It implies $q_0 = q_1 = 0$ and $\tilde{p}_0 = \tilde{p}_1 = 0$; there is no nontrivial solution of the generalized eigenvalue problem with $\mu > 0$, that is $N_n^+ = 0$.

For $\mu = 0$, the equations reduce to

$$\begin{aligned}
&\tau q_0 - q_1 = 0 && \tau p_0 - p_1 + c\sqrt{2}\int_{\mathbb{R}^n}\sigma\varpi_0\,\mathrm{d}z = 0,\\
&-q_0 + \tau q_1 = 0, && -p_0 + \tau p_1 + \tau c\sqrt{2}\int_{\mathbb{R}^n}\sigma\varpi_1\,\mathrm{d}z = 0\\
&2c^2(-\Delta\varphi_0) = 0 && 2c^2(-\Delta\varpi_0) + c\sqrt{2}\sigma p_0 = 0\\
&2c^2(-\Delta\varphi_1) = 0 && 2c^2(-\Delta\varpi_1) + \tau c\sqrt{2}\sigma p_1 = 0
\end{aligned}$$

It yields $\varphi_0 = \varphi_1 = 0$ and $\varpi_0 = -\frac{1}{c\sqrt{2}}(-\Delta)^{-1}\sigma p_0$, $\varpi_1 = -\frac{\tau}{c\sqrt{2}}(-\Delta)^{-1}\sigma p_1$, hence the systems

$$M_0 \begin{pmatrix} q_0 \\ q_1 \end{pmatrix} = 0, \qquad M_\kappa \begin{pmatrix} p_0 \\ p_1 \end{pmatrix} = 0.$$

Solving these systems, we obtain $p_0 = 0 = p_1$ and $q_1 = \tau q_0$. As a consequence X is proportional to $-\mathscr{J} X_0$. Reinterpreting $(\mathbb{K}X, X) = (\mathbb{K}(-\mathscr{J} X_0)| - \mathscr{J} X_0)$ as $(-\mathscr{J} X_0|Y_0)$ with Y_0 given in Theorem 4.3, we obtain $(\mathbb{K}(-\mathscr{J} X_0)| - \mathscr{J} X_0) < 0$ and we conclude that $N_n^0 = 1$.

To sum up, we have the following

$$N_n^0 = 1, N_n^+ = 0 \text{ and } N_n^- = \begin{cases} 0 \text{ if } \tau = -1 \text{ or } \tau = 1 \text{ and } \kappa < 2, \\ 1 \text{ if } \tau = 1 \text{ and } \kappa > 2. \end{cases}.$$

We remind the reader that the spectral stability means that the spectrum of $\mathbb{L}$ is contained in $i\mathbb{R}$. To derive information about $\sigma(\mathbb{L})$, we use the counting argument of [7, Theorem 1] (see also [28]) which asserts that

$$N_n^- + N_n^0 + N_n^+ + N_{C^+} = n(\mathscr{L}).$$

The presence of spectrally unstable directions corresponds to $N_n^- \neq 0$ or $N_{C^+} \neq 0$. Gathering the obtained information, we infer that

- if $\tau = 1$ and $\kappa < 2$, $N_n^- = 0$ and $N_{C^+} = n(\mathscr{L}) - 1 = 0$;
- if $\tau = 1$ and $\kappa > 2$, $N_n^- = 1$ and $N_{C^+} = n(\mathscr{L}) - 1 - 1 = 0$;
- if $\tau = -1$, $N_n^- = 0$ and $N_{C^+} = n(\mathscr{L}) - 1 = 2$.

Accordingly, we conclude with the following claim.

Proposition 4.6 *Suppose* $0 < \kappa < 2$ *and let* $\tau = +1$. *Then, the reference solution* (23) *of* (1)-(2) *is spectrally stable. If* $\tau = -1$ *or* $\tau = +1$ *with* $\kappa > 2$, *the reference solution* (23) *of* (1)-(2) *is spectrally unstable.*

This result is illustrated in Figure 7 (complemented by Figure 8 about the non linear system). Inspired by the asymptotic problem, see Proposition 3.2 and Figure 3, we guess that the linearized stabiltiy requires suitable orthogonality conditions. Indeed, we can check that $\text{Ker}(\mathbb{L}) = \text{Span}(0, 1, 0, \tau, \mathbf{0})$ and $\text{Ker}(\mathbb{L}^*) = \text{Span}(1, 0, \tau, 0, \mathbf{0})$. In particular, denoting $\Psi = (1, 0, \tau, 0, \mathbf{0})$, we have $\frac{\mathrm{d}}{\mathrm{d}t}(X|\Psi) = 0$, and in order to prevent grows of the linearized solution, we select initial data such that $(X_{\text{init}}|\Psi) = 0$, which reduces to $q_{\text{init},0} + \tau q_{\text{init},1} = 0$.

When $\kappa > 2$, a similar statement holds for the solution (24).

Proposition 4.7 *Suppose* $\kappa > 2$. *Then, the reference solution* (25) *of* (1)-(2) *is spectrally stable.*

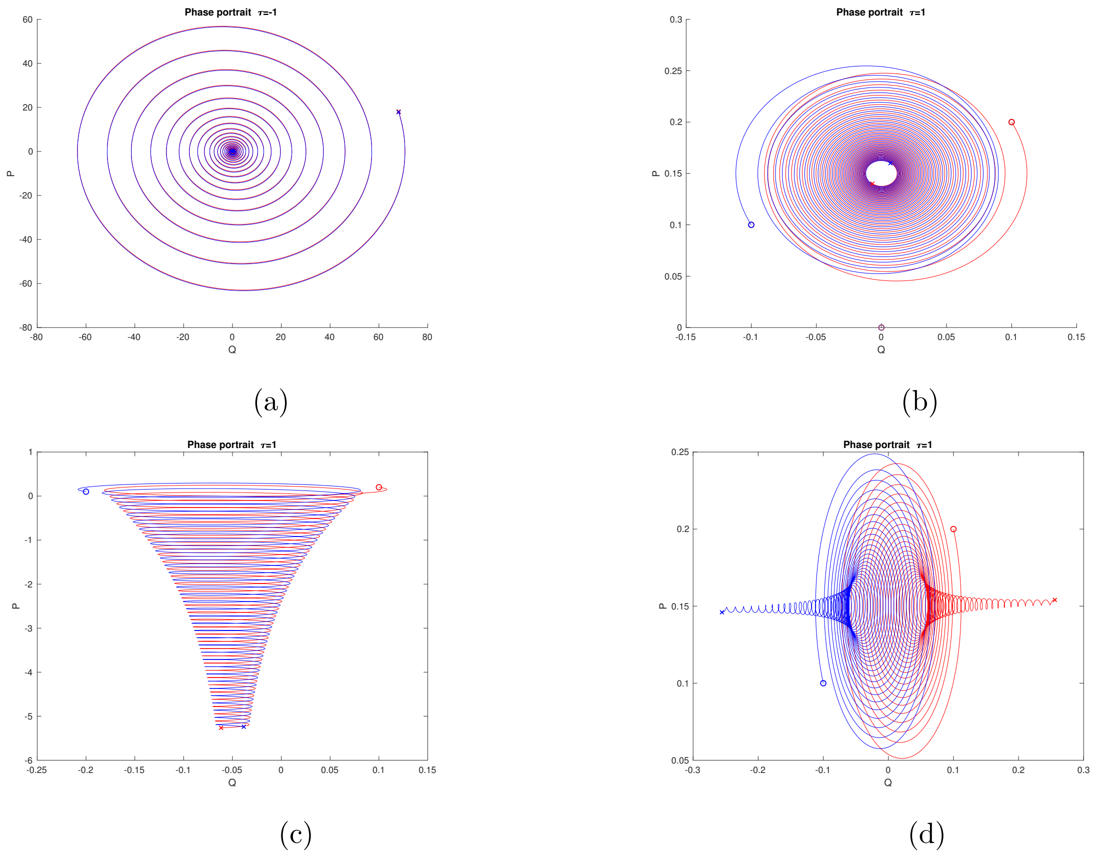
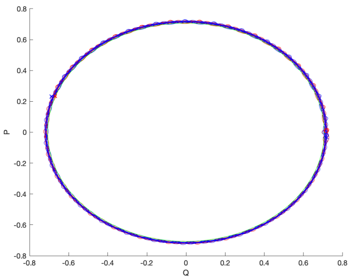


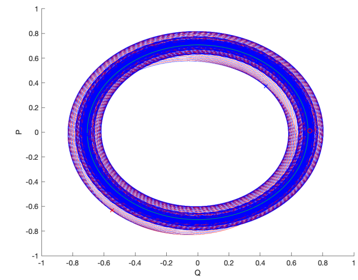
Figure 7: Simulation of the linearized coupled model: phase portrait at $T = 100$, for $\kappa = 1.1$ with $\tau = -1$ (a), with $\tau = 1$ for well-prepared initial data (b), with $\tau = 1$ for ill-prepared initial data (c), and with $\kappa = 2.0688$, $\tau = 1$ at $T = 150$ for well-prepared initial data (d). The circled points indicate the initial state, the cross indicate the final state

Proof. As before, we are concerned with the generalized eigenvalue problem $\mathscr{M}X = \mu\tilde{X}$, $\mathscr{L}\tilde{X} = X$ with $X, \tilde{X} \in (\mathrm{Ker}(\mathscr{L}))^{\perp}$. It now takes the form

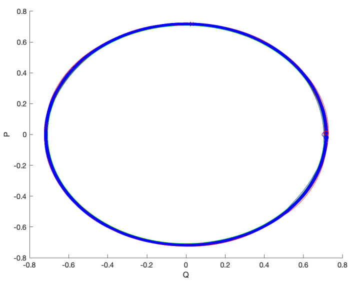
$$\begin{aligned}
&Aq_0 - q_1 = \mu\tilde{q}_0 && Ap_0 - p_1 + 2\alpha c\int_{\mathbb{R}^n} \sigma\varpi_0 \,\mathrm{d}z = \mu\tilde{p}_0,\\
&-q_0 + Bq_1 = \mu\tilde{q}_1, && -p_0 + Bp_1 + 2\beta c\int_{\mathbb{R}^n} \sigma\varpi_1 \,\mathrm{d}z = \mu\tilde{p}_1,\\
&2c^2(-\Delta\varphi_0) = \mu\tilde{\varphi}_0 && 2c^2(-\Delta\varpi_0) + 2\alpha c\sigma p_0 = \mu\tilde{\varpi}_0,\\
&2c^2(-\Delta\varphi_1) = \mu\tilde{\varphi}_1, && 2c^2(-\Delta\varpi_1) + 2\beta c\sigma p_1 = \mu\tilde{\varpi}_1,
\end{aligned}$$



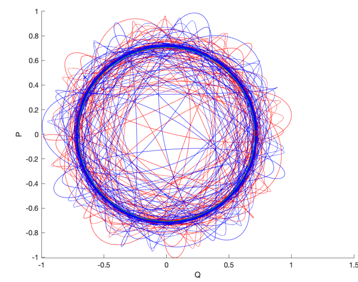
(a)



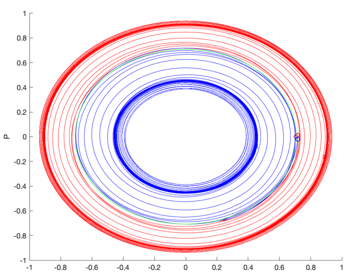
(b)



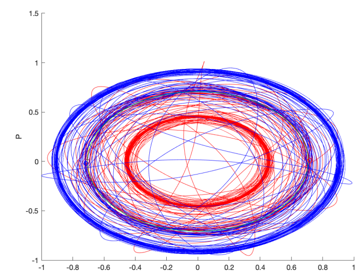
(c)



(d)



(e)



(f)

Figure 8: Simulation of the coupled model: phase portrait at $T = 700$, with $\tau = +1$ (a, c, e), with $\tau = -1$ (b, d, f), and several values of κ: $\kappa = 0.193$ for (a, b), $\kappa = 1.58$ for (c, d), $\kappa = 2.42$ (e, f). The circled points indicate the initial state, the cross indicate the final state

coupled to

$$\begin{aligned}
&A\tilde{q}_0 - \tilde{q}_1 + \alpha \int_{\mathbb{R}^n} \sigma(-\Delta)^{-1/2}\tilde{\varphi}_0 \,\mathrm{d}z = q_0, && A\tilde{p}_0 - \tilde{p}_1 = p_0 \\
&-\tilde{q}_0 + B\tilde{q}_1 + \beta \int_{\mathbb{R}^n} \sigma(-\Delta)^{-1/2}\tilde{\varphi}_1 \,\mathrm{d}z = q_1, && -\tilde{p}_0 + B\tilde{p}_1 = p_1, \\
&\frac{\tilde{\varphi}_0}{2} + \alpha(-\Delta)^{-1/2}\sigma\tilde{q}_0 = \varphi_0, && \frac{\tilde{\varpi}_0}{2} = \varpi_0, \\
&\frac{\tilde{\varphi}_1}{2} + \beta(-\Delta)^{-1/2}\sigma\tilde{q}_1 = \varphi_1, && \frac{\tilde{\varpi}_1}{2} = \varpi_1.
\end{aligned}$$

As before the operator $(-\frac{\mu}{c^2} - \Delta)$ plays a crucial role. In particular, if $\mu > 0$ it cannot be inverted so that the possibility to find nontrivial solutions with $\mu > 0$ is exluded. As a consequence, $N_n^+ = 0$. When $\mu < 0$, we have

$$\left(-\Delta - \frac{\mu}{c^2}\right)\varpi_0 = -\frac{\alpha}{c}\sigma p_0, \qquad \left(-\Delta - \frac{\mu}{c^2}\right)\varpi_1 = -\frac{\beta}{c}\sigma p_1,$$

and

$$\left(-\Delta - \frac{\mu}{c^2}\right)\tilde{\varphi}_0 = -2\alpha(-\Delta)^{1/2}\sigma\tilde{q}_0, \qquad \left(-\Delta - \frac{\mu}{c^2}\right)\tilde{\varphi}_1 = -2\beta(-\Delta)^{1/2}\sigma\tilde{q}_1.$$

Setting $\gamma = -\frac{\mu}{c^2} > 0$ and κ_γ as before leads to the following systems of equations

$$\begin{pmatrix} A - 2\alpha^2\kappa_\gamma & -1 \\ -1 & B - 2\beta^2\kappa_\gamma \end{pmatrix}\begin{pmatrix} p_0 \\ p_1 \end{pmatrix} = \mu\begin{pmatrix} \tilde{p}_0 \\ \tilde{p}_1 \end{pmatrix}, \qquad \begin{pmatrix} A & -1 \\ -1 & B \end{pmatrix}\begin{pmatrix} \tilde{p}_0 \\ \tilde{p}_1 \end{pmatrix} = \begin{pmatrix} p_0 \\ p_1 \end{pmatrix}$$

and

$$\begin{pmatrix} A & -1 \\ -1 & B \end{pmatrix}\begin{pmatrix} q_0 \\ q_1 \end{pmatrix} = \mu\begin{pmatrix} \tilde{q}_0 \\ \tilde{q}_1 \end{pmatrix}, \qquad \begin{pmatrix} A - 2\alpha^2\kappa_\gamma & -1 \\ -1 & B - 2\beta^2\kappa_\gamma \end{pmatrix}\begin{pmatrix} \tilde{q}_0 \\ \tilde{q}_1 \end{pmatrix} = \begin{pmatrix} q_0 \\ q_1 \end{pmatrix}.$$

As before, $\mu < 0$ should be such that

$$\det\left(\begin{pmatrix} A & -1 \\ -1 & B \end{pmatrix}\begin{pmatrix} A - 2\alpha^2\kappa_\gamma & -1 \\ -1 & B - 2\beta^2\kappa_\gamma \end{pmatrix} - \mu\mathbb{I}\right) = 0$$

This condition is equivalent to

$$\begin{aligned}
0 &= \det\left(\begin{pmatrix} A & -1 \\ -1 & B \end{pmatrix}\begin{pmatrix} A - 2\frac{B}{\kappa}\kappa_\gamma & -1 \\ -1 & B - 2\frac{A}{\kappa}\kappa_\gamma \end{pmatrix} - \mu\mathbb{I}\right) \\
&= \det\left(\begin{pmatrix} A\kappa - 2\frac{\kappa_\gamma}{\kappa} & -\kappa + 2A\frac{\kappa_\gamma}{\kappa} \\ -\kappa + 2B\frac{\kappa_\gamma}{\kappa} & B\kappa - 2\frac{\kappa_\gamma}{\kappa} \end{pmatrix} + \gamma c^2\mathbb{I}\right) \\
&= \left(A\kappa - 2\frac{\kappa_\gamma}{\kappa} + \gamma c^2\right)\left(B\kappa - 2\frac{\kappa_\gamma}{\kappa} + \gamma c^2\right) - \left(2B\frac{\kappa_\gamma}{\kappa} - \kappa\right)\left(2A\frac{\kappa_\gamma}{\kappa} - \kappa\right) \\
&= (A+B)\kappa\gamma c^2 - 4\frac{\kappa_\gamma}{\kappa}\gamma c^2 + (\gamma c^2)^2 = \gamma c^2\left(\kappa^2 - 4\frac{\kappa_\gamma}{\kappa} + \gamma c^2\right) \\
&= \gamma c^2\left(\kappa^2 - 4 + 4\left(1 - \frac{\kappa_\gamma}{\kappa}\right) + \gamma c^2\right)
\end{aligned}$$

where we use $A = \beta^2\kappa$, $B = \alpha^2\kappa$ and $A + B = \kappa$. However, since $\kappa > 2$ and $\kappa > \kappa_\gamma$ for any $\gamma > 0$, we have

$$\gamma c^2\left(\kappa^2 - 4 + 4\left(1 - \frac{\kappa_\gamma}{\kappa}\right) + \gamma c^2\right) > 0$$

so that $N_n^- = 0$.

Finally, Theorem 4.5 tells us that $N_n^0 = 1$, while $n(\mathscr{L}) = 1$. Applying the counting argument, we conclude that $N_{C^+} = 0$. ■

4.3 Orbital stability

To discuss the orbital stability of solutions to (1)-(2), it would be useful to write the system in a more convenient way by means of the change of variables

$$u_j = q_j + ip_j, \ \varphi_j = (-\Delta)^{1/2}\psi_j, \ \varpi_j = \frac{\partial_t \psi_j}{c}.$$

Hence, (1)-(2) reads as

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}q_0 = p_0 - p_1 + p_0\int_{\mathbb{R}^n}(-\Delta)^{-1/2}\sigma\varphi_0\,\mathrm{d}z && \partial_t\varphi_0 = c(-\Delta)^{1/2}\varpi_0\\
&\frac{\mathrm{d}}{\mathrm{d}t}p_0 = -q_0 + q_1 - q_0\int_{\mathbb{R}^n}(-\Delta)^{-1/2}\sigma\varphi_0\,\mathrm{d}z && \partial_t\varpi_0 = -c(-\Delta)^{1/2}\varphi_0 - c\sigma(|q_0|^2+|p_0|^2)\\
&\frac{\mathrm{d}}{\mathrm{d}t}q_1 = p_1 - p_0 + p_1\int_{\mathbb{R}^n}(-\Delta)^{-1/2}\sigma\varphi_1\,\mathrm{d}z && \partial_t\varphi_1 = c(-\Delta)^{1/2}\varpi_1\\
&\frac{\mathrm{d}}{\mathrm{d}t}p_1 = -q_1 + q_0 - q_1\int_{\mathbb{R}^n}(-\Delta)^{-1/2}\sigma\varphi_1\,\mathrm{d}z && \partial_t\varpi_1 = -c(-\Delta)^{1/2}\varphi_1 - c\sigma(|q_1|^2+|p_1|^2)
\end{aligned} \tag{47}$$

and it can be written as

$$\partial_t X = \mathscr{J}\nabla\mathscr{H}_{SW}(X) \tag{48}$$

with $X = (S, W) \in \mathbb{R}^4 \times (L^2(\mathbb{R}^n))^4$ as in (41), $\mathscr{J}$ defined by (42) and

$$\begin{aligned}
\mathscr{H}_{SW}(X) = \frac{|q_0-q_1|^2+|p_0-p_1|^2}{2} + \frac{1}{4}\int_{\mathbb{R}^n}(|\varpi_0|^2+|\varpi_1|^2+|\varphi_0|^2+|\varphi_1|^2)\,\mathrm{d}z\\
+\frac{1}{2}\int_{\mathbb{R}^n}(-\Delta)^{1/2}\sigma(\varphi_0(|q_0|^2+|p_0|^2)+\varphi_1(|q_1|^2+|p_1|^2))\,\mathrm{d}z.
\end{aligned} \tag{49}$$

Next, we denote by $F(S) = \frac{|S|^2}{2} = \frac{q_0^2+q_1^2+p_0^2+p_1^2}{2}$ and introduce the functional

$$\mathscr{E}(X) = \mathscr{H}_{SW}(X) + \omega F(S)$$

which is thus conserved by the dynamical system (48). Let $X_* = (S_*, W_*)$ one of the special solutions described in subsection 2.2. In particular, $S_* = (Q_{*0}, 0, Q_{*1}, 0)$ with

$$\begin{pmatrix} Q_{*0}\\ Q_{*1}\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ \tau\end{pmatrix}, \ \omega = \frac{\kappa}{2} + \tau - 1 \text{ for (23)} \tag{50}$$

$$\begin{pmatrix} Q_{*0}\\ Q_{*1}\end{pmatrix} = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}, \ \omega = \kappa - 1 \text{ for (25)} \tag{51}$$

and $W_* = (\varphi_{0*}, 0, \varphi_{1*}, 0)$ where $\varphi_{*j} = -|Q_{*j}|^2(-\Delta)^{-1/2}\sigma$.

Adapting the argument used for the asymptotic model, we consider the level set

$$\mathscr{S} = \{X = (S, W), \ F(S) = F(S_*) = 1/2\}.$$

and its tangent set given by

$$T\mathscr{S} = \{(S,W),\ \nabla F(S_*)\cdot S = 0\}.$$

Note that $(S,W) = (q_0,p_0,q_1,p_1,W) \in T\mathscr{S}$ if and only if $Q_{*0}q_0 + Q_{*1}q_1 = 0$. The orbit associated to X_* is given by

$$\mathscr{O} = \Big\{(S_\theta, -|Q_{*0}|^2(-\Delta)^{-1/2}\sigma, 0, -|Q_{*1}|^2(-\Delta)^{-1/2}\sigma, 0),\ S_\theta = R(\theta)S_*,\ \theta\in\mathbb{R}\Big\}$$

and $(T\mathscr{O})^\perp$ is made of (S,W) with $S = (q_0,p_0,q_1,p_1)$ such that $Q_{*0}p_0 + Q_{*1}p_1 = 0$.

Remark 4.8 *In contrast to the observation made for the asymptotic problem in Remark 3.3, and to a common property of Hamiltonian systems, here the phase invariance property holds in a restricted sense: the energy $\mathscr{H}(X)$, and $\mathscr{E}(X)$ as well, is left unchanged when changing $X = (S,W)$ into $(R(\theta)S,W)$, where the rotation $R(\theta)$ acts only on a part of the variables.*

The Euler-Lagrange relation for the coupled problem can be reformulated as

$$\nabla\mathscr{E}(X_*) = 0 \tag{52}$$

and $\mathscr{L}$, defined in (43) or (45), corresponds to the Hessian of $\mathscr{E}$ evaluated at X_* given by (50) or (51) respectively. We wish to establish a coercivity estimate, on a certain subspace, for the quadratic form $X \mapsto D^2\mathscr{E}(X_*)(X,X)$. This is a crucial property for establishing the orbital stability. A straightforward computation gives, for any $X \in T\mathscr{S}\cap(T\mathscr{O})^\perp$,

$$\begin{aligned}D^2\mathscr{E}(X_*)(X,X) = {}& \frac{Q_{*1}}{Q_{*0}}q_0^2 - q_1q_0 + 2Q_{*0}q_0\int_{\mathbb{R}^n}(-\Delta)^{-1/2}\sigma\varphi_0\,\mathrm{d}z + \frac{Q_{*1}}{Q_{*0}}p_0^2 - p_1p_0\\ &+ \frac{Q_{*0}}{Q_{*1}}q_1^2 - q_0q_1 + 2Q_{*1}q_1\int_{\mathbb{R}^n}(-\Delta)^{-1/2}\sigma\varphi_1\,\mathrm{d}z + \frac{Q_{*0}}{Q_{*1}}p_1^2 - p_0p_1\\ &+ \frac{1}{2}(\|\varphi_0\|^2_{L^2(\mathbb{R}^n)} + \|\varphi_1\|^2_{L^2(\mathbb{R}^n)}) + \frac{1}{2}(\|\varpi_0\|^2_{L^2(\mathbb{R}^n)} + \|\varpi_1\|^2_{L^2(\mathbb{R}^n)})\end{aligned}$$

where we use the fact that $(1-|Q_{*0}|^2\kappa+\omega)Q_{*0} = Q_{*1}$ and $(1-|Q_{*1}|^2\kappa+\omega)Q_{*1} = Q_{*0}$. Now, since $X\in T\mathscr{S}\cap(T\mathscr{O})^\perp$, we have $Q_{*0}q_0 + Q_{*1}q_1 = 0 = Q_{*0}p_0 + Q_{*1}p_1$, so that

$$\begin{aligned}D^2\mathscr{E}(X_*)(X,X) = {}& \left(\frac{1}{Q_{*0}Q_{*1}}\right)(q_0^2+q_1^2) + 2Q_{*0}q_0\int_{\mathbb{R}^n}(-\Delta)^{-1/2}\sigma\varphi_0\,\mathrm{d}z\\ &+ 2Q_{*1}q_1\int_{\mathbb{R}^n}(-\Delta)^{-1/2}\sigma\varphi_1\,\mathrm{d}z + \left(\frac{1}{Q_{*0}Q_{*1}}\right)(p_0^2+p_1^2)\\ &+ \frac{1}{2}(\|\varphi_0\|^2_{L^2(\mathbb{R}^n)} + \|\varphi_1\|^2_{L^2(\mathbb{R}^n)}) + \frac{1}{2}(\|\varpi_0\|^2_{L^2(\mathbb{R}^n)} + \|\varpi_1\|^2_{L^2(\mathbb{R}^n)})\end{aligned}$$

By virtue of the Cauchy-Schwarz inequality,

$$2|Q_{*j}||q_j|\left|\int_{\mathbb{R}^n}(-\Delta)^{-1/2}\sigma\varphi_j\,\mathrm{d}z\right| \leqslant 2|Q_{*j}||q_j|\sqrt{\kappa}\|\varphi_j\|_{L^2} \leqslant \frac{\kappa}{\epsilon}|Q_{*j}|^2q_j^2 + \epsilon\|\varphi_j\|^2_{L^2}$$

for any $\epsilon > 0$. Therefore,

$$\begin{aligned}D^2\mathscr{E}(X_*)(X,X) \geqslant& \left(\frac{1}{Q_{*0}Q_{*1}} - \frac{\kappa}{\epsilon}|Q_{0*}|^2\right) q_0^2 + \left(\frac{1}{Q_{*0}Q_{*1}} - \frac{\kappa}{\epsilon}|Q_{1*}|^2\right) q_1^2 \\ &+ \left(\frac{1}{Q_{*0}Q_{*1}}\right)(p_0^2+p_1^2) + \left(\frac{1}{2}-\epsilon\right)(\|\varphi_0\|^2_{L^2(\mathbb{R}^n)} + \|\varphi_1\|^2_{L^2(\mathbb{R}^n)}) \\ &+ \frac{1}{2}(\|\varpi_0\|^2_{L^2(\mathbb{R}^n)} + \|\varpi_1\|^2_{L^2(\mathbb{R}^n)}). \end{aligned} \tag{53}$$

Assume that X_* is such that (Q_{*0}, Q_{*1}) is given by (50). Then Proposition 4.6 implies that X_* is spectrally stable only for $\tau = 1$ and $0 < \kappa < 2$. In this case, (53) reads as

$$\begin{aligned}D^2\mathscr{E}(X_*)(X,X) \geqslant& \left(2 - \frac{\kappa}{2\epsilon}\right)(q_0^2+q_1^2) + 2(p_0^2+p_1^2) + \left(\frac{1}{2}-\epsilon\right)(\|\varphi_0\|^2_{L^2(\mathbb{R}^n)} + \|\varphi_1\|^2_{L^2(\mathbb{R}^n)}) \\ &+ \frac{1}{2}(\|\varpi_0\|^2_{L^2(\mathbb{R}^n)} + \|\varpi_1\|^2_{L^2(\mathbb{R}^n)}) \geqslant C(\epsilon)\|X\|^2. \end{aligned}$$

with $C(\epsilon) = \min\left\{ \left(2 - \frac{\kappa}{2\epsilon}\right), \left(\frac{1}{2}-\epsilon\right) \right\}$. Note that $C(\epsilon) > 0$ provided ϵ is chosen such that $\frac{\kappa}{4} < \epsilon < \frac{1}{2}$. This leads to the orbital stability of X_* given by (50) when $\tau = 1$ and $0 < \kappa < 2$.

Note that if $\tau = 1$ and $\kappa > 2$ or $\tau = -1$, then the quadratic form $X \mapsto D^2\mathscr{E}(X_*)(X,X)$ has no definite sign on $T\mathscr{S} \cap (T\mathscr{O})^\perp$.

Next, let $\kappa > 2$ and let X_* be such that (Q_{*0}, Q_{*1}) is given by (51). Then Proposition 4.7 implies that X_* is always spectrally stable. In this case, (53) reads as

$$\begin{aligned}D^2\mathscr{E}(X_*)(X,X) \geqslant& \left(\kappa - \frac{\kappa}{\epsilon}\alpha^2\right) q_0^2 + \left(\kappa - \frac{\kappa}{\epsilon}\beta^2\right) q_1^2 + \kappa(p_0^2+p_1^2) \\ &+ \left(\frac{1}{2}-\epsilon\right)(\|\varphi_0\|^2_{L^2(\mathbb{R}^n)} + \|\varphi_1\|^2_{L^2(\mathbb{R}^n)}) + \frac{1}{2}(\|\varpi_0\|^2_{L^2(\mathbb{R}^n)} + \|\varpi_1\|^2_{L^2(\mathbb{R}^n)}) \end{aligned}$$

where we use that $\alpha\beta = \frac{1}{\kappa}$. Next, we consider separately the cases $\tau = 1$ and $\tau = -1$.

If $\tau = 1$, we write $q_1 = -\frac{\alpha}{\beta} q_0$ so that

$$\begin{aligned}D^2\mathscr{E}(X_*)(X,X) \geqslant& \kappa\left(\frac{1}{\beta^2} - \frac{2}{\epsilon}\alpha^2\right) q_0^2 + \kappa(p_0^2+p_1^2) \\ &+ \left(\frac{1}{2}-\epsilon\right)(\|\varphi_0\|^2_{L^2(\mathbb{R}^n)} + \|\varphi_1\|^2_{L^2(\mathbb{R}^n)}) + \frac{1}{2}(\|\varpi_0\|^2_{L^2(\mathbb{R}^n)} + \|\varpi_1\|^2_{L^2(\mathbb{R}^n)}) \\ =& \frac{\kappa}{\beta^2\epsilon}\left(\epsilon - \frac{2}{\kappa^2}\right)\left(1 - \frac{\alpha^2}{\beta^2}\right) q_0^2 + \frac{\kappa}{\beta^2\epsilon}\left(\epsilon - \frac{2}{\kappa^2}\right) q_1^2 + \kappa(p_0^2+p_1^2) \\ &+ \left(\frac{1}{2}-\epsilon\right)(\|\varphi_0\|^2_{L^2(\mathbb{R}^n)} + \|\varphi_1\|^2_{L^2(\mathbb{R}^n)}) + \frac{1}{2}(\|\varpi_0\|^2_{L^2(\mathbb{R}^n)} + \|\varpi_1\|^2_{L^2(\mathbb{R}^n)}). \end{aligned}$$

By choosing $\frac{2}{\kappa^2} < \epsilon < \frac{1}{2}$ which is possible since $\kappa > 2$ and since $\frac{\alpha^2}{\beta^2} < 1$, we obtain $D^2\mathscr{E}(X_*)(X,X) \geqslant C(\epsilon)\|X\|^2$ with $C(\epsilon) > 0$ and for any $X \in T\mathscr{S} \cap (T\mathscr{O})^\perp$.

If $\tau = -1$, we write $q_0 = -\frac{\beta}{\alpha} q_1$ so that

$$D^2\mathscr{E}(X_*)(X,X) \geqslant \frac{\kappa}{\alpha^2\epsilon}\left(\epsilon - \frac{2}{\kappa^2}\right) q_0^2 + \frac{\kappa}{\alpha^2\epsilon}\left(\epsilon - \frac{2}{\kappa^2}\right)\left(1 - \frac{\beta^2}{\alpha^2}\right) q_1^2 + \kappa(p_0^2 + p_1^2)$$
$$+ \left(\frac{1}{2} - \epsilon\right)(\|\varphi_0\|^2_{L^2(\mathbb{R}^n)} + \|\varphi_1\|^2_{L^2(\mathbb{R}^n)}) + \frac{1}{2}(\|\varpi_0\|^2_{L^2(\mathbb{R}^n)} + \|\varpi_1\|^2_{L^2(\mathbb{R}^n)}).$$

and we can conclude as above.

■

4.4 Instability

In this section we study the nonlinear instability of the solution X_* given by (50) whenever $\tau = -1$ or $\tau = 1$ and $\kappa > 2$, *i.e.* whenever X_* is spectrally unstable. To this goal, we use again the same change of variables as in the previous section so that (1)-(2) reads as (47). Note that the reasoning of [29], as described in Section 3.4, can be applied only in the case $\tau = 1$ and $\kappa > 2$, the Morse index of $\mathscr{L}$ being larger than 2 when $\tau = -1$. Hence, we are going to apply the general result described in [32] to treat both cases at the same time. The instability analysis is of different nature than in Section 3.4. In Section 3.4, the method of [29] relies on the spectral property of the self-adjoint operator $\mathscr{L}$; it shows a linear growth of the perturbation by using the energy conservation, but it requires a strong assumption on the dimension of the eigenspace of unstable directions. Here, the arguments of [32], which has been extended to various type of nonlinear Schrödinger equation in [8, 14], uses the fact that $\mathbb{L}$ admits an eigenvalue with a positive real part. This property can be deduced from the counting argument. As in [32], starting from the worst linearly unstable direction, we construct an initial datum close to equilibrium such that the corresponding time evolution exits a neighborhood of this equilibrium in a finite time.

We start by observing that the linearized operator $\mathbb{L}$ satisfies

$$(\mathbb{L}X|X) = -\frac{1}{\sqrt{2}}\int_{\mathbb{R}^n} (-\Delta)^{-1/2}\sigma(\varphi_0 p_0 + \tau\varphi_1 p_1)\,\mathrm{d}z - c\sqrt{2}\int_{\mathbb{R}^n}\sigma(\varpi_0 q_0 + \tau\varpi_1 q_1)\,\mathrm{d}z.$$

The Cauchy-Schwarz inequality yields

$$|(\mathbb{L}X|X)| \leqslant 2(\sqrt{\kappa/2} + c\sqrt{2}\|\sigma\|_{L^2(\mathbb{R}^n)})\|X\|^2.$$

As it will be detailed below, the operator $\lambda - \mathbb{L}$ is onto for sufficiently large (real part of) λ's. Accordingly, we can apply Lumer-Phillips' theorem [31, Th. 12.22] to the linearized equation

$$\partial_t X = \mathbb{L}X.$$

It can be formulated as the existence of the semi-group $t \mapsto e^{\mathbb{L}t}$, which satisfies the continuity estimate: there exists $\Lambda > 0$ such that for any $t \geqslant 0$, $\|e^{\mathbb{L}t}\| \leqslant e^{\Lambda t}$. For further purposes, we denote

$$K_0 = \sup\left\{\|e^{\mathbb{L}t}\|, 0 \leqslant t \leqslant 1\right\}.$$

Then, we consider the evolution of a perturbation of X_* according to the dynamical system (47). More precisely, we set

$$\begin{pmatrix} \tilde{q}_j \\ \tilde{p}_j \end{pmatrix} = R(\omega t) \begin{pmatrix} Q_{*j} + q_j \\ p_j \end{pmatrix}, \ \tilde{\varphi}_j = \varphi_{*j} + \varphi_j, \ \tilde{\varpi}_j = \varpi_j.$$

From (47), we deduce that the perturbation $Y = (q_0, p_0, q_1, p_1, \varphi_0, \varpi_0, \varphi_1, \varpi_1)$ satisfies

$$\partial_t Y = \mathbb{L}Y + F(Y)$$

where the nonlinear remainder reads

$$F(Y) = \begin{pmatrix} p_0 \displaystyle\int_{\mathbb{R}^n} \sigma(-\Delta)^{-1/2} \varphi_0 \, \mathrm{d}z \\ -q_0 \displaystyle\int_{\mathbb{R}^n} \sigma(-\Delta)^{-1/2} \varphi_0 \, \mathrm{d}z \\ p_1 \displaystyle\int_{\mathbb{R}^n} \sigma(-\Delta)^{-1/2} \varphi_1 \, \mathrm{d}z \\ -q_1 \displaystyle\int_{\mathbb{R}^n} \sigma(-\Delta)^{-1/2} \varphi_1 \, \mathrm{d}z \\ 0 \\ -c\sigma(|p_0|^2 + |q_0|^2) \\ 0 \\ -c\sigma(|p_1|^2 + |q_1|^2) \end{pmatrix}.$$

The orbital stability of the solution (50) to (47) is rephrased in the orbital stability of 0 for this problem. More precisely, we shall obtain the critical estimates by using the integral formulation

$$Y(t) = e^{\mathbb{L}t} Y_{\mathrm{init}} + \int_0^t e^{\mathbb{L}(t-s)} F(Y(s)) \, \mathrm{d}s \tag{54}$$

of the problem. The application of the reasonings in [32] relies on the following estimate

Lemma 4.9 *There exists $C_1 > 0$ such that for any X, we have $|F(X)| \leqslant C_1 |X|^2$.*

Proof. In order to estimate $F(Y)$, we make the following quantities appear

$$(|p_j|^2 + |q_j|^2) \left(\int_{\mathbb{R}^n} (-\Delta)^{-1/2} \sigma \varphi_j \, \mathrm{d}z \right)^2 \leqslant \kappa (|p_j|^2 + |q_j|^2) \|\varphi_j\|^2_{L^2(\mathbb{R}^n)}$$

and

$$(|p_j|^2 + |q_j|^2)^2 \int_{\mathbb{R}^n} |\sigma|^2 \, \mathrm{d}z = \|\sigma\|^2_{L^2(\mathbb{R}^n)} (|p_j|^2 + |q_j|^2)^2.$$

It leads to the asserted conclusion with $C_1 = 2(\sqrt{\kappa} + c\|\sigma\|_{L^2(\mathbb{R}^n)})$.

■

Next, we need the following information on the spectrum of $\mathbb{L}$.

Proposition 4.10 $\sigma(e^{\mathbb{L}}) = e^{\sigma(\mathbb{L})}$.

This statement strengthens the embedding $\exp(\sigma(\mathbb{L})) \subset \sigma(e^{\mathbb{L}})$ which always holds. It is not a prerequisite but it simplifies the argument, see [32]. According to Gearhart-Greiner-Herbst-Prüss Spectral Mapping Theorem, see e.g. [30, Prop. 1] (in fact, we use the criterion in the same form as in [15, Section 2]), the proof relies on a uniform estimate on the resolvent $(\lambda - \mathbb{L})^{-1}$, as $\mathrm{Im}(\lambda) \to \pm\infty$ with $\mathrm{Re}(\lambda) \neq 0$ fixed, that we are going to establish. We denote

$$\mathbb{H} = \mathbb{R}^4 \times (L^2(\mathbb{R}^n))^4$$

endowed with the norm

$$\|X\|_{\mathbb{H}} = \sqrt{|q_0|^2 + |p_0|^2 + |q_1|^2 + |p_1|^2 + \|\varphi_0\|^2_{L^2(\mathbb{R}^n)} + \|\varpi_0\|^2_{L^2(\mathbb{R}^n)} + \|\varphi_1\|^2_{L^2(\mathbb{R}^n)} + \|\varpi_1\|^2_{L^2(\mathbb{R}^n)}}.$$

Let $\lambda \in \mathbb{C}\backslash\{0\}$ and for a given data X', we consider the equation

$$(\lambda - \mathbb{L})X = X',$$

that is

$$\begin{aligned}
&\lambda q_0 - \tau p_0 + p_1 = q_0',\\
&\lambda p_0 + \tau q_0 - q_1 + \frac{1}{\sqrt{2}} \int_{\mathbb{R}^n} (-\Delta)^{-1/2} \sigma \varphi_0 \,\mathrm{d}z = p_0',\\
&\lambda q_1 + p_0 - \tau p_1 = q_1',\\
&\lambda p_1 - q_0 + \tau q_1 + \frac{\tau}{\sqrt{2}} \int_{\mathbb{R}^n} (-\Delta)^{-1/2} \sigma \varphi_1 \,\mathrm{d}z = p_1',\\
&\lambda \varphi_0 - c(-\Delta)^{1/2} \varpi_0 = \varphi_0',\\
&\lambda \varpi_0 + c(-\Delta)^{1/2} \varphi_0 + c\sqrt{2}\sigma q_0 = \varpi_0',\\
&\lambda \varphi_1 - c(-\Delta)^{1/2} \varpi_1 = \varphi_1',\\
&\lambda \varpi_1 + c(-\Delta)^{1/2} \varphi_1 + \tau c\sqrt{2}\sigma q_1 = \varpi_1'.
\end{aligned}$$

Therefore, we get

$$\varpi_0 = \frac{(-\Delta)^{-1/2}}{c}(\lambda \varphi_0 - \varphi_0'), \qquad \varpi_1 = \frac{(-\Delta)^{-1/2}}{c}(\lambda \varphi_1 - \varphi_1'),$$

which allows us to write

$$\begin{aligned}
&\Big(\frac{\lambda^2}{c^2} - \Delta\Big)\varphi_0 = \frac{\lambda}{c^2}\varphi_0' + (-\Delta)^{1/2}\varpi_0' - \sqrt{2}(-\Delta)^{1/2}\sigma q_0,\\
&\Big(\frac{\lambda^2}{c^2} - \Delta\Big)\varphi_1 = \frac{\lambda}{c^2}\varphi_1' + (-\Delta)^{1/2}\varpi_1' - \tau\sqrt{2}(-\Delta)^{1/2}\sigma q_1.
\end{aligned}$$

We solve these equations by means of Fourier transform. Note that this makes the symbol $\big(\frac{\lambda^2}{c^2} + \xi^2\big)$ appear. However, it does not vanish outside of the axis $i\mathbb{R}$. Hence, we still can use the function

$$z \in \mathbb{C}\backslash i\mathbb{R} \longmapsto \kappa_z = \int_{\mathbb{R}^n} \frac{|\widehat{\sigma}(\xi)|^2}{z^2 + |\xi|^2} \frac{\mathrm{d}\xi}{(2\pi)^n}.$$

As consequence, we arrive at the reduced system:

$$\begin{aligned}
\lambda q_0 - \tau p_0 + p_1 &= q_0', \\
\lambda p_0 + \tau q_0 - q_1 - \kappa_{\lambda^2/c^2} q_0 &= S_0, \\
\lambda q_1 + p_0 - \tau p_1 &= q_1', \\
\lambda p_1 - q_0 + \tau q_1 - \kappa_{\lambda^2/c^2} q_1 &= S_1,
\end{aligned}$$

where we have set

$$\begin{aligned}
S_0 &= p_0' - \frac{1}{\sqrt{2}} \int_{\mathbb{R}^n} \frac{\widehat{\sigma}(\xi)\widehat{\varpi_0'}(\xi)}{\lambda^2/c^2 + |\xi|^2} \frac{\mathrm{d}\xi}{(2\pi)^n} - \frac{\lambda}{\sqrt{2}c^2} \int_{\mathbb{R}^n} \frac{\widehat{\sigma}(\xi)\widehat{\varphi_0'}(\xi)}{(\lambda^2/c^2 + |\xi|^2)|\xi|} \frac{\mathrm{d}\xi}{(2\pi)^n}, \\
S_1 &= p_1' - \frac{\tau}{\sqrt{2}} \int_{\mathbb{R}^n} \frac{\widehat{\sigma}(\xi)\widehat{\varpi_1'}(\xi)}{\lambda^2/c^2 + |\xi|^2} \frac{\mathrm{d}\xi}{(2\pi)^n} - \tau \frac{\lambda}{\sqrt{2}c^2} \int_{\mathbb{R}^n} \frac{\widehat{\sigma}(\xi)\widehat{\varphi_1'}(\xi)}{(\lambda^2/c^2 + |\xi|^2)|\xi|} \frac{\mathrm{d}\xi}{(2\pi)^n}.
\end{aligned} \tag{55}$$

Since

$$\lambda(q_0 + \tau q_1) = q_0' + \tau q_1',$$

we obtain

$$\begin{aligned}
\lambda p_0 + 2\tau q_0 - \kappa_{\lambda^2/c^2} q_0 &= S_0 + \tau \frac{(q_0' + \tau q_1')}{\lambda}, \\
\lambda p_1 - 2q_0 + \tau \kappa_{\lambda^2/c^2} q_0 &= S_1 - \tau(\tau - \kappa_{\lambda^2/c^2}) \frac{(q_0' + \tau q_1')}{\lambda}.
\end{aligned}$$

It eventually yields

$$(\lambda^2 + 4 - 2\tau\kappa_{\lambda^2/c^2}) q_0 = \lambda q_0' - (S_1 - \tau S_0) + (2 - \tau\kappa_{\lambda^2/c^2}) \frac{(q_0' + \tau q_1')}{\lambda}. \tag{56}$$

In particular, setting $X' = 0$, we obtain the relation (44) introduced above for studying the eigenvalues of $\mathbb{L}$. Next, we are going to use the following elementary claim.

Lemma 4.11 *Let $\lambda = a + ib \in \mathbb{C}$, with a and b reals. If $|b| \geqslant \sqrt{3}|a|$, then, for any $\epsilon \geqslant 0$ we have*

$$\Big| \frac{1}{\lambda^2 + \epsilon} \Big| \leqslant \frac{\sqrt{2}}{|\lambda|^2}, \qquad \Big| \frac{\lambda}{\lambda^2 + \epsilon} \Big| \leqslant \frac{\sqrt{2}}{|\lambda|}.$$

Proof. We write $\lambda = re^{i\theta}$ with $r = \sqrt{a^2 + b^2}$, so that

$$\Big| \frac{1}{\lambda^2 + \epsilon} \Big| = \Big| \frac{1}{e^{i\theta} r^2 + e^{-i\theta}\epsilon} \Big|, \qquad \Big| \frac{\lambda}{\lambda^2 + \epsilon} \Big| = \Big| \frac{1}{e^{i\theta} r + e^{-i\theta}\epsilon/r} \Big|.$$

Now, we re-organize

$$\begin{aligned}
|e^{i\theta} r^2 + e^{-i\theta}\epsilon|^2 &= r^4 + \epsilon^2 + 2r^2\epsilon\cos(2\theta) = (r^2 - \epsilon)^2 + 4r^2\epsilon\cos^2(\theta) \\
&\geqslant \frac{(r^2 - \epsilon)^2}{2} + \frac{r^4 + \epsilon^2}{2} \geqslant \frac{r^4}{2}, \\
|e^{i\theta} r + e^{-i\theta}\epsilon/r|^2 &= r^2 + \frac{\epsilon^2}{r^2} + 2\epsilon\cos(2\theta) = \Big(r - \frac{\epsilon}{r}\Big)^2 + 4\epsilon\cos^2(\theta) \\
&\geqslant \frac{(r - \epsilon/r)^2}{2} + \frac{r^2 + \epsilon^2/r^2}{2} \geqslant \frac{r^2}{2}.
\end{aligned}$$

where we have used that, by assumptions on a, b, $|\cos(\theta)| = \frac{|a|}{\sqrt{a^2+b^2}} \leqslant \frac{1}{2}$. It thus implies

$$\Big|\frac{1}{\lambda^2+\epsilon}\Big| \leqslant \frac{\sqrt{2}}{r^2}, \qquad \Big|\frac{\lambda}{\lambda^2+\epsilon}\Big| \leqslant \frac{\sqrt{2}}{r}.$$

■

This allows us to estimate the resolvent $(\lambda - \mathbb{L})^{-1}$.

Lemma 4.12 *Let $\lambda = a + ib \in \mathbb{C}$, with $a \neq 0$, $|b| \geqslant \sqrt{3}|a|$. Then, there exists a constant $C_a > 0$ such that the quantities S_0, S_1 in* (55) *satisfy*

$$|S_j| \leqslant C_a \|X'\|_{\mathbb{H}}.$$

Proof. The only difficulty is to estimate the integrals involving σ. Owing to Lemma 4.11 and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\Big|\frac{\lambda}{c^2}\int_{\mathbb{R}^n} \frac{\widehat{\sigma}(\xi)\widehat{\varphi_0'}(\xi)}{(\lambda^2/c^2+|\xi|^2)|\xi|}\frac{\mathrm{d}\xi}{(2\pi)^n}\Big| &\leqslant \frac{\sqrt{2}}{|\lambda|}\int_{\mathbb{R}^n} \frac{|\widehat{\sigma}(\xi)|\ |\widehat{\varphi_0'}(\xi)|}{|\xi|}\frac{\mathrm{d}\xi}{(2\pi)^n} \\ &\leqslant \frac{\sqrt{2}}{|\lambda|}\left(\int_{\mathbb{R}^n} \frac{|\widehat{\sigma}(\xi)|^2}{|\xi|^2}\frac{\mathrm{d}\xi}{(2\pi)^n}\right)^{1/2}\left(\int_{\mathbb{R}^n} |\widehat{\varphi}_0'(\xi)|^2\frac{\mathrm{d}\xi}{(2\pi)^n}\right)^{1/2} \\ &\leqslant \frac{\sqrt{2\kappa}}{|\lambda|}\|\varphi_0'\|_{L^2(\mathbb{R}^n)} \leqslant \frac{\sqrt{\kappa}}{\sqrt{2}|a|}\|\varphi_0'\|_{L^2(\mathbb{R}^n)}.\end{aligned}$$

Similarly, we get

$$\begin{aligned}\Big|\frac{1}{c^2}\int_{\mathbb{R}^n} \frac{\widehat{\sigma}(\xi)\widehat{\varpi_0'}(\xi)}{\lambda^2/c^2+|\xi|^2}\frac{\mathrm{d}\xi}{(2\pi)^n}\Big| &\leqslant \frac{\sqrt{2}}{|\lambda|^2}\int_{\mathbb{R}^n} |\widehat{\sigma}(\xi)|\ |\widehat{\varpi_0'}(\xi)|\frac{\mathrm{d}\xi}{(2\pi)^n} \\ &\leqslant \frac{\sqrt{2}}{|\lambda|^2}\|\sigma\|_{L^2(\mathbb{R}^n)}\|\varpi_0'\|_{L^2(\mathbb{R}^n)} \leqslant \frac{\sqrt{2}\|\sigma\|_{L^2(\mathbb{R}^n)}}{4|a|^2}\|\varpi_0'\|_{L^2(\mathbb{R}^n)}.\end{aligned}$$

■

By direct inspection, Lemma 4.11 also yields the following estimate.

Lemma 4.13 *Let $\lambda = a + ib \in \mathbb{C}$, with $a \neq 0$ and $|b| \geqslant \sqrt{3}|a|$. Then, we have*

$$|\kappa_{\lambda^2/c^2}| \leqslant c^2 \frac{\|\sigma\|^2_{L^2(\mathbb{R}^n)}}{2\sqrt{2}|a|^2}.$$

Let $\lambda = a + ib \in \mathbb{C}$. By virtue of Lemma 4.13, when b is large enough, $\lambda^2 + 4 - 2\tau\kappa_{\lambda^2/c^2}$ does not vanish. We can therefore obtain q_0 from the data X' with (56). Moreover, as $b \to \infty$, with $a \neq 0$ fixed, $\frac{1}{\lambda^2+4-2\tau\kappa_{\lambda^2/c^2}}$, and $\frac{\lambda}{\lambda^2+4-2\tau\kappa_{\lambda^2/c^2}}$ both tend to 0. We conclude that we can find some $r > 0$ and $M > 0$ (depending on a, c, σ) such that for any $b \in \mathbb{R}$, $|b| \geqslant r$, we have $\|X\|_{\mathbb{H}} = \|(a + ib - \mathbb{L})^{-1}X'\|_{\mathbb{H}} \leqslant M\|X'\|_{\mathbb{H}}$. This justifies Proposition 4.10. ■

In case of spectral instability, $\mathbb{L}$ admits eigenvalues with positive real value. There is only a finite number of such eigenvalues (as indicated by the counting argument). Since, $\exp(\sigma(\mathbb{L})) \subset \sigma(e^{\mathbb{L}})$ we thus already know that the spectral radius of $e^{\mathbb{L}}$ is larger than 1. In fact, we can use the identity in Proposition 4.10. Let us denote

$$\lambda_* = a_* + ib_* \in \sigma(\mathbb{L}), \quad a_* = \sup\left\{\mathrm{Re}(\lambda),\ \lambda \in \sigma(\mathbb{L})\right\} > 0.$$

Of course, for any $t \geqslant 0$, we have $|e^{\lambda_* t}| = e^{a_* t}$ and the spectral radius of $e^{\mathbb{L}}$ is $e^{a_*} > 1$, see [14] for more details. We are going to use the following claim.

Lemma 4.14 *[32, Lemma 2 & Lemma 3] There exists a constant K_1, such that for any $t \geqslant 0$, there holds $\|e^{t\mathbb{L}}\|_{\mathscr{L}(\mathbb{H})} \leqslant K_1 e^{3a_* t/2}$.*

Let us define $\epsilon > 0$ such that

$$\frac{4K_1(1+C_1)^2}{a_*}\epsilon < 1$$

with C_1 and K_1 defined in Lemma 4.9 and 4.14 respectively. Then, pick an arbitrary $0 < \delta < \epsilon$ and set

$$T_\epsilon = \frac{1}{a_*}\ln\left(\frac{\epsilon}{\delta}\right)$$

Let Y_* be a normalized eigenvector of $\mathbb{L}$ associated to λ_*:

$$\mathbb{L}Y_* = \lambda_* Y_*, \qquad \|Y_*\|_{\mathbb{H}} = 1.$$

It will serve to define the initial perturbation that leads to instability: we start from the perturbation

$$Y\Big|_{t=0} = \delta Y_*,$$

which has thus an arbitrarily small norm. As a matter of fact, (54) becomes

$$Y(t) = \delta e^{\lambda_* t} Y_* + \int_0^t e^{\mathbb{L}(t-s)} F(Y(s))\,\mathrm{d}s.$$

We are going to contradict the orbital stability by showing that $Y(T_\epsilon)$ is at a distance larger than $\kappa\epsilon$, for a certain constant $\kappa > 0$, to the orbit $\mathscr{O}$. Let

$$\tilde{T}_\epsilon = \sup\left\{t \in [0, T_\epsilon],\ \|Y(s)\| \leqslant (1+C_1)\delta e^{a_* s} \text{ for } 0 \leqslant s \leqslant t\right\} \in (0, T_\epsilon].$$

The Duhamel formula (54) yields

$$\|Y(t)\| \leqslant \delta e^{a_* t} + \int_0^t K_1 e^{3a_*(t-s)/2} C_1 \|Y(s)\|^2\,\mathrm{d}s$$

by using Lemma 4.9 and 4.14. Therefore

$$\begin{aligned}\|Y(t)\| &\leqslant \delta e^{a_* t} + K_1 C_1 (1+C_1)^2 \delta^2 \int_0^t e^{3a_*(t-s)/2} e^{2a_* s}\,\mathrm{d}s \\ &\leqslant \delta e^{a_* t} + K_1 C_1 (1+C_1)^2 \delta^2 \frac{2e^{2a_* t}}{a_*} \\ &\leqslant \delta e^{a_* t}\left(1 + \frac{2K_1 C_1 (1+C_1)^2}{a_*}\delta e^{a_* T_\epsilon}\right) \leqslant \delta e^{a_* t}\left(1 + C_1 \frac{2K_1 (1+C_1)^2}{a_*}\epsilon\right)\end{aligned}$$

holds for any $t \in [0, \tilde{T}_\epsilon] \subset [0, T_\epsilon]$. Hence, ϵ is chosen small enough so that this implies

$$\|Y(t)\| < \left(1 + \frac{C_1}{2}\right)\delta e^{a_* t},$$

which would contradict the definition of $\tilde{T}_\epsilon$ if $\tilde{T}_\epsilon < T_\epsilon$. We deduce that

$$\|Y(t)\| \leqslant (1+C_1)\delta e^{a_* t} \leqslant (1+C_1)\epsilon$$

holds for any $t \in [0, T_\epsilon]$. Owing to this estimate, we go back to the Duhamel formula and we obtain, for $0 \leqslant t \leqslant T_\epsilon$,

$$\begin{aligned}\|Y(t) - \delta e^{\lambda_* t} Y_*\| &\leqslant \int_0^t |e^{\mathbb{L}(t-s)} F(Y(s))|\,\mathrm{d}s \leqslant \int_0^t e^{3a_*(t-s)/2} K_1 C_1 \|Y(s)\|^2\,\mathrm{d}s \\ &\leqslant K_1 C_1 (1+C_1)^2 \delta^2 \int_0^t e^{3a_*(t-s)/2} e^{2a_* s}\,\mathrm{d}s \leqslant \frac{2K_1 C_1 (1+C_1)^2}{a_*}\delta^2 e^{2a_* t} \\ &\leqslant \frac{2K_1 C_1 (1+C_1)^2}{a_*}\delta^2 e^{2a_* T_\epsilon} = \frac{2K_1 C_1 (1+C_1)^2}{a_*}\epsilon^2.\end{aligned} \tag{57}$$

We distinguish the components of the solution $X_* = (S_*, W_*)$, $Y(t) = (\tilde{S}(t), \tilde{W}(t))$ and $X(t) = (S(t), W(t)) = (R(\omega t)(S_* + \tilde{S}(t)), W_* + \tilde{W}(t))$. We wish to evaluate

$$\begin{aligned}\Xi_\epsilon &= \inf_\theta \|X(T_\epsilon) - (R(\theta)S_*, W_*)\| \\ &= \inf_\theta \|(R(\omega T_\epsilon)(S_* + \tilde{S}(T_\epsilon)), W_* + \tilde{W}(T_\epsilon)) - (R(\theta)S_*, W_*)\| \\ &= \inf_\theta \|(S_* + \tilde{S}(T_\epsilon), W_* + \tilde{W}(T_\epsilon)) - (R(-\omega T_\epsilon)R(\theta)S_*, W_*)\| \\ &= \inf_{\theta'} \|Y(T_\epsilon) + X_* - (R(\theta')S_*, W_*)\|.\end{aligned}$$

Let θ_ϵ denote the phase which reaches this infimum:

$$\Xi_\epsilon = \|Y(T_\epsilon) + X_* - (R(\theta_\epsilon)S_*, W_*)\|.$$

We observe that

$$\Xi_\epsilon \leqslant \inf_{\theta'}\big(\|Y(T_\epsilon)\| + \|X_* - (R(\theta')S_*, W_*)\|\big) \leqslant \|Y(T_\epsilon)\| \leqslant (1+C_1)\epsilon.$$

Next, we have

$$\|X_* - (R(\theta_\epsilon)S_*, W_*)\| \leqslant \Xi_\epsilon + \|Y(T_\epsilon)\| \leqslant 2(1+C_1)\epsilon,$$

which implies that $\lim_{\epsilon\to 0}\theta_\epsilon = 0$. Hence, a basic Taylor expansion tells us that

$$X_* - (R(\theta_\epsilon)S_*, W_*) = (-\theta_\epsilon \mathscr{J}_S S_*, 0) + \epsilon r_\epsilon, \qquad \lim_{\epsilon\to 0}\|r_\epsilon\| = 0.$$

Now, we are going to use the following splitting of the initial perturbation

$$Y_* = \big(Y_*|(\mathscr{J}_S S_*, 0)\big)\frac{(\mathscr{J}_S S_*, 0)}{\|(\mathscr{J}_S S_*, 0)\|^2} + Y_*^\perp, \qquad \big(Y_*^\perp|(\mathscr{J}_S S_*, 0)\big) = 0.$$

The Cauchy-Schwarz inequality yields

$$\begin{aligned}\Xi_\epsilon\|Y_*^\perp\| &\geqslant \left|\big(Y(T_\epsilon) + X_* - (R(\theta_\epsilon)S_*, W_*)|Y_*^\perp\big)\right| \\ &\geqslant \left|\delta e^{\lambda_* T_\epsilon}\big(Y_*|Y_*^\perp\big) + \big(Y(T_\epsilon) - \delta e^{\lambda_* T_\epsilon}Y_*|Y_*^\perp\big) - \theta_\epsilon \underbrace{\big((\mathscr{J}_S S_*, 0)|Y_*^\perp\big)}_{=0} + \epsilon\big(r_\epsilon|Y_*^\perp\big)\right|\end{aligned}$$

Possibly at the price of choosing a smaller ϵ, coming back to (57), we can make both quantities

$$\left|\big(Y(T_\epsilon) - \delta e^{\lambda_* T_\epsilon}Y_*|Y_*^\perp\big)\right| \leqslant \left\|Y(T_\epsilon) - \delta e^{\lambda_* T_\epsilon}Y_*\right\|\|Y_*^\perp\| \text{ and } \epsilon\left|\big(r_\epsilon|Y_*^\perp\big)\right| \leqslant \epsilon\|r_\epsilon\|\|Y_*^\perp\|$$

smaller than $\frac{\epsilon}{4}\|Y_*^\perp\|^2$. It follows that

$$\begin{aligned}\Xi_\epsilon\|Y_*^\perp\| &\geqslant \left|\delta e^{\lambda_* T_\epsilon}\big(Y_*|Y_*^\perp\big)\right| - \left|\big(Y(T_\epsilon) - \delta e^{\lambda_* T_\epsilon}Y_*|Y_*^\perp\big)\right| - \epsilon\left|\big(r_\epsilon|Y_*^\perp\big)\right| \\ &\geqslant \delta e^{a_* T_\epsilon}\|Y_*^\perp\|^2 - \frac{\epsilon}{2}\|Y_*^\perp\|^2 = \frac{\epsilon}{2}\|Y_*^\perp\|^2.\end{aligned}$$

This estimate is meaningful provided $Y_*^\perp \neq 0$. This is indeed the case because $\mathscr{J}_S S_* = \frac{1}{\sqrt{2}}(0,-1,0,-\tau)$ and we can check that $(\mathscr{J}_S S_*, 0)$ lies in $\mathrm{Ker}(\mathbb{L})$ while $Y_* \in \mathrm{Ker}(\mathbb{L} - \lambda_*)$, with $\lambda_* \neq 0$. ■

A Proof of L^2 and energy conservation properties

The three models can be cast under the general form

$$i\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}u_0\\u_1\end{pmatrix} = \begin{pmatrix}A_0 & -1\\ -1 & A_1\end{pmatrix}\begin{pmatrix}u_0\\u_1\end{pmatrix}$$

where $A_0 = A_1 = 1$ for (4), $A_0 = 1-\kappa|u_0|^2$, $A_1 = 1-\kappa|u_1|^2$ for (5), and $A_0 = 1 + \int_{\mathbb{R}^n}\sigma\psi_0\,\mathrm{d}z$, $A_1 = 1 + \int_{\mathbb{R}^n}\sigma\psi_1\,\mathrm{d}z$ for (1)-(2). In any case, A_0 and A_1 are real. Therefore, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(|u_0|^2 + |u_1|^2) &= \frac{\overline{u_0}}{i}(A_0u_0 - u_1) - \frac{u_0}{i}(A_0\overline{u_0} - \overline{u_1}) + \frac{\overline{u_1}}{i}(A_1u_1 - u_0) - \frac{u_1}{i}(A_1\overline{u_1} - \overline{u_0}) \\ &= \frac{A_0}{i}(\overline{u_0}u_0 - u_0\overline{u_0}) + \frac{A_1}{i}(\overline{u_1}u_1 - u_1\overline{u_1}) + \frac{1}{i}(-\overline{u_0}u_1 + \overline{u_1}u_0 - u_0\overline{u_1} + u_1\overline{u_0}) \\ &= 0,\end{aligned}$$

which proves the conservation of $|u_0|^2+|u_1|^2$.

Moreover, we have

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u_0-u_1|^2 &= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(u_0\overline{u_1}+u_1\overline{u_0})\\
&= -\frac{1}{2}\left(\frac{\overline{u_1}}{i}(A_0u_0-u_1)-\frac{u_1}{i}(A_0\overline{u_0}-\overline{u_1})+\frac{\overline{u_0}}{i}(A_1u_1-u_0)-\frac{u_0}{i}(A_1\overline{u_1}-\overline{u_0})\right)\\
&= -\frac{1}{2i}\left(A_0(\overline{u_1}u_0-u_1\overline{u_0})+A_1(\overline{u_0}u_1-u_0\overline{u_1})\right)=-(A_0-A_1)\mathrm{Im}(u_0\overline{u_1}).
\end{aligned}$$

For (5), this combines to

$$\begin{aligned}
\frac{\kappa}{4}\frac{\mathrm{d}}{\mathrm{d}t}(|u_0|^4+|u_1|^4) &= \frac{\kappa|u_0|^2}{2}\left(\frac{\overline{u_0}}{i}(A_0u_0-u_1)-\frac{u_0}{i}(A_0\overline{u_0}-\overline{u_1})\right)\\
&\quad+\frac{\kappa|u_1|^2}{2}\left(\frac{\overline{u_1}}{i}(A_1u_1-u_0)-\frac{u_1}{i}(A_1\overline{u_1}-\overline{u_0})\right)\\
&= -\frac{\kappa|u_0|^2}{2i}(\overline{u_0}u_1-u_0\overline{u_1})-\frac{\kappa|u_1|^2}{2i}(\overline{u_1}u_0-u_1\overline{u_0})\\
&= \kappa(|u_0|^2-|u_1|^2)\mathrm{Im}(u_0\overline{u_1})=-(A_0-A_1)\mathrm{Im}(u_0\overline{u_1}),
\end{aligned}$$

so that (9) holds. For (1)-(2), we also compute the energy of the vibrational field

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n}\left(\frac{1}{c^2}(|\partial_t\psi_0|^2+|\partial_t\psi_1|^2)+|\nabla\psi_0|^2+|\nabla\psi_1|^2\right)\mathrm{d}z\\
&=\int_{\mathbb{R}^n}\left\{\left(\frac{1}{c^2}\partial_t^2\psi_0-\Delta\psi_0\right)\partial_t\psi_0+\left(\frac{1}{c^2}\partial_t^2\psi_1-\Delta\psi_1\right)\partial_t\psi_1\right\}\mathrm{d}z\\
&=-\int_{\mathbb{R}^n}\sigma\big(|u_0|^2\partial_t\psi_0+|u_1|^2\partial_t\psi_1\big)\,\mathrm{d}z.
\end{aligned}$$

Finally, we compute the evolution of the interaction energy

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n}\sigma(\psi_0|u_0|^2+\psi_1|u_1|^2)\,\mathrm{d}z &= \int_{\mathbb{R}^n}\sigma\big(|u_0|^2\partial_t\psi_0+|u_1|^2\partial_t\psi_1\big)\,\mathrm{d}z\\
&\quad+\int_{\mathbb{R}^n}\sigma\psi_0\left(\frac{\overline{u_0}}{i}(A_0u_0-u_1)-\frac{u_0}{i}(A_0\overline{u_0}-\overline{u_1})\right)\mathrm{d}z\\
&\quad+\int_{\mathbb{R}^n}\sigma\psi_1\left(\frac{\overline{u_1}}{i}(A_1u_1-u_0)-\frac{u_1}{i}(A_1\overline{u_1}-\overline{u_0})\right)\mathrm{d}z\\
&= \int_{\mathbb{R}^n}\sigma\big(|u_0|^2\partial_t\psi_0+|u_1|^2\partial_t\psi_1\big)\,\mathrm{d}z\\
&\quad-\frac{1}{i}\int_{\mathbb{R}^n}\sigma\Big(\psi_0(\overline{u_0}u_1-\overline{u_1}u_0)+\psi_1(\overline{u_1}u_0-\overline{u_0}u_1)\Big)\,\mathrm{d}z\\
&= \int_{\mathbb{R}^n}\sigma\big(|u_0|^2\partial_t\psi_0+|u_1|^2\partial_t\psi_1\big)\,\mathrm{d}z+2(A_0-A_1)\mathrm{Im}(u_0\overline{u_1}).
\end{aligned}$$

Gathering these identities, we arrive at (8).

B Proof of Lemma 4.2.

We extend Σ by 0 on $(-\infty, 0)$ and we assume that Σ is supported in $[-R+\sqrt{\mu}, R+\sqrt{\mu}]$, for some $0 < R < \infty$. Extending the discussion to a function with fast decay at infinity follows from a standard density argument. We start by defining the principal value

$$\text{P.V.} \int_0^{+\infty} \frac{\Sigma(r)}{(r-\sqrt{\mu})(r+\sqrt{\mu})} \,\mathrm{d}r = \lim_{\epsilon \to 0} \int_{-R+\sqrt{\mu}}^{R+\sqrt{\mu}} \mathbf{1}_{|r-\sqrt{\mu}|\geqslant \epsilon} \frac{\Sigma(r)}{(r-\sqrt{\mu})(r+\sqrt{\mu})} \,\mathrm{d}r.$$

We decompose

$$\frac{1}{(r-\sqrt{\mu})(r+\sqrt{\mu})} = \frac{1}{2\sqrt{\mu}(r-\sqrt{\mu})} - \frac{1}{2\sqrt{\mu}(r+\sqrt{\mu})}.$$

There is no difficulty in handling the last term by means of the Lebesgue's dominated convergence theorem (bearing in mind that Σ is supported on $[0, \infty)$) and we obtain

$$\lim_{\epsilon \to 0} \int_{-R+\sqrt{\mu}}^{R+\sqrt{\mu}} \mathbf{1}_{|r-\sqrt{\mu}|\geqslant \epsilon} \frac{\Sigma(r)}{2\sqrt{\mu}(r+\sqrt{\mu})} \,\mathrm{d}r = \int_{-R+\sqrt{\mu}}^{R+\sqrt{\mu}} \frac{\Sigma(r)}{2\sqrt{\mu}(r+\sqrt{\mu})} \,\mathrm{d}r.$$

Next, we make use of parity so that for any $\epsilon > 0$

$$\int_{-R+\sqrt{\mu}}^{R+\sqrt{\mu}} \mathbf{1}_{|r-\sqrt{\mu}|\geqslant \epsilon} \frac{\mathrm{d}r}{r-\sqrt{\mu}} = 0.$$

Hence, we rewrite

$$\begin{aligned} \lim_{\epsilon \to 0} \int_{-R+\sqrt{\mu}}^{R+\sqrt{\mu}} \mathbf{1}_{|r-\sqrt{\mu}|\geqslant \epsilon} \frac{\Sigma(r)}{r-\sqrt{\mu}} \,\mathrm{d}r &= \lim_{\epsilon \to 0} \int_{-R+\sqrt{\mu}}^{R+\sqrt{\mu}} \mathbf{1}_{|r-\sqrt{\mu}|\geqslant \epsilon} \frac{\Sigma(r)-\Sigma(\sqrt{\mu})}{r-\sqrt{\mu}} \,\mathrm{d}r \\ &= \int_{-R+\sqrt{\mu}}^{R+\sqrt{\mu}} \frac{\Sigma(r)-\Sigma(\sqrt{\mu})}{r-\sqrt{\mu}} \,\mathrm{d}r \end{aligned}$$

which is well-defined since the integrand is bounded by $\|\Sigma'\|_{L^\infty}$ and the integral is over a bounded domain. We conclude that

$$\begin{aligned} &\text{P.V.} \int_0^{+\infty} \frac{\Sigma(r)}{(r-\sqrt{\mu})(r+\sqrt{\mu})} \,\mathrm{d}r \\ &= -\int_{-R+\sqrt{\mu}}^{R+\sqrt{\mu}} \frac{\Sigma(r)}{2\sqrt{\mu}(r+\sqrt{\mu})} \,\mathrm{d}r + \int_{-R+\sqrt{\mu}}^{R+\sqrt{\mu}} \frac{\Sigma(r)-\Sigma(\sqrt{\mu})}{2\sqrt{\mu}(r-\sqrt{\mu})} \,\mathrm{d}r. \end{aligned}$$

We split $P(-\mu, B)$ into its real and imaginary parts; it yields

$$P(-\mu, B) = \int_0^\infty \frac{(r^2-\mu)\Sigma(r)}{B^2+(r^2-\mu)^2} \,\mathrm{d}r - iB \int_0^\infty \frac{\Sigma(r)}{B^2+(r^2-\mu)^2} \,\mathrm{d}r.$$

With $B>0$, and the change of variable $r' = \frac{r-\sqrt{\mu}}{B}$, the imaginary parts recasts as

$$\int_0^\infty \frac{\Sigma(r)}{1+((r-\sqrt{\mu})/B)^2(r+\sqrt{\mu})^2}\frac{\mathrm{d}r}{B} = \int_{-\sqrt{\mu}/B}^\infty \frac{\Sigma(Br'+\sqrt{\mu})}{1+r'^2(Br'+2\sqrt{\mu})^2}\,\mathrm{d}r'.$$

A direct application of the Lebesgue's dominated convegence theorem shows that it tends to

$$\Sigma(\sqrt{\mu})\int_{-\infty}^\infty \frac{1}{1+4\mu r'^2}\,\mathrm{d}r' = \frac{\pi\Sigma(\sqrt{\mu})}{2\sqrt{\mu}}$$

as $B\to 0^+$. Taking the limit $B\to 0^-$, changes the sign of this expression.

We proceed in two steps to handle the real part. Let $\epsilon>0$ and compute

$$\begin{aligned}
&\int_{|r-\sqrt{\mu}|\leqslant\epsilon} \frac{(r^2-\mu)\Sigma(r)}{B^2+(r^2-\mu)^2}\,\mathrm{d}r - \int_{|r-\sqrt{\mu}|\leqslant\epsilon} \frac{2\sqrt{\mu}(r-\sqrt{\mu})\Sigma(r)}{B^2+4\mu(r-\sqrt{\mu})^2}\,\mathrm{d}r\\
&\quad= \int_{|r-\sqrt{\mu}|\leqslant\epsilon} \Sigma(r)(r-\sqrt{\mu})\\
&\qquad\times\frac{B^2(r+\sqrt{\mu}-2\sqrt{\mu})+4\mu(r-\sqrt{\mu})^2(r+\sqrt{\mu})-2\sqrt{\mu}(r-\sqrt{\mu})^2(r+\sqrt{\mu})^2}{(B^2+(r^2-\mu)^2)(B^2+4\mu(r-\sqrt{\mu})^2)}\,\mathrm{d}r\\
&\quad= \int_{|r-\sqrt{\mu}|\leqslant\epsilon} \Sigma(r)(r-\sqrt{\mu})^2\frac{B^2+(r-\sqrt{\mu})(r+\sqrt{\mu})(4\mu-2\sqrt{\mu}(r+\sqrt{\mu}))}{(B^2+(r^2-\mu)^2)(B^2+4\mu(r-\sqrt{\mu})^2)}\,\mathrm{d}r\\
&\quad= \int_{|r-\sqrt{\mu}|\leqslant\epsilon} \Sigma(r)(r-\sqrt{\mu})^2\frac{B^2-2\sqrt{\mu}(r-\sqrt{\mu})^2(r+\sqrt{\mu})}{(B^2+(r^2-\mu)^2)(B^2+4\mu(r-\sqrt{\mu})^2)}\,\mathrm{d}r.
\end{aligned}$$

This difference is dominated by

$$\begin{aligned}
&\int_{|r-\sqrt{\mu}|\leqslant\epsilon} \Sigma(r)\left(\frac{B^2(r-\sqrt{\mu})^2}{B^2 4\mu(r-\sqrt{\mu})^2} + \frac{2\sqrt{\mu}(r-\sqrt{\mu})^4(r+\sqrt{\mu})}{4\mu(r-\sqrt{\mu})^4(r+\sqrt{\mu})^2}\right)\mathrm{d}r\\
&\qquad= \int_{|r-\sqrt{\mu}|\leqslant\epsilon} \Sigma(r)\left(\frac{1}{4\mu}+\frac{1}{2\sqrt{\mu}(r+\sqrt{\mu})}\right)\mathrm{d}r.
\end{aligned}$$

Pick $\delta>0$. Since $r\mapsto\Sigma(r)\big(\frac{1}{4\mu}+\frac{1}{2\sqrt{\mu}(r+\sqrt{\mu})}\big)$ is integrable over $(0,\infty)$, this quantity can be made $\leqslant\delta$ by choosing ϵ small enough. Possibly at the price of reducing ϵ, we also suppose that

$$\left|\int_0^{+\infty} \frac{\Sigma(r)}{2\sqrt{\mu}(r+\sqrt{\mu})}\,\mathrm{d}r - \int_{|r-\sqrt{\mu}|\geqslant\epsilon} \frac{\Sigma(r)}{2\sqrt{\mu}(r+\sqrt{\mu})}\,\mathrm{d}r\right| \leqslant \delta$$

holds. Having disposed of this preliminary, we write

$$\begin{aligned}
\int_0^\infty \frac{(r^2-\mu)\Sigma(r)}{B^2+(r^2-\mu)^2}\,\mathrm{d}r &= \int_{|r-\sqrt{\mu}|\leqslant\epsilon} \frac{(r^2-\mu)\Sigma(r)}{B^2+(r^2-\mu)^2}\,\mathrm{d}r + \int_{|r-\sqrt{\mu}|\geqslant\epsilon} \frac{(r^2-\mu)\Sigma(r)}{B^2+(r^2-\mu)^2}\,\mathrm{d}r\\
&= \left(\int_{|r-\sqrt{\mu}|\leqslant\epsilon} \frac{(r^2-\mu)\Sigma(r)}{B^2+(r^2-\mu)^2}\,\mathrm{d}r - \int_{|r-\sqrt{\mu}|\leqslant\epsilon} \frac{2\sqrt{\mu}(r-\sqrt{\mu})\Sigma(r)}{B^2+4\mu(r-\sqrt{\mu})^2}\,\mathrm{d}r\right)\\
&\quad+ \int_{|r-\sqrt{\mu}|\leqslant\epsilon} \frac{2\sqrt{\mu}(r-\sqrt{\mu})\Sigma(r)}{B^2+4\mu(r-\sqrt{\mu})^2}\,\mathrm{d}r + \int_{|r-\sqrt{\mu}|\geqslant\epsilon} \frac{(r^2-\mu)\Sigma(r)}{B^2+(r^2-\mu)^2}\,\mathrm{d}r
\end{aligned}$$

where the first term can be made $\leqslant \delta$, uniformly with respect to B, *i.e.*

$$\sup_{B\in\mathbb{R}}\left|\int_{|r-\sqrt{\mu}|\leqslant\epsilon}\frac{(r^2-\mu)\Sigma(r)}{B^2+(r^2-\mu)^2}\,\mathrm{d}r-\int_{|r-\sqrt{\mu}|\leqslant\epsilon}\frac{2\sqrt{\mu}(r-\sqrt{\mu})\Sigma(r)}{B^2+4\mu(r-\sqrt{\mu})^2}\,\mathrm{d}r\right|\leqslant\delta.$$

The limit of the last integral is identified by applying Lebesgue's dominated convergence theorem

$$\begin{aligned}\lim_{B\to 0}\int_{|r-\sqrt{\mu}|\geqslant\epsilon}\frac{(r^2-\mu)\Sigma(r)}{B^2+(r^2-\mu)^2}\,\mathrm{d}r &= \int_{|r-\sqrt{\mu}|\geqslant\epsilon}\frac{\Sigma(r)}{(r^2-\mu)}\,\mathrm{d}r\\
&=\int_{|r-\sqrt{\mu}|\geqslant\epsilon}\frac{\Sigma(r)}{2\sqrt{\mu}(r-\sqrt{\mu})}\,\mathrm{d}r-\int_{|r-\sqrt{\mu}|\geqslant\epsilon}\frac{\Sigma(r)}{2\sqrt{\mu}(r+\sqrt{\mu})}\,\mathrm{d}r\\
&=\int_{-R+\sqrt{\mu}}^{R+\sqrt{\mu}}\mathbf{1}_{|r-\sqrt{\mu}|\geqslant\epsilon}\frac{\Sigma(r)-\Sigma(\sqrt{\mu})}{2\sqrt{\mu}(r-\sqrt{\mu})}\,\mathrm{d}r-\int_{|r-\sqrt{\mu}|\geqslant\epsilon}\frac{\Sigma(r)}{2\sqrt{\mu}(r+\sqrt{\mu})}\,\mathrm{d}r.\end{aligned}$$

Finally, the second term can be recast as

$$\int_{-R+\sqrt{\mu}}^{R+\sqrt{\mu}}\mathbf{1}_{|r-\sqrt{\mu}|\leqslant\epsilon}\frac{2\sqrt{\mu}(r-\sqrt{\mu})(\Sigma(r)-\Sigma(\sqrt{\mu}))}{B^2+4\mu(r-\sqrt{\mu})^2}\,\mathrm{d}r$$

so that, in the limit as B goes to 0, we get

$$\int_{-R+\sqrt{\mu}}^{R+\sqrt{\mu}}\mathbf{1}_{|r-\sqrt{\mu}|\leqslant\epsilon}\frac{\Sigma(r)-\Sigma(\sqrt{\mu})}{2\sqrt{\mu}(r-\sqrt{\mu})}\,\mathrm{d}r.$$

Therefore, we get

$$\begin{aligned}\lim_{B\to 0}\left(\int_{|r-\sqrt{\mu}|\leqslant\epsilon}\frac{2\sqrt{\mu}(r-\sqrt{\mu})\Sigma(r)}{B^2+4\mu(r-\sqrt{\mu})^2}\,\mathrm{d}r+\int_{|r-\sqrt{\mu}|\geqslant\epsilon}\frac{(r^2-\mu)\Sigma(r)}{B^2+(r^2-\mu)^2}\,\mathrm{d}r\right)\\
=\int_{-R+\sqrt{\mu}}^{R+\sqrt{\mu}}\frac{\Sigma(r)-\Sigma(\sqrt{\mu})}{2\sqrt{\mu}(r-\sqrt{\mu})}\,\mathrm{d}r-\int_{|r-\sqrt{\mu}|\geqslant\epsilon}\frac{\Sigma(r)}{2\sqrt{\mu}(r+\sqrt{\mu})}\,\mathrm{d}r,\end{aligned}$$

which is close, up to δ, to P.V. $\int_0^\infty\frac{\Sigma(r)}{(r-\sqrt{\mu})(r+\sqrt{\mu})}\,\mathrm{d}r$. As a consequence, we conclude that, for any $\delta>0$, we can exhibit $B(\delta)>0$ small enough so that

$$\left|\int_0^\infty\frac{(r^2-\mu)\Sigma(r)}{B^2+(r^2-\mu)^2}\,\mathrm{d}r-\text{P.V.}\int_0^\infty\frac{\Sigma(r)}{(r-\sqrt{\mu})(r+\sqrt{\mu})}\,\mathrm{d}r\right|\leqslant 2\delta$$

holds for any $0<|B|<B(\delta)$. ∎

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