a priori estimates. Energy / Entropy methods

We consider the heat eq:

$$\partial_t u = \Delta u \quad \text{on } (0,\infty)\times\Omega$$
$$u|_{t=0} = u_0 \quad \text{on } \Omega$$

with various type of b.c:

- Dirichlet b.c $\quad u|_{\partial\Omega} = 0$
- Neumann b.c $\quad \partial_\nu u|_{\partial\Omega} = 0$
- Periodic b.c $\quad \Omega = \mathbb{T}^N$
- $\Omega = \mathbb{R}^N$

Multiply by u and integrate by parts: owing to the b.c, there is no boundary contributions

$$\frac{d}{dt}\int \frac{u^2}{2}\,dx = \int u\,\partial_t u\,dx = \int u\,\Delta u\,dx$$
$$= -\int |\nabla u|^2\,dx \leq 0$$

Integrate wrt time:

$$\int \frac{u^2}{2}(t,x)\,dx + \int_0^t \int |\nabla u|^2(s,x)\,dx\,ds = \int |u_0(x)|^2\,dx.$$

We learn two things:

- $u \in L^\infty(0,\infty; L^2(\Omega))$
- $\nabla u \in L^2((0,\infty)\times\Omega)$.

This reasoning applies:

* when replacing Δu by $\nabla\cdot(A\nabla u)$

where $A : (0,\infty)\times\Omega \longrightarrow M_N$
$\qquad\qquad (t,x) \longmapsto A(t,x)$

satisfying a coercivity property:

$$A(t,x)\,\xi\cdot\xi \geq \alpha|\xi|^2$$

for some $\alpha > 0$ (and any $t\geq 0, x\in\Omega, \xi\in\mathbb{R}^N$)

* replacing $\frac{u^2}{2}$ by more general function

$\Phi(u)$ with $\Phi(0) = 0$

$$\frac{d}{dt}\int \phi(u)\,dx = \int \phi'(u)\partial_t u\,dx = \int \phi'(u)\,\Delta u\,dx$$

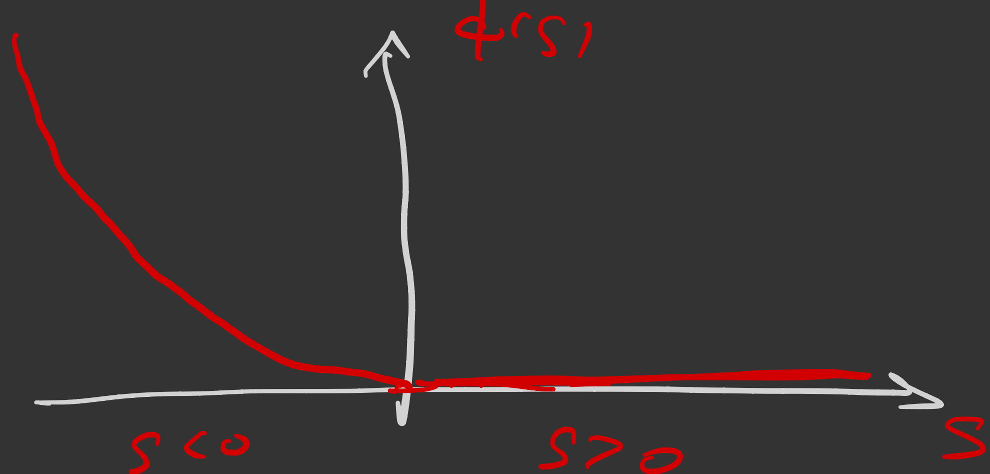
$$= -\int \phi''(u)\,|\nabla u|^2\,dx \leq 0$$

for ϕ convex.

In this way we can show that

- $u \in L^\infty(0,\infty; L^p(\Omega))$ when $u_0 \in L^p(\Omega)$
- maximum principle holds: if $u_0 \geq 0$ then $u \geq 0$

To this end, we should work with



It works with a RHS $f \geq 0$ because $\int f\,\phi'(u)\,dx \leq 0$.

Application: large time behavior of the sol. of the heat eq. with Neumann b.c.:

$$\begin{cases} \partial_t u = \Delta u & t \geq 0,\ x \in \Omega \\ \partial_\nu u|_{\partial\Omega} = 0 \\ u|_{t=0} = u_0 \end{cases}$$

We have:

$$\frac{1}{2}\frac{d}{dt}\int u^2\,dx = -\int |\nabla u|^2\,dx.$$

Poincaré's lemma. There exists $C_\Omega > 0$ such that, for any $u \in H^1(\Omega)$, we have

$$\| u - \langle u \rangle \|_{L^2}^2 \leq C_\Omega \int |\nabla u|^2\,dx$$

$$\langle u \rangle = \frac{1}{|\Omega|}\int u\,dx.$$

Remark that

$$\int |u - \langle u \rangle|^2\,dx = \int u^2\,dx + \langle u \rangle^2 |\Omega| \overbrace{- 2\langle u \rangle \int u\,dx}^{= -\langle u \rangle^2 |\Omega|}$$

$$\int u^2\,dx - |\Omega|\langle u \rangle^2 \geq 0$$

Proof. Ad absurdum. We assume that we can find $u_n \in H^1$ such that

$$\|u_n\|_{L^2} = 1 \ , \quad \frac{1}{n} \geq \int |\nabla u_n|^2 dx$$

$$\langle u_n \rangle = 0$$

We work in the closed subspace of H^1:

$$H^{1,0} = \{ u \in H^1, \int u \, dx = 0 \}$$

$(u_n)_{n \in \mathbb{N}}$ is bounded in H^1; by Rellich's theorem, we can extract a subsequence which converges strongly in L^2:

$$u_{n_k} \xrightarrow[k \to +\infty]{} u \quad \text{in } L^2$$

But, we also have $\|\nabla u_{n_k}\|_{L^2}^2 \leq \frac{1}{n_k} \xrightarrow[k \to 0]{} 0$

Hence, let $\varphi \in C_c^\infty(\Omega)$, we have

$$\int \nabla u_{n_k} \varphi \, dx = -\int u_{n_k} \nabla \varphi \, dx \xrightarrow[k \to +\infty]{} 0 = -\int u \nabla \varphi \, dx$$

which tells us that $\nabla u = 0$

Assuming Ω is connected, it implies

$u(x) = \bar{u}$ is constant.

But, we have $\int u_{n_k}\, dx = 0 \xrightarrow[k\to+\infty]{} \int u\, dx = |\Omega|\, \bar{u}$

which implies $\bar{u} = 0$, which contradicts

$\|u_{n_k}\|_{L^2} = 1$.

Let us go back to the heat eq:

$$\frac{d}{dt}\int u^2\, dx \leq -2C_\Omega \int |u - \langle u\rangle|^2\, dx$$

We remark that

$$\frac{d}{dt}\left(\int |u - \langle u\rangle|^2\, dx\right) = \frac{d}{dt}\left\{\int u^2 - |\Omega|\langle u\rangle^2\right\}$$

where $\langle u\rangle$ does not depend on time

$$\frac{d}{dt}\int u\, dx = \int \Delta u\, dx = 0$$

$$= \int_{\partial\Omega} \partial_\nu u\, d\sigma(x) = 0$$

Therefore, $y(t) = \int |u - \langle u \rangle|^2 dx$

satisfies $$y'(t) \leq -2C_2\, y(t)$$

It implies $$\boxed{y(t) \leq e^{-2C_2 t}\, y_0}$$

$u(t,x) \longrightarrow \langle u \rangle$ the mean value of u (= mean value of u_0) exponentially fast.

Application 2: non linear heat eq:

$$\partial_t u = \Delta u + F(u) \quad \text{on } (0,\infty)\times\Omega$$

$$u|_{t=0} = u_{Init}$$

F is a locally Lipschitz function $\mathbb{R}\to\mathbb{R}$.

Question: existence-uniqueness of sol.?
locally or globally in time?

Idea: Proceed with an iteration scheme

$$\partial_t u_{n+1} = \Delta u_{n+1} + F(u_n), \quad u_{n+1}|_{t=0} = u_{Init}$$

For $n=0$, we start the scheme with $u_0 = 0$.

For instance, on $\mathbb{R}^N$ we have:

$$u_{n+1}(t,x) = \int \frac{e^{-|x-y|^2/4t}}{(4\pi t)^{N/2}} u_{Init}(y)\,dy$$

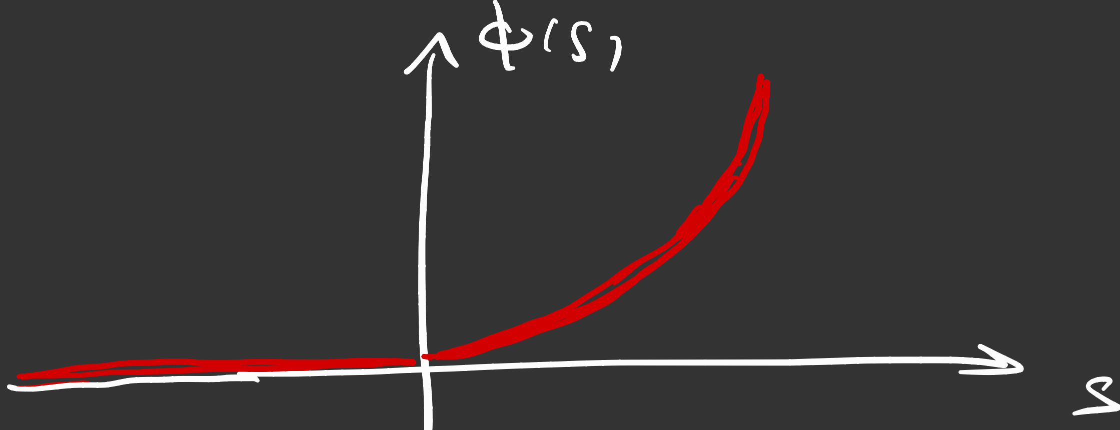
$$+ \int_0^t \int \frac{e^{-|x-y|^2/4(t-s)}}{(4\pi(t-s))^{N/2}} F(u_n)(s,y)\,dy\,ds$$

As already, if $u_0 \geq 0$ then $u_n \geq 0$.

Let us assume $u_0 \in L^2 \cap L^\infty(\Omega)$ with

$$\|u_0\|_{L^\infty} \leq R.$$

Let $C_R = \sup_{0 \leq u \leq R} F(u)$



$$\frac{d}{dt} \int \phi(u - \|u_0\|_\infty - C_R t)\, dx$$

$$= \int \phi'(\dots)(\partial_t u - C_R)\, dx$$

$$= -\int \phi''(\dots)|\nabla u|^2 + \int (F - C_R)\, \phi'(\dots)\, dx$$

There is no boundary term due to the b.c. including with Dirichlet b.c because $\underbrace{\phi'(0 - \|u_0\|_\infty - C_R t)}_{=0}$

We have $\int \underbrace{(F - C_R)}_{\geq 0} \underbrace{\phi'}_{\leq 0} dx \leq 0$.

so that

$$\int \phi \left(u(t,x) - \|u_0\|_\infty - C_R t \right) dx$$

$$\leq \int \phi \left(u_0 - \|u_0\|_\infty \right) dx = 0$$

and actually

$$u(t,x) \leq \|u_0\|_\infty + C_R t .$$

We have obtained an L^∞ estimate on the sol. of $\partial_t u = \Delta u + F$, $u|_{t=0} = u_0$.

It tells us that if $0 \leq u_n(t,x) \leq R$

then $0 \leq u_{n+1}(t,x) \leq \|u_0\|_\infty + C_R T$

on $(0,T) \times \Omega$. We choose T small enough so that $\|u_0\|_\infty + C_R T \leq R$.

We set $\quad L_R = \sup_{0 \le u \le R} |F'(u)|.$

We have

$$\frac{d}{dt} \int \frac{|u_{n+1}-u_n|^2}{2} dx + \int |\nabla(u_{n+1}-u_n)|^2$$

$$= \int \big(F(u_n)-F(u_{n-1})\big)\big(u_{n+1}-u_n\big) dx$$

$$\le L_R \int |u_n-u_{n-1}|\,|u_{n+1}-u_n|\, dx$$

$$\le L_R \left(\int |u_n-u_{n-1}|^2 dx\right)^{1/2} \left(\int |u_{n+1}-u_n|^2 dx\right)^{1/2}$$

$$ab \le \frac{1}{2}(a^2+b^2)$$

$$\le \frac{L_R^2}{\varepsilon} \int |u_{n+1}-u_n|^2 dx$$

$$+ \varepsilon \int |u_n-u_{n-1}|^2 dx$$

Use the Gronwall lemma to deduce an estimate on $\int |u_{n+1}-u_n|^2 dx$

We arrive at for $0 \leq t \leq T$

$$\int |u_{n+1} - u_n|^2 (t,x)\, dx \leq \varepsilon\, e^{L_R^2/\varepsilon\, T}\, T \times \sup_{0 \leq \xi \leq T} \int |u_n - u_{n-1}|^2 (\xi, x)\, dx$$

Therefore, choosing T small enough

$$\mathcal{F} : u_n \longmapsto u_{n+1}$$

is a contraction on $L^\infty(0,T; L^2(\Omega))$

Hence it admits a unique fixed point u, which is a sol. of the nonlinear eq.

Do we have globale existence or not?

Competition between:

- regularizing effect of the heat eq.
- blow-up behavior of the ODE

$$y' = y^2 \qquad \boxed{F(y) = y^2}$$

On the torus $\mathbb{T}^N$:

$$\frac{d}{dt}\Big\{\underbrace{\int \frac{|\nabla u|^2}{2}\,dx - \int \frac{u^3}{3}\,dx}_{\mathcal{E}(u)}\Big\}$$

$$= \int \nabla u\, \nabla \partial_t u\,dx - \int u^2\, \partial_t u\,dx$$

$$= \int \nabla u\, \nabla(\Delta u + u^2) - \int u^2(\Delta u + u^2)\,dx$$

$$= -\int (\Delta u + u^2)^2\,dx \leq 0$$

Therefore, we have $\mathcal{E}(u)(t) \le \mathcal{E}(u_0)$

We assumed implicitly the regularity of the initial data u_0, so that $\mathcal{E}(u_0)$.

But, note that $\mathcal{E}$ has no sign.

- We go back to the basic estimate

$$\frac{d}{dt}\int \frac{u^2}{2}\,dx = -\int |\nabla u|^2 dx + \int u^3\,dx$$

$$= -2\,\mathcal{E}(u)(t) + \frac{1}{3}\int u^3 dx$$

But, we have

$$\int u^2 dx = \int \left(u^3\right)^{2/3} dx \le \left(\int u^3 dx\right)^{2/3} \left(\mathbb{T}^N\right)^{1/3}$$

by Hölder's inequality.

We conclude that

$$\frac{1}{2}\frac{d}{dt}\int u^2 dx \geqslant -2\,\mathcal{E}(u)(t) + C\left(\int u^2 dx\right)^{3/2}$$

we shall use this differential inequality

- If $\mathcal{E}(u_0) < 0$ then by step 1, $\mathcal{E}(u) < 0$ $\forall t \geq 0$ and therefore $Z(t) = \int u^2 dx$ satisfies

$$Z'(t) \geq C\, Z(t)^{3/2}$$

By a comparison argument, it implies (because sol. of $y' = Cy^{3/2}$ blowup in finite time) that

$$Z(t) \xrightarrow[t \to T_*]{} +\infty \text{ with } 0 < T_* < \infty$$

There exists such data. Indeed for $\varphi \in C_c^\infty(\mathbb{R}^N)$, $\varphi \geq 0$ we set $u_0(x) = \lambda \varphi(x)$ so that

$$\mathcal{E}(u_0) = \lambda^2 \int |\nabla \varphi(x)|^2 dx - \frac{\lambda^3}{3} \int \varphi^3 dx$$

$$\xrightarrow[\lambda \to +\infty]{} -\infty$$

On a bounded domain, with Dirichlet b.c, we can also show that the sol. doesn't exist globally (Evans' method)

We need an auxilliary function that uses the properties of the Δ operator: there exists a pair (λ, W) with $\lambda > 0$

and $W : \Omega \longrightarrow (0, \infty)$

($W(x) > 0$ for any $x \in \Omega$) such that

$$-\Delta W = \lambda W, \quad \int W dx = 1$$

We compute

$$\frac{d}{dt} \int u W dx = \int (\Delta u + u^2) W dx$$

$$= \int (u \Delta W + u^2 W) dx$$

$$= -\lambda \int u W \, dx + \int u^2 W dx$$

where $\int u W dx \leq \left(\int u^2 W dx\right)^{1/2} \underbrace{\left(\int W dx\right)^{1/2}}_{=1}$

by Cauchy-Schwarz.

Let us set $\mathcal{E}(t) = \int u W \, dx$

We have

$$\mathcal{E}'(t) \geq -\lambda \mathcal{E}(t) + \mathcal{E}^2(t)$$

or, in other words

$$\frac{d}{dt}\left(e^{\lambda t}\mathcal{E}(t)\right) \geq e^{-\lambda t}\left(e^{\lambda t}\mathcal{E}(t)\right)^2$$

We deduce that

$$e^{\lambda t}\mathcal{E}(t) \geq \frac{1}{1/\mathcal{E}(0) - \frac{1}{\lambda}\left(1-e^{-\lambda t}\right)}$$

$$\geq \frac{\lambda \mathcal{E}(0)}{\lambda - \mathcal{E}(0)\left(1-e^{-\lambda t}\right)}$$

which exhibits a blow up when $\lambda < \mathcal{E}(0)$

(and, again, we can construct such data).

$$\int u W \, dx \xrightarrow[t \to T_*]{} +\infty \quad , \quad 0 < T_* < \infty$$

- Application 3: Galerkin's method

→ theoretical motivation

→ numerical motivation

$$\partial_t u - \nabla\cdot(A\nabla u) + \nabla\cdot(b u) + c u = 0$$

$$u|_{\partial\Omega} = 0 \quad , \; u|_{t=0} = u_{init}.$$

$$b : (t, x) \in (0,\infty)\times\Omega \longrightarrow \mathbb{R}^N$$
$$c : (t,x) \in (0,\infty)\times\Omega \longrightarrow \mathbb{R}.$$
$$A : (t,x) \in (0,\infty)\times\Omega \longrightarrow M_N$$

$$A(t,x)\,\xi\cdot\xi \geq \alpha|\xi|^2$$

A, b, c belong to L^∞

A natural space is $L^\infty(0,\infty; L^2(\Omega)) \cap L^2((0,\infty)\times\Omega)$ due to the energy estimate:

$$\frac{d}{dt}\int \frac{u^2}{2}\,dx + \underbrace{\int A\nabla u\cdot\nabla u\,dx}_{\geq\, \alpha\int|\nabla u|^2 dx} + \int b u\cdot\nabla u\,dx + \int c u^2\,dx = 0$$

The idea is to approach the functional space in which lives the sol.:

- work with a finite dimensional space that makes the pb. easier
- a priori estimates
  - ↳ compactness properties
- pass to the limit.

We expect $u(t) \in V$, Hilbert space ($V = H^1_0(\Omega)$)

We search for a sequence of spaces

$$V_n \subset V$$

$$\dim V_n < \infty, \quad \dim V_n \xrightarrow[n \to +\infty]{} +\infty$$

Ex.: • V admits an hilbertian basis $\{e_n, n \in \mathbb{N}\}$, and we can work with $V_n = \text{Span}\{e_1, \dots, e_n\}$

• take a basis of eigenfunctions w_n of an operator arising in the eq.

Here we can solve the stationary eq

$$\begin{cases} -\nabla \cdot (A \nabla u) = f \\ u_{|\partial\Omega} = 0 \end{cases}$$

It defines an isomorphism $T : H^{-1} \longrightarrow H_0^1$
$f \longmapsto u$

$S = T^{-1}$ is a compact operator on L^2 (by virtue of Rellich's theorem)

Hence it admits a basis of eigenfunctions $w_n \in H_0^1$, $-\nabla \cdot (A \nabla w_n) = \lambda_n w_n$.

Multiply the eq. by w_n:

$$\frac{d}{dt}\int u\, w_n\, dx + \int A\nabla u \cdot \nabla w_n\, dx + \int b u \cdot \nabla w_n\, dx + \int c\, u\, w_n\, dx = 0$$

This motivates to write u as a series

$$u(t,x) = \sum_{n=0}^{\infty} u_n(t)\, w_n(x)$$

and to search an approximation

$$u_N(t,x) = \sum_{n=0}^{N} \alpha_n^N(t)\, w_n(x)$$

where the functions $t \mapsto \alpha_n(t)$ are defined by the ODE system:

$$\sum_{j=0}^{N} \dot{\alpha}_j^N \int w_j w_n\, dx + \sum_{j=0}^{N} \alpha_j^N \left\{ \int A\nabla w_j \cdot \nabla w_n\, dx + \int b w_j \cdot \nabla w_n\, dx + \int c\, w_j w_n\, dx \right\} = 0$$

, and $\alpha_j^N(0) = \int u_{init}\, w_j\, dx$.

This is a linear differential system

$$\begin{cases} A \dot{y} + B y = 0 \\ y(0) = y_{init} \end{cases}$$

$$y = (\alpha_1^N, \dots \alpha_N^N)$$

$$A_{jn} = \int w_j w_n \, dx$$

$$B_{jn} = \int A \nabla w_j \nabla w_n \, dx + \int b \, w_j \nabla w_n \, dx + \int c \, w_j w_n \, dx$$

Therefore u_n is uniquely defined with

$$u_n \in C^0([0,T]; V_N)$$

$$\frac{d}{dt} u_n \in L^2(0,T; V_N).$$

We establish now uniform estimates.

We have:

$$\int u_N^2 \, dx = + \sum_{j,k} \int \alpha_j^N(t)\, w_j(x)\, \alpha_k^N\, w_k(x)\, dx.$$

$$= \sum_{j,k} \alpha_j^N \alpha_k^N \int w_j(x)\, w_k\, dx$$

The eq. tells us that

$$\frac{1}{2}\frac{d}{dt}\int u_N^2 dx = \underbrace{-\int A\nabla u_N \cdot \nabla u_N \, dx}$$

$$+ \underbrace{\int b\, u_N \cdot \nabla u_N \, dx} + \underbrace{\int c\, u_N^2 \, dx}$$

$$\leq \underbrace{-\alpha \int |\nabla u_N|^2 dx}$$

$$+ \underbrace{\|b\|_\infty \|u_N\|_{L^2}} \underbrace{\|\nabla u_N\|_{L^2}^2}$$

$$+ \underbrace{\|c\|_\infty \|u_N\|_{L^2}^2}$$

We have:

$$\|b\|_\infty \|u_N\|_{L^2} \|\nabla u_N\|_{L^2}$$

$$\leq \frac{\alpha}{2} \|\nabla u_N\|_{L^2}^2 + \frac{\|b\|_\infty^2}{2\alpha} \|u_N\|_{L^2}^2$$

by using $st \leq \frac{1}{2} s^2 + \frac{1}{2} t^2$

We arrive at

$$\frac{1}{2} \frac{d}{dt} \|u_N\|^2 + \frac{\alpha}{2} \|\nabla u_N\|_{L^2}^2$$

$$\leq \left(\frac{\|b\|_\infty^2}{2\alpha} + \|c\|_\infty \right) \|u_N\|_{L^2}^2 .$$

We apply Grönwall's lemma:

$$\|u_N(t)\|_{L^2}^2 \leq e^{T(1 + \|c\|_\infty + \|b\|_\infty^2/2\alpha)} \times \|u_{init}\|_{L^2}^2$$

for any $0 \leq t \leq T < \infty$

and $$\int_0^T \int |\nabla u_N|^2 (s,x)\, dx\, ds \leq C_T$$

We conclude that u_n is bounded in

$$L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$$

We can apply the Banach-Alaoglu theorem so that we can assume, up to a subsequence,

$$\begin{cases} u_n \rightharpoonup u & \text{in } L^2(0,T;H_0^1(\Omega)) \\ u_n \overset{*}{\rightharpoonup} u & \text{in } L^\infty(0,T;L^2(\Omega)) \end{cases}$$

In order to conclude, we need "good" properties on the approximation spaces V_N: $\forall \varphi \in C_c^\infty(\Omega)$ there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ such that

$\forall n \in \mathbb{N}$, $\varphi_n \in V_n$ and $\|\varphi - \varphi_n\|_V \xrightarrow[n\to+\infty]{} 0$

We observe that we have, for any $k \in \{1, \dots, N\}$

$$\underbrace{\frac{d}{dt}\int u_N w_k \, dx}$$

$$+ \int \left(A \nabla u_N \cdot \nabla w_k + b u_N \nabla w_k + c u_N w_k \right) dx = 0$$

Consequently it works replacing w_k by the elements φ_ℓ, $\ell \in \{1, \dots, N\}$ of the sequence $(\varphi_\ell)_{\ell \in \mathbb{N}}$ that approaches a given function $\varphi \in C_c^\infty$, $\varphi_\ell \in V_\ell$.

Therefore, we can let N go to ∞ and we obtain, for any $\zeta \in C_c^\infty([0, \infty[)$

$$-\int_0^\infty\int u(t,x)\,\varphi(x)\,\zeta'(t)\,dx\,dt$$

$$+\int_0^T\int \left(A\nabla u\cdot\nabla\varphi + b u\nabla\varphi + c u\varphi\right)\zeta(t)\,dx\,dt$$

$$= \int u_{init}\,\varphi(x)\,dx.$$

This equality holds for any $\varphi \in C_c^\infty(\Omega)$

$\zeta \in C_c^\infty([0,\infty))$

By density, it also holds for $\varphi \in H_0^1(\Omega)$

Hence u satisfies

$$\partial_t u = \nabla\cdot(\underbrace{A\nabla u}_{L^\infty_t L^2_x} + \underbrace{b u}_{L^\infty_t L^2_x}) + \underbrace{c u}_{L^\infty_t L^2_x}$$

in $\mathcal{D}'([0,\infty)\times\Omega)$

$$\in L^2(0,T; \underline{H^{-1}(\Omega)})$$

- Uniqueness is now a consequence of the energy estimate.
- We can treat similarly source terms $f \in L^2((0,T) \times \Omega)$ or even $f \in L^2(0,T; H^{-1}(\Omega))$
- It can be adapted to handle certain non linear problems.
- It handles the general pb:

$$\frac{d}{dt}(u(t), \varphi)_H + a(t, u(t), \varphi) = \langle f(t), \varphi \rangle_{V',V}$$

$\forall \varphi \in V$

in $\mathcal{D}'([0,T])$

$u \in C^0([0,T]; H) \cap L^2(0,T; V)$

$$V \subset H \subset V'$$

$f \in L^2(0,T;V')$

$t \mapsto a(t,u,v)$ is measurable for any $u, v \in V$

$$|a(t,u,v)| \leq M \|u\|_V \|v\|_V$$

$$a(t,u,u) \geq \alpha \|u\|_V^2 - \lambda \|u\|_H^2$$

(we can change unknown by setting $\tilde{u}(t) = e^{kt} u(t)$ for $k > \lambda$)

Ref.: Brézis.
Dautray - Lions.

When the w_n's are eigenfunctions of the stationary eq, orthonormalized in $L^2(\Omega)$, the analysis simplifies: we have

$$\partial_t u_N = \underbrace{\Pi_N}_{\text{projection on } \mathrm{Span}\{w_1, \dots, w_N\}} \left[\nabla \cdot (A \nabla u_N) + b u_N + c u_N \right]$$

$$\| \Pi_N \|_{\mathcal{L}(H^{-1})} \leq C.$$

Theorem Let $a_{ij} \in L^\infty(\Omega)$

$$\sum_{i,j} a_{ij} \xi_j \xi_i \geq \alpha |\xi|^2$$

Then:

- $T: f \mapsto u$ sol. of $-\nabla \cdot (A \nabla u) = f$ is an isomorphism from H^{-1} to H_0^1.
- The inverse is a compact in L^2 which, moreover is positive: if $f \geq 0$, $f \neq 0$, then $u(x) > 0$ in Ω
- There exists a sequence w_n orthogonal and total in L^2, made of eigenfunctions of T.

$$-\nabla \cdot (A \nabla w_n) = \lambda_n w_n$$

- we have $\lambda_n > 0$, $\lambda_n \xrightarrow[n \to +\infty]{} +\infty$
- $\lambda_1 =$ smallest eigenvalue, $w_1 > 0$ in Ω.

This statement combines:

- Lax-Milgram to define $f \mapsto u$
- Rellich theorem provides the compactness
- Positivity comes from the maximum principle
- Spectral decomposition

is a general statement about compact operators:

see Brezis Th. VI-4

- We have:

$$\int A \nabla w_n \cdot \nabla w_n \, dx = \int w_n^2 \, dx \, \lambda_n$$

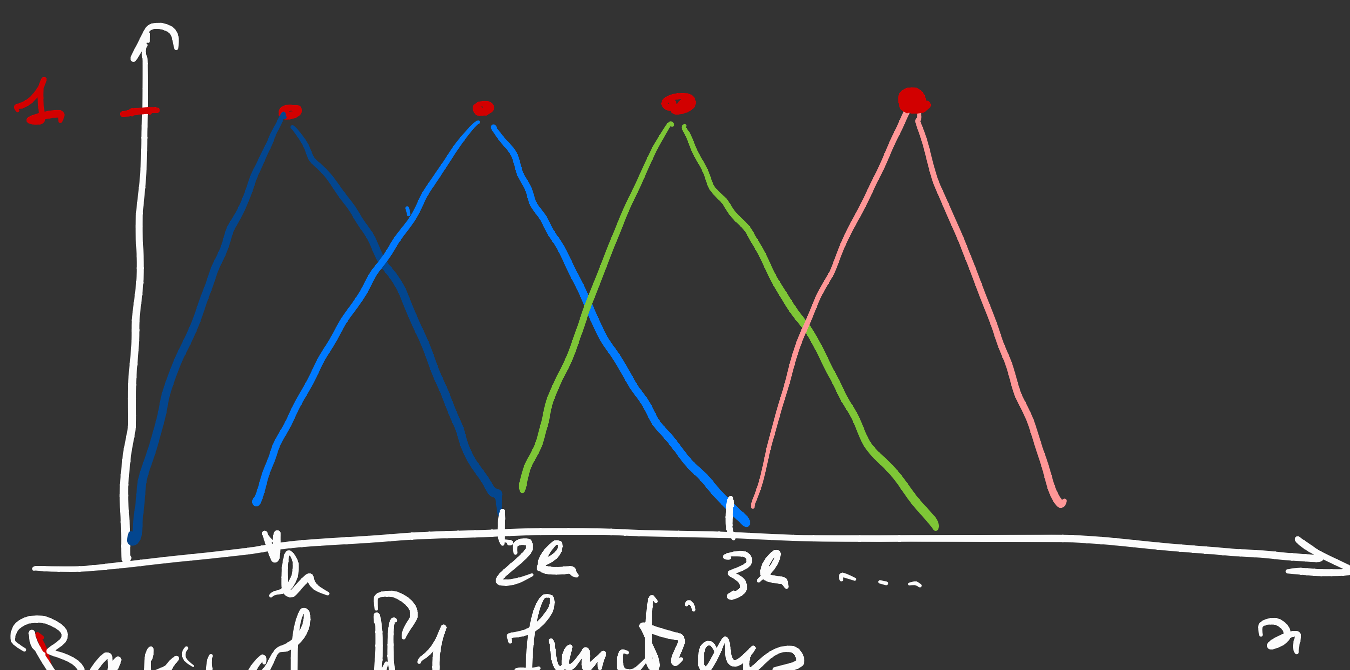
$$\geq \alpha \int |\nabla w_n|^2 dx$$

implies $\lambda_n > 0$.

Another way for constructing an approximation of functional spaces is motivated by numerical purpose.

We work with variable basis, parametrized by a discretization parameter $h > 0$

1D $\Omega = (0,1)$



Basis of $\mathbb{P}_1$ functions

$v_j(jh) = 1, \quad v_j((j \pm 1)h) = 0, \quad v_j(x) = 0$ if $x \notin [(j-1)h, (j+1)h]$

$v_j |_{[jh,(j+1)h] \cup [(j-1)h; jh]} \in \mathbb{P}_1$

$$V_h = \text{Span}\{v_1, \dots, v_{m-1}\}$$

$$m h = 1.$$

$$u \in H^1_0(0,1) \subset C^0([0,1])$$ by Sobolev's theorem in 1D

and we can write

$$u_m(x) = \sum_{j=1}^{m} u(jh)\, v_j(x)$$

$$u_m'(x) = \frac{1}{h}\Big(u(jh) - u((j-1)h)\Big)$$

$$= \frac{1}{h}\int_{(j-1)h}^{jh} u'(y)\, dy$$ $\forall ((j-1)h; jh)$

and we show that $u_m \xrightarrow[m\to\infty]{} u$ in $H^1_0(\Omega)$

This is the viewpoint of the FINITE ELEMENT METHOD