Application of variational and compactness techniques: homogeneization

Thierry Goudon

11 novembre 2020

1 Set up of the problem

The goal is to understand the behavior of the sequence $(u_n)_{n\in\mathbb{N}}$ of solutions of the model PDE

$$-\nabla\cdot(A_n\nabla u_n) = f \quad \text{in } \Omega, \qquad u_n\big|_{\partial\Omega} = 0. \tag{1}$$

Throughout this discussion we assume that

— Ω is a smooth bounded domain in $\mathbb{R}^N$,

— f is a fixed function in $L^2(\Omega)$,

— A_n is a matrix valued function defined on Ω such that, for some $\alpha, M > 0$ we have

$$|[A_n(x)]_{ij}| \leq M, \qquad A_n(x)\xi\cdot\xi \geq \alpha|\xi|^2,$$

for almost every $x \in \Omega$ and any $\xi \in \mathbb{R}^N$. (Note that the estimates are uniform with respect to n.)

We shall see that u_n, or at least a subsequence, converges to u weakly in H^1_0, *where u satisfies an equation of the same type*

$$-\nabla\cdot(A_{\text{eff}}\nabla u) = f \quad \text{in } \Omega, \qquad u_n\big|_{\partial\Omega} = 0. \tag{2}$$

That the limit u satisfied such a PDE is far from obvious. The “effective” diffusion matrix A_{eff} will be determined in a quite indirect way ; we shall see it satisfies L^∞ and coercivity bounds that can be expressed by means of α, M. As the analysis of the one dimensional case with periodic coefficients showed, in general A_{eff} *does not* coincide with the weak limit of A_n.

Based on the assumptions, we already know that, for any $f \in L^2$ and any $n \in \mathbb{N}$, the problem admits a unique solution $u_n \in H_0^1(\Omega)$. We remind the reader that $H_0^1(\Omega)$ is the adherence of $C_c^\infty(\Omega)$ for the norm

$$\|u\|_{H^1}^2 = \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2.$$

It can be considered (up to a suitable definition of the traces of functions in H^1) as the space of functions in H^1 which "vanish" on $\partial\Omega$. By Poincaré's lemma

$$u \longmapsto \|\nabla u\|_{L^2}$$

is a norm on H_0^1, equivalent to $\|u\|_{H^1}$. For further purposes it is convenient to introduce the space H^{-1} of continuous linear forms on H_0^1 endowed with the norm

$$\|\lambda\|_{H^{-1}} = \sup\left\{\frac{|\langle\lambda|u\rangle|}{\|u\|_{H^1}},\ u \in H_0^1,\ u \neq 0\right\}.$$

For instance, if $\phi \in L^2$, then $\nabla\phi$ defined by the relation

$$\langle\nabla\phi|u\rangle = -\int \phi\nabla u\,\mathrm{d}x$$

lies in $\in H^{-1}$. In particular equation (1) can be solved with $f \in H^{-1}$ as well, by means of the variational formulation. Despite the fact that H_0^1 is a Hilbert space, it is not identified with its dual space here ; in fact, we have the embedding

$$H_0^1 \subset L^2 \subset H^{-1} = \left(H_0^1\right)'$$

where the pivot space L^2 identifies to its dual.

As a matter of fact, the variational formulation

$$\int A_n \cdot \nabla u_n \cdot \nabla u_n\,\mathrm{d}x = \int f u_n\,\mathrm{d}x \tag{3}$$

already provides the following a priori estimates on the solutions

$$\|\nabla u_n\|_{L^2} \leq \frac{C_P}{\alpha}\|f\|_{L^2},$$

where C_P is the Poincaré constant that depends on the domain Ω. (More generally, we can replace $\|f\|_{L^2}$ by H^{-1}.) It already tells us that we can extract a subsequence such that $u_{n_k} \rightharpoonup u$ and $\nabla u_{n_k} \rightharpoonup \nabla u$ weakly in L^2. In view of the uniform bound of the coefficients of A_n we can suppose that they converge weakly-$\star$ in L^∞. This is not enough to identify the limit of the sequence $A_n\nabla u_n$. This is the issue we are going to discuss. The analysis relies on abstract arguments of functional analysis. Let us describe these arguments, and then, we shall come back to the homogenization problem.

2 Preliminaries from Functional Analysis

Let V be a separable Hilbert space, and denote by V' its topological dual (we do not identify V and V'), endowed with the norm

$$\|\lambda\|_{V'} = \sup\left\{\frac{|\langle\lambda, v\rangle_{V,V'}|}{\|v\|_V},\ v \in V \setminus \{0\}\right\}.$$

Note that V' is also a separable space (see [2], p. 48 and 78). Let us consider a sequence $\big(T_n\big)_{n\in\mathbb{N}}$ of bounded operators from V to V'. We suppose that the T_n's are both uniformly bounded and uniformly coercive, which means

$$\begin{aligned}&\text{there exist } \alpha, M > 0 \text{ such that for any } n \in \mathbb{N},\ u \in V,\\ &\langle T_n u, u\rangle_{V',V} \geq \alpha\|u\|_V^2, \qquad \|T_n u\|_{V'} \leq M\|u\|_V.\end{aligned} \tag{4}$$

Let us set

$$\begin{array}{rccc} a_n: & V \times V & \longrightarrow \mathbb{R} & \\ & (u, v) & \longmapsto & \langle T_n u, v\rangle_{V',V}\end{array}$$

which is a bilinear continuous and coercive mapping. Therefore, for any $f \in V'$, there exists a unique $u_n \in V$ such that

$$\langle T_n u_n, v\rangle_{V',V} = \langle f, v\rangle_{V',V} \tag{5}$$

holds for all $v \in V$. We note

$$u_n = S_n f.$$

In other words $S_n = T_n^{-1}$. Going back to (1), we have

— $V = H_0^1$, $V' = H^{-1}$,

— $T_n u = \nabla \cdot (A_n \nabla u)$,

— a_n is given by the LHS in (3).

As a warm up we rephrase several estimates in terms of properties of the operator S_n.

Lemma 2.1 *The sequence* $\big(S_n\big)_{n\in\mathbb{N}}$ *is a bounded sequence in* $\mathcal{L}(V', V)$ *; it satisfies*

$$\|S_n\|_{\mathcal{L}(V',V)} \leq 1/\alpha, \tag{6}$$

and

$$\langle f, S_n f\rangle_{V',V} \geq \frac{\alpha}{M^2}\ \|f\|_{V'}^2. \tag{7}$$

Proof. Let $f \in V'$ and set $u_n = S_n f$, $T_n u_n = f$. By using (5) with $v = u_n$ and the assumptions on T_n, we get

$$\alpha\|S_n f\|_V^2 = \alpha\|u_n\|_V^2 \leq \langle T_n u_n, u_n\rangle_{V',V} = \langle f, u_n\rangle_{V',V} \leq \|f\|_{V'}\|u_n\|_V = \|f\|_{V'}\|S_n f\|_V.$$

It proves (6). Next, we remark that

$$|\langle f, v\rangle_{V',V}| = |\langle T_n u_n, v\rangle_{V',V}| \leq M\|u_n\|_V\|v\|_V$$

holds for any $v \in V$. We deduce that

$$\|f\|_{V'} \leq M\|u_n\|_V = M\|S_n f\|_V.$$

Eventually, we get

$$\langle f, S_n f\rangle_{V',V} = \langle f, u_n\rangle_{V',V} = \langle T_n u_n, u_n\rangle_{V',V} \geq \alpha\|u_n\|_V^2 = \alpha\|S_n f\|_V^2 \geq \frac{\alpha}{M^2}\ \|f\|_{V'}^2.$$

■

The result we are interested in states as follows ; this is a compactness property for sequences of operators.

Theorem 2.2 *There exists a subsequence $\{n_k,\ k \in \mathbb{N}\}$ and an operator $T_\infty \in \mathcal{L}(V, V')$ such that for any $f \in V'$ the sequence $\big(u_{n_k}\big)_{k\in\mathbb{N}}$ of the elements of V defined by $T_{n_k} u_{n_k} = f$ converges weakly in V to some u verifying $T_\infty u = f$. Furthermore, we have*

$$\|T_\infty u\|_{V'} \leq \frac{M^2}{\alpha}, \qquad \langle T_\infty u, u\rangle_{V,V'} \geq \alpha\|u\|_V^2.$$

This result will be justified as a consequence of the following claim.

Lemma 2.3 *Let X be a separable Banach space, and Y a reflexive Banach space. Let $\big(S_n\big)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{L}(X, Y)$ such that $\|S_n\|_{\mathcal{L}(X,Y)} \leq C$. Then, there exists a subsequence $\{n_k,\ k \in \mathbb{N}\}$ and an operator $S_\infty \in \mathcal{L}(X, Y)$ such that for any $f \in X$, $S_{n_k} f$ converges to $S_\infty f$ weakly in Y. Furthermore, we have*

$$\|S_\infty\|_{\mathcal{L}(X,Y)} \leq \liminf_{k\to\infty} \|S_{n_k}\|_{\mathcal{L}(X,Y)}.$$

Remark 2.4 *Of course, the result does not hold in general for the whole sequence : the simplest counterexample is given by $X = Y = \mathbb{R}$ and S_n the multiplication by $(-1)^n$!*

Proof. For any $f \in X$, we have $\|S_n f\|_Y \leq C\|f\|_X$, so that $\big(S_n f\big)_{n\in\mathbb{N}}$ is bounded in Y. Since Y is a reflexive Banach space, we can extract a subsequence, depending on f, which converges weakly in Y. We shall construct a subsequence which works for any f by using a diagonal Cantor's argument. Indeed, since X is separable, we can consider a dense denombrable set

$$D = \{\varphi_k,\, k \in \mathbb{N}\}.$$

Then, we reproduce the scheme :

- for $k = 1$, we extract from $\mathbb{N}$ a subsequence $\{\sigma_1(n),\, n \in \mathbb{N}\}$ such that $\big(S_{\sigma_1(n)}\varphi_1\big)_{n\in\mathbb{N}}$ converges weakly in Y,
- for $k = 2$, we extract from $\{\sigma_1(n),\, n \in \mathbb{N}\}$ a subsequence $\{\sigma_1(\sigma_2(n)),\, n \in \mathbb{N}\}$ such that $\big(S_{\sigma_1(\sigma_2(n))}\varphi_2\big)_{n\in\mathbb{N}}$ converges weakly in Y,
- etc...
- for $k \in \mathbb{N}$, we extract from $\{\sigma_1 \circ ... \circ \sigma_{k-1}(n),\, n \in \mathbb{N}\}$ a subsequence $\{\sigma_1 \circ ... \circ \sigma_{k-1} \circ \sigma_k(n),\, n \in \mathbb{N}\}$ such that $\big(S_{\sigma_1\circ...\circ\sigma_k(n)}\varphi_k\big)_{n\in\mathbb{N}}$ converges weakly in Y,...

Then, consider the diagonal sequence $\{\sigma_1 \circ ... \circ \sigma_n(n),\, n \in \mathbb{N}\}$. For $k \leq n$ it is extracted from $\{\sigma_1 \circ ... \circ \sigma_k(n),\, n \in \mathbb{N}\}$ so that $\big(S_{\sigma_1\circ...\circ\sigma_n(n)}\varphi_k\big)_{n\in\mathbb{N}}$ converges weakly in Y. From now on, we simply denote by $\big(S_n\big)_{n\in\mathbb{N}}$ this subsequence and $S_n\varphi_k \rightharpoonup S_\infty\varphi_k$ holds as $n \to \infty$, for any $\varphi_k \in D$. We check that S_∞ is a linear application on D and by lower semi-continuity we have

$$\|S_\infty\varphi_k\|_Y \leq \liminf_{n\to\infty} \|S_n\varphi_k\|_Y \leq C\|\varphi_k\|_X.$$

Since D is dense in X, we can extend S_∞ to the whole space X : this defines $S_\infty \in \mathcal{L}(X, Y)$ (with operator norm $\leq C$).

Now, pick $f \in X$ and $\lambda \in Y'$. We have

$$\begin{aligned} |\langle \lambda, S_n f - S_\infty f\rangle_{Y',Y}| \;\leq\; & |\langle \lambda, S_n f - S_n\varphi\rangle_{Y',Y}| \\ & + |\langle \lambda, S_n\varphi - S_\infty\varphi\rangle_{Y',Y}| + |\langle \lambda, S_\infty\varphi - S_\infty f\rangle_{Y',Y}|. \end{aligned}$$

Let $\epsilon > 0$. Both the first and the third term in the right hand side can be dominated by

$$C\|\lambda\|_{Y'}\|f - \varphi\|_X.$$

Therefore since D is dense in X, we can choose $\varphi = \varphi(\epsilon) \in D$ such that these quantities are both $\leq \epsilon$. Then, using the convergence established for the elements of D, we can exhibit $N(\epsilon, \varphi(\epsilon), \lambda) = N_\epsilon \in \mathbb{N}$ such that for $n \geq N_\epsilon$

$$|\langle \lambda, S_n\varphi - S_\infty\varphi\rangle_{Y',Y}| \leq \epsilon.$$

It follows that

$$|\langle \lambda, S_n f - S_\infty f\rangle_{Y',Y}| \leq 3\epsilon$$

holds when $n \geq N_\epsilon$. ■

Proof of Theorem 2.2. We apply Lemma 2.3 with $X = V'$, $Y = V$. Hence, there exists a subsequence, still labelled by n, and $S_\infty \in \mathcal{L}(V', V)$ such that for any $f \in V'$, $S_n f \rightharpoonup S_\infty f$ weakly in V, with $\|S_\infty\|_{\mathcal{L}(V',V)} \leq 1/\alpha$. Besides, letting $n \to \infty$ in (7) yields

$$\frac{\alpha}{M^2} \|f\|^2_{V'} \leq \langle f, S_\infty f \rangle_{V',V} \leq \|f\|_{V'} \|S_\infty f\|_V.$$

It follows that

— S_∞ is injective
since $S_\infty f = 0$ forces $f = 0$,

— $\mathrm{Ran}(S_\infty)$ is closed,
since if $S_\infty f_n$ tends to some u in V, then $\big(f_n\big)_{n\in\mathbb{N}}$ is a Cauchy sequence and converges to some f ; by continuity it follows that $u = S_\infty f \in \mathrm{Ran}(S_\infty)$.

— S_∞ is surjective,
indeed, let $\lambda \in V'$ such that for any $f \in V$, we have $\langle \lambda, S_\infty f \rangle_{V',V} = 0$. Then, taking $f = \lambda$ leads to $\langle \lambda, S_\infty \lambda \rangle_{V',V} = 0 \geq (\alpha/M^2)\|\lambda\|^2_{V'}$, thus $\lambda = 0$. By the Hahn-Banach theorem (see [2] p. 7 or [10, Corollary 7.6]) this implies that $\overline{\mathrm{Ran}(S_\infty)} = V$.

We conclude that S_∞ is a bounded operator form V' to V which is bijective. The open mapping theorem (see [2] p. 18, 19 or [10, Theorem 5.43]) then tells us that the inverse $S_\infty^{-1} = T_\infty$ is a bounded operator from V to V'. The coercivity inequality $\|S_\infty f\|_V \geq (\alpha/M^2)\|f\|_{V'}$ for S_∞ (used with $u = S_\infty f$) can be recast as a continuity property for $T_\infty = S_\infty^{-1}$

$$(M^2/\alpha)\|u\|_V \geq \|T_\infty u\|_V.$$

For $f \in V'$, let us denote $u_n = S_n f \in V$ the solution of $T_n u_n = f$. It converges weakly to $u = S_\infty f \in V$, which satisfies $T_\infty u = f$. Besides, we have

$$\langle f, u_n \rangle_{V',V} = \langle T_n u_n, u_n \rangle_{V',V} \geq \alpha \|u_n\|^2_V.$$

Therefore, as $n \to \infty$ we obtain

$$\lim_{n\to\infty} \langle f, u_n \rangle_{V',V} = \langle f, u \rangle_{V',V} = \langle T_\infty u, u \rangle_{V',V} \geq \alpha \liminf_{n\to\infty} \|u_n\|^2_V \geq \alpha \|u\|^2_V.$$

It will be convenient to rephrase the statement as follows : any element of V can be reached as a weak limit of a sequence of solutions of problems $T_n u_n = f$, for an appropriate right hand side f.

Corollary 2.5 *There exists a subsequence $\{n_k,\ k \in \mathbb{N}\}$ and an operator $T_\infty \in (V, V')$ such that for any $u \in V$ the sequence $\big(u_{n_k}\big)_{k\in\mathbb{N}}$ of the elements of V defined by $T_{n_k} u_{n_k} = T_\infty u$ converges weakly in V to u.*

3 Homogeneization

We apply the previous statements to the sequences $T_n\cdot = \nabla\cdot(A_n\nabla\cdot)$ and $u_n = T_n^{-1}f = S_n f$. The sequence $(u_n)_{n\in\mathbb{N}}$ converges weakly in H_0^1 to $u = S_\infty f$. We are left with the question of identifying u. It is not even clear that it satisfies an equation of the same type as (1).

We shall use the div-curl lemma :

Lemma 3.1 (Div-Curl Lemma) *Let U_n and V_n be two functions defined on Ω with values in $\mathbb{R}^N$ and such that*

— $U_n \rightharpoonup U$, $V_n \rightharpoonup V$ *weakly in* L^2,

— $\mathrm{div}(U_n) = \nabla\cdot U_n = \sum_{j=1}^N \partial_{x_j} U_{n,j}$ *lies in a compact of* H^{-1},

— $\mathrm{curl}(V_n) = \nabla\times U_n$ *(the skew-symmetric matrix with components* $\partial_{x_j}V_{n,k} - \partial_{x_k}V_{n,j}$*) lies in a compact of* H^{-1}.

Then, we have $U_n\cdot V_n = \sum_{j=1}^N U_{n,j}V_{n,j} \rightharpoonup U\cdot V = \sum_{j=1}^N U_jV_j$ *in* $\mathscr{D}'(\Omega)$.

Indeed, we can suppose that $u_n \rightharpoonup u$ in H^1, which in particular means $\nabla u_n \rightharpoonup \nabla u$ and we already observe that $U_n = A_n\nabla U_n$ and $V_n = \nabla u_n$ satisfy

$$\nabla\cdot U_n = f,\ \nabla\times V_n = 0 \text{ both belong to a compact set of } H^{-1}.$$

We have $V_n \rightharpoonup \nabla u$ and we can suppose $U_n \rightharpoonup U$ in L^2 ; we wish to identify U.

All these observations equally applies to the *adoint equation* where A_n is replaced by $A_n^\intercal$. By using Corollary 2.5, for any $v\in H_0^1$, we can find a sequence $(v_n)_{n\in\mathbb{N}}$ such that

$$v_n\in H_0^1,\ v_n\rightharpoonup v \text{ weakly in } H_0^1,\ -\nabla\cdot(A_n^\intercal\nabla v_n) = g$$

where $g = \mathcal{T}_\infty u$, $\mathcal{T}_\infty = \mathcal{S}_\infty^{-1}\in\mathcal{L}(H_0^1, H^{-1})$ being the limit operator defined from $\mathcal{T}_n\cdot = \nabla\cdot(A_n^\intercal\nabla\cdot)$ by virtue of Lemma 2.3. We can also set $\tilde{U}_n = A_n^\intercal\nabla v_n$ and $\tilde{V}_n = \nabla v_n$ which satisfy the conditions of the div-curl lemma since, again, $\nabla\times\tilde{V}_n = 0$, $\nabla\cdot\tilde{U}_n = -g$ both belong to a compact set of H^{-1}. We have $\tilde{V}_n\rightharpoonup\nabla v$ and we can suppose $\tilde{U}_n\rightharpoonup\tilde{U}$ in L^2.

Let us pick $\varphi\in C_c^\infty(\Omega)$ and consider

$$I_n = \int A_n\nabla u_n\cdot\nabla v_n\varphi\,\mathrm{d}x = a_n(u_n, v_n).$$

It can be cast as

$$I_n = \int U_n\cdot\tilde{V}_n\varphi\,\mathrm{d}x = \int\nabla u_n\cdot A_n^\intercal\nabla v_n\varphi\,\mathrm{d}x = \int V_n\cdot\tilde{U}_n\varphi\,\mathrm{d}x.$$

Hence, this nonlinear quantity passes to the limit, owing to the div-curl lemma :

$$\lim_{n\to\infty} I_n = \int U \cdot \nabla v \varphi \, \mathrm{d}x = \int \nabla u \cdot \tilde{U} \varphi \, \mathrm{d}x.$$

It implies $U \cdot \nabla v = \tilde{U} \cdot \nabla u$.

In order to characterize U the idea would be to take $v \in \{v^1, ..., v^N\}$ such that $\partial_{x_i} v^j = \delta_{ij}$. Unfortunately this amounts to set $v^j(x) = x_j$... which is not an element of H_0^1. We should proceed indirectly to make this idea efficient. We consider a domain Ω' such that $\overline{\Omega} \subset \Omega'$ (strictly). We extend the matrix-valued functions A_n over Ω' by setting

$$B_n(x) = \begin{cases} A_n(x) & \text{if } x \in \Omega, \\ \alpha \mathbb{I} & \text{if } x \in \Omega' \setminus \Omega, \end{cases}$$

for some $\alpha > 0$. Then, we consider the extended operators $\mathcal{T}_n' \cdot = -\nabla \cdot (B_n \nabla \cdot)$ on Ω', endowed with homogeneous Dirichlet boundary conditions. With the same reasoning as above for any $v \in H_0^1(\Omega')$, we can find a sequence $(v_n)_{n\in\mathbb{N}}$ such that

$$v_n \in H_0^1(\Omega'),\ v_n \rightharpoonup v \text{ weakly in } H_0^1(\Omega'),\ -\nabla \cdot (B_n^\intercal \nabla v_n) = g$$

where $g = \mathcal{T}_\infty' u$, $\mathcal{T}_\infty' = (\mathcal{S}_\infty')^{-1} \in \mathcal{L}(H_0^1(\Omega'), H^{-1}(\Omega'))$. We make this construction with the functions v^j, $j \in \{1, ..., N\}$ such that

$$v^j \in H_0^1(\Omega'), \qquad v^j\big|_\Omega = x_j.$$

(For instance we can set $v^j(x) = x_j \chi(x)$ with $\chi \in C_c^\infty(\Omega')$ a cut off function such that $\chi(x) = 1$ for any $x \in \overline{\Omega}$.) Finally, we arrive at

$$U = A_{\text{eff}} \nabla u$$

where the effective matrix $x \mapsto A_{\text{eff}}(x)$ has its coefficients defined by

$$A_{\text{eff},jk} = \lim_{n\to\infty} \left[A_n^\intercal \nabla v_n^j\right]_k = \lim_{n\to\infty} \sum_{\ell=1}^N A_{n,\ell k} \partial_{x\ell} v_n^j$$

where the limit holds weakly in L^2.

The last question consists in finding bounds on the effective coefficients : we started with matrices in the set

$$\mathscr{M}_{\alpha,M} = \big\{A : x \in \Omega \mapsto \mathcal{M}_N,\ |A_{ij}(x)| \le M,\ A(x)\xi \cdot \xi \ge \alpha|\xi|^2\big\}.$$

We wonder whether the effective matrix satisfies such bounds. To this end, we use again the div-curl lemma to pass to the limit in the expression

$$\int A_n \nabla u_n \cdot \nabla u_n \varphi \, \mathrm{d}x = \int U_n \cdot V_n \varphi \, \mathrm{d}x \xrightarrow[n\to\infty]{} \int U \cdot V \varphi \, \mathrm{d}x = \int A_{\text{eff}} \nabla u \cdot \nabla u \varphi \, \mathrm{d}x.$$

Working with a non negative trial function $\varphi \geq 0$, we have

$$\int A_n \nabla u_n \cdot \nabla u_n \varphi \,\mathrm{d}x \geq \alpha \int |\nabla u_n|^2 \varphi \,\mathrm{d}x$$

which yields

$$\int A_{\text{eff}} \nabla u \cdot \nabla u \varphi \,\mathrm{d}x \geq \alpha \int |\nabla u|^2 \varphi \,\mathrm{d}x. \tag{8}$$

We already know (see Theorem 2.2 and its proof) that $S_\infty = T_\infty^{-1}$ is an isomorphism from H^{-1} to H_0^1. Therefore, any $u \in H_0^1$ can be reached as $u = S_\infty f$. Accordingly inequality holds for any $u \in H_0^1$. Given $\varphi \in C_c^\infty(\Omega)$, $\varphi \geq 0$, we construct $u \in H_0^1$ such that

$$u(x)\big|_{\text{supp}(\phi)} = \xi \cdot x,$$

for $\xi \in \mathbb{R}^N$. It follows that $A_{\text{eff}}\xi \cdot \xi \geq \alpha|\xi|^2$. The effective matrix A_{eff} thus satisfies the same coercivity estimate as the A_n's. For the bound on the coefficients we shall obtain a different estimate than the bound on the coefficients of the A_n's. We start with the obvious estimate $|A\xi| \leq M|\xi|$ which yields $|\zeta| \leq M|A^{-1}\zeta|$. Next, $A\xi \cdot \xi \geq \alpha|\xi|^2$ becomes $A^{-1}\zeta \cdot \zeta \geq \alpha|A^{-1}\zeta|^2 \geq \frac{\alpha}{M^2}|\zeta|^2$. Accordingly, for any non negative trial function $\varphi \in C_c^\infty$, we get

$$\int \phi A_n \nabla u_n \cdot \nabla u_n \,\mathrm{d}x \geq \frac{\alpha}{M^2} \int |A_n \nabla u_n|^2 \phi \,\mathrm{d}x.$$

Letting n run to ∞, we get

$$\int \phi A_{\text{eff}} \nabla u \cdot \nabla u \,\mathrm{d}x \geq \liminf_{n\to\infty} \frac{\alpha}{M^2} \int |A_n \nabla u_n|^2 \phi \,\mathrm{d}x \geq \frac{\alpha}{M^2} \int |A_{\text{eff}} \nabla u|^2 \phi \,\mathrm{d}x.$$

As above, we choose $u \in H_0^1$ such that $u(x)\big|_{\text{supp}(\phi)} = \xi \cdot x$. It yields $\frac{\alpha}{M^2}|A_{\text{eff}}\xi|^2 \leq |A_{\text{eff}}\xi|\ |\xi|$ and finally

$$|A_{\text{eff}}\xi| \leq \frac{M^2}{\alpha}|\xi|.$$

When the matrices A_n are symmetric, the effective matrix is symmetric too. In this specific case the coefficients of A_{eff} remain bounded by M.

4 Comments

Basis on functional analysis can be found in the textbooks [2, 6, 10, 12]. An introduction to the modeling viewpoint and numerical issues can be found in [11]. The compensated-compactness approach (div-curl lemma and various extensions), and their applications to the theory of homogeneization date back to F. Murat and L. Tartar [14], see also [13, 16]. The div-curl lemma has also been shown to be an efficient tool to handle conservation laws see [5, 4, 15]. Another applications is concerned with certain hydrodynamic limits from kinetic equations [3]. An overview of compactness issues for analysing non linear PDEs can be found in [7]. The case of *periodic* homogeneization is amenable to specific treatments and more explicit effective formulae can be derived [1]. Recent breakthrough are concerned with the derivation of sharp estimates on the effective coefficients in the case of stochastic homogeneization [8, 9].

Références

[1] G. Allaire. Homogenization and two scale convergence. *SIAM J. Math. Anal.*, 23 :1482–1518, 1992.

[2] H. Brézis. *Analyse fonctionnelle, théorie et applications.* Masson, 1993.

[3] P. Degond, T. Goudon, and F. Poupaud. Diffusion limit for nonhomogeneous and non-micro-reversible processes. *Indiana Univ. Math. J.*, 49(3) :1175–1198, 2000.

[4] R. J. DiPerna. Convergence of approximate solutions to conservation laws. *Arch. Rational Mech. Anal.*, 82(1) :27–70, 1983.

[5] R. J. DiPerna. Convergence of the viscosity method for isentropic gas dynamics. *Comm. Math. Phys.*, 91(1) :1–30, 1983.

[6] R. Edwards. *Functional analysis, theory and applications.* Dover, 1994.

[7] L. C. Evans. *Weak convergence methods for nonlinear partial differential equations*, volume 74 of *CBMS Regional Conf. Series in Math.* AMS, 1990.

[8] A. Gloria and F. Otto. An optimal variance estimate in stochastic homogenization of discrete elliptic equations. *Ann. Probab.*, 39(3) :779–856, 2011.

[9] A. Gloria and F. Otto. An optimal error estimate in stochastic homogenization of discrete elliptic equations. *Ann. Appl. Probab.*, 22(1) :1–28, 2012.

[10] T. Goudon. *Intégration. Intégrale de Lebesgue et introduction à l'analyse fonctionnelle.* Références Sciences. Ellipses, 2011.

[11] T. Goudon. *Mathématiques pour la modélisation et le calcul scientifique*. Coll. Math. & Stat. ISTE, 2017.

[12] P. Lax. *Functional analysis.* John Wiley & Sons, 2002.

[13] F. Murat and L. Tartar. *H*-convergence. In A. Cherkaev and R. V. Kohn, editors, *Topics in the mathematical modelling of composite materials*, volume 31 of *Progress in Nonlinear Differential Equations Appl.*, pages 21–43. Birkhäuser, 1997.

[14] L. Tartar. H-convergence et compacité par compensation, Cours Peccot, Collège de France, march, 1977. Partially written in : F. Murat, *H-convergence* Séminaire d'Analyse Fonctionelle et Numérique 1977-78, Université d'Alger, multicopied 34 p.

[15] L. Tartar. The compensated compactness method applied to systems of conservation laws. In *Systems of nonlinear partial differential equations (Oxford, 1982)*, volume 111 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 263–285. Reidel, Dordrecht, 1983.

[16] L. Tartar. Remarks on homogenization. In *Homogenization and effective moduli of material and media*, IMA Vol. in Math. and Appl., pages 228–246. Springer, 1986.