Fourier transform and PDE: Yukawa potentials, Sobolev spaces

Resolution of $\lambda u - \Delta u = f$

Let $\lambda > 0$. Apply the Fourier transform to

$$\lambda u - \Delta u = f$$

It leads to

$$\widehat{u}(\xi) = \frac{\widehat{f}(\xi)}{\lambda + \xi^2}.$$

Note: if $f \in \mathscr{S}(\mathbb{R}^N)$, $\xi \mapsto \frac{\widehat{f}(\xi)}{\lambda+\xi^2}$ lies in $\mathscr{S}(\mathbb{R}^N)$ too. Since $\xi \mapsto \frac{1}{\lambda+\xi^2}$ takes value in $[0, 1]$, the function $\xi \mapsto \frac{\widehat{f}(\xi)}{\lambda+\xi^2} \in L^2(\mathbb{R}^N)$ when $f \in L^2(\mathbb{R}^N)$.

Resolution of $\lambda u - \Delta u = f$, II

We can thus define

$$u(x) = \mathscr{F}^{-1}_{\xi \to x}\Big(\frac{\widehat{f}(\xi)}{\lambda + \xi^2}\Big)(x).$$

We get a representation formula of u by using the inverse transform. We are led to study

$$u(x) = \mathscr{F}^{-1}_{\xi \to x}\Big(\frac{\widehat{f}(\xi)}{\lambda + \xi^2}\Big)(x) = \int_{\mathbb{R}^N} K_\lambda(x-y) f(y)\,\mathrm{d}y,$$

where the kernel K_λ corresponds to inverse Fourier transform of $\xi \mapsto \frac{1}{\lambda+\xi^2}$.

Dimension 1

For $N = 1$, the function $\xi \mapsto \frac{1}{1+\xi^2}$ is integrable: we split

$$\frac{1}{1+\xi^2} = \frac{1}{2(1+i\xi)} + \frac{1}{2(1-i\xi)}.$$

The two terms are not integrable, but they are image of Fourier transform:

$$\frac{1}{1 \pm i\xi} = \int_0^{\infty} e^{-x(1 \pm i\xi)} \, \mathrm{d}x.$$

Therfore

$$\begin{aligned} \frac{1}{1+\xi^2} &= \frac{1}{2} \int_{-\infty}^{+\infty} e^{-ix\xi} (e^{-x} \mathbf{1}_{x \geq 0} + e^{x} \mathbf{1}_{x \leq 0}) \, \mathrm{d}x \\ &= \frac{1}{2} \int_{-\infty}^{+\infty} e^{-ix\xi} e^{-|x|} \, \mathrm{d}x. \end{aligned}$$

Dimension 1, ctn'd

We rescale to account for λ. We have

$$\frac{1}{\lambda+\xi^2} = \frac{1}{\lambda}\frac{1}{1+(\xi/\sqrt{\lambda})^2} = \frac{1}{\lambda}\mathscr{F}\left(\frac{e^{-|x|}}{2}\right)\left(\frac{\xi}{\sqrt{\lambda}}\right)$$

and

$$\frac{1}{\lambda}\widehat{f}\left(\frac{\xi}{\sqrt{\lambda}}\right) = \frac{1}{\sqrt{\lambda}}\int e^{-ix\xi/\sqrt{\lambda}}f(x)\frac{\mathrm{d}x}{\sqrt{\lambda}} = \frac{1}{\sqrt{\lambda}}\int e^{-iy\xi}f(y\sqrt{\lambda})\,\mathrm{d}y.$$

We conclude with

$$K_\lambda(x) = \frac{e^{-\sqrt{\lambda}|x|}}{2\sqrt{\lambda}}.$$

Note that K_λ and its Fourier transform are both integrable : K_λ belongs to the Wiener algebra.

Dimension 3: Yukawa's potential

We check that

$$K_\lambda(x) = \frac{e^{-\sqrt{\lambda}|x|}}{4\pi|x|}$$

is elementary solution of the PDE : $(\lambda - \Delta)K_\lambda = \delta_0$.

Note: K_λ is integrable over $\mathbb{R}^3$, smooth on $\{|x| \geq \epsilon\}$ for any $\epsilon > 0$, and

$$\begin{aligned}
\langle \Delta K_\lambda | \phi \rangle &= \int_{\mathbb{R}^3} K_\lambda(x)\Delta\phi(x)\,\mathrm{d}x = \lim_{\epsilon \to 0} \int_{|x| \geq \epsilon} K_\lambda(x)\Delta\phi(x)\,\mathrm{d}x \\
&= \lim_{\epsilon \to 0} \left\{ -\int_{|x| \geq \epsilon} \nabla K_\lambda(x) \cdot \nabla\phi(x)\,\mathrm{d}x + \int_{|x| = \epsilon} K_\lambda(x)\nabla\phi(x) \cdot \nu(x)\,\mathrm{d}\sigma(x) \right\} \\
&= \lim_{\epsilon \to 0} \left\{ \int_{|x| \geq \epsilon} \Delta K_\lambda(x)\phi(x)\,\mathrm{d}x - \int_{|x| = \epsilon} \nabla K_\lambda(x) \cdot \nu(x)\phi(x)\,\mathrm{d}\sigma(x) \right. \\
&\qquad \left. + \int_{|x| = \epsilon} K_\lambda(x)\nabla\phi(x) \cdot \nu(x)\,\mathrm{d}\sigma(x) \right\}.
\end{aligned}$$

Dimension 3, ctn'd

When $|x| \geq \epsilon > 0$, we can compute

$$\nabla K_\lambda(x) = -x\,\frac{e^{-\sqrt{\lambda}|x|}}{4\pi|x|^2}\Big(\sqrt{\lambda} + \frac{1}{|x|}\Big),$$

and

$$\begin{aligned}\Delta K_\lambda(x) &= -\frac{e^{-\sqrt{\lambda}|x|}}{4\pi|x|^2}\Big\{3\Big(\sqrt{\lambda}+\frac{1}{|x|}\Big) - x\Big(\sqrt{\lambda}+\frac{1}{|x|}\Big)\cdot\frac{x}{|x|}\,\frac{\sqrt{\lambda}|x|^2+2|x|}{|x|^2} - \frac{1}{|x|^2}\Big\}\\ &= -\frac{e^{-\sqrt{\lambda}|x|}}{4\pi|x|^2}\Big\{\Big(3-\sqrt{\lambda}|x|-2\Big)\Big(\sqrt{\lambda}+\frac{1}{|x|}\Big) - \frac{1}{|x|}\Big\} = \lambda K_\lambda.\end{aligned}$$

Since $\nu(x) = -\frac{x}{|x|} = -\omega \in \mathbb{S}^2$, we get

$$\int_{|x|=\epsilon} K_\lambda(x)\nabla\phi(x)\cdot\nu(x)\,\mathrm{d}\sigma(x) = -\frac{e^{-\sqrt{\lambda}\epsilon}}{4\pi\epsilon}\int_{\mathbb{S}^2}\nabla\phi(\epsilon\omega)\cdot\omega\epsilon^2\,\mathrm{d}\omega = \mathscr{O}(\epsilon)$$

and

$$\begin{aligned}\int_{|x|=\epsilon}\nabla K_\lambda(x)\cdot\nu(x)\phi(x)\,\mathrm{d}\sigma(x) &= \frac{e^{-\sqrt{\lambda}\epsilon}}{4\pi\epsilon^2}\int_{\mathbb{S}^2}\Big(\sqrt{\lambda}+\frac{1}{\epsilon}\Big)\epsilon\omega\cdot\omega\phi(\epsilon\omega)\epsilon^2\,\mathrm{d}\omega\\ &\xrightarrow[\epsilon\to0]{} \phi(0).\end{aligned}$$

General case

Note that

$$\int_{\mathbb{R}^N} \frac{\mathrm{d}\xi}{(\lambda+\xi^2)^p} = C_N \int_0^\infty \frac{r^{N-1}}{(\lambda+r^2)^p}\,\mathrm{d}r$$

is finite for $p > N/2$. In particular $\xi \mapsto \frac{1}{\lambda+\xi^2}$ lies in L^1 only for $N = 1$; it lies in L^2 in dimensions $1, 2, 3$. Nevertheless it still makes sense to write

$$K_\lambda(x) = \lim_{R\to\infty} \underbrace{\frac{1}{(2\pi)^N} \int_{|\xi|\le R} \frac{e^{ix\cdot\xi}}{\lambda+\xi^2}\,\mathrm{d}\xi}_{=K_{\lambda,R}(x)}.$$

We can write

$$\frac{1}{\lambda+\xi^2} = \int_0^\infty e^{-t(\lambda+\xi^2)}\,\mathrm{d}t$$

General case, ctn'd

It allows us to consider $R \to \infty$ in

$$\begin{aligned}\langle K_{\lambda,R}|\phi\rangle &= \frac{1}{(2\pi)^N}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\int_0^\infty \mathbf{1}_{|\xi|\le R}e^{ix\cdot\xi}e^{-t(\lambda+\xi^2)}\phi(x)\,\mathrm{d}t\,\mathrm{d}\xi\,\mathrm{d}x\\ &= \frac{1}{(2\pi)^N}\int_0^\infty\int_{\mathbb{R}^N} e^{-t\lambda}e^{-t\xi^2}\mathbf{1}_{|\xi|\le R}\underbrace{\left(\int_{\mathbb{R}^N} e^{ix\cdot\xi}\phi(x)\,\mathrm{d}x\right)}_{=\widehat{\phi}(-\xi)}\,\mathrm{d}\xi\,\mathrm{d}t.\end{aligned}$$

The integrand is dominated by $e^{-t\lambda}e^{-t\xi^2}|\widehat{\phi}(-\xi)| \in L^1([0,\infty[\times\mathbb{R}^N)$. Lebesgue's theorem yields

$$\begin{aligned}\lim_{R\to\infty}\langle K_{\lambda,R}|\phi\rangle &= \frac{1}{(2\pi)^N}\int_0^\infty\int_{\mathbb{R}^N} e^{-t\lambda}e^{-t\xi^2}\widehat{\phi}(-\xi)\,\mathrm{d}\xi\,\mathrm{d}t\\ &= \frac{1}{(2\pi)^N}\int_0^\infty\int_{\mathbb{R}^N} e^{-t\lambda}\left(\frac{\pi}{t}\right)^{N/2}e^{-x^2/(4t)}\phi(x)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

where we used the exchange formula and the Fourier transform of the Gaussian

$$\mathscr{F}(e^{-t\xi^2})(x) = \left(\frac{\pi}{t}\right)^{N/2}e^{-\xi^2/(4t)}.$$

We concude that

$$K_\lambda(x) = \frac{1}{(4\pi)^{N/2}}\int_0^\infty \frac{e^{-t\lambda}e^{-x^2/(4t)}}{t^{N/2}}\,\mathrm{d}t.$$

Going back to $N = 3$

We have

$$
\begin{aligned}
K_{\lambda,R}(x) &= \frac{1}{(2\pi)^3}\int_0^R\int_0^\pi\int_0^{2\pi}\frac{e^{ir|x|\cos(\theta)}}{\lambda+r^2}r^2\sin(\theta)\,\mathrm{d}\psi\,\mathrm{d}\theta\,\mathrm{d}r \\
&= \frac{1}{(2\pi)^2}\int_0^R\frac{r^2}{\lambda+r^2}\left(\int_0^\pi e^{ir|x|\cos(\theta)}\sin(\theta)\,\mathrm{d}\theta\right)\,\mathrm{d}r \\
&= \frac{1}{(2\pi)^2}\int_0^R\frac{r^2}{\lambda+r^2}\frac{e^{ir|x|}-e^{-ir|x|}}{ir|x|}\,\mathrm{d}r \\
&= -\frac{i}{(2\pi)^2|x|}\int_{-R}^R\frac{re^{ir|x|}}{\lambda+r^2}\,\mathrm{d}r
\end{aligned}
$$

We let $R \to \infty$ by using the residues formula.

Going back to $N = 3$, ctn'd

Indeed $F : z \in \mathbb{C} \mapsto \frac{re^{iz|x|}}{(z-i\sqrt{\lambda})(z+i\sqrt{\lambda})}$ has a single singular point $z = i\sqrt{\lambda}$, in the half-plane $\{\mathrm{Im}(z) \geq 0\}$. Integrate on $\Gamma_R = [-R, R] \cup C_R$, with C_R the half circle with radius R, large enough for the pole to be inside the curve. We get

$$\begin{aligned} \int_{\Gamma_R} F(z)\,\mathrm{d}z &= 2i\pi \mathrm{Res}(F)(i\sqrt{\lambda}) = 2i\pi \frac{i\sqrt{\lambda}e^{-\sqrt{\lambda}|x|}}{2i\sqrt{\lambda}} = i\pi\ e^{-\sqrt{\lambda}|x|} \\ &= \int_{-R}^{R} \frac{re^{ir|x|}}{\lambda + r^2}\,\mathrm{d}r + \int_0^{\pi} \frac{Re^{iRe^{i\theta}|x|}}{R^2e^{2i\theta} + \lambda} R\,\mathrm{d}\theta \end{aligned}$$

We have

$$\begin{aligned} \left| \int_0^{\pi} \frac{Re^{iRe^{i\theta}|x|}}{R^2e^{2i\theta} + \lambda} R\,\mathrm{d}\theta \right| &\leq \frac{R^2}{R^2 - \lambda} \int_0^{\pi} e^{-|x|R\sin(\theta)}\,\mathrm{d}\theta = \frac{2R^2}{R^2 - \lambda} \int_0^{\pi/2} e^{-|x|R\sin(\theta)}\,\mathrm{d}\theta \\ &\leq \frac{2R^2}{R^2 - \lambda} \int_0^{\pi/2} e^{-2|x|R\theta/\pi}\,\mathrm{d}\theta = \frac{\pi R}{|x|(R^2 - \lambda)} \xrightarrow[R\to\infty]{} 0 \end{aligned}$$

since $\theta \mapsto \frac{\sin(\theta)}{\theta}$ in non increasing over $[0, \pi/2]$ which implies $\sin(\theta) \geq \frac{2\theta}{\pi}$ pour $0 \leq \theta \leq \pi/2$. We deduce

$$K_\lambda(x) = \lim_{R\to\infty} K_{\lambda,R}(x) = \frac{e^{-\sqrt{\lambda}|x|}}{4\pi|x|}.$$

Sobolev spaces

We wish to construct a hierarchy of functional spaces measuring the regularity of solutions of PDEs. We already know the spaces C^k, but it is convenient to construct spaces based on the notion of weak derivatives instead.

Let

$$H^1(\mathbb{R}^N) = \{u : \mathbb{R}^N \to \mathbb{R} \text{ such that } u \in L^2(\mathbb{R}^N) \text{ and } \nabla u \in L^2(\mathbb{R}^N)\}.$$

That $u \in L^2$ lies in H^1 means that its weak derivative ∇u also lies in L^2, By virtue of Riesz theorem, it means that we can find $C > 0$ such that for any $\phi \in C_c^\infty(\mathbb{R}^N)$, we have

$$|\langle \nabla u | \phi \rangle| = \left| \int_{\mathbb{R}^N} u(x) \nabla \phi(x) \, \mathrm{d}x \right| \leq C \|\phi\|_{L^2}.$$

Sobolev spaces, ctn'd

Next, we can play with higher derivatives and define recursively the spaces $H^k(\mathbb{R}^N)$, $k \in \mathbb{N}$. We can work with L^p norms as well, defining the spaces $W^{k,p}$ by imposing that derivatives up to order k belong to L^p). The interest of the L^2 framework is to offer an hilbertian structure with the inner product

$$(u|v)_{H^k} = \sum_{|\alpha| \leq k} \int_{\mathbb{R}^N} \partial^\alpha u(x) \overline{\partial^\alpha v(x)} \, \mathrm{d}x.$$

Sobolev spaces, Fourier viewpoint

Since $\widehat{\nabla u}(\xi) = i\xi\widehat{u}(\xi)$, we can also define H^1 as

$$H^1(\mathbb{R}^N) = \left\{u : \mathbb{R}^N \to \mathbb{R} \text{ such that } \int_{\mathbb{R}^N} (1+\xi^2)|\widehat{u}(\xi)|^2 \,\mathrm{d}\xi < \infty\right\}.$$

More generally, given $s \geq 0$,

$$H^s(\mathbb{R}^N) = \left\{u : \mathbb{R}^N \to \mathbb{R} \text{ telle que } \int_{\mathbb{R}^N} (1+\xi^2)^s|\widehat{u}(\xi)|^2 \,\mathrm{d}\xi < \infty\right\},$$

an Hilbert space for the inner product

$$(u|v)_{H^s} = \int_{\mathbb{R}^N} (1+\xi^2)^s \widehat{u}(\xi)\overline{\widehat{v}(x)} \,\mathrm{d}\xi.$$

Finally, we can consider negative $s < 0$, dealing with tempered distributions.

Sobolev spaces, Fourier viewpoint, ctn'd

Observe that $(1-\Delta)^{-1}$, which associate to f the sol. u of $(1-\Delta)u = f$ is given by $\widehat{u}(\xi) = \frac{\widehat{f}(\xi)}{1+\xi^2}$. This defines an isomorphism from $H^{-1}(\mathbb{R}^N)$ to $H^1(\mathbb{R}^N)$.

We have, for $0 < s < t$,

$$C_c^\infty \subset \mathscr{S} \subset H^t \subset H^s \subset L^2 \subset H^{-s} \subset H^{-t} \subset \mathscr{S}' \subset \mathscr{D}'.$$

Exercices

Since $\widehat{\delta_0} = 1$ and $\frac{1}{(1+\xi^2)^s} \in L^1(\mathbb{R}^N)$ for $s > N/2$, we observe that $\delta_0 \in H^s(\mathbb{R}^N)$ pour tout $s < -N/2$.

The Heaviside function satisfies $\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{1}_{x\geq 0} = \delta_0$, which yields $i\xi\widehat{\mathbf{1}_{x\geq 0}}(\xi) = 1$. Since elements of $H^s(\mathbb{R})$ are locally square integrable, $\mathbf{1}_{x\geq 0} \notin H^s(\mathbb{R})$ for any $s \in \mathbb{R}$ because $1/|\xi| \notin L^1(B(0,r))$ for any $r > 0$.

Sobolev's embedding theorem

Theorem. Let $s > N/2$. Then the elements of $H^s(\mathbb{R}^N)$ are continuous and bounded functions.

We are going to show that $\widehat{u} \in L^1$ when $u \in H^s$ with $s > N/2$. Then the conclusion follows since we can apply the integral expression of the inverse Fourier transform. Cauchy-Schwarz inequality indeed yields

$$\int |\widehat{u}(\xi)| \, \mathrm{d}\xi = \int \frac{(1+\xi^2)^{s/2} |\widehat{u}(\xi)|}{(1+\xi^2)^{s/2}} \, \mathrm{d}\xi \leq \sqrt{\int \frac{\mathrm{d}\xi}{(1+\xi^2)^s}} \|u\|_{H^s},$$

where $\xi \mapsto \frac{1}{(1+\xi^2)^s} \in L^1(\mathbb{R}^N)$ when $s > N/2$.

Further properties of Sobolev spaces

Theorem. The following assertions hold

- For any $\phi \in \mathscr{S}(\mathbb{R}^N)$ and $s \in \mathbb{R}$, the application $T \mapsto \phi T$ is a linear and continuous from $H^s(\mathbb{R}^N)$ to $H^s(\mathbb{R}^N)$.
- If $s > N/2$, $H^s(\mathbb{R}^N)$ is an algebra.

The proof uses the result (Peetre's lemma)

$$\begin{aligned}&\text{for any } s \in \mathbb{R} \text{ there exists } C_s > 0 \text{ such that for any } \xi, \zeta \in \mathbb{R}^N\\ &(1+\xi^2)^s \le C_s (1+\zeta^2)^s (1+|\xi-\zeta|^2)^{|s|}\end{aligned} \tag{1}$$

Indeed, for $\phi, u \in \mathscr{S}$, we get

$$\int (1+\xi^2)^s |\widehat{\phi u}(\xi)|^2 \, \mathrm{d}\xi = \int (1+\xi^2)^s \left| \int \widehat{\phi}(\xi-\zeta)\widehat{u}(\zeta)\, \mathrm{d}\zeta \right|^2 \mathrm{d}\xi$$
$$\le C \int \left| \int (1+|\xi-\zeta|^2)^{|s|/2} |\widehat{\phi}(\xi-\zeta)| \; (1+\zeta^2)^{s/2} |\widehat{u}(\zeta)| \, \mathrm{d}\zeta \right|^2 \mathrm{d}\xi$$

the RHS can be written $\|F \star G\|_{L^2}^2$, with

$$F(\xi) = (1+\xi^2)^{|s|/2} \; |\widehat{\phi}(\xi)|, \qquad G(\xi) = (1+\xi^2)^{s/2} \; |\widehat{u}(\xi)|.$$

By definition of H^s, we have $G \in L^2$ with $\|G\|_{L^2} = \|u\|_{H^s}$, while $\phi \in \mathscr{S}$ implies $\widehat{\phi} \in \mathscr{S}$, and $F \in \mathscr{S} \subset L^1$. We conclude with a density argument from

$$\|\phi u\|_{H^s} \le C \|F\|_{L^1} \|G\|_{L^2} \le C(\phi) \|u\|_{H^s}. \tag{2}$$