Lax-Milgram Theorem

Let E be a Hilbert space. Let $a : E \times E \to \mathbb{C}$ be a sesquilinear form[1] such that

$$\begin{aligned} &\text{continuity}: \exists C > 0 \text{ s. t. } \forall x, y \in E \text{ one has } |a(x,y)| \leq C\|x\| \; \|y\|, \\ &\text{coercivity}: \exists \alpha > 0 \text{ s. t. } \forall x \in E \text{ one has } \mathrm{Re}\; a(x,x) \geq \alpha\|x\|^2. \end{aligned}$$

Let $\ell : E \to \mathbb{C}$ be an anti-linear form[2] continuous on E.[3] Then there exists a unique $x \in E$ such that for any $y \in E$, $a(x,y) = \ell(y)$. If a is **hermitian**, then x is characterized by

$$\frac{1}{2}a(x,x) - \ell(x) = \inf\left\{\frac{1}{2}a(z,z) - \mathrm{Re}\;\ell(z),\; z \in E\right\}.$$

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[1] For any $x, x' \in E$, $y, y' \in E$, $\lambda \in \mathbb{C}$, we have $a(x+x', y) = a(x,y) + a(x',y)$, $a(\lambda x, y) = \lambda a(x,y)$, $a(x, y+y') = a(x,y) + a(x,y')$, $a(x, \lambda y) = \bar{\lambda} a(x,y)$.

[2] For any $x, x' \in E$, $\lambda \in \mathbb{C}$, one has $\ell(x+x') = \ell(x) + \ell(x')$ et $\ell(\lambda x) = \bar{\lambda}\ell(x)$.

[3] There exists $C > 0$ such that for any $x \in E$, one has $|\ell(x)| \leq C\|x\|$.

Proof

Hermitian case. Apply Riesz' theorem with norm $N(x) = \sqrt{a(x,x)}$, noting that N and $\|\cdot\|$ are equivalent norms on E.

General case. $y \in E \longmapsto a(x,y) \in \mathbb{C}$ is a anti-linear form on E. Riesz' theorem allows us to define

$$\begin{array}{cccc} A: & E & \longrightarrow & E \\ & x & \longmapsto & Ax \end{array}$$

such that

$$a(x,y) = (Ax, y).$$

Check that $A \in \mathcal{L}(E)$ with $\|A\| \leq C$. Similarly Riesz' theorem identifies ℓ with a vector $f \in E$: for any $y \in E$

$$\ell(y) = (f, y), \qquad \|f\| = \|\ell\|.$$

The problem becomes

$$\text{To find } x \in E \text{ such that } y \in E, \ (Ax - f, y) = 0.$$

We search for $x \in E$ such that $Ax = f$.

Proof, ctn'd

Let

$$\begin{array}{cccc} T: & E & \longrightarrow & E \\ & x & \longmapsto & x - \rho(Ax - f) \end{array}$$

with $\rho > 0$ to be determined.

Idea: x, solution of $Ax = f$, is found as a fixed point fixe of T.

We have

$$\begin{aligned}
&\|T(x) - T(x')\|^2 \\
&= \|x - \rho(Ax - f) - x' + \rho(Ax' - f)\|^2 = \|x - x' - \rho A(x - x')\|^2 \\
&= \|x - x'\|^2 + \rho^2 \|A(x - x')\|^2 \\
&\qquad -\rho\big((x - x', A(x - x')) + (A(x - x'), x - x')\big) \\
&= \|x - x'\|^2 + \rho^2 \|A(x - x')\|^2 - 2\rho \,\mathrm{Re}\, (A(x - x'), x - x') \\
&= \|x - x'\|^2 + \rho^2 \|A(x - x')\|^2 - 2\rho \,\mathrm{Re}\, a(x - x', x - x') \\
&\leq \|x - x'\|^2 (1 + \rho^2 \|A\|^2 - 2\rho\alpha).
\end{aligned}$$

For $\rho(\rho\|A\|^2 - 2\alpha) < 0$, T is a contraction. Banach's theorem tells us that T has a unique fixed point.

Hermitian case

Let $\mathcal{J}(x) = \frac{1}{2}a(x,x) - \mathrm{Re}\,(f,x) \in \mathbb{R}$. Note that $A = A^\star$ since $a(x,y) = (Ax,y) = (x, A^\star y) = \overline{a(y,x)} = \overline{(Ay,x)} = (x, Ay)$.
Hence

$$\mathcal{J}(x+h) = \mathcal{J}(x) + \mathrm{Re}\,(Ax - f, h) + \frac{1}{2}a(h,h)$$

where $a(h,h) \geq \alpha \|h\|^2 \geq 0$. If x vérifie $Ax = f$, then $\mathcal{J}(x+h) \geq \mathcal{J}(x)$. Reciprocally, if x minimizes $\mathcal{J}$ then with $h = \pm\alpha k$ and $h = \pm i\alpha k$, $\alpha > 0$ going to 0, we obtain $Ax = f$.

A crucial remark

Let $x \in E$ solution of $a(x, y) = \ell(y)$ for any $y \in E$. Choosing $y = x$ we get

$$\alpha \|x\| \leq \|f\| = \|\!|\ell|\!\|.$$

We can define the inverse opreator $A^{-1} : E \to E$ par $A^{-1}f = x$. We have $A^{-1} \in \mathcal{L}(E)$ with

$$\|\!|A^{-1}|\!\| \leq 1/\alpha.$$

Estimates for elliptic equations

We go back to the model problem

$$\lambda u - \nabla \cdot (A\nabla u) = f$$

with Dirichlet b. c.
The matrix valued function A is supposed to have bounded coefficients such that $A(x)\xi \cdot \xi \geq \alpha|\xi|^2$ for some $\alpha > 0$.
Variational formulation: for any $v \in H_0^1$,

$$\lambda \int uv \, \mathrm{d}x + \int A\nabla u \cdot \nabla v \, \mathrm{d}x = \int fv \, \mathrm{d}x.$$

In particular, we can use this with $v = u$. It leads to

$$\lambda \|u\|_{L^2}^2 + \alpha \|\nabla u\|_{L^2}^2 \leq \|f\|_{L^2} \|u\|_{L^2}.$$

In any case (including $\lambda = 0$ with Poincaré's inequality), it yields

$$\|u\|_{H^1} \leq C\|f\|_{L^2}.$$

Consequence: the operator $f \mapsto u$ is compact on L^2.

Maximum principle: Stampacchia's method

We can find further useful estimates. Idea: work with $v = \Phi(u)$ with suitable Φ.

Lemma. *Let $\Phi \in C^1(\mathbb{R})$ with $\Phi(0) = 0$ and $|\Phi'(z)| \leq M$ for any $z \in \mathbb{R}$. For any $u \in H^1$, we have*

$$\Phi(u) \in H^1, \qquad \nabla\Phi(u) = \Phi'(u)\nabla u.$$

Since $|\Phi(u)| \leq M|u|$ and $|\Phi'(u)\nabla u| \leq |\nabla u|$, these two functions belong to L^2. Consider a sequence u_n of functions in C_c^∞ which converges to u in H^1 and a. e.. For any $\psi \in C_c^\infty$, we have

$$\int \Phi(u_n)\nabla\psi \, \mathrm{d}x = -\int \Phi'(u_n)\nabla u_n \psi \, \mathrm{d}x$$

Then, by dominated convergence $\Phi(u_n)$ and $\Phi'(u_n)\nabla u_n$ converge to $\Phi(u)$ and $\Phi(u)\nabla u$ in L^2, respectively, which allows us to identify $\nabla\Phi(u)$.

The statement still applies with Φ uniformly lipschitzian ($\Phi \in W^{1,\infty}$).

Maximum principle: Stampacchia's method

The idea is to work with $v = \Phi(u)$.
Claim: if $f \geq 0$ then $u \geq 0$.
Work with $\Phi(z) = 0$ when $z \geq 0$, $\Phi(z) < 0$ when $z < 0$, Φ non decreasing. (Ex.: typically $[z]_- = \min(z, 0)$).
We get

$$\lambda \int u\Phi(u)\,\mathrm{d}x + \int \Phi'(u) A\nabla u \cdot \nabla u\,\mathrm{d}x = \int f\Phi(u)\,\mathrm{d}x \leq 0.$$

In the LHS all terms are ≥ 0. If $\lambda > 0$, we can conclude directly: $u\Phi(u)$ vanishes a. e. which means $u \geq 0$ a. e. Otherwise, we should work with

$$\Psi(z) = \int_0^z \sqrt{\Phi'(s)}\,\mathrm{d}s$$

We have $\nabla\Psi(u) = 0$ and $\Psi(u) \in H_0^1$ so that $\Psi(u) = 0$ a. e., hence $u \geq 0$.

Application: homogeneization in 1D

$$-\frac{d}{dx}\Big(a(x)\frac{d}{dx}u(x)\Big) = f(x), \qquad x \in]0,1[\tag{1}$$

with Dirichlet b. c.

$$u(0) = 0 = u(1). \tag{2}$$

Suppose

$$a, f \in C^0([0,1]), \qquad 0 < a_\star \leq a(x) \leq a^\star < \infty.$$

We integrate by hand

$$u(x) = \int_0^x \frac{1}{a(y)} \left(C - \int_0^y f(z)\,\mathrm{d}z \right) \mathrm{d}y,$$

$$C = \int_0^1 \frac{1}{a(y)} \left(\int_0^y f(z)\,\mathrm{d}z \right) \mathrm{d}y \times \left(\int_0^1 \frac{1}{a(y)}\,\mathrm{d}y \right)^{-1}.$$

Application: homogeneization in 1D, ctn'd

Consider a **sequence** $0 < a_\star \leq a_n(x) \leq a^\star < \infty$.
The sequences $u_n(x)$ and $u_n'(x) = \frac{d}{dx}u_n(x)$ are uniformly bounded wrt $x \in [0,1]$ and $n \in \mathbb{N}$ by a constant depending only on $a_\star, a^\star$ and $\sup_{x\in[0,1]} |f(x)|$.
By Arzela-Ascoli's theorem we can extract a subsequence $\left(u_{n_k}\right)_{k\in\mathbb{N}}$ which converges uniformly to u on $[0,1]$.
What is the eq. satisfied by the limit $u(x)$?

The variational framework provides directly L^2 estimates (by virtue of Poincaré's lemma):

$$a_\star \|u_n'\|_{L^2} \leq C\|f\|_{L^2}$$

u_n is bounded in H_0^1, u_n' is bounded in L^2. We can assume, for a subsequence, that $u_n \to u$ strongly and $u_n' \rightharpoonup u'$ weakly in L^2.

Naive guess

u could satisfy the same equation with the constant coefficient $\int_0^1 \alpha(y)\,\mathrm{d}y$, an intuition based on

Lemma. Let $b_n(x) = \beta(nx)$ where β is bounded and 1-périodique. Let $\bar{\beta} = \int_0^1 \beta(y)\,\mathrm{d}y$. Then, for any $\varphi \in C_c^0(]0,1[)$ we have

$$\lim_{n\to\infty} \int_0^1 b_n(x)\varphi(x)\,\mathrm{d}x = \int_0^1 \bar{\beta}\varphi(x)\,\mathrm{d}x.$$

b_n converges weakly to $\bar{\beta}$

cf. Riemann-Lebesgue theorem: $e^{i2\pi nx}$ converges weakly to 0 in $L^2((0,1))$.

Proof

We have

$$\int_0^1 a_n(x)\varphi(x)\,dx - \overline{\alpha}\int_0^1 \varphi(x)\,dx = \sum_{k=0}^{n-1}\int_{k/n}^{(k+1)/n} (\alpha(nx)-\overline{\alpha})\varphi(x)\,dx$$

$$= \sum_{k=0}^{n-1}\int_0^1 (\alpha(y)-\overline{\alpha})\varphi((y+k)/n)\,dy/n$$

$$= \int_0^1 (\alpha(y)-\overline{\alpha})\frac{1}{n}\sum_{k=0}^{n-1}\varphi((y+k)/n)\,dy.$$

Observe that

$$\frac{1}{n}\sum_{k=0}^{n-1}\varphi((y+k)/n)-\int_0^1\varphi(z)\,\mathrm{d}z = \sum_{k=0}^{n-1}\int_{k/n}^{(k+1)/n}\big(\varphi((y+k)/n)-\varphi(z)\big)\,\mathrm{d}z$$

is dominated, for φ in C^1, by

$$\sup_{\xi\in(0,1)}|\varphi'(\xi)|\sum_{k=0}^{n-1}\int_{k/n}^{(k+1)/n}|y/n+k/n-z|\,\mathrm{d}z \leq 2\sup_{\xi\in(0,1)}|\varphi'(\xi)|/n.$$

This quantity thus tends to 0 as $n\to\infty$ and is dominated.

A product lemma

Let $(v_n)_{n\in\mathbb{N}}$ and $(w_n)_{n\in\mathbb{N}}$ be two sequences of continuous functions, uniformly bounded. Assume that v_n converges weakly to v and w_n converges uniformly to w. Then $v_n w_n$ converges weakly to vw.

Indeed, we have

$$\int_0^1 (v_n w_n - vw)\varphi(x)\,dx = \int_0^1 (v_n - v)w\varphi(x)\,dx + \int_0^1 v_n(w_n - w)\varphi(x)\,dx$$

The 1st integral tends to 0 as $n \to \infty$ since $v_n \rightharpoonup v$ the 2nd is dominated by $\sup_m |v_m|_{L^\infty} \|u_n - u\|_{L^\infty}\| \, \|\varphi\|_{L^1}$, which equally tends to 0.

Similar conclusion with strong L^2 convergence instead of uniform convergence.

Passage to the limit

This statement does not apply for our purposes: neither $\left(a_n\right)_{n\in\mathbb{N}}$ nor $\left(u_n'\right)_{n\in\mathbb{N}}$ converge uniformly.

Idea: consider $w_n(x) = a_n(x)u_n'(x)$
The equation tells us it satisfies Arzela-Ascoli's criterion. We can thus suppose (subsequence) that $\left(w_{n_k}\right)_{k\in\mathbb{N}}$ converges uniformly to some w on $[0,1]$. We get

$$u_{n_k}'(x) = \frac{1}{a_{n_k}(x)}\ w_{n_k}(x) \text{ converges weakly to } \int_0^1 \frac{dy}{\alpha(y)}\ w(x).$$

Effective coefficient

Let $\varphi \in C_c^\infty((0,1))$.

$$\int_0^1 u_n'\varphi(x)\,dx = -\int_0^1 u_n\varphi'(x)\,dx$$

tends to

$$\int_0^1 \overline{1/\alpha}\; w(x)\varphi(x)\,dx = -\int_0^1 u\varphi'(x)\,dx.$$

Set $U(x) = \int_0^x \overline{1/\alpha}\; w(y)\,dy \in C^1([0,1])$ so that

$$\int_0^1 \big(u(x) - U(x)\big)\varphi'(x)\,dx = 0.$$

It holds for any $\varphi \in C_c^\infty((0,1))$, and we deduce $u(x) = U(x)$ a. e. $u'(x) = \overline{1/\alpha}\; w(x)$.

Similarly, letting $k \to \infty$ in the eq. yields

$$-w'(x) = f(x) = -\frac{d}{dx}\Big(a_{\text{eff}}\frac{d}{dx}u(x)\Big), \qquad a_{\text{eff}} = \left(\int_0^1 \frac{dy}{\alpha(y)}\right)^{-1}$$

Continuous and L^2 framework

In dimension $N = 1$, we can also go back to the explicit formula

$$u_n(x) = \int_0^x \frac{1}{a_n(y)} \left(C_n - \int_0^y f(z)\,\mathrm{d}z \right) \mathrm{d}y,$$

$$C_n = \int_0^1 \frac{1}{a_n(y)} \left(\int_0^y f(z)\,\mathrm{d}z \right) \mathrm{d}y \times \left(\int_0^1 \frac{1}{a_n(y)} \mathrm{d}y \right)^{-1}.$$

We remark that C_n tends to

$$C = \int_0^1 \int_0^1 \frac{\mathrm{d}y}{a(y)} \left(\int_0^y f(z)\,\mathrm{d}z \right) \mathrm{d}y \times \left(\int_0^1 \frac{1}{a(y)} \mathrm{d}y \right)^{-1}.$$

and we can directly check that u_n converges to

$$u(x) = \int_0^x \int_0^1 \frac{\mathrm{d}y'}{a(y')} \left(C - \int_0^y f(z)\,\mathrm{d}z \right) \mathrm{d}y.$$

This being said, we verify that this u satisfies

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(a_{\text{eff}} \frac{\mathrm{d}}{\mathrm{d}x} u \right) = f$$

Continuous and L^2 framework

The proof presented here uses "elementary" functional arguments. The proof can be adapted to the L^2/H^1 framework, using weak and strong convergences in L^2 instead on weak and uniform convergence.

This cannot be avoided in higher dimensions.

Homogeneization in higher dimensions

$$-\nabla \cdot (A_n \nabla u_n) = f \text{ and Dirichlet b. c.}$$

where $\sup_{i,j,x} |[A_n]_{ij}(x)| \leq a^\star$ and $A(x)\xi \cdot \xi \geq a_\star |\xi|^2$,
$0 < a_\star < a^\star < \infty$.
We still have $\|\nabla u_n\|_{L^2} \leq C\|f\|_{L^2}$. We can thus assume that

$$u_n \to u, \qquad \nabla u_n \rightharpoonup \nabla u.$$

Compensated Compactness

Let $U_n = (U_n^1, \dots, U_n^N)$ and $V_n = (V_n^1, \dots, V_n^N)$ be sequences of $\mathbb{R}^N$ vector fields with components bounded in $L^2(\mathbb{R}^N)$. We assume that

$$U_n^i \rightharpoonup U^i, \qquad V_n^i \rightharpoonup V^i \qquad \text{weakly in } L^2(\mathbb{R}^N), \text{ for any } i \in \{1, \dots, N\}.$$

Then, we address the question of the behavior of the **inner product** $U_n \cdot V_n = \sum_{i=1}^N U_n^i V_n^i$.

Of course, by using the Rellich theorem, the limit is $U \cdot V$ when we have additional bounds on the whole derivative of one of the vector fields, say $\partial_j U_n^i$ bounded in $L^2(\mathbb{R}^N)$ for any $i, j \in \{1, \dots, N\}$. In such a very favorable case, we actually have convergence componentwise $U_n^i V_n^i \rightharpoonup U^i V^i$ in $\mathcal{D}'(\mathbb{R}^N)$.

The div-curl lemma needs information on certain first order derivatives of U_n combined with, in some sense complementary, (first order) derivatives of V_n, so that the inner product passes to the limit, even if convergences do not hold componentwise.

Compensated Compactness

Let us define the scalar quantity

$$\mathrm{div}(U_n) = \sum_{i=1}^{N} \partial_i U_n^i$$

and the matrix valued quantity

$$\big(\mathrm{curl}(V_n)\big)_{ij} = \partial_i V_n^j - \partial_j V_n^i.$$

Lemma [F. Murat and L. Tartar]. Suppose furthermore that

$$\mathrm{div}(U_n) \text{ and } \mathrm{curl}(V_n) \text{ are compact in } H^{-1}(\mathbb{R}^N).$$

Then, $U_n \cdot V_n \sum_{i=1}^N U_n^i V_n^i \rightharpoonup \sum_{i=1}^N U^i V^i = U \cdot V$ in $\mathcal{D}'(\mathbb{R}^N)$.

The proof is actually very simple by means of Fourier transform where we see how oscillations compensates. Without loss of generality, we can suppose that $U = 0 = V$.

Proof

Then, our task is to show that, given $\psi \in C_c^\infty(\mathbb{R}^N)$,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} U_n \cdot V_n \, \psi \, \mathrm{d}x = 0.$$

Let us denote I_n the integral under consideration. Pick $\varphi \in C_c^\infty(\mathbb{R}^N)$ such that $0 \le \varphi(x) \le 1$ on $\mathbb{R}^N$ with $\varphi(x) = 1$ on $\mathrm{supp}(\psi)$, so that

$$I_n = \int_{\mathbb{R}^N} U_n\varphi \cdot V_n\psi \, \mathrm{d}x.$$

Let us set $\widetilde{U}_n = U_n\varphi$ and $\widetilde{V}_n = V_n\psi$ that both tend weakly to 0 in $L^2(\mathbb{R}^N)$ and are compactly supported. We deduce that the Fourier transform

$$\widehat{\widetilde{U}_n}(\xi) = \int_{\mathbb{R}^N} e^{-ix\cdot\xi} U_n(x) \, \mathrm{d}x \xrightarrow[n\to\infty]{} 0 \qquad a.e$$

with the uniform estimate

$$|\widehat{\widetilde{U}_n}(\xi)| \le \|U_n\|_{L^2(\mathbb{R}^N)} \, \mathrm{meas}(\mathrm{supp}(\varphi)) \le C < \infty.$$

Similar conclusions hold for $\widehat{\widetilde{V}_n}$.

Proof

Then, by the Plancherel formula, I_n can be recast as

$$I_n = \int_{\mathbb{R}^N} \widehat{\widetilde{U}_n}(\xi) \cdot \overline{\widehat{\widetilde{V}_n}(\xi)} \, \mathrm{d}\xi.$$

Now, the additional assumption means for any $i, j \in \{1, \dots, N\}$,

$$\frac{\sum_{i=1}^{N} \xi_i \; \widehat{\widetilde{U}_n^i}}{\sqrt{1+|\xi|^2}} \qquad \text{and} \qquad \frac{\xi_i \; \widehat{\widetilde{V}_n^j} - \xi_j \; \widehat{\widetilde{V}_n^i}}{\sqrt{1+|\xi|^2}} \xrightarrow[n\to\infty]{} 0 \qquad \text{strongly in } L^2(\mathbb{R}^N).$$

We write

$$\begin{cases} I_n = J_n + K_n, \\ J_n = \displaystyle\int_{\mathbb{R}^N} \frac{1}{1+|\xi|^2} \; \widehat{\widetilde{U}_n}(\xi) \cdot \overline{\widehat{\widetilde{V}_n}(\xi)} \, \mathrm{d}\xi, \\ K_n = \displaystyle\int_{\mathbb{R}^N} \frac{|\xi|^2}{1+|\xi|^2} \; \widehat{\widetilde{U}_n}(\xi) \cdot \overline{\widehat{\widetilde{V}_n}(\xi)} \, \mathrm{d}\xi \end{cases}$$

Then, we split $J_n = \int_{|\xi|\le R} \dots \, \mathrm{d}\xi + \int_{|\xi|\ge R} \dots \, \mathrm{d}\xi.$

Proof

We readily show that J_n tends to 0 as $n \to \infty$, since on the one hand

$$\left| \int_{|\xi| \geq R} \dots \, \mathrm{d}\xi \right| \leq \|\psi\|_{L^\infty(\mathbb{R}^N)} \frac{\|U_n\|_{L^2(\mathbb{R}^N)} \, \|V_n\|_{L^2(\mathbb{R}^N)}}{1 + R^2} \leq \frac{C}{1 + R^2}$$

can be made arbitrarily small by choosing R large enough, while, for any fixed $0 < R < \infty$, the pointwise convergences and bounds discussed above yield

$$\lim_{n \to \infty} \int_{|\xi| \leq R} \dots \, \mathrm{d}\xi = 0,$$

by virtue of the Lebesgue theorem.

Proof

Next, we get

$$
\begin{aligned}
K_n &= \sum_{i,j=1}^{N} \int_{\mathbb{R}^N} \frac{\xi^i\ \xi^i\ \widehat{\widetilde{U}_n^j}\ \overline{\widehat{\widetilde{V}_n^j}}}{1+|\xi|^2} \\
&= \sum_{i,j=1}^{N} \int_{\mathbb{R}^N} \frac{\xi^i}{\sqrt{1+|\xi|^2}}\ \widehat{\widetilde{U}_n^j}\ \frac{\xi^i\,\overline{\widehat{\widetilde{V}_n^j}} - \xi^j\,\overline{\widehat{\widetilde{V}_n^i}}}{\sqrt{1+|\xi|^2}}\,\mathrm{d}\xi \\
&\quad + \sum_{i=1}^{N} \int_{\mathbb{R}^N} \frac{\xi^i}{\sqrt{1+|\xi|^2}}\ \frac{\displaystyle\sum_{j=1}^{N} \xi^j\,\widehat{\widetilde{U}_n^j}}{\sqrt{1+|\xi|^2}}\ \overline{\widehat{\widetilde{V}_n^i}}\,\mathrm{d}\xi
\end{aligned}
$$

Then, the integrand in both integral of the right hand side reads as the product of the bounded quantity $\xi^i/\sqrt{1+|\xi|^2} \in L^\infty(\mathbb{R}^N)$ times a sequence that is bounded in $L^2(\mathbb{R}^N)$ times a sequence that converges strongly to 0 in $L^2(\mathbb{R}^N)$. Accordingly, K_n goes to 0 as $n \to 0$, which ends the proof.