Uniqueness for the wave equation (1d)

Wave eq.

$$(\partial_{tt}^2 - c^2 \partial_{xx}^2) u = 0 \tag{1}$$

endowed with Cauchy data

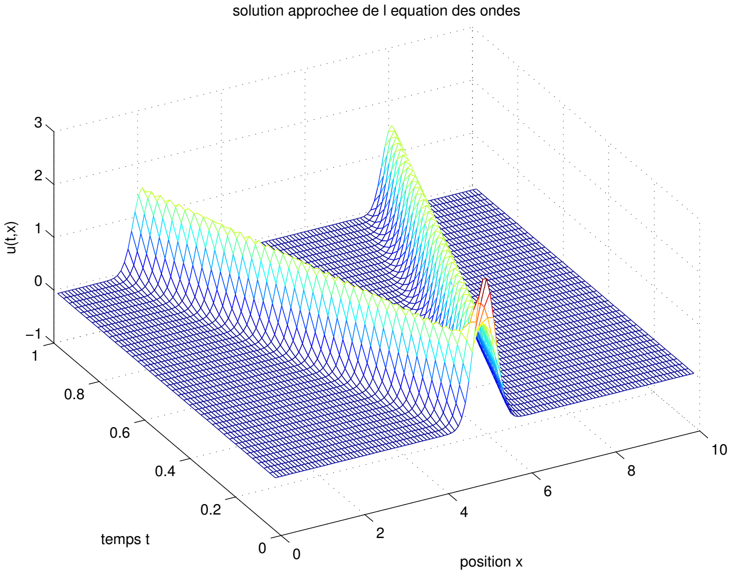
$$u(0,x) = g(x), \qquad \partial_t u(0,x) = h(x).$$

D'Alembert's formula

$$u(t,x) = \frac{1}{2}\Big(g(x-ct) + g(x+ct) + \frac{1}{c}\int_{x-ct}^{x+ct} h(y)\,\mathrm{d}y\Big). \tag{2}$$

With $h \in C^0$, $g \in C^1$ it provides a <u>weak</u> C^1 sol. of (1)
Crucial observation: **Finite speed of propagation** The sol. at time t and position x only depends on the data in the domain $[x - c|t|, x + c|t|]$. If the data is supported in $[-R, +R]$, the sol. is supported in $[R - c|t|, R + c|t|]$.

Finite speed of propagation



Uniqueness

We suppose v sol. with data $g = 0$, $h = 0$. We wish to show that

$$v = 0 \qquad a.e.$$

Reasoning by duality: we are going to show that

$$\int_{\mathbb{R}} v(T, y)\zeta(y)\,\mathrm{d}y = 0$$

for any time T and any trial function $\zeta \in C_c^\infty(\mathbb{R})$.
Let

$$\psi(t, x) = \frac{1}{2c}\int_{x-c(t-T)}^{x+c(t-T)} \zeta(y)\,\mathrm{d}y.$$

This is a solution of $(\partial_{tt}^2 - c^2\partial_{xx}^2)\psi = 0$, with

$$\psi(T, x) = 0, \qquad \partial_t\psi(T, x) = \zeta(x).$$

Uniqueness, II

Let

$$I(t) = \int_{\mathbb{R}} \partial_t v(t,x)\psi(t,x)\,\mathrm{d}x - \int_{\mathbb{R}} v(t,x)\partial_t\psi(t,x)\,\mathrm{d}x.$$

It is well defined since ζ is at least C^2 and assuming $\mathrm{supp}(\zeta) \subset [-R,+R]$, for any t fixed, $x \mapsto \psi(t,x)$ is supported in $[-R - c|t-T|, R + c|t-T|]$.
Therefore $I \in C^0(\mathbb{R})$ with

$$I(0) = 0, \qquad I(T) = -\int v(T,x)\zeta(x)\,\mathrm{d}x.$$

Goal: show that $I(t) = 0$ for any t.

Uniqueness, III

I is only continuous: we cannot take $\frac{\mathrm{d}}{\mathrm{d}t}$.
Let $\theta \in C_c^\infty(\mathbb{R})$. With integration by parts:

$$\int_{\mathbb{R}} I(t)\theta'(t)\,\mathrm{d}t = \iint_{\mathbb{R}\times\mathbb{R}} \partial_t v(t,x)\psi(t,x)\theta'(t)\,\mathrm{d}x\,\mathrm{d}t - \iint_{\mathbb{R}\times\mathbb{R}} v(t,x)\partial_t\psi(t,x)\theta'(t)\,\mathrm{d}x\,\mathrm{d}t$$

$$= \iint_{\mathbb{R}\times\mathbb{R}} \partial_t v(t,x)\partial_t\left(\psi(t,x)\theta(t)\right)\mathrm{d}x\,\mathrm{d}t - \iint_{\mathbb{R}\times\mathbb{R}} \partial_t v(t,x)\partial_t\psi(t,x)\theta(t)\,\mathrm{d}x\,\mathrm{d}t$$

$$- \iint_{\mathbb{R}\times\mathbb{R}} v(t,x)\partial_t\psi(t,x)\theta'(t)\,\mathrm{d}x\,\mathrm{d}t$$

$$= \iint_{\mathbb{R}\times\mathbb{R}} \partial_t v(t,x)\partial_t\left(\psi(t,x)\theta(t)\right)\mathrm{d}x\,\mathrm{d}t + \iint_{\mathbb{R}\times\mathbb{R}} v(t,x)\partial_t\left(\partial_t\psi(t,x)\theta(t)\right)\mathrm{d}x\,\mathrm{d}t$$

$$- \iint_{\mathbb{R}\times\mathbb{R}} v(t,x)\partial_t\psi(t,x)\theta'(t)\,\mathrm{d}x\,\mathrm{d}t$$

$$= \iint_{\mathbb{R}\times\mathbb{R}} \partial_t v(t,x)\partial_t\left(\psi(t,x)\theta(t)\right)\mathrm{d}x\,\mathrm{d}t + \iint_{\mathbb{R}\times\mathbb{R}} v(t,x)\partial_{tt}^2\psi(t,x)\theta(t)\,\mathrm{d}x\,\mathrm{d}t$$

$$= \iint_{\mathbb{R}\times\mathbb{R}} \partial_t v(t,x)\partial_t\left(\psi(t,x)\theta(t)\right)\mathrm{d}x\,\mathrm{d}t + \iint_{\mathbb{R}\times\mathbb{R}} v(t,x)c^2\partial_{xx}^2\psi(t,x)\theta(t)\,\mathrm{d}x\,\mathrm{d}t$$

$$= \iint_{\mathbb{R}\times\mathbb{R}} (\partial_t - c\partial_x)v(t,x)(\partial_t + c\partial_x)\left(\psi(t,x)\theta(t)\right)\mathrm{d}x\,\mathrm{d}t = 0.$$

Uniqueness, IV

Work with

$$\theta(t) = \int_0^t \left(h(\tau) - \kappa(\tau) \int_0^T h(s)\,\mathrm{d}s \right) \mathrm{d}\tau$$

with h and κ continuous, supported in $[0, T]$, with $\int_0^T \kappa(s)\,\mathrm{d}s = 1$.
Thus θ is C^1 on $[0, T]$ with $\theta(0) = 0$ and $\theta(T) = 0$.
Since $\theta'(t) = h(t) - \kappa(t) \int_0^T h(s)\,\mathrm{d}s$, we get

$$\int_{\mathbb{R}} I(t)\theta'(t)\,\mathrm{d}t = \int_0^T I(t)\theta'(t)\,\mathrm{d}t = 0 = \int_0^T \left(I(t) - \int_0^T I(s)\kappa(s)\,\mathrm{d}s \right) h(t)\,\mathrm{d}t.$$

It holds for arbitrary trial function h, thus $I(t)$ is constant on $[0, T]$. Since $I(0) = 0$, we have $I(t) = 0$ on $[0, T]$, and finally $v(t, x) = 0$ a.e.