Selected Topics in Information Theory

INFO5147 Lecture Notes
École Normale Supérieure de Lyon

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Foreword

These notes are written for the third edition of the course *INFO5147: Selected Topics in Information Theory* that I am teaching at École Normale Supérieure (ENS) de Lyon during the fall of 2020. The course is divided in two parts: theoretical foundations and applications. These lecture notes cover only the first part.

The objective of the first part is to level the ground to study information theory outside the classical framework of communications theory. The motivation for studying information theory outside its most prominent application domain is to widen and strengthen its connections with other disciplines and mathematical theories, in particular, real analysis, measure theory, probability theory, optimization, game theory, and statistics. In my opinion, this choice provides a more general look to the theory and might inspire new applications in different fields. Certainly, by adopting this choice, information theory can be truly appreciated and embraced as a developing mathematical theory whose impact on pure and applied sciences is yet to be discovered.

These lecture notes are certainly incomplete and do not pretend to be a monograph on information theory. In the current form, they are probably useful only for having a written support for my lectures. This said, the course covers a variety of elementary topics which turn out to be part of two essential building blocks of information theory. The first building block is *la théorie de la mesure* (measure theory), which developed around a problem formulated by Henri Léon Lebesgue during his studies on integration at ENS: *le problème de la mesure* (the problem of measure). To tackle the problem of measure and establish the foundations of measure theory, the first lecture is devoted to the *algebra of sets* and *integration* from the point of views of Darboux, Riemann and Lebesgue. The second lecture extends the notions of measure developed by Lebesgue beyond Euclidian spaces to measurable spaces in which a general integration theory is presented. The central object of study in this lecture is the Radon-Nikodym derivative, which is a corner stone in the definition of most information measures.

The second building block is probability theory and thus, the third lecture consists in an introduction to probability theory from the perspective of measure theory. That is, real-valued random variables are defined as measurable functions with respect to abstract measurable spaces and the Borel σ-field in $\mathbb{R}$. The focus is on fundamental notions of independence, expectation, conditional independence and their connection to Lebesgue's integral. Using this connection, the notions of *exponential families* and *exponential tilting* are reviewed. The fourth lecture concentrates on central limit theorems and saddle point approximations for calculating probability density functions (pdf) and cumulative distribution functions (cdf) independently on whether or not a closed-form exists for the pdf.

After the reviews on measure theory and probability theory, the course focuses on topics that were introduced by Claude E. Shannon in his seminal paper *a mathematical theory of communications*, published in 1948. The fifth lecture discusses the notion of information and wedge this notion to a positive real-valued function. Using this function, often referred to as the *information function*, this lecture presents a thorough exposition of information, relative information, information density, entropy and relative entropy. Mutual information is presented as a special case of relative entropy. The sixth lecture introduces the notion of concentration inequality and reviews classical and recent results in this topic. Using concentration inequalities, the concept of typicality and joint-typicality, which are due to Shannon, are presented in their most general forms. Both asymptotic and non-asymptotic typicality are reviewed.

Armed with the knowledge of information measures and knowledge about phenomena such as concentration of measure, the last two lectures of the first part are devoted to hypothesis testing and the analysis of the probability of error on hypothesis discrimination. The problem is studied considering both a finite and countable set of observations. In both cases, fundamental limits on the probability of error are presented.

Part of these lecture notes are inspired on scribed notes taken by some students during the lectures. Nonetheless, those scribed notes were only a starting point and have been entirely rewritten. I am particularly thankful to Quentin Deschamps, Julien Devevey, Nemo Fournier, Jean-Yves Franceschi, Charles Gassot, Rémy Grünblatt, Victor Mollimard, Jérémy Petithomme, Pegah Pournajafi, Denis Rochette, Xuan Thang, Herménégilde Valentin, and Lucas Venturi. I would also like to thank the PhD students at *l'École Doctorale de Lyon* and Postdocs at INRIA who have provided comments on these lecture notes. I am thankful to Dadja Anade, Selma Belhadj-Amor, Lélio Chetot, Nizar Khalfet, David Kibloff, and Victor Quintero.

During the reading of this notes, you will certainly bump into errors, typos and unclear statements that are certainly my fault. Please let me know about this. Thank you!

Enjoy the course, enjoy the reading, enjoy ENS.

Samir M. Perlaza
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Part I

Theoretical Foundations

1 Algebra of Sets

1.1 Notation

A *set* is a collection of objects referred to as *elements*. In the following, sets are denoted by caligraphic letters, e.g., $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$ and the elements of a given set are listed within braces "$\{\}$". When the number of elements in a set is finite, they can be listed explicitly, e.g., $\{0, 1\}$ is the set of binary digits. Note that the use of ellipses "$\ldots$" is rather common when the elements follow a particular pattern, e.g., $\{0, 1, 2, 3, \ldots, 9\}$ denotes the set of decimal digits, which contains ten elements. Some particular notations, different from calligraphic letters, are also used to denote some special sets. For instance,

- $\emptyset \triangleq \{\}$ is the empty set, a set without elements;
- $\mathbb{R}$ is the set of all real numbers;
- $\mathbb{N} \triangleq \{1, 2, \ldots, \}$ is the set of natural numbers; and
- $\mathbb{Z} \triangleq \{\ldots, -2, -1, 0, 1, 2, \ldots, \}$ is the set of integers.

Given a set, there exists a specific notation that allows specifying whether or not an element belongs to the set. This notation establishes a relation of membership between elements and sets.

> DEFINITION 1.1 (Membership). An element a that is in $\mathcal{A}$ is said to belong to $\mathcal{A}$, which is denoted by $a \in \mathcal{A}$. The opposite is denoted by $a \notin \mathcal{A}$.

From Definition 1.1, it follows that $0 \notin \mathbb{N}$; $1 \in \mathbb{N}$; and $\pi \notin \mathbb{N}$.

When the number of elements of a set is too big for explicitly listing all the elements, using ellipses is not necessarily a good choice. This is due to the fact that identifying the right pattern of the elements is left up to the reader, who might guess the right pattern or any other. Consider for instance the set $\{3, 5, 7, \ldots, \}$, which might be interpreted as the set of odd natural numbers bigger than two; or the set of prime numbers. This said, an alternative consists in using an explicit description of the elements such that any ambiguity is eliminated, e.g., $\{x :$ "description of x" $\}$. In this case, it is always recommended to explicit a set $\mathcal{O}$ containg all possible elements to which the "description" applies, e.g., $\{x \in \mathcal{O} :$ "description of x" $\}$. Using this notation, some other special sets, for which notations different from calligraphic letters are used, can be defined:

- $\mathbb{C} \triangleq \{a + \sqrt{-1}b : a \in \mathbb{R} \text{ and } b \in \mathbb{R}\}$ is the set of complex numbers; and
- $\mathbb{Q} \triangleq \left\{\frac{p}{q} \in \mathbb{R} : p \in \mathbb{Z}, q \in \mathbb{Z} \text{ and } q \neq 0\right\}$ is the set of rational numbers.

A special notation is used for some subsets in $\mathbb{R}$, namely, the intervals. That is, given two real numbers a and b such that $a < b$, an interval is a set of one of the following forms:

$$[a, b] \triangleq \{x \in \mathbb{R} : a \leqslant x \leqslant b\}, \tag{1.1}$$

$$]a, b] \triangleq \{x \in \mathbb{R} : a < x \leqslant b\}, \tag{1.2}$$

$$[a, b[\triangleq \{x \in \mathbb{R} : a \leqslant x < b\}, \text{ and} \tag{1.3}$$

$$]a, b[\triangleq \{x \in \mathbb{R} : a < x < b\}. \tag{1.4}$$

Intervals and more elaborated subsets of $\mathbb{R}$ are studied in Section 1.3.

The cardinality of a set, which is a measure on the number of elements in the set, can be finite or infinite.

> Definition 1.2 (Cardinality). The cardinality of a set $\mathcal{A}$ is a measure on the number of elements, denoted by $|\mathcal{A}|$, and satisfies either $|\mathcal{A}| \in \mathbb{N}$, $|\mathcal{A}| = 0$, or $|\mathcal{A}| = +\infty$.

Without any surprise, sets whose cardinality is finite or infinite are referred to as *finite sets* or *infinite sets*, respectively. The case of the empty set is an example of a finite set, i.e., $|\emptyset| = 0$. The set of natural numbers satisfies $|\mathbb{N}| = \infty$, whereas the set of binary digits $|\{0, 1\}| = 2$. The notion of cardinality implies that the elements of some sets can be counted. This holds clearly when the cardinality is finite, nonetheless, even when the cardinality is infinite in some cases the elements of a set can be counted. This observation leads to distinguishing between two types of sets: *countable* and *uncountable* sets.

> Definition 1.3 (Countable and uncountable sets). A set $\mathcal{A}$ is said to be countable if and only if there exists an injective function $f : \mathcal{A} \to \mathbb{N}$. When such a function f exists and it is also bijective, the set $\mathcal{A}$ is said to be countably infinite. Otherwise, the set $\mathcal{A}$ is said to be uncountable.

Note that the sets $\emptyset$, $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ are countable. More specifically, $\emptyset$ is finite, whereas $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ are countably infinite. Alternatively, the sets $\mathbb{R}$ and $\mathbb{C}$ are uncountable, and thus infinite. Every finite set is countable and thus, the designation “finite” is preferred instead of “countable” in this case. Nonetheless, the designation “countable” is often reserved to mean both finite and “countably infinite” sets. Every uncountable set is infinite and thus, the designation “uncountably infinite” is often avoided to make room for “uncountable”.

Two sets can be compared in a similar way as two real numbers are compared. The following definition introduces the notation for these comparisons.

Definition 1.4 (Comparison). Given two sets $\mathcal{A}$ and $\mathcal{B}$,

- the set $\mathcal{A}$ is said to be a **subset** of $\mathcal{B}$, denoted by $\mathcal{A} \subseteq \mathcal{B}$ or $\mathcal{B} \supseteq \mathcal{A}$, if and only if for all $a \in \mathcal{A}$, it holds that $a \in \mathcal{B}$;
- the set $\mathcal{A}$ is said to be a **proper subset** of $\mathcal{B}$, denoted by $\mathcal{A} \subset \mathcal{B}$ or $\mathcal{B} \supset \mathcal{A}$, if and only if $\mathcal{A} \subseteq \mathcal{B}$ and there exists at least one element $b \in \mathcal{B}$ such that $b \notin \mathcal{A}$;
- the set $\mathcal{A}$ is said to be **identical** to $\mathcal{B}$, denoted by $\mathcal{A} = \mathcal{B}$, if and only if $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{A} \supseteq \mathcal{B}$. The opposite is denoted by $\mathcal{A} \neq \mathcal{B}$.

From Definition 1.4, the following holds:

$$\emptyset \subset \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}. \tag{1.5}$$

1.2 Basic Operations

1.2.1 Unions and Intersections

Union and intersection are two operations performed between two sets. These operations are analogous to operations such as addition and subtraction in $\mathbb{R}$. These operations are defined hereunder.

Definition 1.5 (Unions and Intersections). Given two subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{O}$,

- the **union** of the sets $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{A} \cup \mathcal{B}$, is a set that contains all the elements of $\mathcal{A}$ and $\mathcal{B}$, i.e.,

$$\mathcal{A} \cup \mathcal{B} \triangleq \{a \in \mathcal{O} : a \in \mathcal{A} \vee a \in \mathcal{B}\}\text{; and} \tag{1.6}$$

- the **intersection** of the sets $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{A} \cap \mathcal{B}$, is a set that contains the common elements between $\mathcal{A}$ and $\mathcal{B}$, i.e.,

$$\mathcal{A} \cap \mathcal{B} \triangleq \{a \in \mathcal{O} : a \in \mathcal{A} \wedge a \in \mathcal{B}\}. \tag{1.7}$$

The union and the intersection of sets satisfy the following properties.

Theorem 1.6 (Properties). *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be some sets. Then, the following holds:*

- *Commutative Property*

$$\mathcal{A} \cup \mathcal{B} = \mathcal{B} \cup \mathcal{A} \text{ and} \tag{1.8}$$
$$\mathcal{A} \cap \mathcal{B} = \mathcal{B} \cap \mathcal{A}. \tag{1.9}$$

- *Associative Property*

$$\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} = (\mathcal{A} \cup \mathcal{B}) \cup \mathcal{C} = \mathcal{A} \cup (\mathcal{B} \cup \mathcal{C}) \quad \textit{and} \tag{1.10}$$

$$\mathcal{A} \cap \mathcal{B} \cap \mathcal{C} = (\mathcal{A} \cap \mathcal{B}) \cap \mathcal{C} = \mathcal{A} \cap (\mathcal{B} \cap \mathcal{C}) \,. \tag{1.11}$$

- *Distributive Property*

$$(\mathcal{A} \cup \mathcal{B}) \cap \mathcal{C} = (\mathcal{A} \cap \mathcal{C}) \cup (\mathcal{B} \cap \mathcal{C}) \quad \textit{and} \tag{1.12}$$

$$(\mathcal{A} \cap \mathcal{B}) \cup \mathcal{C} = (\mathcal{A} \cup \mathcal{C}) \cap (\mathcal{B} \cup \mathcal{C}) \,. \tag{1.13}$$

- *Idempotent Property*

$$\mathcal{A} \cap \mathcal{A} = \mathcal{A} \tag{1.14}$$

$$\mathcal{A} \cup \mathcal{A} = \mathcal{A} \tag{1.15}$$

Proof The proof of these statements follow immediately from the fact that conjunction and disjunction are logic operations that exhibit the commutative, associative, distributive and idempotent properties.

Proof of (1.8): Let x be an element of $\mathcal{A} \cup \mathcal{B}$. Then,

$$x \in \mathcal{A} \cup \mathcal{B} \Longleftrightarrow x \in \mathcal{A} \vee x \in \mathcal{B} \tag{1.16}$$

$$\Longleftrightarrow x \in \mathcal{B} \vee x \in \mathcal{A} \tag{1.17}$$

$$\Longleftrightarrow x \in \mathcal{B} \cup \mathcal{A}, \tag{1.18}$$

where the implication in (1.17) holds given the fact that disjunction is a commutative operation.

Proof of (1.9): Let x be an element of $\mathcal{A} \cap \mathcal{B}$. Then,

$$x \in \mathcal{A} \cap \mathcal{B} \Longleftrightarrow x \in \mathcal{A} \wedge x \in \mathcal{B} \tag{1.19}$$

$$\Longleftrightarrow x \in \mathcal{B} \wedge x \in \mathcal{A} \tag{1.20}$$

$$\Longleftrightarrow x \in \mathcal{B} \cap \mathcal{A}, \tag{1.21}$$

where the implication in (1.20) holds given the fact that conjunction is a commutative operation.

Proof of (1.10): Let x be an element of $(\mathcal{A} \cup \mathcal{B}) \cup \mathcal{C}$. Then,

$$x \in (\mathcal{A} \cup \mathcal{B}) \cup \mathcal{C} \Longleftrightarrow (x \in \mathcal{A} \cup \mathcal{B}) \vee x \in \mathcal{C} \tag{1.22}$$

$$\Longleftrightarrow (x \in \mathcal{B} \vee x \in \mathcal{A}) \vee x \in \mathcal{C} \tag{1.23}$$

$$\Longleftrightarrow x \in \mathcal{A} \vee x \in \mathcal{B} \vee x \in \mathcal{C} \tag{1.24}$$

$$\Longleftrightarrow x \in \mathcal{A} \vee (x \in \mathcal{B} \vee x \in \mathcal{C}) \tag{1.25}$$

$$\Longleftrightarrow x \in \mathcal{A} \vee x \in \mathcal{B} \cup \mathcal{C} \tag{1.26}$$

$$\Longleftrightarrow x \in \mathcal{A} \cup (\mathcal{B} \cup \mathcal{C}). \tag{1.27}$$

Note that (1.24) implies also that $x \in (\mathcal{A} \cup \mathcal{B}) \cup \mathcal{C} \Longleftrightarrow x \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$. Hence, $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} = (\mathcal{A} \cup \mathcal{B}) \cup \mathcal{C} = \mathcal{A} \cup (\mathcal{B} \cup \mathcal{C})$.

Proof of (1.11): Let x be an element of $(\mathcal{A}\cap\mathcal{B})\cap\mathcal{C}$. Then,

$$\begin{aligned} x\in(\mathcal{A}\cap\mathcal{B})\cap\mathcal{C} &\iff (x\in\mathcal{A}\cap\mathcal{B})\wedge x\in\mathcal{C} && (1.28)\\ &\iff (x\in\mathcal{B}\wedge x\in\mathcal{A})\wedge x\in\mathcal{C} && (1.29)\\ &\iff x\in\mathcal{B}\wedge x\in\mathcal{A}\wedge x\in\mathcal{C} && (1.30)\\ &\iff x\in\mathcal{A}\wedge(x\in\mathcal{B}\wedge x\in\mathcal{C}) && (1.31)\\ &\iff x\in\mathcal{A}\wedge x\in\mathcal{B}\cap\mathcal{C} && (1.32)\\ &\iff x\in\mathcal{A}\cap(\mathcal{B}\cap\mathcal{C}). && (1.33)\end{aligned}$$

Note that (1.30) implies also that $x\in(\mathcal{A}\cap\mathcal{B})\cap\mathcal{C}\iff x\in\mathcal{A}\cap\mathcal{B}\cap\mathcal{C}$. Hence, $\mathcal{A}\cap\mathcal{B}\cap\mathcal{C}=(\mathcal{A}\cap\mathcal{B})\cap\mathcal{C}=\mathcal{A}\cap(\mathcal{B}\cap\mathcal{C})$.

Proof of (1.12): Let x be an element of $(\mathcal{A}\cup\mathcal{B})\cap\mathcal{C}$. Then,

$$\begin{aligned} x\in(\mathcal{A}\cup\mathcal{B})\cap\mathcal{C} &\iff x\in(\mathcal{A}\cup\mathcal{B})\wedge x\in\mathcal{C} && (1.34)\\ &\iff (x\in\mathcal{A}\vee x\in\mathcal{B})\wedge x\in\mathcal{C} && (1.35)\\ &\iff (x\in\mathcal{A}\wedge x\in\mathcal{C})\vee(x\in\mathcal{B}\wedge x\in\mathcal{C}) && (1.36)\\ &\iff (x\in\mathcal{A}\cap\mathcal{C})\wedge(x\in\mathcal{B}\cap\mathcal{C}) && (1.37)\\ &\iff x\in(\mathcal{A}\cap\mathcal{C})\cup(\mathcal{B}\cap\mathcal{C}). && (1.38)\end{aligned}$$

Proof of (1.13): Let x be an element of $(\mathcal{A}\cap\mathcal{B})\cup\mathcal{C}$. Then,

$$\begin{aligned} x\in(\mathcal{A}\cap\mathcal{B})\cup\mathcal{C} &\iff x\in(\mathcal{A}\cap\mathcal{B})\vee x\in\mathcal{C} && (1.39)\\ &\iff (x\in\mathcal{A}\wedge x\in\mathcal{B})\vee x\in\mathcal{C} && (1.40)\\ &\iff (x\in\mathcal{A}\vee x\in\mathcal{C})\wedge(x\in\mathcal{B}\vee x\in\mathcal{C}) && (1.41)\\ &\iff (x\in\mathcal{A}\cup\mathcal{C})\wedge(x\in\mathcal{B}\cup\mathcal{C}) && (1.42)\\ &\iff x\in(\mathcal{A}\cup\mathcal{C})\cap(\mathcal{B}\cup\mathcal{C}). && (1.43)\end{aligned}$$

Proof of (1.14): Let x be an element of $\mathcal{A}\cap\mathcal{A}$. Then,

$$\begin{aligned} x\in\mathcal{A}\cap\mathcal{A} &\iff x\in\mathcal{A}\wedge x\in\mathcal{A} && (1.44)\\ &\iff x\in\mathcal{A}. && (1.45)\end{aligned}$$

Proof of (1.15): Let x be an element of $\mathcal{A}\cup\mathcal{A}$. Then,

$$\begin{aligned} x\in\mathcal{A}\cup\mathcal{A} &\iff x\in\mathcal{A}\vee x\in\mathcal{A} && (1.46)\\ &\iff x\in\mathcal{A}, && (1.47)\end{aligned}$$

which completes the proof. □

1.2.2 Complements and Differences

Often, operations among sets are performed with respect to a set that contains all the elements involved in the operation. Such a "reference" set is known as the universal set and it is often denoted by $\mathcal{O}$. Taking this into account, operations such as the complement of a set can be defined.

Definition 1.7 (Complements). Given two subsets $\mathcal{A}$ and $\mathcal{B}$ of the set $\mathcal{O}$,

- the **complement** of the set $\mathcal{A}$ with respect to $\mathcal{O}$, denoted by $\mathcal{A}^{\mathsf{c}}$, is a set that contains all the elements in $\mathcal{O}$ except those in $\mathcal{A}$, i.e.,

$$\mathcal{A}^{\mathsf{c}} \triangleq \{a \in \mathcal{O} : a \notin \mathcal{A}\}\,; \tag{1.48}$$

- the **difference** of the sets $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{A} \setminus \mathcal{B}$, is a set that contains all the elements of $\mathcal{A}$ except those in $\mathcal{B}$, i.e.,

$$\mathcal{A} \setminus \mathcal{B} \triangleq \{a \in \mathcal{A} : a \notin \mathcal{B}\}\,; \tag{1.49}$$

Definition 1.7 highlights the relevance of determining a universal set for calculating the complement of a set. In the following, unless it is clear from the context, a universal set is always specified. Note for instance that the complement of $\mathcal{A}$ with respect to the universal set is $\mathcal{A}^{\mathsf{c}} = \mathcal{O} \setminus \mathcal{A}$, whereas with respect to $\mathcal{B}$, it is $\mathcal{B} \setminus \mathcal{A}$.

The simple operations of unions, intersections and complements establish the foundations of the algebra of sets. The following results are easily obtained from Definition 1.4, Definition 1.5 and Definition 1.7. Nonetheless, for the sake of completeness, a proof is provided.

Theorem 1.8. *Given a subset $\mathcal{A}$ of a set $\mathcal{O}$, it holds that $a \in \mathcal{A}$ if and only if $a \notin \mathcal{A}^{\mathsf{c}}$, where the complement is with respect to $\mathcal{O}$.*

Proof The proof is an *argumentum ad absurdum*. That is, let a be an element of $\mathcal{A}$. Then, if $a \in \mathcal{A}^{\mathsf{c}}$, then, $a \in \mathcal{A} \cap \mathcal{A}^{\mathsf{c}} = \emptyset$, which is an absurdity. On the other hand, let a be an element of $\mathcal{A}^{\mathsf{c}}$. Then, if $a \in \mathcal{A}$, then, $a \in \mathcal{A} \cap \mathcal{A}^{\mathsf{c}} = \emptyset$, which is also an absurdity as in the previous case. Hence, $a \in \mathcal{A}$ if and only if $a \notin \mathcal{A}^{\mathsf{c}}$. This completes the proof. □

Theorem 1.9. *Given two subsets $\mathcal{A}$ and $\mathcal{B}$ of a set $\mathcal{O}$, such that $\mathcal{A} \subseteq \mathcal{B}$, it follows that $\mathcal{A}^{\mathsf{c}} \supseteq \mathcal{B}^{\mathsf{c}}$, where the complement is with respect to $\mathcal{O}$.*

Proof Let x be an element of $\mathcal{A}$. Hence, from the assumption that $\mathcal{A} \subseteq \mathcal{B}$, the following implications hold:

$$x \in \mathcal{A} \Rightarrow x \in \mathcal{B} \text{ and} \tag{1.50}$$

$$x \notin \mathcal{B} \Rightarrow x \notin \mathcal{A}. \tag{1.51}$$

Now, assume that $x \in \mathcal{B}^{\mathsf{c}}$. Hence, from Theorem 1.8, it holds that $x \notin \mathcal{B}$. Hence, from (1.51), it follows that $x \notin \mathcal{A}$, which implies $x \in \mathcal{A}^{\mathsf{c}}$ (Theorem 1.8). Therefore $\mathcal{B}^{\mathsf{c}} \subseteq \mathcal{A}^{\mathsf{c}}$, which completes the proof. □

THEOREM 1.10. *Given two subsets $\mathcal{A}$ and $\mathcal{B}$ of a set $\mathcal{O}$, it holds that*

$$\mathcal{A} \setminus \mathcal{B} = \mathcal{A} \cap \mathcal{B}^c, \tag{1.52}$$

where the complement is with respect to $\mathcal{O}$.

Proof From Definition 1.7, the following holds:

$$\begin{aligned} \mathcal{A} \setminus \mathcal{B} &= \{a \in \mathcal{A} : a \notin \mathcal{B}\} && (1.53)\\ &= \{x \in \mathcal{O} : x \in \mathcal{A} \wedge x \notin \mathcal{B}\} \text{ follows from } \mathcal{A} \subseteq \mathcal{O} && (1.54)\\ &= \{x \in \mathcal{O} : x \in \mathcal{A} \wedge x \in \mathcal{B}^c\} \text{ follows from Theorem 1.8} && (1.55)\\ &= \mathcal{A} \cap \mathcal{B}^c, && (1.56) \end{aligned}$$

which completes the proof. □

The difference of sets is not commutative, nonetheless, there exists an interesting connection between difference of sets and complements, as shown by the following theorem.

THEOREM 1.11. *Given the subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{O}$, it holds that*

$$\mathcal{A} \setminus \mathcal{B} \triangleq \mathcal{B}^c \setminus \mathcal{A}^c, \tag{1.57}$$

where the complement is with respect to $\mathcal{O}$.

Proof Using Theorem 1.10, the following holds:

$$\begin{aligned} \mathcal{A} \setminus \mathcal{B} &= \mathcal{A} \cap \mathcal{B}^c && (1.58)\\ &= \mathcal{B}^c \cap (\mathcal{A}^c)^c && (1.59)\\ &= \mathcal{B}^c \setminus \mathcal{A}^c, && (1.60) \end{aligned}$$

which completes the proof. □

1.2.3 De Morgan's Laws

The following identities were introduced by Augustus de Morgan and play a key role in the algebra of sets.

THEOREM 1.12 (de Morgan Laws). *Let $\mathcal{A}$ and $\mathcal{B}$ be two subsets of $\mathcal{O}$. Then,*

$$\mathcal{A} \cup \mathcal{B} = (\mathcal{A}^c \cap \mathcal{B}^c)^c \text{ and} \tag{1.61}$$

$$\mathcal{A} \cap \mathcal{B} = (\mathcal{A}^c \cup \mathcal{B}^c)^c, \tag{1.62}$$

where the complement is with respect to $\mathcal{O}$.

Proof Let x be an element of $(\mathcal{A}^\mathsf{c} \cap \mathcal{B}^\mathsf{c})^\mathsf{c}$. Then,

$$\begin{aligned} x \in (\mathcal{A}^\mathsf{c} \cap \mathcal{B}^\mathsf{c})^\mathsf{c} &\iff x \notin \mathcal{A}^\mathsf{c} \cap \mathcal{B}^\mathsf{c} && (1.63)\\ &\iff \neg(x \in \mathcal{A}^\mathsf{c} \cap \mathcal{B}^\mathsf{c}) && (1.64)\\ &\iff \neg(x \in \mathcal{A}^\mathsf{c} \wedge x \in \mathcal{B}^\mathsf{c}) && (1.65)\\ &\iff \neg(x \in \mathcal{A}^\mathsf{c}) \vee \neg(x \in \mathcal{B}^\mathsf{c}) && (1.66)\\ &\iff (x \notin \mathcal{A}^\mathsf{c}) \vee (x \notin \mathcal{B}^\mathsf{c}) && (1.67)\\ &\iff (x \in \mathcal{A}) \vee (x \in \mathcal{B}) && (1.68)\\ &\iff x \in \mathcal{A} \cup \mathcal{B}. && (1.69) \end{aligned}$$

Hence, $x \in (\mathcal{A}^\mathsf{c} \cap \mathcal{B}^\mathsf{c})^\mathsf{c}$ if and only if $x \in \mathcal{A} \cup \mathcal{B}$, which proves the equality in (1.61). The proof of the second equality uses similar arguments. Let x be an element of $(\mathcal{A}^\mathsf{c} \cup \mathcal{B}^\mathsf{c})^\mathsf{c}$. Then,

$$\begin{aligned} x \in (\mathcal{A}^\mathsf{c} \cup \mathcal{B}^\mathsf{c})^\mathsf{c} &\iff x \notin \mathcal{A}^\mathsf{c} \cup \mathcal{B}^\mathsf{c} && (1.70)\\ &\iff \neg(x \in \mathcal{A}^\mathsf{c} \cup \mathcal{B}^\mathsf{c}) && (1.71)\\ &\iff \neg(x \in \mathcal{A}^\mathsf{c} \vee x \in \mathcal{B}^\mathsf{c}) && (1.72)\\ &\iff \neg(x \in \mathcal{A}^\mathsf{c}) \wedge \neg(x \in \mathcal{B}^\mathsf{c}) && (1.73)\\ &\iff (x \notin \mathcal{A}^\mathsf{c}) \wedge (x \notin \mathcal{B}^\mathsf{c}) && (1.74)\\ &\iff (x \in \mathcal{A}) \wedge (x \in \mathcal{B}) && (1.75)\\ &\iff x \in \mathcal{A} \cap \mathcal{B}, && (1.76) \end{aligned}$$

Hence, $x \in (\mathcal{A}^\mathsf{c} \cup \mathcal{B}^\mathsf{c})^\mathsf{c}$ if and only if $x \in \mathcal{A} \cap \mathcal{B}$, which completes the proof. □

The relevance of Theorem 1.12 is that it allows expressing the union of sets in terms of complements and intersections; and the intersection of sets in terms of complements and unions. This might appear trivial but it actually plays a central role in many of the proofs presented in this work.

1.2.4 Symmetric Difference

The symmetric difference is defined as follows.

Definition 1.13 (Symmetric Difference). Given two subsets $\mathcal{A}$ and $\mathcal{B}$ of the set $\mathcal{O}$, the symmetric difference between $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{A} \triangle \mathcal{B}$ contains the elements that belong only to either $\mathcal{A}$ or $\mathcal{B}$. That is,

$$\mathcal{A} \triangle \mathcal{B} \triangleq \{a \in \mathcal{O} : a \in \mathcal{A} \setminus \mathcal{B} \vee a \in \mathcal{B} \setminus \mathcal{A}\}. \quad (1.77)$$

The following theorem provides an enlightening interpretation of the symmetric property.

Theorem 1.14. *Given the subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{O}$, it holds that*

$$\mathcal{A} \triangle \mathcal{B} = \mathcal{B} \triangle \mathcal{A} = (\mathcal{A} \cup \mathcal{B}) \cap (\mathcal{A} \cap \mathcal{B})^{\mathsf{c}}, \tag{1.78}$$

where the complement is with respect to $\mathcal{O}$.

Proof From Definition 1.13, it follows that

$$\begin{aligned} \mathcal{A} \triangle \mathcal{B} &= (\mathcal{A} \setminus \mathcal{B}) \cup (\mathcal{B} \setminus \mathcal{A}) & (1.79) \\ &= \mathcal{B} \triangle \mathcal{A}. & (1.80) \end{aligned}$$

Hence, from Theorem 1.10 and Theorem 1.12, the following holds:

$$\begin{aligned} (\mathcal{A} \setminus \mathcal{B}) \cup (\mathcal{B} \setminus \mathcal{A}) &= (\mathcal{A} \cap \mathcal{B}^{\mathsf{c}}) \cup (\mathcal{B} \cap \mathcal{A}^{\mathsf{c}}) & (1.81) \\ &= (\mathcal{A} \cup (\mathcal{B} \cap \mathcal{A}^{\mathsf{c}})) \cap (\mathcal{B}^{\mathsf{c}} \cup (\mathcal{B} \cap \mathcal{A}^{\mathsf{c}})) & (1.82) \\ &= ((\mathcal{A} \cup \mathcal{B}) \cap (\mathcal{A} \cup \mathcal{A}^{\mathsf{c}})) \cap ((\mathcal{B}^{\mathsf{c}} \cup \mathcal{B}) \cap (\mathcal{B}^{\mathsf{c}} \cup \mathcal{A}^{\mathsf{c}})) & (1.83) \\ &= ((\mathcal{A} \cup \mathcal{B}) \cap \mathcal{O}) \cap (\mathcal{O} \cap (\mathcal{B}^{\mathsf{c}} \cup \mathcal{A}^{\mathsf{c}})) & (1.84) \\ &= (\mathcal{A} \cup \mathcal{B}) \cap (\mathcal{B}^{\mathsf{c}} \cup \mathcal{A}^{\mathsf{c}}) & (1.85) \\ &= (\mathcal{A} \cup \mathcal{B}) \cap (\mathcal{B} \cap \mathcal{A})^{\mathsf{c}}, & (1.86) \end{aligned}$$

which completes the proof. □

1.2.5 Disjoint Sets

Two sets are disjoint if they do not possess elements in common.

Definition 1.15 (Disjoint Sets). Given two sets $\mathcal{A}$ and $\mathcal{B}$, they are said to be disjoint if and only if

$$\mathcal{A} \cap \mathcal{B} = \emptyset. \tag{1.87}$$

The following theorem expresses the union of two sets by an equivalent union of two disjoints sets. This trick reveals to be particularly useful in the following chapters.

Theorem 1.16. *The union of any two subsets $\mathcal{A}$ and $\mathcal{B}$ of a set $\mathcal{O}$ can be expressed as the union of two disjoint sets: $\mathcal{A}$ and $\mathcal{A}^{\mathsf{c}} \cap \mathcal{B}$. That is,*

$$\mathcal{A} \cup \mathcal{B} = \mathcal{A} \cup (\mathcal{A}^{\mathsf{c}} \cap \mathcal{B}), \tag{1.88}$$

where the complement is with respect to $\mathcal{O}$.

Proof The proof follows from verifying that $\mathcal{A}$ and $\mathcal{A}^{\mathsf{c}} \cap \mathcal{B}$ are disjoint sets, that is,

$$\begin{aligned} \mathcal{A} \cap (\mathcal{A}^{\mathsf{c}} \cap \mathcal{B}) &= (\mathcal{A} \cap \mathcal{A}^{\mathsf{c}}) \cap \mathcal{B} \\ &= \emptyset \cap \mathcal{B} \\ &= \emptyset, & (1.89) \end{aligned}$$

and the fact that,

$$\begin{aligned}\mathcal{A} \cup (\mathcal{A}^c \cap \mathcal{B}) &= (\mathcal{A} \cup \mathcal{A}^c) \cap (\mathcal{A} \cup \mathcal{B}) \\ &= \mathcal{O} \cap (\mathcal{A} \cup \mathcal{B}) \\ &= \mathcal{A} \cup \mathcal{B}, \end{aligned} \tag{1.90}$$

where $\mathcal{O}$ is the set containing the elements of both $\mathcal{A}$ and $\mathcal{B}$. This completes the proof. □

Theorem 1.16 can be generalized to countable unions. Given a sequence of subsets $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \dots$ of a set $\mathcal{O}$, it follows from Theorem 1.16 that their union satisfies

$$\mathcal{A}_1 \cup \mathcal{A}_2 \cup \dots = \mathcal{A}_1 \cup (\mathcal{A}_1^c \cap \mathcal{A}_2) \cup (\mathcal{A}_1^c \cap \mathcal{A}_2^c \cap \mathcal{A}_3) \cup \dots, \tag{1.91}$$

where the complement is with respect to the set $\mathcal{O}$. Note that the sets $\mathcal{A}_1$, $(\mathcal{A}_1^c \cap \mathcal{A}_2)$, $(\mathcal{A}_1^c \cap \mathcal{A}_2^c \cap \mathcal{A}_3)$, ... are disjoint sets.

1.2.6 Cartesian Products

Some sets are formed by elements that are tuples. More specifically, each component of a tuple might be an element of a given set. Sets whose elements are tuples can be obtained by an operation referred to as Cartesian product.

Definition 1.17 (Cartesian Products). Given two subsets $\mathcal{A}$ and $\mathcal{B}$, their Cartesian products are denoted by $\mathcal{A} \times \mathcal{B}$ and $\mathcal{B} \times \mathcal{A}$ such that

$$\mathcal{A} \times \mathcal{B} \triangleq \{(a, b) : a \in \mathcal{A} \text{ and } b \in \mathcal{B}\} \text{ and} \tag{1.92}$$

$$\mathcal{B} \times \mathcal{A} \triangleq \{(a, b) : a \in \mathcal{B} \text{ and } b \in \mathcal{A}\}. \tag{1.93}$$

Note that the Cartesian product of $\mathcal{A}$ and $\mathcal{B}$ is a set whose elements are ordered pairs and thus, when $\mathcal{A} \neq \mathcal{B}$ it holds that $\mathcal{A} \times \mathcal{B} \neq \mathcal{B} \times \mathcal{A}$. Consider a sequence of sets $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$. Hence, the Cartesian product $\mathcal{A}_1 \times \mathcal{A}_2 \times \dots \times \mathcal{A}_n$ is often denoted by $\prod_{s=1}^{n} \mathcal{A}_s$ and

$$\prod_{s=1}^{n} \mathcal{A}_s \triangleq \{(a_1, a_2, \dots, a_n) : \forall t \in \{1, 2, \dots, n\}, a_t \in \mathcal{A}_t\}. \tag{1.94}$$

When all sets are identical, i.e., $\mathcal{A}_s = \mathcal{A}$ for all $s \in \{1, 2, \dots, n\}$, the notation can be simplified to

$$\mathcal{A}^n \triangleq \prod_{t=1}^{n} \mathcal{A} = \{(a_1, a_2, \dots, a_n) : \forall t \in \{1, 2, \dots, n\}, a_t \in \mathcal{A}\}. \tag{1.95}$$

Given a Cartesian product of the sets $\mathcal{A}_1$, $\mathcal{A}_2$, ..., $\mathcal{A}_n$, a subset of such Cartesian product often implies a relation between the elements of $\mathcal{A}_1$, $\mathcal{A}_2$, ..., $\mathcal{A}_{n-1}$ and $\mathcal{A}_n$. The following definition formalizes this intuition in the case of two sets.

DEFINITION 1.18 (Binary Relations). Given two subsets $\mathcal{A}$ and $\mathcal{B}$, a binary relation between $\mathcal{A}$ and $\mathcal{B}$ is determined by a set $\mathcal{C} \subseteq \mathcal{A} \times \mathcal{B}$. The elements $a \in \mathcal{A}$ and $b \in \mathcal{B}$ are said to be related if and only if $(a, b) \in \mathcal{C}$.

Binary relations are useful to describe several mathematical objects, for instance functions, as discussed in Section 1.7.

1.2.7 The Empty Set and the Power Set

The empty set $\emptyset$ has been defined in Section 1.1 as the set that does not contain any element. It has also naturally appeared in previous calculations, e.g., the intersection of two disjoints sets.

Alternatively, given a set $\mathcal{A}$, the power set of $\mathcal{A}$ is denoted by $2^{\mathcal{A}}$ and it is defined hereunder.

DEFINITION 1.19 (Power Set). Given a set $\mathcal{A}$, the power set of $\mathcal{A}$, denoted by $2^{\mathcal{A}}$, is the set of all possible subsets of $\mathcal{A}$.

Some of the properties of the empty set are listed by the following theorem.

THEOREM 1.20. *Let $\mathcal{A}$ be a subset of $\mathcal{O}$. Hence, the following holds:*

$$|\emptyset| = 0; \tag{1.96}$$

$$\emptyset \subseteq \mathcal{A}; \tag{1.97}$$

$$\emptyset \cup \mathcal{A} = \mathcal{A} \cup \emptyset = \mathcal{A}; \tag{1.98}$$

$$\mathcal{A} \cap \emptyset = \emptyset \cap \mathcal{A} = \emptyset; \tag{1.99}$$

$$2^{\emptyset} = \{\emptyset\}; \tag{1.100}$$

$$\emptyset \times \mathcal{A} = \mathcal{A} \times \emptyset = \emptyset; \text{ and} \tag{1.101}$$

$$\mathcal{A} \subseteq \emptyset \iff \mathcal{A} = \emptyset. \tag{1.102}$$

Proof See Homework 1 □

1.3 Subsets of $\mathbb{R}^n$

1.3.1 Balls

Given a point in $\boldsymbol{x} \in \mathbb{R}^n$, with $n \in \mathbb{N}$, and a positive real $r < \infty$, a ball is a set that contains all the points whose Euclidian distance to $\boldsymbol{x}$ is bounded by r. The point $\boldsymbol{x}$ is often called *the center of the ball* and the real r is referred to as the *radius*. When $n = 3$, the resulting ball is a geometric object that is as familiar as the shape of an orange or a volley ball. Hence, it is easy to see the justification of this denomination for these sets. Nonetheless, in the case when

$n = 1$ or $n = 2$, these sets are simply intervals and disks, respectively. In higher dimensions, these sets receive the generic name *ball*. Balls in $\mathbb{R}^n$ can be open or closed.

DEFINITION 1.21 (Open ball in $\mathbb{R}^n$). Given a point $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, with $n \in \mathbb{N}$, and a real r, with $0 < r < \infty$, the set

$$\mathcal{B}(\boldsymbol{x}, r) \triangleq \{\boldsymbol{y} \in \mathbb{R}^n : ||\boldsymbol{x} - \boldsymbol{y}||_2 < r\} \tag{1.103}$$

is an open ball.

Open balls are properly defined only for $0 < r < \infty$. The case $r = 0$ leads to an implication in which the norm $||\boldsymbol{x} - \boldsymbol{y}||_2$ in (1.103) is negative, which leads to the equality $\mathcal{B}(\boldsymbol{x}, 0) = \emptyset$, for all $\boldsymbol{x} \in \mathbb{R}^n$. The case in which $r = \infty$ corresponds to a degenerate ball that is equivalent to the whole Euclidian space $\mathbb{R}^n$.

DEFINITION 1.22 (Closed ball in $\mathbb{R}^n$). Given a point $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, with $n \in \mathbb{N}$, and a positive real r, with $0 \leqslant r < \infty$, the set

$$\bar{\mathcal{B}}(\boldsymbol{x}, r) \triangleq \{\boldsymbol{y} \in \mathbb{R}^n : ||\boldsymbol{x} - \boldsymbol{y}||_2 \leqslant r\} \tag{1.104}$$

is a closed ball.

Contrary to the case of open balls, closed balls with $r = 0$ are nonempty. The case $r = 0$ leads to the equality $\bar{\mathcal{B}}(\boldsymbol{x}, 0) = \{\boldsymbol{x}\}$, which is a singleton. Moreover, it holds that for all $\boldsymbol{x} \in \mathbb{R}^n$, with $n \in \mathbb{N}$, and for all $r \in [0, +\infty[$,

$$\mathcal{B}(\boldsymbol{x}, r) \subset \bar{\mathcal{B}}(\boldsymbol{x}, r). \tag{1.105}$$

1.3.2 Boxes

Boxes are subsets of $\mathbb{R}^n$, with $n \in \mathbb{N}$, whose denomination is also due to their geometry. When $n = 3$, boxes are regular polyhedra whose faces are pair-wise pararell and parallel to one of the axes of the Cartesian coordinates. This is despite the fact that when $n = 1$, a box is simply an interval; and when $n = 2$, a box is a rectangle. For higher dimensions, these sets are generally referred to as boxes.

DEFINITION 1.23 (Generic Box in $\mathbb{R}^n$). Given n intervals $\mathcal{R}_1$, $\mathcal{R}_2$, ..., $\mathcal{R}_n$ in $\mathbb{R}$, such that for all $i \in \{1, 2, \ldots, n\}$, interval $\mathcal{R}_i$ is of the form $[a_i, b_i]$, $]a_i, b_i[$, $]a_i, b_i]$, or $[a_i, b_i[$, with $-\infty < a_i < b_i < +\infty$, the set

$$\mathcal{R} \triangleq \mathcal{R}_1 \times \mathcal{R}_2 \times \ldots \times \mathcal{R}_n, \tag{1.106}$$

is a generic box, or simply a box.

As in the case of balls, boxes can be open or closed. An open box satisfies the following definition.

DEFINITION 1.24 (Open Box in $\mathbb{R}^n$). Given n open intervals in $\mathbb{R}$ of the form $]a_1, b_1[$, $]a_2, b_2[$, ..., $]a_n, b_n[$, with $n \in \mathbb{N}$ and $-\infty < a_i < b_i < +\infty$ for all $i \in \{1, 2, \ldots, n\}$, the set

$$\mathcal{R} \triangleq]a_1, b_1[\times]a_2, b_2[\times \ldots \times]a_n, b_n[, \tag{1.107}$$

is an open box.

Alternatively, a closed box can be defined as follows.

DEFINITION 1.25 (Closed Box in $\mathbb{R}^n$). Given n closed intervals in $\mathbb{R}$ of the form $[a_1, b_1]$, $[a_2, b_2]$, ..., $[a_n, b_n]$, with $n \in \mathbb{N}$ and $-\infty < a_i < b_i < +\infty$ for all $i \in \{1, 2, \ldots, n\}$, the set

$$\mathcal{R} \triangleq [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n], \tag{1.108}$$

is a closed box.

1.3.3 Elementary Sets

An elementary set is a subset of $\mathbb{R}^n$, with $n \in \mathbb{N}$, that can be obtained by finite union of boxes.

DEFINITION 1.26 (Elementary Sets). An elementary set $\mathcal{A}$ is a subset of $\mathbb{R}^n$, with $n \in \mathbb{N}$, for which there always exists a sequence of generic boxes $\mathcal{A}_1$, $\mathcal{A}_2$, ..., $\mathcal{A}_m$, with $m < \infty$, such that $\mathcal{A} = \bigcup_{t=1}^{m} \mathcal{A}_t$.

Elementary sets exhibit an interesting property. The fundamental operations between elementary sets, e.g., union, intersection, set difference and set symmetric difference, lead to elementary sets. The following theorem formalizes this statement.

THEOREM 1.27 (Properties of elementary sets). *Let $\mathcal{A}$ and $\mathcal{B}$ be two elementary subsets of $\mathbb{R}^n$, with $n \in \mathbb{N}$. Then, the following sets $\mathcal{A} \cup \mathcal{B}$; $\mathcal{A} \cap \mathcal{B}$; $\mathcal{A} \setminus \mathcal{B}$; and $\mathcal{A} \triangle \mathcal{B}$, are elementary sets.*

Proof See Homework 1. □

1.3.4 Open Sets and Closed Sets

The reader is certainly acquainted with the notions of open and closed intervals, which are respectively open and closed subsets in $\mathbb{R}$. In the previous section, open and closed intervals were used to build open and closed boxes, which are examples of open and closed subsets in $\mathbb{R}^n$, with $n \in \mathbb{N}$. Other examples are open balls and closed balls in $\mathbb{R}^n$. In this section, the definitions of open and closed boxes (and balls) are extended to build a formal definition of both open and closed sets in $\mathbb{R}^n$.

DEFINITION 1.28 (Open sets in $\mathbb{R}^n$). A subset $\mathcal{A}$ of $\mathbb{R}^n$, with $n \in \mathbb{N}$, is said to be open if for all $x \in \mathcal{A}$, there a exists a real $r > 0$ such that

$$\mathcal{B}(\boldsymbol{x}, r) \subset \mathcal{A}. \tag{1.109}$$

Note that the Definition 1.28 is stated in terms of open balls, but the same effect is obtained if such a definition is made in terms of open boxes. The disavantage of such an alternative is the need of specifying many more parameters. More specifically, describing a box requires n intervals ($2n$ real numbers) instead of a center for the ball, which is a point in $\mathbb{R}^n$ (n real numbers), and the radius (one real number). From this perspective, Definition 1.28 is not unique but it is certainly one of the simplest definitions.

The definition of a closed set is given in terms of its complement.

DEFINITION 1.29 (Closed sets in $\mathbb{R}^n$). A subset $\mathcal{A}$ of $\mathbb{R}^n$, with $n \in \mathbb{N}$, is said to be closed if $\mathcal{A}^c$ is open, where the complement is with respect to $\mathbb{R}^n$.

The following theorem shows that countable unions of open sets form open sets.

THEOREM 1.30 (Unions of open sets). *Let $\mathcal{C}$ be a countable set such that for all $t \in \mathcal{C}$, $\mathcal{A}_t$ is an open set. Hence, the union*

$$\bigcup_{t \in \mathcal{C}} \mathcal{A}_t \tag{1.110}$$

is an open set.

Proof See Homework 1. □

Alternatively, countably infinite intersections of open sets do not necessarily form open sets.

EXAMPLE 1.31. Consider a pair $(a, b) \in \mathbb{R}^2$, with $-\infty < a < b < +\infty$. For all $n \in \mathbb{N}$, let $\mathcal{A}_n \triangleq \left]a - \frac{1}{n}, b + \frac{1}{n}\right[$ be an open set. Then, note that $\bigcap_{t=1}^{\infty} \mathcal{A}_t = [a, b]$, which is a closed set.

Nonetheless, finite intersections of open sets form an open set.

THEOREM 1.32 (Intersections of open sets). *Let $\mathcal{A}_1$, $\mathcal{A}_2$, $\ldots$, $\mathcal{A}_k$, be an* *arbitrary sequence of open sets, with $k < \infty$. Hence, the intersection*

$$\bigcap_{t=1}^{k} \mathcal{A}_t \tag{1.111}$$

form an open set.

Proof See Homework 1. □

Using Theorem 1.30, the following shows that countable intersections of closed sets form closed sets.

THEOREM 1.33 (Intersections of closed sets). *Let $\mathcal{C}$ be a countable set such that for all $t \in \mathcal{C}$, $\mathcal{A}_t$ is a closed set. Hence, the intersection*

$$\bigcap_{t\in\mathcal{C}} \mathcal{A}_t \tag{1.112}$$

is a closed set.

Proof Note that for all $t \in \mathcal{C}$, $\mathcal{A}_t^{\mathsf{c}}$ is an open set. Hence, from Theorem 1.30, it holds that $\bigcup_{t\in\mathcal{C}} \mathcal{A}_t^{\mathsf{c}}$ is an open set, and therefore, its complement is closed. Thus, using Theorem 1.12, it follows that

$$\left(\bigcup_{t\in\mathcal{C}} \mathcal{A}_t^{\mathsf{c}}\right)^{\mathsf{c}} = \bigcap_{t\in\mathcal{C}} \mathcal{A}_t \tag{1.113}$$

is closed, which completes the proof. □

Alternatively, countably infinite unions of closed sets do not necessarily form closed sets.

EXAMPLE 1.34. Given a pair $(a, b) \in \mathbb{R}^2$, with $a < b - 2$, let for all $n \in \mathbb{N}$, $\mathcal{A}_n \triangleq \left[a + \frac{1}{n}, b - \frac{1}{n}\right]$ be a closed set. Note that $\bigcup_{t=1}^{\infty} \mathcal{A}_t =]a, b[$, which is an open set.

Nonetheless, finite unions of closed sets form a closed set.

THEOREM 1.35 (Unions of closed sets). *Let $\mathcal{A}_1$, $\mathcal{A}_2$, ..., $\mathcal{A}_k$, be an arbitrary sequence of closed sets, with $k < \infty$. Hence, the union*

$$\bigcup_{t=1}^{k} \mathcal{A}_t \tag{1.114}$$

forms a closed set.

Proof Note that $\mathcal{A}_1^{\mathsf{c}}$, $\mathcal{A}_2^{\mathsf{c}}$, ..., $\mathcal{A}_k^{\mathsf{c}}$ is a finite sequence of open sets. Hence, from

Theorem 1.32, it holds that $\bigcap_{t=1}^{k} \mathcal{A}_t^{\mathsf{c}}$ is an open set, and therefore, its complement is closed. Thus, using Theorem 1.12, it follows that

$$\left(\bigcap_{t=1}^{k} \mathcal{A}_t^{\mathsf{c}}\right)^{\mathsf{c}} = \bigcup_{t=1}^{k} \mathcal{A}_t \tag{1.115}$$

is closed, which completes the proof. □

Note that subsets of $\mathbb{R}^n$, with $n \in \mathbb{N}$, might not necessarily be open or closed. In $\mathbb{R}$, an interval of the form $[0, 1[$ is neither closed nor open. It is said to be *closed to the left* and *open to the right*. The case of the empty set is even more interesting as shown by the following theorem.

THEOREM 1.36. *The empty set in $\mathbb{R}^n$, with $n \in \mathbb{N}$, is both closed and open.*

Proof See Homework 1. □

1.3.5 Bounded Sets and Compact Sets

A set is said to be bounded in $\mathbb{R}^n$, with $n \in \mathbb{N}$, if it is a subset of a ball centered somewhere, e.g., at the origin, and whose radius is finite.

DEFINITION 1.37 (Bounded Sets). A set $\mathcal{A} \subset \mathbb{R}^n$, with $n \in \mathbb{N}$, is said to be bounded if there exists a real $r < \infty$ such that

$$\mathcal{A} \subseteq \mathcal{B}(\mathbf{0}, r), \tag{1.116}$$

where $\mathbf{0} = (0, 0, \ldots, 0) \in \mathbb{R}^n$.

Sets that are both closed and bounded form a particular class of sets in $\mathbb{R}^n$, i.e., compact sets.

DEFINITION 1.38 (Compact sets). A set $\mathcal{A} \subset \mathbb{R}^n$, with $n \in \mathbb{N}$, is said to be compact if it is closed and bounded.

Note that the following theorem is an immediate consequence of Definition 1.38, Theorem 1.33, and Theorem 1.35.

THEOREM 1.39 (Union and Intersection of Compact Sets). *A finite union of compact sets forms a compact set. A countable intersection of compact sets forms a compact set.*

1.3.6 Interior, Closure, and Boundary

In order to define the interior of a set in $\mathbb{R}^n$, with $n \in \mathbb{N}$, consider first the definition of an interior point.

Definition 1.40 (Interior Point). Given a set $\mathcal{A} \subset \mathbb{R}^n$, with $n \in \mathbb{N}$, the point $\boldsymbol{x} \in \mathbb{R}^n$ is said to be an interior point of $\mathcal{A}$ if there exists an $r > 0$ such that

$$\mathcal{B}(\boldsymbol{x}, r) \subset \mathcal{A}. \tag{1.117}$$

From Definition 1.28, it follows that all elements of an open set are interior points. In a nutshell, a point is interior to a set if it is always possible to center an open ball in such a point and ensure that the ball is a subset of the set. The union of all interior points of a set form its interior.

Definition 1.41 (Interior of a set). The interior of a set $\mathcal{A} \subset \mathbb{R}^n$, with $n \in \mathbb{N}$, denoted by $\text{int}\mathcal{A}$, is

$$\text{int}\mathcal{A} \triangleq \{\boldsymbol{x} \in \mathcal{A} : \exists r > 0,\ \mathcal{B}(\boldsymbol{x}, r) \subset \mathcal{A}\} \tag{1.118}$$

Given a nonempty set $\mathcal{A} \subset \mathbb{R}^n$, with $n \in \mathbb{N}$, it holds that $\text{int}\mathcal{A}$ is a proper subset of $\mathcal{A}$ if $\mathcal{A}$ is closed. On the other hand, $\text{int}\mathcal{A}$ is identical to $\mathcal{A}$ if $\mathcal{A}$ is open.

The definition of the closure of a set is in terms of the definition of points of closure, also known as closure points or adherent points.

Definition 1.42 (Closure Point). Given a set $\mathcal{A} \subset \mathbb{R}^n$, with $n \in \mathbb{N}$, the point $\boldsymbol{x} \in \mathbb{R}^n$ is said to be a closure point of $\mathcal{A}$ if for all $r > 0$, there exists a point $\boldsymbol{y} \in \mathcal{A}$, such that

$$\boldsymbol{y} \in \mathcal{B}(\boldsymbol{x}, r). \tag{1.119}$$

A closure point of a set is a point that is arbitrarily close to at least one element of the set. This said, any interior point is a closure point. The reunion of all closure points of a given set forms its closure.

Definition 1.43 (Closure of a set). The closure of a set $\mathcal{A} \subset \mathbb{R}^n$, with $n \in \mathbb{N}$, denoted by $\text{clo}\mathcal{A}$, is

$$\text{clo}\mathcal{A} \triangleq \{\boldsymbol{x} \in \mathbb{R}^n : \forall r > 0\ \exists \boldsymbol{y} \in \mathcal{A},\ \boldsymbol{y} \in \mathcal{B}(\boldsymbol{x}, r)\} \tag{1.120}$$

Given a nonempty set $\mathcal{A} \subset \mathbb{R}^n$, with $n \in \mathbb{N}$, it holds that $\mathcal{A} \subseteq \text{clo}\mathcal{A}$, with strict inclusion if $\mathcal{A}$ is open. On the other hand, $\text{clo}\mathcal{A}$ is identical to $\mathcal{A}$ if $\mathcal{A}$ is closed. The following theorem strengthen this observation.

Theorem 1.44. *Consider a set $\mathcal{A} \subset \mathbb{R}^n$, with $n \in \mathbb{N}$. Then, $\mathcal{A}$ is closed if and only if $\mathcal{A} = \text{clo}\mathcal{A}$.*

Proof See Homework 1. □

EXAMPLE 1.45. Consider for instance, a subset $\mathcal{A}$ of $\mathbb{R}^n$, with $n \in \mathbb{N}$, that satisfies $|\mathcal{A}| = k$, with $k < \infty$. From Definition 1.41 and Definition 1.43, it follows that $\text{int}\mathcal{A} = \emptyset$ and $\text{clo}\mathcal{A} = \mathcal{A}$.

The following theorem formalizes this observation.

THEOREM 1.46. *Given a set $\mathcal{A} \subset \mathbb{R}^n$, with $n \in \mathbb{N}$, it holds that* $\text{int}\mathcal{A}$ *is always open;* $\text{clo}\mathcal{A}$ *is always closed; and*

$$\text{int}\mathcal{A} \subseteq \text{clo}\mathcal{A}. \tag{1.121}$$

Proof See Homework 1. □

Example 1.45 highlights a special class of points in $\mathbb{R}^n$ often referred to as isolated points.

DEFINITION 1.47 (Isolated Point). Given a set $\mathcal{A} \subset \mathbb{R}^n$, with $n \in \mathbb{N}$, the point $\boldsymbol{x} \in \mathbb{R}^n$ is said to be an isolated point of $\mathcal{A}$ if there exists an $r > 0$ such that

$$\mathcal{A} \cap \mathcal{B}(\boldsymbol{x}, r) = \{\boldsymbol{x}\}. \tag{1.122}$$

The sets that do not contain isolated points form a particular class of sets.

DEFINITION 1.48 (Perfect Set). A subset of $\mathbb{R}^n$, with $n \in \mathbb{N}$, that does not contain isolated points is said to be perfect.

The definition of boundary is given in terms of the definition of boundary point.

DEFINITION 1.49 (Boundary Point). Given a set $\mathcal{A} \subset \mathbb{R}^n$, with $n \in \mathbb{N}$, the point $\boldsymbol{x} \in \mathbb{R}^n$ is said to be a boundary point of $\mathcal{A}$ if

$$\boldsymbol{x} \in \text{clo}\mathcal{A} \setminus \text{int}\mathcal{A}. \tag{1.123}$$

Using Definition 1.49, the definition of boundary can be stated as follows.

DEFINITION 1.50 (Boundary of a set). The boundary of a set $\mathcal{A} \subset \mathbb{R}^n$, with $n \in \mathbb{N}$, denoted by $\text{bou}\mathcal{A}$, is

$$\text{bou}\mathcal{A} \triangleq \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} \in \text{clo}\mathcal{A} \setminus \text{int}\mathcal{A}\} \tag{1.124}$$

Using the notion of boundary of a set, a relaxation of the definition of disjoints sets (Definition 1.15) can be formalized.

DEFINITION 1.51 (Almost disjoint sets). Two subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathbb{R}^n$, with

$n \in \mathbb{N}$, are almost disjoint if

$$\text{int}\mathcal{A} \cap \text{int}\mathcal{B} = \emptyset. \tag{1.125}$$

1.4 Partitions and Covers

A partition of a set is essentially a collection of disjoints subsets that satisfy the following definition.

DEFINITION 1.52 (Partition). Given a set $\mathcal{A}$, let $\mathcal{C}$ be a set such that for all $c \in \mathcal{C}$, $\mathcal{B}_c$ is a non-empty subset of $\mathcal{A}$. These subsets form a partition of $\mathcal{A}$ if for all pairs $(i, j) \in \mathcal{C}^2$, with $i \neq j$, $\mathcal{B}_i \cap \mathcal{B}_j = \emptyset$; and

$$\bigcup_{c \in \mathcal{C}} \mathcal{B}_c = \mathcal{A}. \tag{1.126}$$

The empty set has exactly one partition, which corresponds to the empty set itself. A trivial partition of a nonempty set $\mathcal{A}$ is the set $\mathcal{A}$ itself. The smallest partition of $\mathcal{A}$, containing the proper subset $\mathcal{B}$ is formed by the sets $\mathcal{B}$ and $\mathcal{B} \setminus \mathcal{A}$.

The largest partition of a set can be constructed as follows. Consider for instance a set $\mathcal{A}$ and define for all $a \in \mathcal{A}$ the subset $\mathcal{B}_a = \{a\}$. Hence, $\bigcup_{a \in \mathcal{A}} \mathcal{B}_a = \mathcal{A}$. Thus, these subsets form the biggest partition of $\mathcal{A}$.

When the set $\mathcal{C}$ in Definition 1.52 is countable, the corresponding partition is said to be a *countable partition*. Otherwise, the partition is said to be an *uncountable partition*.

EXAMPLE 1.53. Given a set $\mathcal{A} = [a, b]$, with $(a, b) \in \mathbb{R}^2$ and $a < b$, let $\mathcal{C} = [0, 1]$, and for all $t \in \mathcal{C}$, let $\mathcal{A}_t = \{a + t(b - a)\}$ be subsets of $\mathcal{A}$. These subsets form an uncountable partition of $\mathcal{A}$.

Alternatively, a cover can be defined as follows.

DEFINITION 1.54 (Covers and Exact Cover). Given a set $\mathcal{A}$ that satisfies

$$\mathcal{A} \subseteq \bigcup_{c \in \mathcal{C}} \mathcal{B}_c, \tag{1.127}$$

the sets $\mathcal{B}_c$, for all $c \in \mathcal{C}$, form a cover of $\mathcal{A}$. A cover that satisfies

$$\bigcup_{c \in \mathcal{C}} \mathcal{B}_c \subseteq \mathcal{A}, \tag{1.128}$$

is an exact cover.

From Definition 1.52 and Definition 1.54, it follows that every partition of a given set forms an exact cover of such a set. Nonetheless, the opposite is not true as sets forming a cover are not necessarily disjoint. Essentially, the sets $\mathcal{B}_c$, for all

$c \in \mathcal{C}$, form a cover of $\mathcal{A}$ if for all $a \in \mathcal{A}$, there exists at least one $j \in \mathcal{C}$ such that $a \in \mathcal{B}_j$. When the set $\mathcal{C}$ in Definition 1.54 is countable, the corresponding cover is said to be a *countable cover*. Otherwise, the cover is said to be an *uncountable cover*.

Example 1.55. Given a set $\mathcal{A} = [a, b]$, with $(a, b) \in \mathbb{R}^2$ and $a < b$, let $\mathcal{C} = [0, 1]$ and for all $t \in \mathcal{C}$, let $\mathcal{A}_t = [a, a + t(b - a)]$ be subsets of $\mathcal{A} = [a, b]$. These subsets form an uncountable cover of $\mathcal{A}$.

Covers of compact sets in $\mathbb{R}^n$, with $n \in \mathbb{N}$, exhibit a unique property.

Theorem 1.56 (Heine-Borel). *Let $\mathcal{A} \in \mathbb{R}^n$, with $n \in \mathbb{N}$, be a compact set, and assume that*

$$\mathcal{A} \subset \bigcup_{t \in \mathcal{C}} \mathcal{B}_t \tag{1.129}$$

where $\mathcal{C}$ is a set such that for all $t \in \mathcal{C}$, $\mathcal{B}_t$ is an open set. Then, there exists a finite $k \in \mathbb{N}$ such that

$$\mathcal{A} \subset \bigcup_{j=1}^{k} \mathcal{B}_{t_j}, \tag{1.130}$$

where, for all $j \in \{1, 2, \ldots, k\}$, $t_j \in \mathcal{C}$.

Proof See Homework 1. □

Theorem 1.56 states that from every cover (Definition 1.54) formed by infinitely many open sets of a compact set, it is always possible to obtain a cover formed by a finite number of those open sets. The first statement of Theorem 1.56 is attributed to Eduard Heine. Nonetheless, the first formal proof, in the case in which $\mathcal{C}$ is countable, is attributed to Émile Borel in 1895. The current form of this theorem is due to contributions of Pierre Cousin, Henri Lebesgue, and Arthur Moritz Schoenflies.

1.5 Sequences of Sets

1.5.1 Monotonic Sequences of Sets

Monotonic sequences of sets are either increasing or decreasing. These can be formally defined as follows.

Definition 1.57 (Increasing/Decreasing Sequences). Given a set $\mathcal{A}$, a countable sequence of sets $\mathcal{A}_1, \mathcal{A}_2, \ldots$ is said to form an *increasing sequence* whose limit is $\mathcal{A}$, if and only if

(a) $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \mathcal{A}_3 \subset \ldots$ and

(b) $\bigcup_{t=1}^{\infty} \mathcal{A}_t = \mathcal{A}$.

This is denoted by $\mathcal{A}_n \uparrow \mathcal{A}$. Alternatively, they are said to form a *decreasing sequence* whose limit is $\mathcal{A}$, if and only if

(c) $\mathcal{A}_1 \supset \mathcal{A}_2 \supset \mathcal{A}_3 \supset \dots$ and

(d) $\bigcap_{t=1}^{\infty} \mathcal{A}_t = \mathcal{A}$.

This is denoted by $\mathcal{A}_n \downarrow \mathcal{A}$.

The following examples show an increasing sequence of closed intervals and a decreasing sequence of open intervals, respectively.

EXAMPLE 1.58. Given a pair $(a, b) \in \mathbb{R}^2$, with $a < b - 2$, the open interval $]a, b[$ can be shown to be the limit of an increasing sequence of closed intervals. Assume for instance that for all $n \in \mathbb{N}$, $\mathcal{A}_n \triangleq \left[a + \frac{1}{n}, b - \frac{1}{n}\right]$. Note that $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \mathcal{A}_3 \subset \dots$ and $\bigcup_{t=1}^{\infty} \mathcal{A}_t =]a, b[$. Thus,

$$\left[a + \frac{1}{n}, b - \frac{1}{n}\right] \uparrow \,]a, b[. \tag{1.131}$$

EXAMPLE 1.59. Given a pair $(a, b) \in \mathbb{R}^2$, with $a < b$, the closed interval $[a, b]$ can be shown to be the limit of a decreasing sequence of open intervals. Assume for instance that for all $n \in \mathbb{N}$, $\mathcal{A}_n \triangleq \left]a - \frac{1}{n}, b + \frac{1}{n}\right[$. Then, note that $\mathcal{A}_1 \supset \mathcal{A}_2 \supset \mathcal{A}_3 \supset \dots$ and $\bigcap_{t=1}^{\infty} \mathcal{A}_t = [a, b]$. Thus,

$$\left]a - \frac{1}{n}, b + \frac{1}{n}\right[\downarrow [a, b]. \tag{1.132}$$

The De Morgan's laws (Theorem 1.12) lead to the following implications.

THEOREM 1.60. *Consider an infinite sequence of sets $\mathcal{A}_1, \mathcal{A}_2, \dots$. Then,*

(i) If $\mathcal{A}_n \uparrow \mathcal{A}$, then $\mathcal{A}_n^c \downarrow \mathcal{A}^c$; and
(ii) If $\mathcal{A}_n \downarrow \mathcal{A}$, then $\mathcal{A}_n^c \uparrow \mathcal{A}^c$.

Proof To prove (i), note that if $\mathcal{A}_n \uparrow \mathcal{A}$, it follows that $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \mathcal{A}_3 \subset \dots$ and $\bigcup_{t=1}^{\infty} \mathcal{A}_t = \mathcal{A}$. From Theorem 1.9, the former implies that $\mathcal{A}_1^c \supset \mathcal{A}_2^c \supset \mathcal{A}_3^c \supset \dots$.

Hence, from Theorem 1.12, it follows that

$$\mathcal{A}^{\mathsf{c}} = \left(\bigcup_{n=1}^{\infty} \mathcal{A}_n \right)^{\mathsf{c}} = \bigcap_{n=1}^{\infty} \mathcal{A}_n^{\mathsf{c}}. \tag{1.133}$$

This leads to the conclusion that $\mathcal{A}_n^{\mathsf{c}} \downarrow \mathcal{A}^{\mathsf{c}}$.

To prove (*ii*), note that if $\mathcal{A}_n \downarrow \mathcal{A}$, it follows that $\mathcal{A}_1 \supset \mathcal{A}_2 \supset \mathcal{A}_3 \supset \ldots$ and $\bigcap_{t=1}^{\infty} \mathcal{A}_t = \mathcal{A}$. From Theorem 1.9, the former implies that $\mathcal{A}_1^{\mathsf{c}} \subset \mathcal{A}_2^{\mathsf{c}} \subset \mathcal{A}_3^{\mathsf{c}} \subset \ldots$, whereas the latter, from Theorem 1.12, implies that

$$\mathcal{A}^{\mathsf{c}} = \left(\bigcap_{n=1}^{\infty} \mathcal{A}_n \right)^{\mathsf{c}} = \bigcup_{n=1}^{\infty} \mathcal{A}_n^{\mathsf{c}}. \tag{1.134}$$

This leads to the conclusion that $\mathcal{A}_n^{\mathsf{c}} \uparrow \mathcal{A}^{\mathsf{c}}$ and completes the proof. □

1.5.2 Limits of Sequences of Sets

The notion of a limit in a sequence of sets is analogous to the notion of limit in a sequence of real numbers. The following definition unveils this analogy.

> Definition 1.61. Consider a countable sequence of sets $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \ldots$. Then, the lower-limit of the sequence is
>
> $$\liminf_n \mathcal{A}_n \triangleq \bigcup_{m=1}^{\infty} \bigcap_{k=m}^{\infty} \mathcal{A}_k \tag{1.135}$$
>
> and the upper-limit of the sequence is
>
> $$\limsup_n \mathcal{A}_n \triangleq \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} \mathcal{A}_k. \tag{1.136}$$

Given a countable sequence of sets $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \ldots$ and a set $\mathcal{B}$ such that $\mathcal{B} \subseteq \liminf_n \mathcal{A}_n$, then there always exists a $k \in \mathbb{N}$, such that for all $n > k$, it holds that $\mathcal{B} \subset \mathcal{A}_n$. More specifically, $\mathcal{B} \subseteq \liminf_n \mathcal{A}_n$ if and only if for all $n \in \mathbb{N} \setminus \mathcal{N}$ it holds that $\mathcal{B} \subset \mathcal{A}_n$, with $\mathcal{N} \subset \mathbb{N}$, a finite subset.

Alternatively, given a countable sequence of sets $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \ldots$ and a set $\mathcal{B}$ such that $\mathcal{B} \subseteq \limsup_n \mathcal{A}_n$, then for all $k \in \mathbb{N}$, there always exists an integer $n > k$ such that $\mathcal{B} \subset \mathcal{A}_n$. More specifically, $\mathcal{B} \subseteq \limsup_n \mathcal{A}_n$ if and only if for all $n \in \mathcal{N}$ it holds that $\mathcal{B} \subset \mathcal{A}_n$, with $\mathcal{N} \subset \mathbb{N}$ an infinite subset.

> Example 1.62. Consider a countable sequence of sets $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \ldots$ such

that for all $n \in \mathbb{N}$,

$$\mathcal{A}_n = \begin{cases} \left]\frac{-1}{n}, 1\right] & \text{if } n \text{ is odd} \\ \left]-1, \frac{1}{n}\right] & \text{if } n \text{ is even.} \end{cases} \tag{1.137}$$

Hence, for all $m \in \mathbb{N}$, the following holds:

$$\bigcup_{n=m}^{\infty} \mathcal{A}_n = \bigcup_{n=0}^{\infty} (\mathcal{A}_{m+2n} \cup \mathcal{A}_{m+2n+1}) \tag{1.138}$$

$$= \left(\bigcup_{n=0}^{\infty} \mathcal{A}_{m+2n}\right) \cup \left(\bigcup_{n=0}^{\infty} \mathcal{A}_{m+2n+1}\right) \text{ and} \tag{1.139}$$

$$\bigcap_{n=m}^{\infty} \mathcal{A}_n = \bigcap_{n=0}^{\infty} (\mathcal{A}_{m+2n} \cap \mathcal{A}_{m+2n+1}) \tag{1.140}$$

$$= \left(\bigcap_{n=0}^{\infty} \mathcal{A}_{m+2n}\right) \cap \left(\bigcap_{n=0}^{\infty} \mathcal{A}_{m+2n+1}\right). \tag{1.141}$$

Then, if m is even,

$$\bigcup_{n=m}^{\infty} \mathcal{A}_n = \left(\bigcup_{n=0}^{\infty} \left]-1, \frac{1}{m+2n}\right]\right) \cup \left(\bigcup_{n=0}^{\infty} \left]\frac{-1}{m+2n+1}, 1\right]\right) \tag{1.142}$$

$$= \left]-1, \frac{1}{m}\right] \cup \left]-\frac{1}{m+1}, 1\right] \tag{1.143}$$

$$=]-1, 1] \text{ and} \tag{1.144}$$

$$\bigcap_{n=m}^{\infty} \mathcal{A}_n = \left(\bigcap_{n=0}^{\infty} \left]-1, \frac{1}{m+2n}\right]\right) \cap \left(\bigcap_{n=0}^{\infty} \left]\frac{-1}{m+2n+1}, 1\right]\right) \tag{1.145}$$

$$=]-1, 0] \cap [0, 1] \tag{1.146}$$

$$= \{0\}. \tag{1.147}$$

Alternatively, if m is odd,

$$\bigcup_{n=m}^{\infty} \mathcal{A}_n = \left(\bigcup_{n=0}^{\infty} \left]\frac{-1}{m+2n}, 1\right]\right) \cup \left(\bigcup_{n=0}^{\infty} \left]-1, \frac{1}{m+2n+1}\right]\right) \tag{1.148}$$

$$= \left]-\frac{1}{m}, 1\right] \cup \left]-1, \frac{1}{m+1}\right] \tag{1.149}$$

$$=]-1, 1] \text{ and} \tag{1.150}$$

$$\bigcap_{n=m}^{\infty} \mathcal{A}_n = \left(\bigcap_{n=0}^{\infty} \left]\frac{-1}{m+2n}, 1\right]\right) \cap \left(\bigcap_{n=0}^{\infty} \left]-1, \frac{1}{m+2n+1}\right]\right) \tag{1.151}$$

$$= [0, 1] \cap]-1, 0] \tag{1.152}$$

$$= \{0\}. \tag{1.153}$$

This implies that for all $m \in \mathbb{N}$,

$$\bigcup_{n=m}^{\infty} \mathcal{A}_n =]-1,1], \text{ and} \tag{1.154}$$

$$\bigcap_{n=m}^{\infty} \mathcal{A}_n = \{0\}. \tag{1.155}$$

Thus, the upper-limit of the sequence is

$$\limsup_n \mathcal{A}_n = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \mathcal{A}_n = \bigcap_{m=1}^{\infty}]-1,1] =]-1,1], \tag{1.156}$$

and the lower limit of the sequence is

$$\liminf_n \mathcal{A}_n = \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \mathcal{A}_n = \bigcup_{m=1}^{\infty} \{0\} = \{0\}. \tag{1.157}$$

In general, the upper and lower limits satisfy the following identities.

Theorem 1.63 (Complements of Limits). *Consider a countable sequence of sets $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \ldots$. Then,*

$$\left(\limsup_n \mathcal{A}_n\right)^{\mathsf{c}} = \liminf_n \mathcal{A}_n^{\mathsf{c}}, \text{ and} \tag{1.158}$$

$$\left(\liminf_n \mathcal{A}_n\right)^{\mathsf{c}} = \limsup_n \mathcal{A}_n^{\mathsf{c}}. \tag{1.159}$$

Proof The proof is obtained using the De Morgan's identities (Theorem 1.12). That is,

$$\left(\limsup_n \mathcal{A}_n\right)^{\mathsf{c}} = \left(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \mathcal{A}_n\right)^{\mathsf{c}} \tag{1.160}$$

$$= \bigcup_{m=1}^{\infty} \left(\bigcup_{n=m}^{\infty} \mathcal{A}_n\right)^{\mathsf{c}} \tag{1.161}$$

$$= \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \mathcal{A}_n^{\mathsf{c}} \tag{1.162}$$

$$= \liminf_n \mathcal{A}_n^{\mathsf{c}} \tag{1.163}$$

and

$$\left(\liminf_n \mathcal{A}_n\right)^{\mathsf{c}} = \left(\bigcup_{m=1}^{\infty}\bigcap_{n=m}^{\infty} \mathcal{A}_n\right)^{\mathsf{c}} \tag{1.164}$$

$$= \bigcap_{m=1}^{\infty}\left(\bigcap_{n=m}^{\infty} \mathcal{A}_n\right)^{\mathsf{c}} \tag{1.165}$$

$$= \bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty} \mathcal{A}_n^{\mathsf{c}} \tag{1.166}$$

$$= \limsup_n \mathcal{A}_n^{\mathsf{c}}, \tag{1.167}$$

which completes the proof. □

The following Theorem shows that the lower limit is a subset of the upper limit.

THEOREM 1.64 (Inclusions). *Consider a countable sequence of sets $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, ... Then,*

$$\liminf_n \mathcal{A}_n \subseteq \limsup_n \mathcal{A}_n. \tag{1.168}$$

Proof Note that if $\mathcal{B} \subset \liminf_n \mathcal{A}_n$, it follows that there exits an $n \in \mathbb{N}$ such that for all $k > n$, $\mathcal{B} \subset \mathcal{A}_k$. This implies that for all $n \in \mathbb{N}$, there exists at least one $k > n$ such that $\mathcal{B} \subset \mathcal{A}_k$, which implies that $\mathcal{B} \subset \limsup_n \mathcal{A}_n$. This shows that $\liminf_n \mathcal{A}_n \subseteq \limsup_n \mathcal{A}_n$. □

When the upper and lower limit are identical, it is said that a limit exits. The following theorem introduces a couple of cases in which a limit exits.

THEOREM 1.65. *Consider a countable sequence of sets $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, ... Then, if $\mathcal{A}_n \uparrow \mathcal{A}$ or $\mathcal{A}_n \downarrow \mathcal{A}$, it follows that*

$$\liminf_n \mathcal{A}_n = \limsup_n \mathcal{A}_n. \tag{1.169}$$

Proof Consider that $\mathcal{A}_n \uparrow \mathcal{A}$. Then, it follows that $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \mathcal{A}_3 \subset \ldots$, which implies that for all $m > 0$,

(a) $\bigcup_{n=m}^{\infty} \mathcal{A}_n = \mathcal{A}$; and

(b) $\bigcap_{n=m}^{\infty} \mathcal{A}_n = \mathcal{A}_m$.

From (a), it follows that

$$\limsup_n \mathcal{A}_n = \bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty} \mathcal{A}_n = \bigcap_{m=1}^{\infty} \mathcal{A} = \mathcal{A}, \tag{1.170}$$

and from (b), it follows that

$$\liminf_n \mathcal{A}_n = \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \mathcal{A}_n = \bigcup_{m=1}^{\infty} \mathcal{A}_m = \mathcal{A}. \tag{1.171}$$

The proof in the case in which $\mathcal{A}_n \downarrow \mathcal{A}$ follows similar steps, and this completes the proof. □

1.6 Set Fields and σ-fields

In order to introduce the notion of set fields and σ-fields, some new notations must be introduced. In the following, sets whose elements are sets are denoted by calligraphic script letters, e.g., $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$, ..., $\mathscr{Z}$. Thus, the notation $\mathcal{A} \in \mathscr{A}$ denotes that the set $\mathcal{A}$ is an element of the set $\mathscr{A}$. Moreover, both $\mathcal{A}$ and $\mathscr{A}$ are referred to as sets and further distinction is made only when needed.

Given a set $\mathcal{O}$, a set field or a set algebra is a set $\mathscr{F}$ of subsets of $\mathcal{O}$ that satisfy the axiom of closure under complements and under finite unions. A formal definition is provided hereunder.

DEFINITION 1.66 (Set Field). Let $\mathscr{F}$ be a set of subsets of $\mathcal{O}$. Then, $\mathscr{F}$ is said to be a set field if it is closed under complements and finite unions, that is,

- $\mathcal{O} \in \mathscr{F}$;
- $\forall \mathcal{A} \in \mathscr{F}$, $\mathcal{A}^{\mathsf{c}} \in \mathscr{F}$; and
- for all sequences of subsets $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ in $\mathscr{F}$,

$$\bigcup_{t=1}^{n} \mathcal{A}_t \in \mathscr{F}, \tag{1.172}$$

where $n < \infty$, and complements are with respect to $\mathcal{O}$.

Note that if $\mathscr{F}$ is a set field of $\mathcal{O}$, it holds from the definition that $\mathcal{O} \in \mathscr{F}$ and thus, $\mathcal{O}^{\mathsf{c}} = \emptyset \in \mathscr{F}$. That is, the empty set is part of any set field.

Note that set fields are also closed under finite intersections. Consider for instance the sets $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ in $\mathscr{F}$, with $n < \infty$, then

$$\bigcap_{i=1}^{n} \mathcal{A}_i = \left(\bigcup_{i=1}^{n} \mathcal{A}_i^{\mathsf{c}} \right)^{\mathsf{c}} \in \mathscr{F}, \tag{1.173}$$

which follows from the fact that set fields are closed under complements and finite unions.

A σ-field is a set field whose elements satisfy the axiom of closure under countable unions.

> Definition 1.67 (σ-field). Let $\mathscr{F}$ be a set of subsets of $\mathcal{O}$. Then, $\mathscr{F}$ is said to be a σ-field (or set σ-algebra) if it is closed under complements and countably infinite unions, that is,
>
> - $\mathcal{O} \in \mathscr{F}$;
> - $\forall \mathcal{A} \in \mathscr{F}$, $\mathcal{A}^c \in \mathscr{F}$; and
> - for all countable sequences of subsets $\mathcal{A}_1, \mathcal{A}_2, \ldots$ in $\mathscr{F}$, $\bigcup_{t=1}^{\infty} \mathcal{A}_t \in \mathscr{F}$.

Following the same reasoning as above, it follows that every σ-field contains the empty set; and every σ-field is closed under countable intersections.

The largest σ-field on a set $\mathcal{O}$ is the collection of all possible subsets of $\mathcal{O}$, Often this collection is referred to as the *power set* (Definition 1.19) of $\mathcal{O}$ and it is denoted by $2^{\mathcal{O}}$. Alternatively, the smallest σ-field on a set $\mathcal{O}$ is the collection of two sets: $\mathcal{O}$ and the empty set $\emptyset$.

Given a subset $\mathcal{A} \subset \mathcal{O}$, the smallest σ-field $\mathscr{F}$ on $\mathcal{O}$ containing $\mathcal{A}$ is the collection $\{\mathcal{A}, \mathcal{A}^c, \mathcal{O}, \emptyset\}$. Note that if $\mathscr{G}$ is a σ-field on $\mathcal{O}$ that contains $\mathcal{A}$, then it also contains $\mathcal{A}^c, \mathcal{O}$ and $\emptyset$, and thus, $\mathscr{F} \subset \mathscr{G}$. Hence, the σ-field $\mathscr{F}$ is contained in any σ-field that contains $\mathcal{A}$. That is, $\mathscr{F}$ is the smallest σ-field on $\mathcal{O}$ containing $\mathcal{A}$.

Given a collection $\mathscr{S}$ of subsets of $\mathcal{O}$, the smallest σ-field containing $\mathscr{S}$ is referred to as the σ-field *induced by* $\mathscr{S}$, and it is denoted by $\sigma(\mathscr{S})$.

Given two σ-fields $\mathscr{F}$ and $\mathscr{G}$, with $\mathscr{G} \subset \mathscr{F}$, it is said that $\mathscr{G}$ is a *sub σ-field* of $\mathscr{F}$ and $\mathscr{F}$ is a *refinement* of $\mathscr{G}$.

A σ-field that plays a key role in the following chapters is the Borel σ-field.

> Definition 1.68 (Borel σ-Field). The Borel σ-field on $\mathbb{R}^n$, with $n \in \mathbb{N}$, is the smallest σ-field on $\mathbb{R}^n$ containing all open subsets of $\mathbb{R}^n$.

Note that in Section 1.5.1, it has been shown that in $\mathbb{R}$, intervals of the form $[a, b]$, $]a, b]$, $[a, b[$, and $]a, b[$, with $(a, b) \in \mathbb{R}^2$ and $a < b$, can be obtained as the limit of decreasing sequences of open sets. Similarly, by the closeness under complements, it could be verified that $\mathscr{B}(\mathbb{R})$ also contains the sets $]-\infty, a[$, $]-\infty, a]$, $]b, \infty[$ and $[b, \infty[$.

Borel σ-fields can be defined in any subset of $\mathbb{R}$. The Borel σ-field in a specific interval $\mathcal{A} \in \mathscr{B}(\mathbb{R})$ is denoted by

$$\mathscr{B}(\mathcal{A}) \triangleq \{\mathcal{A} \cap \mathcal{B} : \mathcal{B} \in \mathscr{B}(\mathbb{R})\}. \tag{1.174}$$

Hence, $\mathscr{B}(\mathcal{A})$ is a σ-field on $\mathcal{A}$.

Set operations among σ-fields might form other σ-fields. This is the case of the intersection, but not necessarily the case of unions. The following theorems highlight these observations.

THEOREM 1.69. *Let $\mathscr{F}$ and $\mathscr{G}$ be two σ-fields of $\mathcal{O}$. Then, $\mathscr{F} \cap \mathscr{G}$ is also a σ-field of $\mathcal{O}$.*

Proof First, note that $\mathcal{O} \in \mathscr{F}$ and $\mathcal{O} \in \mathscr{G}$ due to the assumptions that $\mathscr{F}$ and $\mathscr{G}$ are both σ-fields. Hence, $\mathcal{O} \in \mathscr{F} \cap \mathscr{G}$. Second, note that for all $\mathcal{A} \in \mathscr{F} \cap \mathscr{G}$, it holds that $\mathcal{A} \in \mathscr{F}$ and $\mathcal{A} \in \mathscr{G}$. From the assumption that both $\mathscr{F}$ and $\mathscr{G}$ are σ-fields, it holds that $\mathcal{A}^{\mathsf{c}} \in \mathscr{F}$ and $\mathcal{A}^{\mathsf{c}} \in \mathscr{G}$. Therefore, $\mathcal{A}^{\mathsf{c}} \in \mathscr{F} \cap \mathscr{G}$.

Finally, note that for all sequences of subsets $\mathcal{A}_1, \mathcal{A}_2, ...$ in $\mathscr{F} \cap \mathscr{G}$, it holds that $\forall t \in \mathbb{N}$, $\mathcal{A}_t \in \mathscr{F}$ and $\mathcal{A}_t \in \mathscr{G}$. This implies $\bigcup_{t=1}^{\infty} \mathcal{A}_t \in \mathscr{F}$ and $\bigcup_{t=1}^{\infty} \mathcal{A}_t \in \mathscr{G}$. Therefore $\bigcup_{t=1}^{\infty} \mathcal{A}_t \in \mathscr{F} \cap \mathscr{G}$. This verifies that $\mathscr{F} \cap \mathscr{G}$ satisfies the conditions in Definition 1.67, which completes the proof. □

THEOREM 1.70. *Let $\mathscr{F}$ and $\mathscr{G}$ be two σ-fields of $\mathcal{O}$, with $|\mathcal{O}| > 1$. Then, $\mathscr{F} \cup \mathscr{G}$ is not necessarily a σ-field of $\mathcal{O}$.*

Proof The proof is a simple counter example in which the union of two σ-fields of $\mathcal{O}$, denoted by $\mathscr{F}$ and $\mathscr{G}$, does not form a σ-field. Assume that $\mathcal{A}$ and $\mathcal{B}$ are two nonempty proper subsets of $\mathcal{O}$ such that $\mathcal{A} \cup \mathcal{B} \neq \mathcal{A}$; $\mathcal{A} \cup \mathcal{B} \neq \mathcal{B}$; $\mathcal{B}^{\mathsf{c}} \neq \mathcal{A}$; and $\mathcal{A}^{\mathsf{c}} \neq \mathcal{B}$. These assumptions ensure that $\mathcal{A} \cup \mathcal{B} \neq \emptyset$; and $\mathcal{A} \cup \mathcal{B} \neq \mathcal{O}$. Moreover, note that $\mathcal{A} \cup \mathcal{B} \neq \mathcal{A}^{\mathsf{c}}$ and $\mathcal{A} \cup \mathcal{B} \neq \mathcal{B}^{\mathsf{c}}$.

The following σ-fields:

$$\mathscr{F} \triangleq \{\mathcal{A}, \mathcal{A}^{\mathsf{c}}, \mathcal{O}, \emptyset\}, \tag{1.175}$$

and

$$\mathscr{G} \triangleq \{\mathcal{B}, \mathcal{B}^{\mathsf{c}}, \mathcal{O}, \emptyset\}, \tag{1.176}$$

satisfy that $\mathscr{F} \cup \mathscr{G} = \{\mathcal{A}, \mathcal{A}^{\mathsf{c}}, \mathcal{B}, \mathcal{B}^{\mathsf{c}}, \mathcal{O}, \emptyset\}$. Hence, $\mathscr{F} \cup \mathscr{G}$ is not a σ-field, because $\mathcal{A} \in \mathscr{F} \cup \mathscr{G}$ and $\mathcal{B} \in \mathscr{F} \cup \mathscr{G}$, but $\mathcal{A} \cup \mathcal{B} \notin \mathscr{F} \cup \mathscr{G}$. This completes the proof. □

1.7 Set-Valued Functions

Given two sets $\mathcal{A}$ and $\mathcal{B}$, a function f is a binary relation (Definition 1.18) between $\mathcal{A}$ and $\mathcal{B}$ that assigns an element of $\mathcal{B}$ to each element of $\mathcal{A}$. That is, for all $a \in \mathcal{A}$, there is an element of $\mathcal{B}$ assigned to a, denoted by $f(a) \in \mathcal{B}$. When $\mathcal{B} \subset \mathbb{R}$, the function f is said to be a *real-valued function*. More specifically, when $\mathcal{B} \subset [0, \infty[$ or $\mathcal{B} \subset]-\infty, 0]$, the function f is said to be nonnegative or nonpositive, respectively. Otherwise, when $\mathcal{B} \subset]0, \infty[$ or $\mathcal{B} \subset] - \infty, 0[$, the function f is said to be positive or negative, respectively.

Often, the elements of $\mathcal{A}$ and $\mathcal{B}$ are respectively said to be the arguments and the values of the function f. The set $\mathcal{A}$ is often referred to as the domain of f, whereas the set $\mathcal{B}$ is referred to as *co-domain*, *range*, or *image* of f. In the following sections, functions are defined in two steps. First, domain and image sets are defined using the notation $f : \mathcal{A} \to \mathcal{B}$. Second, given an argument a in $\mathcal{A}$, the correspoding element $f(a)$ in $\mathcal{B}$ is determined by a mathematical expression. Consider for instance the quadratic function, e.g., $f : \mathbb{R} \to \mathbb{R}$ and for all $x \in \mathbb{R}$, $f(x) = x^2$.

For all $\mathcal{C} \subseteq \mathcal{A}$, the notation $f(\mathcal{C})$ describes the following set:

$$f(\mathcal{C}) \triangleq \{b \in \mathcal{B} : b = f(a), a \in \mathcal{C}\}, \tag{1.177}$$

which is referred to as the image of $\mathcal{C}$ via the function f. Note that $f(\mathcal{C})$ is an abuse of notation given that the function f accepts as arguments the elements of $\mathcal{A}$ instead of subsets of $\mathcal{A}$. Clarifications are going to be provided when needed, otherwise, this abuse of notation will be accepted and often used in the remaining sections.

On the other hand, the functional inverse of the function $f : \mathcal{A} \to \mathcal{B}$ is denoted by $f^{-1} : \mathcal{B} \to 2^{\mathcal{A}}$, where $2^{\mathcal{A}}$ denotes the power set of $\mathcal{A}$ (Defintion 1.19). That is, given an element b in the image of f, that is, $b \in \mathcal{B}$, the following holds:

$$f^{-1}(b) = \{a \in \mathcal{A} : f(a) = b\}, \tag{1.178}$$

which is a set. In general functions whose image is a set of sets are referred to as set-valued functions. In particular, the inverse function f^{-1} is a set-valued function. With an abuse of notation, given a subset $\mathcal{D} \subseteq \mathcal{B}$, the notation $f^{-1}(\mathcal{D})$ denotes the following set:

$$f^{-1}(\mathcal{D}) \triangleq \{a \in \mathcal{A} : f(a) \in \mathcal{D}\}, \tag{1.179}$$

which is referred to as the *pre-image* of $\mathcal{D}$ via the function f.

The inverse function satisfies some properties that reveal useful in the next chapters.

THEOREM 1.71. *Given two sets $\mathcal{A}$ and $\mathcal{B}$, consider a function $f : \mathcal{A} \to \mathcal{B}$. Then, the inverse function $f^{-1} : \mathcal{B} \to 2^{\mathcal{A}}$ satifies:*

- *For all $a \in \mathcal{A}$, $a \in f^{-1}(f(a))$;*
- *$f^{-1}(\emptyset) = \emptyset$;*
- *$f^{-1}(\mathcal{B}) = \mathcal{A}$;*
- *Given a subset $\mathcal{C} \subseteq \mathcal{B}$, $f^{-1}(\mathcal{B} \setminus \mathcal{C}) = f^{-1}(\mathcal{B}) \setminus f^{-1}(\mathcal{C})$*
- *Given a collection $\mathcal{B}_1$, $\mathcal{B}_2$, ..., $\mathcal{B}_n$ of subsets of $\mathcal{B}$,*

$$f^{-1}\left(\bigcup_{j=1}^{n} \mathcal{B}_j\right) = \bigcup_{j=1}^{n} f^{-1}(\mathcal{B}_j) \text{ and} \tag{1.180}$$

$$f^{-1}\left(\bigcap_{j=1}^{n} \mathcal{B}_j\right) = \bigcap_{j=1}^{n} f^{-1}(\mathcal{B}_j) \tag{1.181}$$

Proof See Homework 1. □

Given two functions $f : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{B} \to \mathcal{C}$, the function $h : \mathcal{A} \to \mathcal{C}$ defined by $h(x) = g(f(x))$, with $x \in \mathcal{A}$, is the *composition* of g and f. The function h is also denoted by $g \circ f$.

THEOREM 1.72. *Given the sets $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, consider the functions $f : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{B} \to \mathcal{C}$. Then, the composition of g with f, denoted by $g \circ f$, satisfies:*

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}. \tag{1.182}$$

Proof See Homework 1. □

2 Integration

The development of the theory of integration has been motivated by different objectives. On one hand, integration can be seen as a tool for the calculation of areas of shapes that are formed under the curve of certain functions. More specifically, given a positive bounded function $f : [a, b] \to \mathbb{R}$, with $-\infty < a < b < +\infty$, it might be of particular interest to calculate the area of the surface formed by the points in the set

$$\{(x, y) \in \mathbb{R}^2 : a \leqslant x \leqslant b, 0 \leqslant y \leqslant f(x)\}, \tag{2.1}$$

which is often referred to as the area under the curve of f. Nonetheless, this problem is not different from the calculation of lengths, areas and volumes of geometric figures, which dates back to the ancient Greece (500 - 200 B.C.). In this regard, this chapter focuses only on relatively recent contributions during the XIX century. In particular, this chapter shows that the formulations of the integral proposed by Darboux and Riemann successfully contribute to solving the problem of areas under certain functions. Darboux's and Riemann's integrals, when they exist are equivalent to each other, and are equivalent to the area under the curve in the case of positive continuous functions, for instance.

On the other hand, integration can be seen as the inverse operation of differentiation. This perspective is due to the contributions of Isaac Newton and Leibniz during the last years of the XVII and the dawn of the XVIII centuries. Darboux's and Riemann's integrals reveal exciting connections between integration and differentiation. Let $F : \mathbb{R} \to \mathbb{R}$ be a continuous function and denote its derivative by $f : \mathbb{R} \to \mathbb{R}$. When the Riemann's integral of f on $[a, b]$, with $-\infty < a < b < +\infty$, exists, it follows that

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a). \tag{2.2}$$

This is in line with the intuition that the integral of the derivative of a function must be equal to the original function. Nonetheless, in general, the Riemman's integral of the derivative of an arbitrary function does not necessarily exists. The notion of integral in which the equality in (2.2) holds always true was proposed by Arnaud Denjoy (Denjoy 1912). Nonetheless, this definition of integral is known today under the name Henstock–Kurzweil integral, due to the important contributions of Jaroslav Kurzweil and Ralph Henstock, who extensively developed this theory.

The most impactful contribution to the theory of integration was undoubtedly made by Henri Lebesgue during his thesis in 1902. The impact of the notion of integral proposed by Lebesgue stems from the fact that it allows integrating a larger class of functions; and unifies the problem of finding a function knowing its derivative; and the problem of calculating the area under its curve. Despite the generality of Lebesgue integral, it is always possible to find a function whose derivative is not integrable in the sense of Riemman and Lebesgue, but in the sense of Henstock–Kurzweil. This highlights the fact that there is no general theory of integration, or at least there is no definition of integral that unifies all what one can expect of such an operation.

Finally, it is important to highlight that, the contributions of Lebesgue opened a new field in mathematical analysis, the theory of measure, which is studied later in this chapter.

2.1 Notation

In order to introduce the definition of integral, in the sense of Darboux, Riemann, and Lebesgue, two new objects are introduced: subdivisions and tagged subdivisions.

A subdivision is a set of points in $\mathbb{R}$ that can be used to form a finite number of subsets within a given interval.

DEFINITION 2.1 (Subdivision). Given an interval $[a, b] \in \mathbb{R}$, with $\infty < a < b < \infty$, a subdivision $\mathcal{R} = \{t_0, t_1, \ldots, t_n\}$ on $[a, b]$, with $n < \infty$, is a subset of $\mathbb{R}$ whose elements satisfy

$$a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b. \tag{2.3}$$

Infinitely many subdivisions can be formed on an interval $[a, b]$, with $a < b$, while no subdivision can be formed if $a = b$. Subdivisions on a given interval are defined under the assumption that such interval is both closed and bounded, which implies that subdivisions are defined only on compact subsets of $\mathbb{R}$ (Definition 1.38).

Every subdivision $\mathcal{R}$ on a compact interval $[a, b]$ induces a collection of perfect subsets (Definition 1.48) of $[a, b]$ of the form, $[t_0, t_1]$, $[t_1, t_2]$, …, $[t_{n-1}, t_n]$, which are *almost disjoint* (Definition 1.51). These sets $[t_0, t_1]$, $[t_1, t_2]$, …, $[t_{n-1}, t_n]$, form an exact cover of $[a, b]$ (Definition 1.54) given that

$$[a, b] \subseteq \bigcup_{j=1}^{n} [t_{j-1}, t_j] \subseteq [a, b]. \tag{2.4}$$

A parameter that describes a subdivision is the length of its widest interval. This parameter is referred to as the *mesh* or *norm* of the subdivision.

Definition 2.2 (Mesh or Norm). Given a subdivision $\mathcal{R} = \{x_0, x_1, \ldots, x_n\}$ on a compact set $[a, b]$, with $a < b$, the mesh or norm of $\mathcal{R}$ is a positive real denoted by $\delta(\mathcal{R})$, and

$$\delta(\mathcal{R}) = \max_{t \in \{1,2,\ldots,n\}} (x_t - x_{t-1}). \tag{2.5}$$

The larger the cardinality of a subdivision, the finer intervals it induces. From this perspective, a subdivision $\mathcal{R}$ can be said to be a refinement of another subdivision $\mathcal{R}'$ if it satisfies the following definition.

Definition 2.3 (Refinement). A subdivision $\mathcal{R}$ on a subset $[a, b] \in \mathbb{R}$, with $a < b$, is said to be the refinement of another subdivision $\mathcal{R}'$ if $\mathcal{R}' \subset \mathcal{R}$, with strict inclusion.

Definition 2.3 leads to the following theorem.

Theorem 2.4 (Meshs and Refinements). *Given two subdivisions $\mathcal{R}$ and $\mathcal{R}'$ on a subset $[a, b] \in \mathbb{R}$, with $a < b$ and $\mathcal{R}$ a refinement of $\mathcal{R}'$, it holds that*

$$\delta(\mathcal{R}) \leqslant \delta(\mathcal{R}'). \tag{2.6}$$

Proof See Homework 1. □

When each of the intervals $[t_{j-1}, t_j]$ in (2.4) is associated with a real q_j, such that $q_j \in [t_{j-1}, t_j]$, a *tagged subdivision* is formed.

Definition 2.5 (Tagged Subdivision). Given an interval $[a, b] \in \mathbb{R}$, with $-\infty < a < b < \infty$, a tagged subdivision is a tuple $(\mathcal{R}, \mathcal{Q})$, where $\mathcal{R}$ is a subdivision on $[a, b]$ of the form $\mathcal{R} = \{x_0, x_1, \ldots, x_n\}$; and $\mathcal{Q} = \{q_1, q_2, \ldots, q_n\}$ is a set such that for all $i \in \{1, 2, \ldots, n\}$,

$$q_i \in [x_{i-1}, x_i]. \tag{2.7}$$

Note that given a tagged subdivision $(\mathcal{R}, \mathcal{Q})$ it holds that $|\mathcal{Q}| = |\mathcal{R}| - 1 = n$, for some integer $n < \infty$.

2.2 Darboux's Integral

The definition of integral proposed by Jean-Gaston Darboux in 1875 (Darboux 1875) arrived after Georg Friedrich Bernhard Riemann had already proposed his own definition in 1854, which was published in 1868 (Riemann 1868). Later in this chapter, it is shown that Darboux's formulation of the integral is a special case of Riemann's general formulation. Nonetheless, through the following sections, it will become evident that Darboux's formulation is easier to treat. More interestingly, it is shown that these two definitions are implications of each other.

In order to introduce the definition of the integral of Darboux, consider first his definitions of lower and upper sums.

Definition 2.6 (Darboux's Sum). Given a bounded function $f : [a, b] \to \mathbb{R}$, with $-\infty < a < b < \infty$, and a subdivision $\mathcal{R} = \{x_0, x_1, \ldots, x_n\}$ on $[a, b]$, the following sums

$$\bar{D}_f(\mathcal{R}) \triangleq \sum_{j=1}^{n} (x_j - x_{j-1}) \left(\sup_{y \in [x_{j-1}, x_j]} f(y) \right) \text{ and} \tag{2.8}$$

$$\underline{D}_f(\mathcal{R}) \triangleq \sum_{j=1}^{n} (x_j - x_{j-1}) \left(\inf_{y \in [x_{j-1}, x_j]} f(y) \right), \tag{2.9}$$

are Darboux's upper and lower sums.

The fact that Darboux's sums are defined over bounded functions, ensures that both $\sup_{y \in [x_{t-1}, x_t]} f(y)$ and $\inf_{y \in [x_{t-1}, x_t]} f(y)$ in (2.8) are finite for all intervals induced by the subdivision $\mathcal{R}$. The following theorem formalizes this observation.

Theorem 2.7. *Given a bounded function $f : [a, b] \to \mathbb{R}$, with $-\infty < a < b < \infty$, and a subdivision $\mathcal{R} = \{x_0, x_1, \ldots, x_n\}$ on $[a, b]$, it follows that*

$$-\infty < (b-a) \left(\inf_{y \in [a,b]} f(y) \right) \leqslant \underline{D}_f(\mathcal{R}) \leqslant \bar{D}_f(\mathcal{R}) \leqslant (b-a) \left(\sup_{y \in [a,b]} f(y) \right) < \infty. \tag{2.10}$$

Proof See Homework 1. □

The lower and upper sums of Darboux exhibit some properties and two of them are central for the definition of Darboux's integral. The first property is on the effect of refinements on the value of the sums.

Theorem 2.8. *Consider a bounded function $f : [a, b] \to \mathbb{R}$, with $-\infty < a < b < \infty$, and let $\mathcal{R}$ and $\mathcal{R}'$ be two subdivisions on $[a, b]$, such that $\mathcal{R}$ is a refinement of $\mathcal{R}'$. Then, it holds that*

$$\underline{D}_f(\mathcal{R}') \leqslant \underline{D}_f(\mathcal{R}) \leqslant \bar{D}_f(\mathcal{R}) \leqslant \bar{D}_f(\mathcal{R}'). \tag{2.11}$$

Proof See Homework 1. □

The second property is on the apparent trivial fact that any upper sum is bigger than any lower sum.

Theorem 2.9. *Consider a bounded function $f : [a, b] \to \mathbb{R}$, with $-\infty < a < b < \infty$, and let $\mathcal{R}$ and $\mathcal{R}'$ be any two arbitrary subdivisions on $[a, b]$. Then,* *it holds that*

$$\underline{D}_f(\mathcal{R}) \leqslant \bar{D}_f(\mathcal{R}'). \tag{2.12}$$

Proof See Homework 1. □

Theorem 2.8 implies that by further dividing the interval $[a, b]$ through refinements of the initial subdivision, the Darboux's lower and upper sums approach to each other. Alternatively, Theorem 2.9 establishes that any Darboux's upper sum is not smaller that any Darboux's lower sum.

Some functions $f : [a, b] \to \mathbb{R}$ satisfy that the lower and upper Darboux's sums $\underline{D}_f(\mathcal{R})$ and $\bar{D}_f(\mathcal{R})$ can be made arbitrarily close for some relatively coarse subdivisions. These functions form a special class of real-valued functions known as *elementary simple functions.*

Definition 2.10 (Elementary Simple Functions). Consider a function $f : [a, b] \to \mathbb{R}$, with $-\infty < a < b < \infty$, for which there exists at least one partition of $[a, b]$ formed by elementary sets $\mathcal{A}_1$, $\mathcal{A}_2$, ..., $\mathcal{A}_m$, with $m < \infty$ and for all $x \in [a, b]$,

$$f(x) = \sum_{t=1}^{m} a_t \mathbb{1}_{\{x \in \mathcal{A}_t\}}, \tag{2.13}$$

where for all $i \in \{1, 2, \ldots, m\}$, $a_i \in \mathbb{R}$. Then, the function f is said to be an elementary simple function.

Elementary simple functions exhibit an important property.

Theorem 2.11. *Let* $f : [a, b] \to \mathbb{R}$*, with* $-\infty < a < b < \infty$*, be an elementary simple function of the form in* (2.13)*. Then, for all* $\epsilon > 0$*, there always exists a subdivision* $\mathcal{R} = \{x_0, x_1, \ldots, x_n\}$ *on* $[a, b]$*, with* $n < \infty$*, such that*

$$\bar{D}_f(\mathcal{R}) - \underline{D}_f(\mathcal{R}) < \epsilon. \tag{2.14}$$

Proof See Homework 1. □

The definition of Darboux's integral is introduced in terms of both the *Darboux's lower integral* and *Darboux's upper integral.*

Definition 2.12. Given a bounded function $f : [a, b] \to \mathbb{R}$, with $-\infty < a < b < \infty$, the values

$$\bar{D}_f^* \triangleq \inf\{\bar{D}_f(\mathcal{R}) : \mathcal{R} \text{ is a partition on } [a, b]\} \text{ and} \tag{2.15}$$

$$\underline{D}_f^* \triangleq \sup\{\underline{D}_f(\mathcal{R}) : \mathcal{R} \text{ is a partition on } [a, b]\}, \tag{2.16}$$

where the supremum and the infimum are with respect to all subdivisions on $[a, b]$, are respectively the upper and lower integrals of Darboux.

Using Definition 2.12, Darboux's integral of the function $f : [a, b] \to \mathbb{R}$, which is denoted by

$$\int_a^b f(x)\mathrm{d}x, \tag{2.17}$$

can be defined as follows.

DEFINITION 2.13. Given a bounded function $f : [a, b] \to \mathbb{R}$, with $-\infty < a < b < \infty$, for which

$$\underline{D}_f^* = \bar{D}_f^*, \tag{2.18}$$

the Darboux's integral is

$$\int_a^b f(x)\mathrm{d}x \triangleq \underline{D}_f^* = \bar{D}_f^*. \tag{2.19}$$

The condition $\underline{D}_f^* = \bar{D}_f^*$ is known as *integrability condition.* The functions that satisfy Darboux's integrability condition are said to be *integrable functions* in the sense of Darboux. A more formal definition of integrability is the following.

DEFINITION 2.14. A function $f : [a, b] \to \mathbb{R}$, with $-\infty < a < b < \infty$, is said to be integrable in the sense of Darboux if for all $\epsilon > 0$, there exists a subdivision $\mathcal{R}$ such that

$$\bar{D}_f(\mathcal{R}) - \underline{D}_f(\mathcal{R}) < \epsilon. \tag{2.20}$$

From Definition 2.14 and Theorem 2.11, the following corollary formalizes an important observation.

COROLLARY 2.15. *All elementary simple functions are integrable in the sense of Darboux.*

Elementary simple functions play a central role in the construction and analysis of the notion of integral. The following theorem shows that all integrable functions can be approximated with arbitrary precision by elementary simple functions.

THEOREM 2.16. *Given a bounded function* $f : [a, b] \to \mathbb{R}$, *with* $-\infty < a < b < \infty$, *and a subdivision* $\mathcal{R} = \{x_0, x_1, \ldots, x_m\}$ *on* $[a, b]$, *with* $m < \infty$, *the following functions*

$$\bar{f}_{\mathcal{R}}(x) = \sum_{t=1}^{m} \mathbb{1}_{\{x \in [x_{t-1}, x_t[\}} \sup_{y \in [x_{t-1}, x_t[} f(y), \text{ and} \tag{2.21}$$

$$\underline{f}_{\mathcal{R}}(x) = \sum_{t=1}^{m} \mathbb{1}_{\{x \in [x_{t-1}, x_t[\}} \inf_{y \in [x_{t-1}, x_t[} f(y), \tag{2.22}$$

satisfy the following inequalities for all $x \in [a, b]$,

$$\underline{f}_{\mathcal{R}}(x) \leqslant f(x) \leqslant \bar{f}_{\mathcal{R}}(x). \tag{2.23}$$

Moreover, for all $\epsilon > 0$, there exists a subdivision $\mathcal{R}'$ such that:

$$\int_a^b \bar{f}_{\mathcal{R}'}(x)\mathrm{d}x - \int_a^b \underline{f}_{\mathcal{R}'}(x)\mathrm{d}x < \epsilon. \tag{2.24}$$

Proof See Homework 1. □

Using Theorem 2.16, Darboux's lower and upper integrals of an arbitrary bounded function f can be alternatively defined in terms of the Darboux's lower and upper integrals of elementary simple functions that approximate f.

$$\underline{D}_f^* = \sup\left\{\int_a^b g(x)\mathrm{d}x \in \mathbb{R} : g \text{ is elementary simple, } \forall x \in [a,b],\ g(x) \leqslant f(x)\right\}; \tag{2.25}$$

and

$$\bar{D}_f^* = \inf\left\{\int_a^b g(x)\mathrm{d}x \in \mathbb{R} : g \text{ is elementary simple, } \forall x \in [a,b],\ f(x) \leqslant g(x)\right\}. \tag{2.26}$$

The equalities in (2.25) and (2.26) lead to an interpretation of Darboux's lower and upper integrals of f in terms of the area under the curve of f. This interpretation is developed in the following section.

2.2.1 Area under the Curve

This section shows that if the Darboux integral of a positive function exists, its value is identical to the area under its curve. In the case of arbitrary functions, it is shown that if the Darboux integral exists, then its value is exactly the difference between two values that can be associated to the areas under the curve of certain functions.

Case of Positive Elementary Simple Functions

Corollary 2.15 leads to an insightful connection between the area under the curve of an elementary simple function and its Darboux's integral. Consider an elementary simple function $f : [a, b] \to \mathbb{R}$, with $-\infty < a < b < \infty$, such that for all $x \in [a, b]$,

$$f(x) = \sum_{t=1}^{m} a_t \mathbb{1}_{\{x \in \mathcal{A}_t\}}, \tag{2.27}$$

where for all $i \in \{1, 2, \ldots, m\}$, with $m < \infty$, $0 \leqslant a_i < \infty$; and $\mathcal{A}_1$, $\mathcal{A}_2$, ..., $\mathcal{A}_m$ form a partition of $[a, b]$. Without any loss of generality, assume that $\mathcal{A}_1$, $\mathcal{A}_2$,

..., $\mathcal{A}_m$ are convex intervals. Consider also a subdivision $\mathcal{R} = \{x_0, x_1, \ldots, x_p\}$ on $[a, b]$, with $m < p < \infty$ satisfying the following conditions. For all $i \in \{1, 2, \ldots, m\}$, $\max\{a, \inf A_i - \delta\} \in \mathcal{R}$ and $\min\{b, \sup A_i + \delta\} \in \mathcal{R}$, with $\delta > 0$ being a constant chosen arbitrarily small. Hence, Darboux's integral satisfies the following:

$$\left| \int_a^b f(x)\mathrm{d}x - \sum_{t=1}^{p} (x_t - x_{t-1}) f\left(\frac{x_t + x_{t-1}}{2}\right) \right| < 2p\delta \sup_{y\in[a,b]} f(y). \tag{2.28}$$

where for all $j \in \{1, 2, \ldots, p\}$, $f\left(\frac{x_j + x_{j-1}}{2}\right) = a_{s_j}$, for some $s_j \in \{1, 2, \ldots, m\}$. Therefore, $\sum_{t=1}^{p}(x_t - x_{t-1})a_{s_t}$ is the sum of the areas of p rectangles. Note that for all $t \in \{1, 2, \ldots, p\}$, the t-th rectangle has a base of length $(x_t - x_{t-1})$ and height a_{s_t}. Note that since δ in (2.28) can be chosen arbitrarily small, the area of the rectangles whose base is 2δ have limited impact in the value of Darboux's integral. Hence, Darboux integral of a non-negative elementary simple function $f : [a, b] \to \mathbb{R}$, with $-\infty < a < b < \infty$, corresponds exactly to the area of the following set in $\mathbb{R}^2$,

$$\{(x, y) \in \mathbb{R}^2 : a \leqslant x \leqslant b, 0 \leqslant y \leqslant f(x)\}, \tag{2.29}$$

which is the area under the curve of f.

Case of Elementary Simple Functions

Following the same order of ideas, the same analysis holds for arbitrary elementary simple functions. In this case, the Darboux integral of a elementary simple function $f : [a, b] \to \mathbb{R}$, with $-\infty < a < b < \infty$, corresponds exactly to the difference of the areas of the following sets in $\mathbb{R}^2$,

$$\left\{(x, y) \in \mathbb{R}^2 : a \leqslant x \leqslant b, 0 \leqslant y \leqslant \max\{0, f(x)\}\right\} \text{ and} \tag{2.30}$$

$$\left\{(x, y) \in \mathbb{R}^2 : a \leqslant x \leqslant b, \min\{0, f(x)\} \leqslant y \leqslant 0\right\}. \tag{2.31}$$

To formalize this observation, consider the positive and negative parts of a given real-valued function f.

Definition 2.17. Given a real-valued function $f : [a, b] \to \mathbb{R}$, with $-\infty < a < b < \infty$, its positive part and negative part are non-negative functions respectively denoted by $f^+ : [a, b] \to \mathbb{R}_+$ and $f^- : [a, b] \to \mathbb{R}_+$ such that for all $x \in [a, b]$,

$$f^+(x) \triangleq \max\{f(x), 0\} \text{ and} \tag{2.32}$$

$$f^-(x) \triangleq -\min\{f(x), 0\}. \tag{2.33}$$

Using Definition 2.17, the discussion above can be trivially formalized as follows.

Theorem 2.18. *Let $f : [a, b] \to \mathbb{R}$, with $-\infty < a < b < \infty$, be an elemen-*

tary simple function. Then, the positive and negative parts of f satisfy

$$\int_a^b f(x)\mathrm{d}x = \int_a^b f^+(x)\mathrm{d}x - \int_a^b f^-(x)\mathrm{d}x. \tag{2.34}$$

Proof See Homework 1. □

Note that both f^- and f^+ in (2.34) non-negative elementay simple functions whose integrals are respectively the areas of the sets in (2.30) and in (2.31).

Case of Positive Bounded Functions

The connections between Darboux's integral and the area under the curve can be extended to the general case of bounded functions. Consider first the case in which the function f in Theorem 2.16 is positive. Hence, the functions $\underline{f}_{\mathcal{R}}$ and $\bar{f}_{\mathcal{R}}$ in (2.16) are both positive elementary simple functions and thus integrable. From the assumption that f is positive and integrable, it follows that

$$\int_a^b \underline{f}_{\mathcal{R}}(x)\mathrm{d}x \leqslant \int_a^b f(x)\mathrm{d}x \leqslant \int_a^b \bar{f}_{\mathcal{R}}(x)\mathrm{d}x. \tag{2.35}$$

In this case, the inequalities in (2.35) imply that the Darboux integral of the function f is: (a) lower bounded by the integral of a simple function $\underline{f}_{\mathcal{R}}$ that is always smaller than f; and (b) upper bounded by the integral of a simple function $\bar{f}_{\mathcal{R}}$ that is always bigger than f. Hence, Darboux's integral of a positive bounded function is a positive real number lower bounded by the area under the curve of $\underline{f}_{\mathcal{R}}$; and upper bounded by the area under the curve of $\bar{f}_{\mathcal{R}}$. From Theorem 2.16, it follows that the subdivision $\mathcal{R}$ in (2.35) can be refined such that the area under the curve of $\underline{f}_{\mathcal{R}}$ and $\bar{f}_{\mathcal{R}}$ are arbitrarily close to each other. This implies that the Darboux's integral of a positive bounded function, if it exists, it is equivalent to the area under the curve of such function.

Case of Bounded Functions

Using Definition 2.17, the discussion above can be trivially extended to the case of arbitrary bounded functions.

THEOREM 2.19. *Let $f : [a, b] \to \mathbb{R}$, with $-\infty < a < b < \infty$, be a bounded function, integrable in the sense of Darboux. Then, the positive and negative parts of f satisfy*

$$\int_a^b f(x)\mathrm{d}x = \int_a^b f^+(x)\mathrm{d}x - \int_a^b f^-(x)\mathrm{d}x. \tag{2.36}$$

Proof See Homework 1. □

From Theorem (2.19), it holds that the Darboux integral of an arbitrary bounded function is the difference between the areas under the curve of its corresponding positive and negative parts.

2.2.2 Geometric Interpretation of Darboux's Integrability

2.3 Riemann's Integral

The Riemman's integral, as introduced in (Riemann 1868), is defined in terms of Riemman's sums. In contrast to Darboux's sums, which uses subdivisions, the Riemann's sum uses tagged subdivisions.

Definition 2.20 (Riemann's Sum). Given a bounded function $f : [a, b] \rightarrow \mathbb{R}$ and a tagged subdivision $(\mathcal{R}, \mathcal{Q})$ on $[a, b]$, with $-\infty < a < b < \infty$; $\mathcal{R} = \{x_0, x_1, \ldots, x_n\}$; and $\mathcal{Q} = \{q_1, q_2, \ldots, q_n\}$, the following sum

$$R_f\left(\mathcal{R}, \mathcal{Q}\right) \triangleq \sum_{j=1}^{n} (x_{j-1} - x_j) f(q_j), \tag{2.37}$$

is a Riemann's sum.

Riemann's sums are defined over bounded functions, which ensures that given a tagged subdivision $(\mathcal{R}, \mathcal{Q})$ on $[a, b]$, it holds that for all $q \in \mathcal{Q}$ in Definition 2.20, $f(q) < \infty$. More precisely, for all $t \in \{1, 2, \ldots, n\}$, it holds that

$$\inf_{y \in [x_{t-1}, x_t]} f(y) \leqslant f(q_t) \leqslant \sup_{y \in [x_{t-1}, x_t]} f(y). \tag{2.38}$$

The inequalities in (2.38) suggest that the Riemann's sum in (2.37) would be a Darboux's lower or upper sum if the tagged subdivision is such that for all $i \in \{1, 2, \ldots, n\}$,

$$q_i = \arg \inf_{y \in [x_{t-1}, x_t]} f(y), \text{ or} \tag{2.39}$$

$$q_i = \arg \sup_{y \in [x_{t-1}, x_t]} f(y), \tag{2.40}$$

respectively. Nonetheless, the implications of this observation are more profound.

Theorem 2.21 (Sums of Darboux and Riemann). *Let $f : [a, b] \rightarrow \mathbb{R}$, with $-\infty < a < b < \infty$, be a bounded function and let $\mathcal{R} = \{x_0, x_1, \ldots, x_n\}$ be a subdivision on $[a, b]$. Then, for all tagged subdivisions of the form $(\mathcal{R}, \mathcal{Q})$, it holds that*

$$\underline{D}_f(\mathcal{R}) \leqslant R_f\left(\mathcal{R}, \mathcal{Q}\right) \leqslant \bar{D}_f(\mathcal{R}). \tag{2.41}$$

Proof See Homework 1. □

The Riemman's integral of a bounded function $f : [a, b] \rightarrow \mathbb{R}$, with $-\infty < a < b < \infty$, which is denoted by

$$\int_a^b f(x) \mathrm{d}x, \tag{2.42}$$

is defined as the limit, if it exists, of the Riemann's sum $R_f\left(\mathcal{R}, \mathcal{Q}\right)$ when the mesh of $\mathcal{R}$ tends to zero.

Definition 2.22 (Riemann's Integral). Given a bounded function $f : [a, b] \to \mathbb{R}$, with $-\infty < a < b < \infty$, its Riemann's integral is

$$\int_a^b f(x)\mathrm{d}x \triangleq \lim_{\delta(\mathcal{R})\to 0} R_f(\mathcal{R}, \mathcal{Q}), \tag{2.43}$$

when the limit exists.

From Definition 2.22, it follows that a condition for the existence of the integral is the existence of the limit in (2.43). A function for which such a limit exists is said to be integrable in the sense of Riemann, which leads to the following definition.

Definition 2.23 (Integrability). Given a bounded function $f : [a, b] \to \mathbb{R}$, with $-\infty < a < b < \infty$, it is said to be integrable in the sense of Riemann, with integral equal to $\int_a^b f(x)\mathrm{d}x$, if and only if for all $\epsilon > 0$, there exist a tagged subdivision $(\mathcal{R}, \mathcal{Q})$ such that

$$\left| \int_a^b f(x)\mathrm{d}x - R_f(\mathcal{R}, \mathcal{Q}) \right| < \epsilon. \tag{2.44}$$

Given that by adopting certain choices in the definition of the tagged subdivision in a Riemann's sum, Darboux's lower and upper sums can be obtained as special cases, Riemann's sums can be seen as a more general definition. Nonetheless, the notion of Riemann's integral is not more general than Darboux integral. More interestingly, the existence of either integral implies the existence of the other. When they exist, the values of these integrals are identical. The following theorem formalizes these statements.

Theorem 2.24. *Consider the function $f : [a, b] \to \mathbb{R}$, with $-\infty < a < b < \infty$. Then, the following statements are equivalent:*
(i) The function f is integrable in the sense of Riemman;
(ii) The function f is integrable in the sense of Darboux; and
(iii) $\int_a^b f(x)\mathrm{d}x = \underline{D}_f^* = \bar{D}_f^* = \lim_{\delta(\mathcal{R})\to 0} R_f(\mathcal{R}, \mathcal{Q})$.

Proof See Homework 1. □

2.4 Riemann Integrable Functions

In the previous section, it was shown that elementary simple functions are integrable in the sense of Darboux, and thus, are integrable in the sense of Riemann as well. Given that Darboux's integrability and Riemann's integrability conditions have been shown to be identical, in the following, the distinction between these integrals is dropped. This said, the following theorems introduce more general classes of functions that satisfy the integrability condition.

Theorem 2.25. *Let $f : [a, b] \to \mathbb{R}$, with $-\infty < a < b < \infty$, be a continuous function. Then, the function f is integrable.*

Proof See Homework 1. □

The result in Theorem 2.25 can be generalized to functions that are piece-wise continuous. These functions are defined hereunder.

Definition 2.26 (Piece-wise Continuity). Given a function $f : [a, b] \to \mathbb{R}$, with $-\infty < a < b < \infty$, it is said to be piece-wise continuous if there exists a subdivision $\mathcal{R} = \{x_0, x_1, \ldots, x_m\}$ on $[a, b]$, with $m < \infty$, such that for all $t \in \{1, 2, \ldots, m\}$, the restriction of f on the interval $]x_{t-1}, x_t[$ can be extended to a continuous function on $[x_{t-1}, x_t]$.

Theorem 2.27. *Let $f : [a, b] \to \mathbb{R}$, with $-\infty < a < b < \infty$, be a piece-wise continuous function. Then, the function f is integrable.*

Proof See Homework 1. □

The continuity condition for integrability can be replaced by a milder condition on the shape of the function. The following definitions introduce some conditions on the shape of functions.

Definition 2.28 (Increasing and decreasing functions). Given a function $f : [a, b] \to \mathbb{R}$, with $a < b$, it is said to be increasing or decreasing if for all pairs $(x_1, x_2) \in [a, b]^2$, with $x_1 < x_2$, it holds that $f(x_1) < f(x_2)$, or $f(x_1) > f(x_2)$, respectively.

Denominations of the form *nondecreasing* or *nonincreasing* are often used. The former refers to functions for which given a pair $(x_1, x_2) \in [a, b]^2$, with $x_1 < x_2$, it holds that $f(x_1) \leqslant f(x_2)$, whereas the latter, refers to functions for which $f(x_1) \geqslant f(x_2)$.

Definition 2.29 (Monotonic functions). A function $f : [a, b] \to \mathbb{R}$, with $a < b$, is said to be monotonic if it is either increasing or decreasing.

The integral of monotonic functions exhibits the following property.

Theorem 2.30. *Let $f : [a, b] \to \mathbb{R}$, with $-\infty < a < b < \infty$, be a monotonic function. Then, the function f is integrable.*

Proof See Homework 1. □

The composition of integrable functions is integrable under certain conditions.

THEOREM 2.31. *Let $f : [a,b] \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$, with $-\infty < a < b < \infty$, be an integrable function and a continuous function, respectively. Then, the composition $g \circ f$ is integrable on $[a,b]$.*

Proof See Homework 1. □

2.5 Properties of Riemann's Integral

THEOREM 2.32. *Let $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$, with $-\infty < a < b < \infty$, be two integrable functions. Then, the following holds:*

- *For all $c \in \mathbb{R}$,*

$$\int_a^b cf(x)\mathrm{d}x = c\int_a^b f(x)\mathrm{d}x; \text{ and} \tag{2.45}$$

$$\int_a^b f(x) + g(x)\mathrm{d}x = \int_a^b f(x)\mathrm{d}x + \int_a^b g(x)\mathrm{d}x. \tag{2.46}$$

- *If for all $x \in [a,b]$, $f(x) \leqslant g(x)$, then*

$$\int_a^b f(x)\mathrm{d}x \leqslant \int_a^b g(x)\mathrm{d}x. \tag{2.47}$$

- *For all $c \in [a,b]$,*

$$\int_c^c f(x)\mathrm{d}x = 0; \text{ and} \tag{2.48}$$

$$\int_a^b f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x. \tag{2.49}$$

Proof See Homework 1. □

2.6 The Extended Real Numbers

In the previous sections, the integral has been defined only for bounded functions defined within a compact interval. In Section 2.2.1, it was argued that the integral of the function $f : [a,b] \to \mathbb{R}$ can be approximated by the integral of an elementary simple function that approximates the function f. This approximation, which can be made arbitrarily precise for all integrable functions, highlights the fact that the integral is a sum of infinitesimally small signed areas of rectangles. The base of those rectangles is defined by the partition on $[a,b]$ that defines the elementary simple function, whereas, their heights are determined by the value of the function f. Their areas are said to be signed as they carry the positive or negative sign of the corresponding values of f. In the case in which

f is unbounded, there might be at least one rectangle whose area is arbitrarily large. This translates into a sum of some finite numbers and some numbers that are arbitrarily away from zero. To study these sums, the set of real numbers must be equipped with some additional elements.

In previous sections, the symbols used to identify the real numbers whose absolute values are arbitrarily large are $-\infty$ and $+\infty$. That is, for all $x \in \mathbb{R}$, $-\infty < x < +\infty$, with strict inequalities. Hence, the set of extended real numbers, denoted by $\bar{\mathbb{R}}$, is defined as follows:

$$\bar{\mathbb{R}} \triangleq \mathbb{R} \cup \{+\infty, -\infty\}. \tag{2.50}$$

This said, for all $a \in \mathbb{R}$, the intervals $[a, +\infty]$, $]a, +\infty]$, $[-\infty, a]$, and $[-\infty, a[$ are proper subsets of $\bar{\mathbb{R}}$, and $\bar{\mathbb{R}} = [-\infty, +\infty]$. The new elements, $-\infty$ and $+\infty$, are adopted under the following assumptions, for all $a \in \mathbb{R}$:

$$a \,+\, +\infty = +\infty \,+\, a = +\infty; \text{ and} \tag{2.51}$$

$$a \,+\, -\infty = -\infty \,+\, a = -\infty. \tag{2.52}$$

Moreover, for all $a \in \mathbb{R} \setminus \{0\}$,

$$|a| \,\cdot\, +\infty = +\infty \,\cdot\, |a| = +\infty; \tag{2.53}$$

$$-|a| \,\cdot\, +\infty = +\infty \,\cdot\, -|a| = -\infty; \tag{2.54}$$

$$|a| \,\cdot\, -\infty = -\infty \,\cdot\, |a| = -\infty; \tag{2.55}$$

$$-|a| \,\cdot\, -\infty = -\infty \,\cdot\, -|a| = \infty; \tag{2.56}$$

$$0 \,\cdot\, +\infty = +\infty \,\cdot\, 0 = 0; \text{ and} \tag{2.57}$$

$$0 \,\cdot\, -\infty = -\infty \,\cdot\, 0 = 0. \tag{2.58}$$

Despite these rules, some operations remain undetermined. For instance, $+\infty - +\infty$, $-\infty + +\infty$, $+\infty + -\infty$, $\frac{+\infty}{+\infty}$, $\frac{-\infty}{-\infty}$, $\frac{-\infty}{+\infty}$ and $\frac{+\infty}{-\infty}$ are undetermined quantities. These mathematical indeterminations constraint conclusions that are obvious when dealing with finite real numbers. For instance, nothing can be concluded from the inequality $a + +\infty < b + +\infty$ in terms of either $a < b$ or $a > b$.

This said, the integral of a positive unbounded function can be zero or $+\infty$. The former arises when the set $\mathcal{A} = f^{-1}(+\infty) \subset \mathbb{R}$ is finite. This is explained from the fact that all those rectangles whose height is arbitrarily large have a base of length zero. Thus, given that $0 \cdot +\infty = 0$, the areas of those rectangles do not have any impact in the sum of areas, which remains finite. The latter arises when $\mathcal{A} = f^{-1}(+\infty) \subset \mathbb{R}$ and $\text{int}\mathcal{A}$ is not empty. In this case, there must be at least one rectangle whose height is $+\infty$ and whose base is bounded away from zero. In this case, the sum contains at least one term that is $+\infty$, and thus, the integral is $+\infty$. In either case, the integral is said to exist, even if it is equal to ∞.

Alternatively, arbitrary functions $f : [a, b] \to \mathbb{R}$ for which their integral is the difference of the integral of f^+ and the integral of f^- (Theorem 2.19), the result might be an undetermined form, e.g., $-\infty + +\infty$ or $+\infty + -\infty$. In this case, the integral is said not to exist.

In a nutshell, the integral of positive functions always exists. This is an immediate consequence of the following observation. Given a (countable or uncountable) set $\mathcal{A}$, for all $\alpha \in \mathcal{A}$, let $x_\alpha \in [0, +\infty]$ be fixed. Hence, it follows that

$$\sum_{\alpha \in \mathcal{A}} x_\alpha \in [0, +\infty]. \tag{2.59}$$

More interestingly, the sum can be reordered and it does not affect the result. That is, let $f : \mathcal{B} \to \mathcal{A}$ be a bijective function. Then, the following holds:

$$\sum_{\beta \in \mathcal{B}} x_{f(\beta)} = \sum_{\alpha \in \mathcal{A}} x_\alpha. \tag{2.60}$$

The equality in (2.60) does not necessarily hold if the condition $x_\alpha \in [0, +\infty]$ is replaced by $x_\alpha \in \mathbb{R}$, for all $\alpha \in \mathcal{A}$, when $|\mathcal{A}| = \infty$. The following theorem, due to Riemann, formalizes this observation.

Theorem 2.33 (Riemann's reordering theorem). *Let $a_1, a_2, \ldots$ be a sequence of real numbers such that*

$$-\infty < \sum_{t=1}^{+\infty} a_t < +\infty, \text{ and} \tag{2.61}$$

$$\sum_{t=1}^{+\infty} |a_t| = +\infty. \tag{2.62}$$

Then, for all $M \in \bar{\mathbb{R}}$, there always exists a bijection $f : \mathbb{N} \to \mathbb{N}$, such that

$$\sum_{t=1}^{+\infty} a_{f(t)} = M. \tag{2.63}$$

Proof See Homework 1. □

Under the conditions of Theorem 2.33, rearrangements of the sum of signed reals cannot be associated to a unique value. Therefore, in this case the sum is undetermined. This observation plays a fundamental role in the case in which integration is over functions whose domains are not compact, e.g., intervals of the form $]-\infty, a]$, with $a \in \bar{\mathbb{R}}$. These cases are discussed further in this chapter.

2.7 Limitations of Riemann Integral

2.8 Henstock–Kurzweil Integral

2.9 The Problem of Measure

Consider an increasing continuous function $f : [a, b] \to [\alpha, \beta]$, with $-\infty < a < b < +\infty$ and $-\infty < \alpha < \beta < +\infty$, and note that any subdivision

$$\mathcal{X} = \{x_0, x_1, \ldots, x_n\} \tag{2.64}$$

on $[a, b]$, with $n < \infty$, induces a subdivision

$$\mathcal{Z} = \{z_0, z_1, \ldots, z_n\} \tag{2.65}$$

on $[\alpha, \beta]$, such that for all $t \in \{0, 1, 2, \ldots, n\}$, $z_t = f(x_t)$. Hence, from the assumption that f is increasing, it holds that:

$$a = x_0 < x_1 < \ldots < x_n = b \tag{2.66}$$

$$\alpha = z_0 < z_1 < \ldots < z_n = \beta. \tag{2.67}$$

The function f is integrable in the sense of Riemann if the Darboux lower and upper integrals,

$$\sum_{t=1}^{n} z_{t-1}(x_t - x_{t-1}) \text{ and } \sum_{t=1}^{n} z_t(x_t - x_{t-1}), \tag{2.68}$$

become identical as the mesh of the subdivision $\mathcal{X}$ or $\mathcal{Z}$ tends to zero. Considering that the mesh of $\mathcal{X}$ tends to zero was the idea used to define both Riemann and Darboux integrals. Alternatively, considering that the mesh of $\mathcal{Z}$ tends to zero is rather a new approach, which implies that the area under the curve can also be calculated by subdividing the y-axis instead of the x-axis.

Note that the intervals $[z_0, z_1]$, $[z_1, z_2]$, ..., $[z_{n-1}, z_n]$, which form an exact cover of $[\alpha, \beta]$, induce an exact cover on $[a, b]$ formed by the sets

$$f^{-1}([z_{t-1}, z_t]) = \{x \in [a, b] : z_{t-1} \leqslant f(x) \leqslant z_t\} \tag{2.69}$$

$$\triangleq [x_{t-1}, x_t], \tag{2.70}$$

with $x_t \triangleq f^{-1}(z_t)$ and $t \in \{1, 2, \ldots, n\}$.

Using this notation, two sequences of n rectangles are formed. In the first sequence, the t-th rectangle has height z_t and base $(x_t - x_{t-1})$. In the second sequence, the t-th rectangle has height z_{t-1} and base $(x_t - x_{t-1})$. The area under the curve, and thus the integral, is the limit of the Darboux's lower and upper sums in (2.68) when the mesh of the subdivision $\mathcal{Z}$ tends to zero. In this case, the base of the rectangles was easy to identify thanks to the assumptions that the function f was continuous, bounded and increasing. Essentially, the base of the rectangles induced by the subdivision $\mathcal{Z}$ on $[\alpha, \beta]$ is equal to the length of the closed and convex intervals $[x_{t-1}, x_t]$, e.g., $(x_t - x_{t-1})$.

When the assumption that the function f is increasing is dropped, the sets $f^{-1}([z_{t-1}, z_t])$ are such that it is less clear how to assume the base of the rectangles to calculate the areas under the curve.

Consider for instance the case of a concave function $f : [-1, 1] \to [0, 1]$ for which for all $x \in [-1, 1]$, $f(x) = x^2$. Then, for any arbitrary subdivision $\mathcal{Z} = \{z_0, z_1, \ldots, z_n\}$ of the interval $[0, 1]$, it holds for all $t \in \{1, 2, \ldots, n\}$,

$$\begin{aligned} f^{-1}([z_{t-1}, z_t]) &= \{x \in [a, b] : z_{t-1} \leqslant f(x) \leqslant z_t\} && (2.71) \\ &\triangleq [-\sqrt{z_t}, -\sqrt{z_{t-1}}] \cup [\sqrt{z_{t-1}}, \sqrt{z_t}]. && (2.72) \end{aligned}$$

In this case, each interval $[z_{t-1}, z_t]$ induces four vertical rectangles. Two rectangles, one located on the interval $[-\sqrt{z_t}, -\sqrt{z_{t-1}}]$ and one on the interval $[\sqrt{z_{t-1}}, \sqrt{z_t}]$, have heights equal to z_{t-1} and bases equal to $\sqrt{z_t} - \sqrt{z_{t-1}}$. The other two rectangles, whose locations are identical to the previous ones, have heights equal to z_t and bases equal to $\sqrt{z_t} - \sqrt{z_{t-1}}$. Darboux's lower and upper sums can be written in the following form:

$$\sum_{t=1}^{n} 2\left(\sqrt{z_t} - \sqrt{z_{t-1}}\right) z_{t-1} \quad \text{and} \quad \sum_{t=1}^{n} 2\left(\sqrt{z_t} - \sqrt{z_{t-1}}\right) z_t, \qquad (2.73)$$

respectively. The limit of both sums as the mesh of the subdivision $\mathcal{Z}$ tends to zero exist and are identical to the Riemann integral. Intuitively, Darboux lower sum in (2.73) can be interpreted as the sum of the areas of n rectangles whose bases are $2\left(\sqrt{z_1} - \sqrt{z_0}\right)$, $2\left(\sqrt{z_2} - \sqrt{z_1}\right)$, ..., $2\left(\sqrt{z_{n-1}} - \sqrt{z_n}\right)$ and their heights are z_0, z_1, ..., z_{n-1}, respectively. A similar observation can be made for Darboux upper sum in (2.73).

In a nutshell, the set $f^{-1}([z_{t-1}, z_t])$ is the union of two compact sets, which are either disjoint or almost-disjoint sets, and thus, the base of the vertical rectangle can be assumed to be the sum of the lengths of these two intervals. This justifies the factor of two in (2.73).

Generalizing this method to arbitrary functions leads to the problem of finding the corresponding length of the basis of the vertical rectangles induced by a subdivision of the range of a function f, which can be $]-\infty, +\infty[$. This is essentially equivalent to finding the length of the intervals $f^{-1}([z_{t-1}, z_t])$ induced by $\mathcal{Z}$. This problem is an instance of the *problem of measure*, which consists in associating a positive real to intervals of $\mathbb{R}$. This positive real is the measure of the set. This problem was formulated by Henri Lebesgue building on previous works on measures, namely by Camille Jordan and Émile Borel. This said, the problem of measure can be formulated for any type of sets, not necessarily subsets of $\mathbb{R}^n$, with $n \in \mathbb{N}$, but any kind of sets.

2.10 Jordan Measure and Jordan Measurable Sets

The Jordan measure applies to bounded subsets in $\mathbb{R}^n$, with $n \in \mathbb{N}$, that can be approximated by elementary sets with arbitrary precision. The Jordan measure of an elementary set, which can be expressed as a finite union of almost disjoint boxes, is the sum of the volumes of such composing boxes.

Let $\mathscr{E}_n$ denote the set of all elementary sets in $\mathbb{R}^n$. Hence, the volume of elementary sets can be measured as follows. Let $v_n : \mathscr{E}_n \to \mathbb{R}$ be a positive function, such that for all bounded generic boxes $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2 \times \ldots \times \mathcal{B}_n \in \mathscr{E}_n$, with $\mathcal{B}_t \subset \mathbb{R}$ an interval of the form $[a_t, b_t]$, $]a_t, b_t[$, $[a_t, b_t[$, or $]a_t, b_t]$ and $-\infty < a_t \leqslant b_t < +\infty$, for all $t \in \{1, 2, \ldots, n\}$, it holds that

$$v_n(\mathcal{B}) = \prod_{t=1}^{n}(b_t - a_t), \tag{2.74}$$

which is the volume of $\mathcal{B}$. In the case of elementary sets, it holds from Definition 1.26 that for all $\mathcal{E} \in \mathscr{E}_n$, there always exists a finite partition of $\mathcal{E}$ formed by the sets $\mathcal{A}_1$, $\mathcal{A}_2$, …, $\mathcal{A}_k$, with $k \in \mathbb{N}$. Thus,

$$v_n(\mathcal{E}) = \sum_{t=1}^{k} v_n(\mathcal{A}_t), \tag{2.75}$$

which is the volume of $\mathcal{E}$. When, the argument of the function v_n is the empty set, it holds that:

$$v_n(\emptyset) = 0. \tag{2.76}$$

In a nutshell, the volume of a generic box $\mathcal{A}$ is the product of the lengths of its sides. The volume of an elementary set $\mathcal{A}$ is the sum of the volumes of its composing generic boxes. And finally, the empty set has volume zero. It is interesting to note that an elementary set can be formed by infinitely many different finite collections of disjoint boxes. That is, two different finite collections of disjoint boxes might be such that their unions form the same set. Nonetheless, the sum of the volumes of such sets is identical to the volume of the set they form.

Theorem 2.34. *Let $\mathcal{E}_1$, $\mathcal{E}_2$, …, $\mathcal{E}_k$ and $\mathcal{D}_1$, $\mathcal{D}_2$, …, $\mathcal{D}_p$ be two different finite sequences of disjoint boxes of $\mathbb{R}^n$, with $(k, n, p) \in \mathbb{N}^3$, such that*

$$\mathcal{A} = \bigcup_{t=1}^{k} \mathcal{E}_t = \bigcup_{t=1}^{p} \mathcal{D}_t. \tag{2.77}$$

Then, the following holds,

$$v_n(\mathcal{A}) = \sum_{t=1}^{k} v_n(\mathcal{D}_t) = \sum_{t=1}^{p} v_n(\mathcal{E}_t). \tag{2.78}$$

Proof See Homework 1. □

Essentially, Theorem 2.34 states that the volume of an elementary set is independent of the boxes that are chosen to measure it.

These observations lead to the following properties:

THEOREM 2.35. *Let $\mathcal{E} \subset \mathbb{R}_n$, with $n \in \mathbb{N}$, be an elementary set. Then, $v_n(\mathcal{E}) \geqslant 0$.*

Proof See Homework 1. □

THEOREM 2.36. *Let $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_k$ be a finite sequence of almost disjoint elementary subsets of $\mathbb{R}^n$, with $(k, n) \in \mathbb{N}^2$. Then,*

$$v_n\left(\bigcup_{t=1}^{k} \mathcal{E}_t\right) = \sum_{t=1}^{k} v_n(\mathcal{E}_t). \tag{2.79}$$

Proof See Homework 1. □

THEOREM 2.37. *Let $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_k$ be a finite sequence of elementary subsets of $\mathbb{R}^n$, with $(k, n) \in \mathbb{N}^2$ and $\mathcal{E}_1 \subseteq \mathcal{E}_2 \subseteq \ldots \subseteq \mathcal{E}_k$. Then, the following holds,*

$$v_n(\mathcal{E}_1) \leqslant v_n(\mathcal{E}_2) \leqslant \ldots \leqslant v_n(\mathcal{E}_k). \tag{2.80}$$

Proof See Homework 1. □

THEOREM 2.38. *Let $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_k$ be a finite sequence of elementary subsets of $\mathbb{R}^n$, with $(k, n) \in \mathbb{N}^2$. Then, the following holds,*

$$v_n\left(\bigcup_{t=1}^{k} \mathcal{E}_t\right) \leqslant \sum_{t=1}^{k} v_n(\mathcal{E}_t). \tag{2.81}$$

Proof See Homework 1. □

Using these properties of the volume of elementary sets, a formal definition of the Jordan measure is given in terms of the inner and outer Jordan measures.

DEFINITION 2.39 (Inner and Outer Jordan Measures). Let $\mathscr{E}_n$ denote the set of all elementary sets in $\mathbb{R}^n$, with $n \in \mathbb{N}$, and let $\mathcal{A} \in \mathscr{E}_n$. Then, the inner Jordan measure of $\mathcal{A}$ is

$$\underline{m}_n(\mathcal{A}) \triangleq \sup_{\mathcal{E} \in \{\mathcal{D} \in \mathscr{E}_n : \mathcal{D} \subseteq \mathcal{A}\}} v_n(\mathcal{E}); \text{ and} \tag{2.82}$$

the outer measure of $\mathcal{A}$ is

$$\bar{m}_n(\mathcal{A}) \triangleq \inf_{\mathcal{E} \in \{\mathcal{D} \in \mathscr{E}_n : \mathcal{A} \subseteq \mathcal{D}\}} v_n(\mathcal{E}). \tag{2.83}$$

THEOREM 2.40. *Let $\mathcal{A}$ be an arbitrary subset of $\mathbb{R}^n$, with $n \in \mathbb{N}$. Then, the following holds,*

$$\underline{m}_n(\mathcal{A}) \leqslant \bar{m}_n(\mathcal{A}). \tag{2.84}$$

Proof See Homework 1. □

DEFINITION 2.41 (Jordan Measure). An arbitrary subset $\mathcal{A}$ of $\mathbb{R}^n$, with $n \in \mathbb{N}$, is said to be Jordan measurable if $\underline{m}_n(\mathcal{A}) = \bar{m}_n(\mathcal{A})$. When $\mathcal{A}$ is Jordan measurable, its Jordan measure is:

$$m_n(\mathcal{A}) \triangleq \underline{m}_n(\mathcal{A}) = \bar{m}_n(\mathcal{A}). \tag{2.85}$$

THEOREM 2.42. *Let $\mathcal{A}$ be an elementary subset of $\mathbb{R}^n$, with $n \in \mathbb{N}$. Then, $\mathcal{A}$ is Jordan measurable. Moreover, $m_n(\mathcal{A}) = v_n(\mathcal{A})$.*

Proof See Homework 1. □

THEOREM 2.43. *Let $\mathcal{A}$ be a bounded subset of $\mathbb{R}^n$, with $n \in \mathbb{N}$. Then, the following statements are equivalent:*
(i) $\mathcal{A}$ is Jordan measurable; and
(ii) For all $\epsilon > 0$, there always exist two elementary sets $\mathcal{D}$ and $\mathcal{E}$ that satisfy $\mathcal{D} \subseteq \mathcal{A} \subseteq \mathcal{E}$, such that:

$$v_n(\mathcal{E} \setminus \mathcal{D}) < \epsilon. \tag{2.86}$$

Proof See Homework 1. □

The Jordan measure and the Riemann integral exhibit a relation that is worth highlighting.

THEOREM 2.44. *Let $f : [a, b] \to \mathbb{R}$, with $-\infty < a < b < \infty$, be a bounded function and let the sets*

$$\mathcal{A}_+ = \{(x, y) \in \mathbb{R}^2 : x \in [a, b], 0 \leqslant y \leqslant f(x)\} \text{ and} \tag{2.87}$$

$$\mathcal{A}_- = \{(x, y) \in \mathbb{R}^2 : x \in [a, b], f(x) \leqslant y \leqslant 0)\}. \tag{2.88}$$

Then, the function f is Riemann integrable if and only if the sets $\mathcal{A}_+$ and $\mathcal{A}_-$ are Jordan measurable. Moreover,

$$\int_a^b f(x)\mathrm{d}x = m_2(\mathcal{A}_+) - m_2(\mathcal{A}_-). \tag{2.89}$$

Proof See Homework 1. □

2.11 Lebesgue Outer Measure

The Lebesgue measure is a generalization of the Jordan measure in the sense that it applies to sets that can be approximated by infinitely countable unions of elementary sets with arbitrary precision. Before introducing the formal definition of Lebesgue measure, consider the notion of an outer measure. That is, a function that approximates the measure of a set by measuring the smallest set that contains it.

DEFINITION 2.45 (Lebesgue outer measure). Given an arbitrary subset $\mathcal{E}$ of $\mathbb{R}^n$, with $n \in \mathbb{N}$, the Lebesgue outer measure is

$$\mu^* (\mathcal{E}) = \inf \sum_{t=1}^{\infty} v_n (\mathcal{A}_t) \tag{2.90}$$

where the infimum is with respect to all countable covers on $\mathcal{E}$ of the form $\mathcal{A}_1, \mathcal{A}_2, \ldots$, such that

$$\mathcal{E} \subseteq \bigcup_{t=1}^{\infty} \mathcal{A}_j, \tag{2.91}$$

and for all $t \in \mathbb{N}$, $\mathcal{A}_t$ is a closed box.

The Lebesgue outer measure has the following three properties. First, it is non-negative, but it can be infinite.

THEOREM 2.46. *Let $\mathcal{A}$ be an arbitrary subset of $\mathbb{R}^n$, with $n \in \mathbb{N}$. Then,*

$$0 \leqslant \mu^* (\mathcal{A}) \leqslant \infty. \tag{2.92}$$

Proof See Homework 1. □

An example of a set whose Lebesgue outer measure is infinity is $\mathbb{R}^n$.

THEOREM 2.47. *The Lebesgue outer measure of $\mathbb{R}^n$, with $n \in \mathbb{N}$, satisfies:*

$$\mu^* (\mathbb{R}^n) = +\infty. \tag{2.93}$$

Proof See Homework 1. □

Alternatively, generic boxes have finite Lebesgue outer measure.

THEOREM 2.48. *Let $\mathcal{E}$ be a generic box of $\mathbb{R}^n$, with $n \in \mathbb{N}$. Then,*

$$\mu^* (\mathcal{E}) = v_n (\mathcal{E}), \tag{2.94}$$

which is the volume of the box.

Proof See Homework 1. □

The second and third property of the Lebesgue outer measure are monotonicity and subadditivity.

THEOREM 2.49. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two arbitrary subsets of $\mathbb{R}^n$, with $n \in \mathbb{N}$, such that $\mathcal{A}_1 \subset \mathcal{A}_2$. Then,*

$$\mu^* (\mathcal{A}_1) \leqslant \mu^* (\mathcal{A}_2) . \tag{2.95}$$

Proof See Homework 1. □

THEOREM 2.50. *Let $\mathcal{A}_1$, $\mathcal{A}_2$, ..., form a countable sequence of arbitrary subsets of $\mathbb{R}^n$, with $n \in \mathbb{N}$. Then, the following holds:*

$$\mu^* \left(\bigcup_{t=1}^{\infty} \mathcal{A}_t \right) \leqslant \sum_{t=1}^{\infty} \mu^* (\mathcal{A}_t) . \tag{2.96}$$

Proof See Homework 1. □

An outer measure is not a measure. In particular, it is possible to find pathological subsets of $\mathbb{R}^n$ in which disjoint sets have elements that are so close to each other that the Lebesgue outer measure of their union is different from the sum of measures of each of its subsets. The following theorems present some cases in which the Lebesgue outer measure of the union of some subsets of $\mathbb{R}^n$ is identical to the sum of the Lebesgue outer measure of each of these subsets.

THEOREM 2.51. *Let $\mathcal{A}$ and $\mathcal{B}$ be two arbitrary subsets of $\mathbb{R}^n$, with $n \in \mathbb{N}$, such that*

$$\inf_{(\boldsymbol{a},\boldsymbol{b}) \in \mathcal{A} \times \mathcal{B}} ||\boldsymbol{a} - \boldsymbol{b}||_2 > 0. \tag{2.97}$$

Then,

$$\mu^* (\mathcal{A} \cup \mathcal{B}) = \mu^* (\mathcal{A}) + \mu^* (\mathcal{B}) . \tag{2.98}$$

Proof See Homework 1. □

THEOREM 2.52. *Let $\mathcal{A}_1$, $\mathcal{A}_2$, ..., form a countable sequence of almost disjoint closed boxes, and let also*

$$\mathcal{A} = \bigcup_{t=1}^{\infty} \mathcal{A}_t, \tag{2.99}$$

be their union. Then, the following holds:

$$\mu^* \left(\bigcup_{t=1}^{\infty} \mathcal{A}_t \right) = \sum_{t=1}^{\infty} \mu^* (\mathcal{A}_t) . \tag{2.100}$$

Proof See Homework 1. □

2.12 Lebesgue Measure and Lebesgue Measurable Sets

THEOREM 2.53. *Let $\mathscr{O}_n$ be the set of all possible open subsets of $\mathbb{R}^n$, and let $\mathcal{A}$ be an arbitrary subset of $\mathbb{R}^n$, with $n \in \mathbb{N}$. Then,*

$$\mu^*(\mathcal{A}) = \inf_{\mathcal{D} \in \{\mathcal{B} \in \mathscr{O}_n : \mathcal{A} \subset \mathcal{B}\}} \mu^*(\mathcal{D}). \tag{2.101}$$

Proof See Homework 1. □

DEFINITION 2.54 (Lebesgue Measurable Sets). Let $\mathcal{A}$ be an arbitrary subset of $\mathbb{R}^n$, with $n \in \mathbb{N}$. Then, $\mathcal{A}$ is said to be Lebesgue measurable if for all $\epsilon > 0$, there always exists an open subset $\mathcal{O} \subset \mathbb{R}^n$, such that $\mathcal{A} \subset \mathcal{O}$ and

$$\mu^*(\mathcal{O} \setminus \mathcal{A}) \leqslant \epsilon. \tag{2.102}$$

This definition leads immediately to an important class of Lebesgue measurable sets.

THEOREM 2.55. *Let $\mathcal{A}$ be an open subset of $\mathbb{R}^n$, with $n \in \mathbb{N}$. Then, the set $\mathcal{A}$ is Lebesgue measurable.*

Proof Note that from the assumption that $\mathcal{A}$ is open and $\mathcal{A} \subseteq \mathcal{A}$, the condition of existence of an open set containing $\mathcal{A}$ is satisfied. Moreover, $\mu^*(\mathcal{A} \setminus \mathcal{A}) = \mu^*(\emptyset) = 0 < \epsilon$, for all $\epsilon > 0$, which completes the proof. □

Using Definition 2.54, the definition of Lesbesgue measure can be introduced as follows.

DEFINITION 2.56 (Lebesgue Measure). Let $\mathcal{A}$ be a Lebesgue measurable subset of $\mathbb{R}^n$, with $n \in \mathbb{N}$. Then, the Lebesgue measure of $\mathcal{A}$, denoted by $\mu(\mathcal{A})$, satisfies

$$\mu(\mathcal{A}) \triangleq \mu^*(\mathcal{A}). \tag{2.103}$$

Definition 2.56 implies that the Lebesgue outer measure becomes a measure when it is restricted to Lebesgue measurable sets (Definition 2.54). That is, as shown in the sequel of this section, the Lebesgue measure inherits all the properties of the Lebesgue outer measure, namely, positivity, monotonicity, and subadditivity. The additivity property is shown later, in Theorem 2.62.

For the moment, using Definition 2.54 and the fact that open sets are Lebesgue measurable, the objective is to identify other classes of sets that are Lebesgue measurable. The following theorem shows that all sets that possess Lebesgue outer measure zero are Lebesgue measurable.

THEOREM 2.57. *Let $\mathcal{A}$ be an arbitrary subset of $\mathbb{R}^n$, with $n \in \mathbb{N}$, such that $\mu^*(\mathcal{A}) = 0$. Then, $\mathcal{A}$ is Lebesgue measurable. Moreover, all subsets $\mathcal{B}$ of $\mathcal{A}$ are Lebesgue measurable.*

Proof From Theorem 2.53, it holds that for all $\epsilon > 0$, there always exists an open set $\mathcal{O}$ such that $\mathcal{A} \subseteq \mathcal{O}$ and $\mu^*(\mathcal{O}) < \epsilon$. Moreover, given that $\mathcal{O} \setminus \mathcal{A} \subseteq \mathcal{O}$, it holds from the monotonicity of the Lebesgue outer measure that $\mu^*(\mathcal{O} \setminus \mathcal{A}) < \mu^*(\mathcal{O}) < \epsilon$. Note that the same holds for any subset of $\mathcal{A}$, which completes the proof □

The following theorem shows that countable unions of Lebesgue measurable sets form Lebesgue measurable sets.

THEOREM 2.58. *Let $\mathcal{A}_1$, $\mathcal{A}_2$, ..., form a countable sequence of Lebesgue measurable subsets of $\mathbb{R}^n$, with $n \in \mathbb{N}$. Then, the countable union*

$$\mathcal{A} = \bigcup_{t=1}^{\infty} \mathcal{A}_t, \tag{2.104}$$

is Lebesgue measurable, and

$$\mu(\mathcal{A}) \leqslant \sum_{t=1}^{\infty} \mu(\mathcal{A}_t). \tag{2.105}$$

Proof From the assumption that $\mathcal{A}_t$ is Lebesgue measurable for all $t \in \mathbb{N}$, it holds that for all $\epsilon_t > 0$, it is always possible to find an open set $\mathcal{O}_t$, such that $\mathcal{A}_t \subseteq \mathcal{O}_t$ and

$$\mu^*(\mathcal{O}_t \setminus \mathcal{A}_t) < \epsilon_t. \tag{2.106}$$

For the ease of presentation, let $\epsilon_t \triangleq \frac{\epsilon}{2^t}$, for some $\epsilon > 0$. Let $\mathcal{O}$ be the set

$$\mathcal{O} \triangleq \bigcup_{t=1}^{+\infty} \mathcal{O}_t, \tag{2.107}$$

which is open, and thus Lebesgue measurable, and verifies that $\mathcal{A} \subseteq \mathcal{O}$. Moreover, it also holds that:

$$\mathcal{O} \setminus \mathcal{A} \subseteq \bigcup_{t=1}^{+\infty} \mathcal{O}_t \setminus \mathcal{A}_t, \tag{2.108}$$

which implies due to the monotonicity of the Lebesgue outer measure that

$$\mu^*(\mathcal{O} \setminus \mathcal{A}) \leqslant \sum_{t=1}^{+\infty} \mu^*(\mathcal{O}_t \setminus \mathcal{A}_t) \tag{2.109}$$

$$\leqslant \sum_{t=1}^{+\infty} \frac{\epsilon}{2^t} \tag{2.110}$$

$$= \epsilon, \tag{2.111}$$

which proves the Lebesgue measurability of $\mathcal{A}$, and thus, $\mu(\mathcal{A}) = \mu^*(\mathcal{A})$. Finally, the inequality in (2.105) follows from the subadditivity of the Lebesgue outer measure (Theorem 2.50). □

The following theorem shows that closed sets of $\mathbb{R}^n$, with $n \in \mathbb{N}$, are Lebesgue measurable.

THEOREM 2.59. *Let $\mathcal{A}$ be a closed subset of $\mathbb{R}^n$, with $n \in \mathbb{N}$. Then, $\mathcal{A}$ is Lebesgue measurable.*

Proof See Homework 1. □

The following theorem shows that the complement of a Lebesgue measurable set is Lebesgue measurable.

THEOREM 2.60. *Let $\mathcal{A}$ be a Lebesgue measurable subset of $\mathbb{R}^n$, with $n \in \mathbb{N}$. Then, $\mathcal{A}^c$ is Lebesgue measurable.*

Proof From the assumption that $\mathcal{A}$ is measurable, it holds that for all $t \in \mathbb{N}$, there always exists an open set $\mathcal{O}_t$ such that $\mathcal{A} \subseteq \mathcal{O}_t$ and

$$\mu^*(\mathcal{O}_t \setminus \mathcal{A}) \leqslant \frac{1}{t}. \tag{2.112}$$

Note that $\mathcal{O}_t^{\mathsf{c}}$ is closed, and thus Lebesgue measurable (Theorem 2.59); and $\mathcal{O}_t^{\mathsf{c}} \subseteq \mathcal{A}^{\mathsf{c}}$ (Theorem 1.9). Let the set $\mathcal{C}$ be

$$\mathcal{C} \triangleq \bigcup_{t=1}^{\infty} \mathcal{O}_t^{\mathsf{c}}, \tag{2.113}$$

which is also measurable (Theorem 2.58) and satisfies $\mathcal{C} \subseteq \mathcal{A}^{\mathsf{c}}$. Thus, the following holds:

$$\mathcal{A}^{\mathsf{c}} = \mathcal{C} \cup \mathcal{A}^{\mathsf{c}} \setminus \mathcal{C}. \tag{2.114}$$

Hence, given that the union of two Lebesgue measurable sets is Lebesgue measurable (Theorem 2.58), the problem boils down to prove the Lebesgue measurability of the set $\mathcal{A}^{\mathsf{c}} \setminus \mathcal{C}$. First, note that from (2.113), it holds that $\mathcal{O}_t^{\mathsf{c}} \subseteq \mathcal{C}$, which implies that $\mathcal{C}^{\mathsf{c}} \subseteq \mathcal{O}_t$ (Theorem 1.9), and

$$\mathcal{C}^{\mathsf{c}} \setminus \mathcal{A} \subseteq \mathcal{O}_t \setminus \mathcal{A}. \tag{2.115}$$

Second, note that $\mathcal{C}^{\mathsf{c}} \setminus \mathcal{A} = \mathcal{A}^{\mathsf{c}} \setminus \mathcal{O}_t$ (Theorem 1.9), which together with the monotonicity of the Lebesgue outer measure (Theorem 2.49), yields from (2.112) and (2.115),

$$\mu^*(\mathcal{C}^{\mathsf{c}} \setminus \mathcal{A}) \leqslant \mu^*(\mathcal{O}_t \setminus \mathcal{A}) \tag{2.116}$$

$$\leqslant \frac{1}{t}. \tag{2.117}$$

Letting t tend to infinity yields $\mu^*(\mathcal{C}^{\mathsf{c}} \setminus \mathcal{A}) = 0$, which implies the Lebesgue measurability of $\mathcal{C}^{\mathsf{c}} \setminus \mathcal{A}$ (Theorem 2.57), and completes the proof. □

Finally, using the previous results, the following theorem shows that the set of all Lebesgue measurable sets in $\mathbb{R}^n$, with $n \in \mathbb{N}$, form a σ-field.

THEOREM 2.61. *The set of all Lebesgue measurable sets in* $\mathbb{R}^n$*, with* $n \in \mathbb{N}$*, form a* σ*-field.*

Proof See Homework 1. □

In the following, the set σ-field formed by all the Lebesgue measurable sets in $\mathbb{R}^n$, with $n \in \mathbb{N}$, is referred to as the Lebesgue σ-field, and it is denoted by $\mathscr{L}(\mathbb{R}^n)$. From Definition 1.68 and Theorem 2.61 it follows that the Borel σ-field, i.e., $\mathscr{B}(\mathbb{R}^n)$, is a subset of $\mathscr{L}(\mathbb{R}^n)$. Nonetheless, it is important to highlight that this inclusion is strict. That is,

$$\mathscr{B}(\mathbb{R}^n) \subset \mathscr{L}(\mathbb{R}^n), \tag{2.118}$$

as there are Lebesgue measurable sets that are not in the Borel σ-field.

This section ends by showing the additivity of the Lebesgue measure.

THEOREM 2.62. *Let* $\mathcal{A}_1, \mathcal{A}_2, \ldots,$ *form a countable sequence of disjoint Lebesgue measurable subsets of* $\mathbb{R}^n$*, with* $n \in \mathbb{N}$*, and let also*

$$\mathcal{A} = \bigcup_{t=1}^{+\infty} \mathcal{A}_t. \tag{2.119}$$

Then,

$$\mu(\mathcal{A}) = \sum_{t=1}^{+\infty} \mu(\mathcal{A}_t). \tag{2.120}$$

Proof See Homework 1. □

2.13 Lebesgue Measurable Functions

The definition of Lebesgue measurable functions can be stated in different equivalent forms. A useful definition is often the one that is expressed in terms of conditions that are easy to verify. The following definition is one of these.

DEFINITION 2.63 (Lebesgue Measurable Function). A function $f : \mathcal{E} \to \bar{\mathbb{R}}$, with $\mathcal{E} \subseteq \mathbb{R}^n$ a Lebesgue measurable set and $n \in \mathbb{N}$, is said to be Lebesgue measurable, if for all $a \in \mathbb{R}$, it holds that $\{x \in \mathcal{E} : f(x) < a\}$ is a Lebesgue measurable set.

An immediate consequence of Definition 2.63 is an important property of continuous functions.

THEOREM 2.64. *A continuous function* $f : \mathbb{R}^n \to \mathbb{R}$, *with* $n \in \mathbb{N}$, *is Lebesgue measurable.*

Proof See Homework 2. □

2.13.1 Alternative Definitions of Lebesgue Measurable Functions

The following theorem provide equivalent definitions of measurability of functions that take values in the extended reals.

THEOREM 2.65. *Consider a function* $f : \mathbb{R}^n \to \bar{\mathbb{R}}$, *with* $n \in \mathbb{N}$. *Then, the following statements are equivalent:*
(i) *The function* f *is Lebesgue measurable;*
(ii) *For all* $a \in \mathbb{R}$, *the set* $f^{-1}([-\infty, a[) = \{x \in \mathbb{R}^n : f(x) < a\}$ *is Lebesgue measurable;*
(iii) *For all* $a \in \mathbb{R}$, *the set* $f^{-1}([-\infty, a]) = \{x \in \mathbb{R}^n : f(x) \leqslant a\}$ *is Lebesgue measurable;*
(iv) *For all* $a \in \mathbb{R}$, *the set* $f^{-1}(]a, +\infty]) = \{x \in \mathbb{R}^n : f(x) > a\}$ *is Lebesgue measurable; and*
(v) *For all* $a \in \mathbb{R}$, *the set* $f^{-1}([a, +\infty]) = \{x \in \mathbb{R}^n : f(x) \geqslant a\}$ *is Lebesgue measurable.*

Proof See Homework 2. □

THEOREM 2.66. *Consider a function* $f : \mathbb{R}^n \to \bar{\mathbb{R}}$, *with* $n \in \mathbb{N}$. *Then,* f *is Lebesgue measurable if and only if*
(i) *The sets* $f^{-1}(-\infty)$ *and* $f^{-1}(\infty)$ *are Lebesgue measurable; and*
at least one of the following conditions hold:
(ii) *For all open subsets* $\mathcal{O}$ *of* $\mathbb{R}$, *the set* $f^{-1}(\mathcal{O}) \subseteq \mathbb{R}^n$ *is a Lebesgue measurable set; or*
(iii) *For all closed subsets* $\mathcal{C}$ *of* $\mathbb{R}$, *the set* $f^{-1}(\mathcal{C}) \subseteq \mathbb{R}^n$ *is a Lebesgue measurable set.*

Proof See Homework 2. □

In the case of finite functions, the following theorems present alternative tools to verify Lebesgue measurability.

THEOREM 2.67. *Consider a function* $f : \mathbb{R}^n \to \mathbb{R}$, *with* $n \in \mathbb{N}$. *Then the following statements are equivalent:*
(i) *The function* f *is Lebesgue measurable;*
(ii) *For all pairs* $(a, b) \in \mathbb{R}^2$, *with* $a < b$, *the set* $f^{-1}(]a, b[) = \{x \in \mathbb{R}^n : a < f(x) < b\}$ *is Lebesgue measurable;*
(iii) *For all pairs* $(a, b) \in \mathbb{R}^2$, *with* $a < b$, *the set* $f^{-1}([a, b[) = \{x \in \mathbb{R}^n :$

$a \leqslant f(x) < b\}$ *is Lebesgue measurable;*
(iv) *For all pairs* $(a, b) \in \mathbb{R}^2$, *with* $a < b$, *the set* $f^{-1}(]a, b]) = \{x \in \mathbb{R}^n :$ $a < f(x) \leqslant b\}$ *is Lebesgue measurable; and*
(v) *For all pairs* $(a, b) \in \mathbb{R}^2$, *with* $a < b$, *the set* $f^{-1}([a, b]) = \{x \in \mathbb{R}^n :$ $a \leqslant f(x) \leqslant b\}$ *is Lebesgue measurable.*

Proof See Homework 2. □

THEOREM 2.68. *Consider a function* $f : \mathbb{R}^n \to \mathbb{R}$, *with* $n \in \mathbb{N}$. *Then,* f *is Lebesgue measurable if and only if*
(i) *For all open subsets* $\mathcal{O}$ *of* $\mathbb{R}$, *the set* $f^{-1}(\mathcal{O}) \subseteq \mathbb{R}^n$ *is a Lebesgue measurable set; or*
(ii) *For all closed subsets* $\mathcal{C}$ *of* $\mathbb{R}$, *the set* $f^{-1}(\mathcal{C}) \subseteq \mathbb{R}^n$ *is a Lebesgue measurable set.*

Proof See Homework 2. □

2.13.2 Lebesgue Measurable Simple Functions

The definition of an elementary simple function (Definition 2.10) is generalized by that of Lebesgue measurable simple functions, which are central in the definition of Lebesgue integral.

DEFINITION 2.69 (Lebesgue Measurable Simple Functions). A function $f : \mathcal{E} \to \bar{\mathbb{R}}$, with $\mathcal{E} \subseteq \mathbb{R}^n$ and $n \in \mathbb{N}$, is said to be a Lebesgue measurable simple function if there always exists a partition on $\mathcal{E}$ formed by Lebesgue measurable sets $\mathcal{A}_1$, $\mathcal{A}_2$, …, $\mathcal{A}_m$, with $m < \infty$, such that for all $x \in \mathcal{E}$,

$$f(x) = \sum_{t=1}^{m} a_t \mathbb{1}_{\{x \in \mathcal{A}_t\}}, \tag{2.121}$$

with $a_i \in \bar{\mathbb{R}}$ for all $i \in \{1, 2, \ldots, m\}$.

From Definition 2.69, it holds that every elementary simple function (Definition 2.10) is a Lebesgue measurable simple function, but the converse is not necessarily true.

There might exist many expressions of the form in (2.121) to describe the same Lebesgue measurable simple function. For instance, consider an alternative description of the function f in (2.121) such that a partition of $\mathcal{E}$ is formed by measurable sets $\mathcal{B}_1$, $\mathcal{B}_2$, …, $\mathcal{B}_p$, with $p < \infty$, and for all $x \in \mathbb{E}$,

$$f(x) = \sum_{t=1}^{m} a_t \mathbb{1}_{\{x \in \mathcal{A}_t\}} = \sum_{t=1}^{p} c_t \mathbb{1}_{\{x \in \mathcal{B}_t\}} \tag{2.122}$$

for some extended reals $c_1, c_2, \ldots, c_p$. In this case, for all pairs $(i, j) \in \{1, 2, \ldots, m\} \times \{1, 2, \ldots, p\}$, it holds that for all $x \in \mathcal{A}_i \cap \mathcal{B}_j$, $f(x) = a_i = c_j$.

To add some extra generality, the condition that the sets $\mathcal{A}_1$, $\mathcal{A}_2$, ..., $\mathcal{A}_m$ in Definition 2.69 are disjoint can be neglected. Nonetheless, in order to avoid an indetermination, it must be ensured that the sum in (2.121) does not include simultaneously the terms $+\infty$ and $-\infty$.

2.13.3 Properties of Lebesgue Measurable Functions

The addition and the product of bounded measurable functions form measurable functions.

THEOREM 2.70. *Let* $f : \mathcal{E} \rightarrow \mathbb{R}$ *and* $g : \mathcal{E} \rightarrow \mathbb{R}$*, with* $\mathcal{E}$ *a Lebesgue measurable subset of* $\mathbb{R}^n$ *and* $n \in \mathbb{N}$*, be two Lebesgue measurable functions. Then, the functions formed by the addition* $f + g$*; the* k*-th power* f^k*; and the product* $f \cdot g$ *are Lebesgue measurable functions.*

Proof See Homework 2. □

The composition of two Lebesgue measurable functions is measurable under certain conditions.

THEOREM 2.71. *Let* $f : \mathbb{R}^n \rightarrow \mathbb{R}$ *and* $g : \mathbb{R} \rightarrow \mathbb{R}$*, with* $n \in \mathbb{N}$*, be a Lebesgue measurable function and a continuous function, respectively. Then, the composition* $g \circ f$ *is measurable.*

Proof See Homework 2. □

Note that under the conditions of Theorem 2.71, the composition $f \circ g$ is not necessarily Lebesgue measurable.

THEOREM 2.72. *For all* $t \in \mathbb{N}$*, let* $f_t : \mathbb{R}^n \rightarrow \mathbb{R}$*, with* $n \in \mathbb{N}$*, be a measurable function. Then, the following functions* $g_i : \mathbb{R}^n \rightarrow \bar{\mathbb{R}}$*, with* $i \in \{1, 2\}$*, such that for all* $x \in \mathbb{R}^n$*,*

$$g_1(x) = \sup_{t \in \mathbb{N}} f_t(x), \quad \text{and} \tag{2.123}$$

$$g_2(x) = \inf_{t \in \mathbb{N}} f_t(x), \tag{2.124}$$

are Lebesgue measurable.

Proof Consider the subset of the form $[-\infty, a[$, with $a \in \mathbb{R}$. Then, the following holds

$$g_1^{-1}\big([-\infty,a[\big) = \{x \in \mathbb{R}^n : g_1(x) < a\} \tag{2.125}$$
$$= \left\{x \in \mathbb{R}^n : \sup_{t>0} f_t(x) < a\right\} \tag{2.126}$$
$$= \bigcap_{t=1}^{+\infty} \{x \in \mathbb{R}^n : f_t(x) < a\} \tag{2.127}$$
$$= \bigcap_{t=1}^{+\infty} f_t^{-1}\big(]-\infty,a[\big); \text{ and} \tag{2.128}$$

$$g_2^{-1}\big([-\infty,a[\big) = \{x \in \mathbb{R}^n : g_2(x) < a\} \tag{2.129}$$
$$= \left\{x \in \mathbb{R}^n : \inf_{t>0} f_t(x) < a\right\} \tag{2.130}$$
$$= \bigcup_{t=1}^{+\infty} \{x \in \mathbb{R}^n : f_t(x) < a\} \tag{2.131}$$
$$= \bigcup_{t=1}^{+\infty} f_t^{-1}\big([-\infty,a[\big). \tag{2.132}$$

From the assumption that for all $t \in \mathbb{N}$, the function f_t is Lebesgue measurable, it follows that $f_t^{-1}\big([-\infty,a[\big)$ is Lebesgue measurable. Hence, from the equalities in (2.128) and (2.132), it holds that $g_1^{-1}\big([-\infty,a[\big)$ and $g_2^{-1}\big([-\infty,a[\big)$ are sets respectively formed by countable intersections and countable unions of measurable sets, and thus, both are Lebesgue measurable. This implies that both g_1 and g_2 are Lebesgue measurable functions. □

A further implication of Theorem 2.72 is described hereunder.

THEOREM 2.73. *For all $t \in \mathbb{N}$, let $f_t : \mathbb{R}^n \to \mathbb{R}$, with $n \in \mathbb{N}$, be a Lebesgue measurable function. Then, the following functions $g_i : \mathbb{R}^n \to \bar{\mathbb{R}}$, with $i \in \{1,2\}$, such that for all $x \in \mathbb{R}^n$,*

$$g_1(x) = \lim_{t\to+\infty} \sup_{k>t} f_k(x), \text{ and} \tag{2.133}$$
$$g_2(x) = \lim_{t\to+\infty} \inf_{k>t} f_k(x), \tag{2.134}$$

are Lebesgue measurable.

Proof See Homework 2. □

The following theorem is an immediate implication of Theorem 2.73.

THEOREM 2.74. *For all $t \in \mathbb{N}$, let $f_t : \mathbb{R}^n \to \mathbb{R}$, with $n \in \mathbb{N}$, be a Lebesgue measurable function. Assume that there exists a function $f : \mathbb{R}^n \to \mathbb{R}$ such* *that for all* $x \in \mathbb{R}^n$*,*

$$\lim_{t \to +\infty} f_t(x) = f(x). \tag{2.135}$$

Then, the function f *is Lebesgue measurable.*

Proof From the assumption in (2.135), it follows that the limit exists for all $x \in \mathbb{R}^n$, and

$$\lim_{t \to +\infty} f_t(x) = g_1(x) = g_2(x), \tag{2.136}$$

with g_1 and g_2 the functions defined in (2.133) and (2.134), respectively. Hence, given that the functions g_1 and g_2 are Lebesgue measurable, the function f is also a Lebesgue measurable function. □

The argument of the proof of Theorem 2.74 can be extended to functions that might differ in certain intervals under the condition that such intervals are of Lebesgue measure zero. This functions are said to be equal almost everywhere.

Definition 2.75. Given two functions $f : \mathcal{E} \to \mathbb{R}$ and $g : \mathcal{E} \to \mathbb{R}$, with $\mathcal{E} \subseteq \mathbb{R}^n$ and $n \in \mathbb{N}$, they are said to be equal *almost everywhere*, if the set

$$\{x \in \mathcal{E} : f(x) \neq g(x)\} \tag{2.137}$$

is of Lebesgue measure zero, i.e., $\mu(\{x \in \mathcal{E} : f(x) \neq g(x)\}) = 0$.

In general, a property on a given point $x \in \mathcal{E}$, with $\mathcal{E} \subset \mathbb{R}^n$ and $n \in \mathbb{N}$, denoted by $C(x)$, is said to hold *almost everywhere* or to hold for almost every $x \in \mathcal{E}$, if the set

$$\mathcal{C} = \{x \in \mathcal{E} : C(x) \text{ does not hold}\} \tag{2.138}$$

satisfies $\mu(\mathcal{C}) = 0$. That is, the property $C(x)$ is verified on all elements of $\mathcal{E}$ except on a subset of Lebesgue measure zero. Using this notion, the following holds.

Theorem 2.76. *Let* $f : \mathcal{E} \to \mathbb{R}$ *and* $g : \mathcal{E} \to \mathbb{R}$*, with* $\mathcal{E} \subseteq \mathbb{R}^n$ *and* $n \in \mathbb{N}$*, be two functions such that* f *is Lebesgue measurable and* f *and* g *are equal almost everywhere. Then, the function* g *is Lebesgue measurable.*

Proof See Homework 2. □

Finally, the most prominent property of Lebesgue measurable functions is presented in two steps. First, it is shown for non-negative Lebesgue measurable functions in the following theorem; and later, it is shown for arbitrary Lebesgue measurable functions in Theorem 2.78.

Theorem 2.77. *Let* $f : \mathcal{E} \to [0, +\infty]$*, with* $\mathcal{E} \subseteq \mathbb{R}^n$ *and* $n \in \mathbb{N}$*, be a non-negative Lebesgue measurable function. Then, there always exist an in-*

creasing sequence of non-negative Lebesgue measurable simple functions f_1, f_2, ..., with $f_t : \mathcal{E} \to [0, +\infty]$ for all $t \in \mathbb{N}$, that converge point-wise to f. That is, for all $x \in \mathcal{E}$, the following holds:

$$f_i(x) \leqslant f_{i+1}(x) \quad \text{and} \quad \lim_{t \to +\infty} f_t(x) = f(x), \tag{2.139}$$

with $i \in \mathbb{N}$.

Proof See Homework 2. □

THEOREM 2.78. *Let $f : \mathcal{E} \to \bar{\mathbb{R}}$, with $\mathcal{E} \subseteq \mathbb{R}^n$ and $n \in \mathbb{N}$, be a Lebesgue measurable function. Then, there always exist a sequence of Lebesgue measurable simple functions f_1, f_2, ..., with $f_t : \mathcal{E} \to \mathbb{R}$ for all $t \in \mathbb{N}$, that satisfies for all for all $x \in \mathcal{E}$:*

$$|f_i(x)| \leqslant |f_{i+1}(x)| \quad \text{and} \quad \lim_{t \to +\infty} f_t(x) = f(x), \tag{2.140}$$

with $i \in \mathbb{N}$.

Proof Note that the function f can be written in terms of non-negative functions as follows. For all $x \in \mathcal{E}$, $f(x) = f^+(x) - f^-(x)$. From Theorem 2.77, it follows that there exist two increasing sequences of non-negative Lebesgue measurable simple functions g_1, g_2, ... and h_1, h_2, ..., with $g_t : \mathcal{E} \to [0, +\infty]$ and $h_t : \mathcal{E} \to [0, +\infty]$ for all $t \in \mathbb{N}$, that converge point-wise to f^+ and f^-, respectively. For all $t \in \mathbb{N}$, let the function f_t be such that for all $x \in \mathcal{E}$, $f_t(x) = g_t(x) - h_t(x)$, which satisfies $\lim_{t \to +\infty} f_t(x) = f(x)$.

Finally, note that for all $x \in \mathcal{E}$ and for all $t \in \mathbb{N}$, $|f_t(x)| = g_t(x) + h_t(x)$. Thus, the sequence of functions $|f_1|$, $|f_2|$, ..., is increasing, which completes the proof. □

2.14 Lebesgue Integral

The Lebesgue integral is defined only for Lebesgue measurable functions, yet this is by no means restrictive. In this section, it is shown that a larger class of functions are integrable in the sense of Lebesgue than in the sense of Riemann-Darboux. In order to study these subtleties of this more general theory of integration, the definition of Lebesgue integral is built into three steps. First, the Lebesgue integral is defined for Lebesgue measurable non-negative simple functions (Definition 2.69). Second, the definition is extended to Lebesgue measurable simple functions; and finally, the integral is defined for functions that satisfy the absolute integrability condition, which is introduced later in Definition 2.94

2.14.1 Case of Non-Negative Lebesgue Measurable Simple Functions

The Lebesgue integral of a non-negative Lebesgue measurable simple function can be described as follows:

DEFINITION 2.79. Consider a non-negative Lebesgue measurable simple function $f : \mathcal{E} \to [0, +\infty]$, with $\mathcal{E} \subseteq \mathbb{R}^n$ and $n \in \mathbb{N}$, such that for all $x \in \mathcal{E}$

$$f(x) = \sum_{t=1}^{m} a_t \mathbb{1}_{\{x \in \mathcal{A}_t\}}, \tag{2.141}$$

where for all $i \in \{1, 2, \ldots, m\}$ and $m < \infty$, it holds that $a_i \in [0, +\infty]$ and $\mathcal{A}_1$, $\mathcal{A}_2$, ..., $\mathcal{A}_m$ form a partition of $\mathcal{E}$. The Lebesgue integral of the function f is

$$\int_{\mathcal{E}} f(x) \mathrm{d}\mu(x) \triangleq \sum_{t=1}^{m} a_t \mu(\mathcal{A}_t). \tag{2.142}$$

Note that the Lebesgue integral of non-negative Lebesgue measurable simple functions is independent of the description of the function. The following theorem formalizes this property.

THEOREM 2.80. *Consider a non-negative measurable simple function $f : \mathcal{E} \to [0, +\infty]$, with $\mathcal{E} \subseteq \mathbb{R}^n$ and $n \in \mathbb{N}$, such that for all $x \in \mathcal{E}$*

$$f(x) = \sum_{t=1}^{m} a_t \mathbb{1}_{\{x \in \mathcal{A}_t\}} = \sum_{t=1}^{p} c_t \mathbb{1}_{\{x \in \mathcal{B}_t\}}, \tag{2.143}$$

where for all $(i, j) \in \{1, 2, \ldots, m\} \times \{1, 2, \ldots, p\}$, with $m < \infty$ and $p < \infty$, it holds that $a_i \in [0, +\infty]$, $c_j \in [0, +\infty]$, and $\mathcal{A}_1$, $\mathcal{A}_2$, ..., $\mathcal{A}_m$ and $\mathcal{B}_1$, $\mathcal{B}_2$, ..., $\mathcal{B}_p$ form two different exact covers of $\mathcal{E}$. Then,

$$\int_{\mathcal{E}} f(x) \mathrm{d}\mu(x) = \sum_{t=1}^{m} a_t \mu(\mathcal{A}_t) = \sum_{t=1}^{p} c_t \mu(\mathcal{B}_t). \tag{2.144}$$

Proof See Homework 2. □

The sum in (2.144) is positive but not necessarily finite. The following theorem sheds some light into this observation.

THEOREM 2.81. *Consider a non-negative Lebesgue measurable simple function $f : \mathcal{E} \to [0, +\infty]$, with $\mathcal{E} \subseteq \mathbb{R}^n$ and $n \in \mathbb{N}$. Then,*

$$\int_{\mathcal{E}} f(x) \mathrm{d}\mu(x) < +\infty \tag{2.145}$$

if and only if the function f is finite almost everywhere and

$$\mu\left(\{x \in \mathcal{E} : f(x) > 0\}\right) < +\infty. \tag{2.146}$$

Proof See Homework 2. □

The Lebesgue integral of a non-negative Lebesgue measurable simple function can be performed over a subset of the domain of such function. This can be done by noticing that the product of non-negative simple functions and the indicator function is a simple function.

DEFINITION 2.82. Consider a non-negative Lebesgue measurable simple function $f : \mathcal{E} \to [0, +\infty]$, with $\mathcal{E} \subseteq \mathbb{R}^n$ and $n \in \mathbb{N}$. The Lebesgue integral of the function f on the subset $\mathcal{A} \subseteq \mathcal{E}$ is

$$\int_{\mathcal{A}} f(x)\mathrm{d}\mu(x) \triangleq \int_{\mathcal{E}} f(x)\mathbb{1}_{\{x\in\mathcal{A}\}}\mathrm{d}\mu(x). \tag{2.147}$$

The integral of non-negative Lebesgue measurable simple functions possesses the following properties.

THEOREM 2.83. *Consider two non-negative Lebesgue measurable simple functions $f : \mathcal{E} \to \mathbb{R}$ and $g : \mathcal{E} \to \mathbb{R}$, with $\mathcal{E} \subseteq \mathbb{R}^n$ and $n \in \mathbb{N}$. Then, the following holds:*

(*i*) *For all pairs $(\alpha, \beta) \in [0, +\infty]^2$,*

$$\int_{\mathcal{E}} (\alpha f(x) + \beta g(x))\, \mathrm{d}\mu(x) = \alpha \int_{\mathcal{E}} f(x)\mathrm{d}\mu(x) + \beta \int_{\mathcal{E}} g(x)\mathrm{d}\mu(x) \tag{2.148}$$

(*ii*) *Given two disjoint measurable subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{E}$,*

$$\int_{\mathcal{A}\cup\mathcal{B}} f(x)\mathrm{d}\mu(x) = \int_{\mathcal{A}} f(x)\mathrm{d}\mu(x) + \int_{\mathcal{B}} f(x)\mathrm{d}\mu(x); \tag{2.149}$$

(*iii*)

$$\int_{\mathcal{E}} f(x)\mathrm{d}\mu(x) = 0, \tag{2.150}$$

if and only if $f(x) = 0$ for almost every $x \in \mathcal{E}$;

(*iv*) *If $f(x) \leqslant g(x)$ for almost every $x \in \mathcal{E}$,*

$$\int_{\mathcal{E}} f(x)\mathrm{d}\mu(x) \leqslant \int_{\mathcal{E}} g(x)\mathrm{d}\mu(x); \text{ and} \tag{2.151}$$

(*v*) *If $f(x) = g(x)$ for almost every $x \in \mathcal{E}$,*

$$\int_{\mathcal{E}} f(x)\mathrm{d}\mu(x) = \int_{\mathcal{E}} g(x)\mathrm{d}\mu(x). \tag{2.152}$$

Proof See Homework 2. □

2.14.2 Case of Absolutely Integrable Simple Functions

Absolute integrability in the case of Lebesgue measurable simple functions can be described as follows.

DEFINITION 2.84 (Absolutely Integrable Simple Functions). A Lebesgue measurable simple function $f : \mathcal{E} \to \mathbb{R}$, with $\mathcal{E} \subseteq \mathbb{R}^n$ and $n \in \mathbb{N}$, is said to be absolutely integrable if

$$\int_{\mathcal{E}} |f(x)| \, \mathrm{d}\mu(x) < \infty. \tag{2.153}$$

The Lebesgue integral of an absolutely integrable simple function is defined as follows.

DEFINITION 2.85 (Lebesgue Integral of Absolutely Integrable Simple Functions). The Lebesgue integral of an absolutely integrable simple function $f : \mathcal{E} \to \mathbb{R}$, with $\mathcal{E} \subseteq \mathbb{R}^n$ and $n \in \mathbb{N}$, is

$$\int_{\mathcal{E}} f(x)\mathrm{d}\mu(x) = \int_{\mathcal{E}} f^+(x)\mathrm{d}\mu(x) - \int_{\mathcal{E}} f^-(x)\mathrm{d}\mu(x). \tag{2.154}$$

Note that from Definition 2.84, it follows that absolute integrability implies that the function has a finite Lebesgue integral.

THEOREM 2.86. *Let $f : \mathcal{E} \to \mathbb{R}$, with $\mathcal{E} \subseteq \mathbb{R}^n$ and $n \in \mathbb{N}$, be an absolutely integrable simple function. Then, it holds that the Lebesgue integral of f is finite.*

Proof The proof follows from the fact that for all $x \in \mathcal{E}$, $|f(x)| = f^+(x)+f^-(x)$, and thus,

$$+\infty > \int_{\mathcal{E}} |f(x)| \, \mathrm{d}\mu(x) = \int_{\mathcal{E}} f^+(x)\mathrm{d}\mu(x) + \int_{\mathcal{E}} f^-(x)\mathrm{d}\mu(x), \tag{2.155}$$

where f^+ and f^- are both non-negative simple functions. Therefore,

$$\int_{\mathcal{E}} f^+(x)\mathrm{d}\mu(x) < +\infty \text{ and} \tag{2.156}$$

$$\int_{\mathcal{E}} f^-(x)\mathrm{d}\mu(x) < +\infty, \tag{2.157}$$

which yields,

$$\int_{\mathcal{E}} f(x)\mathrm{d}\mu(x) = \int_{\mathcal{E}} f^+(x)\mathrm{d}\mu(x) - \int_{\mathcal{E}} f^-(x)\mathrm{d}\mu(x) < +\infty, \tag{2.158}$$

and completes the proof. □

Absolutely integrable simple functions exhibit a particular property regarding the measure of their support. The support of a function is formally defined hereunder.

Definition 2.87 (Support of a Function). Given a Lebesgue measurable function $f : \mathcal{E} \to \mathbb{R}$, with $\mathcal{E} \subseteq \mathbb{R}^n$ and $n \in \mathbb{N}$, the support of f is

$$\operatorname{supp} f \triangleq \{x \in \mathcal{E} : f(x) \neq 0\}. \quad (2.159)$$

A Lebesgue measurable function $f : \mathcal{E} \to \mathbb{R}$, with $\mathcal{E} \subseteq \mathbb{R}^n$ and $n \in \mathbb{N}$, is said to be concentrated in a Lebesgue measurable subset $\mathcal{A} \subset \mathcal{E}$, if for all $x \in \mathcal{E} \setminus \mathcal{A}$, $f(x) = 0$.

Definition 2.88 (Functions with Finite-Measure Support). A Lebesgue measurable function $f : \mathcal{E} \to \mathbb{R}$, with $\mathcal{E} \subseteq \mathbb{R}^n$ and $n \in \mathbb{N}$, is said to have finite-measure support if

$$\mu(\operatorname{supp} f) < +\infty. \quad (2.160)$$

Theorem 2.89. *A Lebesgue measurable simple function $f : \mathcal{E} \to \mathbb{R}$, with $\mathcal{E} \subseteq \mathbb{R}^n$ and $n \in \mathbb{N}$, is an absolutely integrable function if and only if it has a finite measure support.*

Proof See Homework 2. □

The integral of absolutely integrable simple functions possesses the following properties.

Theorem 2.90. *Consider two absolutely integrable simple functions $f : \mathcal{E} \to \mathbb{R}$ and $g : \mathcal{E} \to \mathbb{R}$, with $\mathcal{E} \subseteq \mathbb{R}^n$ and $n \in \mathbb{N}$. Then, the following holds:*

(i) For all pairs $(\alpha, \beta) \in \mathbb{R}^2$,

$$\int_{\mathcal{E}} (\alpha f(x) + \beta g(x))\, \mathrm{d}\mu(x) = \alpha \int_{\mathcal{E}} f(x)\mathrm{d}\mu(x) + \beta \int_{\mathcal{E}} g(x)\mathrm{d}\mu(x) \quad (2.161)$$

(ii) Given two disjoint Lebesgue measurable subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{E}$,

$$\int_{\mathcal{A}\cup\mathcal{B}} f(x)\mathrm{d}\mu(x) = \int_{\mathcal{A}} f(x)\mathrm{d}\mu(x) + \int_{\mathcal{B}} f(x)\mathrm{d}\mu(x); \quad (2.162)$$

(iii) If $f(x) = g(x)$ for almost every $x \in \mathcal{E}$,

$$\int_{\mathcal{E}} f(x)\mathrm{d}\mu(x) = \int_{\mathcal{E}} g(x)\mathrm{d}\mu(x); \quad (2.163)$$

Proof See Homework 2. □

2.14.3 Case of Non-Negative Lebesgue Measurable Functions

The Lebesgue integral of non-negative Lebesgue measurable functions is defined as follows.

Definition 2.91 (Lebesgue Integral of Non-Negative Lebesgue Measurable Functions). The Lebesgue integral of a non-negative Lebesgue measurable function $f : \mathcal{E} \to \mathbb{R}$, with $\mathcal{E} \subseteq \mathbb{R}^n$ and $n \in \mathbb{N}$, is

$$\int_{\mathcal{E}} f(x)\mathrm{d}\mu(x) \triangleq \sup \Big\{ \int_{\mathcal{E}} g(x)\mathrm{d}\mu(x) : g \text{ is Lebesgue measurable simple and } 0 \leqslant g(x) \leqslant f(x) \text{ for almost every } x \in \mathcal{E} \text{ w.r.t. } \mu \Big\}. \quad (2.164)$$

The integral of non-negative Lebesgue measurable functions possesses the following properties.

Theorem 2.92. *Consider two non-negative Lebesgue measurable functions $f : \mathcal{E} \to \mathbb{R}$ and $g : \mathcal{E} \to \mathbb{R}$, with $\mathcal{E} \subseteq \mathbb{R}^n$ and $n \in \mathbb{N}$. Then, the following holds:*

(*i*) *For all pairs $(\alpha, \beta) \in [0, +\infty[^2$,*

$$\int_{\mathcal{E}} (\alpha f(x) + \beta g(x))\, \mathrm{d}\mu(x) = \alpha \int_{\mathcal{E}} f(x)\mathrm{d}\mu(x) + \beta \int_{\mathcal{E}} g(x)\mathrm{d}\mu(x) \quad (2.165)$$

(*ii*) *Given two disjoint measurable subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{E}$,*

$$\int_{\mathcal{A}\cup\mathcal{B}} f(x)\mathrm{d}\mu(x) = \int_{\mathcal{A}} f(x)\mathrm{d}\mu(x) + \int_{\mathcal{B}} f(x)\mathrm{d}\mu(x); \quad (2.166)$$

(*iii*)

$$\int_{\mathcal{E}} f(x)\mathrm{d}\mu(x) = 0, \quad (2.167)$$

if and only if $f(x) = 0$ for almost every $x \in \mathcal{E}$;

(*iv*) *If $f(x) \leqslant g(x)$ for almost every $x \in \mathcal{E}$,*

$$\int_{\mathcal{E}} f(x)\mathrm{d}\mu(x) \leqslant \int_{\mathcal{E}} g(x)\mathrm{d}\mu(x); \text{ and} \quad (2.168)$$

(*v*) *If $f(x) = g(x)$ for almost every $x \in \mathcal{E}$,*

$$\int_{\mathcal{E}} f(x)\mathrm{d}\mu(x) = \int_{\mathcal{E}} g(x)\mathrm{d}\mu(x); \quad (2.169)$$

Proof See Homework 2. □

Theorem 2.93. *Consider a non-negative Lebesgue measurable function $f : \mathcal{E} \to [0, \infty]$, with $\mathcal{E} \subseteq \mathbb{R}^n$ and $n \in \mathbb{N}$ and assume that*

$$\int_{\mathcal{E}} f(x)\mathrm{d}\mu(x) < +\infty. \quad (2.170)$$

Then, it holds that $f(x) < +\infty$ for almost every $x \in \mathcal{E}$. The converse is not necessarily true.

Proof See Homework 2. □

2.14.4 Case of Absolutely Integrable Measurable Functions

Absolute integrability in the case of Lebesgue measurable functions can be described as follows.

DEFINITION 2.94 (Absolutely Integrable Functions). A Lebesgue measurable function $f : \mathcal{E} \to \mathbb{R}$, with $\mathcal{E} \subseteq \mathbb{R}^n$ and $n \in \mathbb{N}$, is said to be absolutely integrable if

$$\int_{\mathcal{E}} |f(x)| \, \mathrm{d}\mu(x) < \infty. \tag{2.171}$$

The Lebesgue integral of an absolutely integrable function is defined as follows.

DEFINITION 2.95 (Lebesgue Integral of Absolutely Integrable Functions). The Lebesgue integral of an absolutely integrable measurable function $f : \mathcal{E} \to \mathbb{R}$, with $\mathcal{E} \subseteq \mathbb{R}^n$ and $n \in \mathbb{N}$, is

$$\int_{\mathcal{E}} f(x)\mathrm{d}\mu(x) = \int_{\mathcal{E}} f^+(x)\mathrm{d}\mu(x) - \int_{\mathcal{E}} f^-(x)\mathrm{d}\mu(x). \tag{2.172}$$

The integral of absolutely integrable measurable functions possesses the following properties.

THEOREM 2.96. *Consider two absolutely integrable measurable functions $f : \mathcal{E} \to \mathbb{R}$ and $g : \mathcal{E} \to \mathbb{R}$, with $\mathcal{E} \subseteq \mathbb{R}^n$ and $n \in \mathbb{N}$. Then, the following holds:*

(*i*) *For all pairs $(\alpha, \beta) \in \mathbb{R}^2$,*

$$\int_{\mathcal{E}} (\alpha f(x) + \beta g(x)) \, \mathrm{d}\mu(x) = \alpha \int_{\mathcal{E}} f(x)\mathrm{d}\mu(x) + \beta \int_{\mathcal{E}} g(x)\mathrm{d}\mu(x) \tag{2.173}$$

(*ii*) *Given two disjoint measurable subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{E}$,*

$$\int_{\mathcal{A}\cup\mathcal{B}} f(x)\mathrm{d}\mu(x) = \int_{\mathcal{A}} f(x)\mathrm{d}\mu(x) + \int_{\mathcal{B}} f(x)\mathrm{d}\mu(x); \tag{2.174}$$

(*iii*) *For all subsets $\mathcal{A}$ of $\mathcal{E}$, the following holds*

$$\int_{\mathcal{A}} f(x)\mathrm{d}\mu(x) = \int_{\mathcal{E}} f(x)\mathbb{1}_{\{x\in\mathcal{A}\}}\mathrm{d}\mu(x); \tag{2.175}$$

(*iv*) *If* $f(x) = g(x)$ *for almost every* $x \in \mathcal{E}$,

$$\int_{\mathcal{E}} f(x)\mathrm{d}\mu(x) = \int_{\mathcal{E}} g(x)\mathrm{d}\mu(x); \text{ and} \tag{2.176}$$

(*v*)

$$\left|\int_{\mathcal{E}} f(x)\mathrm{d}\mu(x)\right| \leqslant \int_{\mathcal{E}} |f(x)|\,\mathrm{d}\mu(x). \tag{2.177}$$

Proof See Homework 2. □

3 Abstract Measure Theory

3.1 Measurable Spaces and Measurable Functions

DEFINITION 3.1 (Measurable Space). Given a set $\mathcal{O}$ and a σ-field $\mathscr{F}$ on $\mathcal{O}$, the pair $(\mathcal{O}, \mathscr{F})$ is said to be a measurable space.

DEFINITION 3.2 (Product of Measurable Spaces). Let $(\mathcal{A}, \mathscr{F})$ and $(\mathcal{B}, \mathscr{G})$ be two measurable spaces. The product of these measurable spaces is a measurable space denoted by $(\mathcal{A}, \mathscr{F}) \times (\mathcal{B}, \mathscr{G})$ such that

$$(\mathcal{A}, \mathscr{F}) \times (\mathcal{B}, \mathscr{G}) \triangleq (\mathcal{A} \times \mathcal{B}, \sigma(\mathscr{F} \times \mathscr{G})), \tag{3.1}$$

where $\sigma(\mathscr{F} \times \mathscr{G})$ is the smallest σ-field on $\mathcal{A} \times \mathcal{B}$ containing $\mathscr{F} \times \mathscr{G}$.

DEFINITION 3.3 (Measurable Function). Let $(\mathcal{A}, \mathscr{F})$ and $(\mathcal{B}, \mathscr{G})$ be two measurable spaces. The function $f : \mathcal{A} \to \mathcal{B}$ is said to be measurable relative to $(\mathcal{A}, \mathscr{F})$ and $(\mathcal{B}, \mathscr{G})$ if for all $\mathcal{G} \in \mathscr{G}$,

$$f^{-1}(\mathcal{G}) \in \mathscr{F}. \tag{3.2}$$

The verification of whether or not a function is measurable might be tedious and thus, the following theorem eases this task in the case in which the target σ-field is induced by a particular collection of sets.

THEOREM 3.4. *Let $(\mathcal{A}, \mathscr{F})$ and $(\mathcal{B}, \mathscr{G})$ be two measurable spaces, such that $\mathscr{G} = \sigma(\mathscr{D})$, for some $\mathscr{D}$. Then, a function $f : \mathcal{A} \to \mathcal{B}$ is measurable relative to $(\mathcal{A}, \mathscr{F})$ and $(\mathcal{B}, \mathscr{G})$ if and only if for all $\mathcal{D} \in \mathscr{D}$,*

$$f^{-1}(\mathcal{D}) \in \mathscr{F}. \tag{3.3}$$

Proof Consider that f is measurable relative to $(\mathcal{A}, \mathscr{F})$ and $(\mathcal{B}, \mathscr{G})$. Then, from Definition 3.3, it holds that for all $\mathcal{D} \in \mathscr{D} \subseteq \mathscr{G}$, $f^{-1}(\mathcal{D}) \in \mathscr{F}$, which proves the sufficiency.

To prove the necessity, assume that for all $\mathcal{D} \in \mathscr{D}$,

$$f^{-1}(\mathcal{D}) \in \mathscr{F}. \tag{3.4}$$

Under this assumption, proving that for all $\mathcal{G} \in \mathscr{G}$ it holds that $f^{-1}(\mathcal{G}) \in \mathscr{F}$ boils down to proving that $\mathscr{G} \subseteq \mathscr{M} \triangleq \{\mathcal{D} \subseteq \mathcal{B} : f^{-1}(\mathcal{D}) \in \mathscr{F}\}$. This is done in two steps. First, it is shown that $\mathscr{M}$ is a σ-field on $\mathcal{B}$. To prove that $\mathscr{M}$ is a σ-field on $\mathcal{B}$, given Definition 1.67, it suffices to verify that:

(a) Given that $f^{-1}(\mathcal{B}) = \mathcal{A} \in \mathscr{F}$, it holds that $\mathcal{B} \in \mathscr{M}$;

(b) For all $\mathcal{M} \in \mathscr{M}$, it holds that $f^{-1}(\mathcal{M}^{\mathsf{c}}) = (f^{-1}(\mathcal{M}))^{\mathsf{c}} \in \mathscr{F}$, given that $f^{-1}(\mathcal{M}) \in \mathscr{F}$; and

(c) For all $i \in \mathbb{N}$, let $\mathcal{M}_i$ be a subset of $\mathscr{M}$. Hence, $f^{-1}(\mathcal{M}_i) \in \mathscr{F}$ and $f^{-1}\left(\bigcup_{i\in\mathbb{N}} \mathcal{M}_i\right) = \bigcup_{i\in\mathbb{N}} f^{-1}\left(\mathcal{M}_i\right) \in \mathscr{F}$.

Second, from the assumption in (3.4), it holds that for all $\mathcal{D} \in \mathscr{D}$, $\mathcal{D} \subseteq \mathscr{M}$. Hence, given that $\mathscr{G} = \sigma(\mathscr{D})$ and $\mathscr{M}$ are both σ-fields, it holds that $\mathscr{G} \subseteq \mathscr{M}$. This completes the proof. □

THEOREM 3.5. *Consider a measurable function f relative to $(\mathcal{A}, \mathscr{E})$ and $(\mathcal{B}, \mathscr{F})$. Consider also a measurable function g relative to $(\mathcal{B}, \mathscr{F})$ and $(\mathcal{C}, \mathscr{G})$. Then, the composition $g \circ f$ is measurable relative to $(\mathcal{A}, \mathscr{E})$ and $(\mathcal{C}, \mathscr{G})$.*

Proof Let $\mathcal{G}$ be an arbitrary set in $\mathscr{G}$ and let $\mathcal{F} = g^{-1}(\mathcal{G})$ be the pre-image of $\mathcal{G}$ through g. Hence, given that the function g is measurable relative to $(\mathcal{B}, \mathscr{F})$ and $(\mathcal{C}, \mathscr{G})$, it follows that $\mathcal{F} = g^{-1}(\mathcal{G}) \in \mathscr{F}$. Alternatively, given that the function f is measurable relative to $(\mathcal{A}, \mathscr{E})$ and $(\mathcal{B}, \mathscr{F})$, it follows that $\mathcal{E} \triangleq f^{-1}(\mathcal{F}) \in \mathscr{E}$. Finally, $(g \circ f)^{-1}(\mathcal{G}) = f^{-1}(g^{-1}(\mathcal{G})) = f^{-1}(\mathcal{F}) = \mathcal{E} \in \mathscr{E}$, which completes the proof.

□

DEFINITION 3.6 (Borel Measurable Functions). A function f that is measurable relatively to $(\mathcal{A}, \mathscr{F})$ and $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ is said to be Borel measurable relative to $(\mathcal{A}, \mathscr{F})$. Moreover, when $\mathcal{A} = \mathbb{R}^k$ and $\mathscr{F} = \mathscr{B}\left(\mathbb{R}^k\right)$, for some $k > 0$, the function f is said to be Borel measurable.

Note that all Borel measurable functions are Lebesgue measurable (Definition 2.63), but the converse is not necessarily true.

DEFINITION 3.7 (Positive and Negative Parts). Given an arbitrary function $f : \mathcal{O} \to \mathbb{R}$, Borel measurable on $(\mathcal{O}, \mathscr{F})$, its positive part and *negative part* are non-negative functions denoted by $f^+ : \mathcal{O} \to \mathbb{R}_+$ and $f^- : \mathcal{O} \to \mathbb{R}_+$, respectively, satisfy for all $x \in \mathcal{O}$,

$$f^+(x) \triangleq \max\{f(x), 0\} \text{ and} \tag{3.5}$$

$$f^-(x) \triangleq -\min\{f(x), 0\}. \tag{3.6}$$

The positive and negative parts of a Borel measurable function relative to given measurable space satisfies the following property.

THEOREM 3.8. *Let f be an arbitrary Borel measurable function on $(\mathcal{O}, \mathscr{F})$. Then, the functions f^+ and f^- are both Borel measurable functions on $(\mathcal{O}, \mathscr{F})$.*

Proof From Definition 1.68, it holds that given the set $\mathcal{T} = \{]a, +\infty[: a \in \mathbb{R}\}$, the Borel σ-field satisfies $\mathscr{B}(\mathbb{R}) = \sigma(\mathcal{T})$. From Theorem 3.4, it holds that to prove that f^+ and f^- are both Borel measurable functions, it is enough to prove that the inverse of each of the intervals $]a, \infty[$ is in $\mathscr{F}$.

The function f^+ is a positive function, and thus, for all $a \in \mathbb{R}$, with $a < 0$, it holds that $(f^+)^{-1}(]a, \infty)) = \mathcal{O} \in \mathscr{F}$. Alternatively, for all $a \in \mathbb{R}$, with $a \geqslant 0$, it holds that $(f^+)^{-1}(]a, \infty[) = \{x \in \mathcal{O} : f^+(x) > a\} = \{x \in \mathcal{O} : \max\{f(x), 0\} > a\} = \{x \in \mathcal{O} : f(x) > a\} = f^{-1}(]a, \infty[) \in \mathscr{F}$, where the last inclusion holds because f is Borel measurable relative to $(\mathcal{O}, \mathscr{F})$.

Similarly, the function f^- is positive. Hence, for all $a \in \mathbb{R}$, with $a < 0$, it holds that $(f^-)^{-1}(]a, \infty[) = \mathcal{O} \in \mathscr{F}$. Alternatively, for all $a \in \mathbb{R}$, with $a \geqslant 0$, it holds that $(f^-)^{-1}(]a, \infty[) = \{x \in \mathcal{O} : f^-(x) > a\} = \{x \in \mathcal{O} : -\min\{f(x), 0\} > a\} = \{x \in \mathcal{O} : -f(x) > a\} = \{x \in \mathcal{O} : f(x) < -a\} = f^{-1}(] - \infty, -a[) \in \mathscr{F}$, where the last inclusion holds because f is Borel measurable relative to $(\mathcal{O}, \mathscr{F})$.

□

DEFINITION 3.9 (Isomorphic Measurable Spaces). Two measurable spaces $(\mathcal{A}, \mathscr{F})$ and $(\mathcal{B}, \mathscr{G})$ are said to be isomorphic if there exists a bijective function $f : \mathcal{A} \to \mathcal{B}$ that is measurable relative to $(\mathcal{A}, \mathscr{F})$ and $(\mathcal{B}, \mathscr{G})$ and its functional inverse f^{-1} is measurable relative to $(\mathcal{B}, \mathscr{G})$ and $(\mathcal{A}, \mathscr{F})$. If it exists, f is referred to as an isomorphism of $(\mathcal{A}, \mathscr{F})$ and $(\mathcal{B}, \mathscr{G})$.

DEFINITION 3.10 (Standard Measurable Spaces). A measurable space is said to be standard if it is isomorphic to a measurable space $(\mathcal{A}, \mathscr{B}(\mathcal{A}))$, with $\mathcal{A} \in \mathscr{B}(\mathbb{R})$.

Given a measure space $(\mathcal{O}, \mathscr{F})$, the elements of $\mathcal{O}$ are referred to as *outcomes*, whereas those in $\mathscr{F}$ are referred to as *events*. These denominations are often related to the fact that measurable spaces are the building blocks of *probability theory*. From this perspective, given an experiment, the set $\mathcal{O}$ contains all the "outcomes" that might be observed after the experiment. A particular "event" is a subset of "outcomes". More specifically, it is a set in $\mathscr{F}$. In order to determine whether or not an "event" $\mathcal{A} \in \mathscr{F}$ has taken place, all the corresponding outcomes must be verified. That is, all the outcomes of the experiment must be elements of the event $\mathcal{A}$. The intuition for the requirement of closeness under complementations follows from the fact that if a given event is verifiable so is the same event not taking place. The intuition for closeness under unions stems from the fact that events can be jointly verified.

A refinement of these intuitions leads to the notion of measure, which is reminiscent to the notion of a distance in a metric space, for instance.

3.2 Measures

DEFINITION 3.11 (Measure). A measure on a σ-field $\mathscr{F}$ is a non-negative real-valued function $\nu : \mathscr{F} \to [0, +\infty]$ such that

$$\nu(\emptyset) = 0; \tag{3.7}$$

and for all countable sequences of disjoint sets $\mathcal{A}_1, \mathcal{A}_2, \ldots$ in $\mathscr{F}$,

$$\nu\left(\bigcup_{t=1}^{\infty} \mathcal{A}_t\right) = \sum_{t=1}^{+\infty} \nu(\mathcal{A}_t). \tag{3.8}$$

Consider a measurable space $(\mathcal{O}, \mathscr{F})$, with $\mathcal{O}$ a finite set. The function $\nu : \mathscr{F} \to \mathbb{N} \cup \{0, +\infty\}$ such that for all $\mathcal{A} \in \mathscr{F}$,

$$\nu(\mathcal{A}) = |\mathcal{A}|, \tag{3.9}$$

is a measure. More specifically, it is said to be a *counting measure*, as it is a measure of the number of elements in the subsets of $\mathscr{F}$.

A more general exemple can be constructed in a measurable space $(\mathcal{O}, \mathscr{F})$, with $\mathcal{O}$ an arbitrary set. Let a be an element of $\mathcal{O}$, then the function $\delta_a : \mathscr{F} \to \{0, 1\}$ such that for all $\mathcal{A} \in \mathscr{F}$,

$$\delta_a(\mathcal{A}) = \begin{cases} 1 & \text{if} \quad a \in \mathcal{A} \\ 0 & \text{if} \quad a \notin \mathcal{A}, \end{cases} \tag{3.10}$$

is a measure on $(\mathcal{O}, \mathscr{F})$. Often, it is referred to as the *Dirac measure* on $(\mathcal{O}, \mathscr{F})$ with respect to a.

Note that a measure is always positive but it is not necessarily finite. This observation is formalized by the following definitions.

DEFINITION 3.12 (Finite Measure). Given a measure ν on the measurable space $(\mathcal{O}, \mathscr{F})$, it is said to be a finite measure if $\nu(\mathcal{O}) < \infty$.

A particular example of finite measures is that of probability measures. A measure ν on a σ-field $\mathscr{F}$ of elements of $\mathcal{O}$ is said to be a *probability measure* if it satisfies $\nu(\mathcal{O}) = 1$.

DEFINITION 3.13 (σ-Finite Measures). Given a measure ν on the measurable space $(\mathcal{O}, \mathscr{F})$, it is said to be a σ-finite if there exists an infinite sequence $\mathcal{A}_1, \mathcal{A}_2, \ldots,$ of sets in $\mathscr{F}$ such that $\bigcup_{t=1}^{\infty} \mathcal{A}_t = \mathcal{O}$ and for all $n \in \mathbb{N}$, $\nu(\mathcal{A}_n) < \infty$.

An exemple of a σ-finite measure is the Lebesgue measure (Definition 2.56) on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$, with $\mathscr{B}(\mathbb{R})$ being the Borel σ-field in $\mathbb{R}$ (Definition 1.68).

DEFINITION 3.14 (Concentration). A measure ν on the measurable space $(\mathcal{O}, \mathscr{F})$ is said to be *concentrated on* $\mathcal{A}$, if $\nu(\mathcal{A}^{c}) = 0$.

DEFINITION 3.15 (Measure Space). Given a measurable space $(\mathcal{O}, \mathscr{F})$ and a measure ν on $\mathscr{F}$, the triplet $(\mathcal{O}, \mathscr{F}, \nu)$ is referred to as a *measure space*.

A measure space $(\mathcal{O}, \mathscr{F}, \nu)$ whose measure ν is a probability measure is called a *probability space*.

The following theorem introduces some properties of measures.

THEOREM 3.16. *Let* $(\mathcal{O}, \mathscr{F}, \nu)$ *be a measure space. Then,*

(*i*) *For all pairs* $(\mathcal{A}, \mathcal{B}) \in \mathscr{F}^2$,

$$\nu(\mathcal{A} \cup \mathcal{B}) + \nu(\mathcal{A} \cap \mathcal{B}) = \nu(\mathcal{A}) + \nu(\mathcal{B})\,; \text{ and} \tag{3.11}$$

(*ii*) *For all pairs* $(\mathcal{A}, \mathcal{B}) \in \mathscr{F}^2$, *with* $\mathcal{A} \subset \mathcal{B}$,

$$\nu(\mathcal{B}) = \nu(\mathcal{A}) + \nu(\mathcal{B} \setminus \mathcal{A})\,. \tag{3.12}$$

Proof The proof of (*i*) is based on the fact that

$$\mathcal{A} = (\mathcal{A} \setminus \mathcal{B}) \cup (\mathcal{A} \cap \mathcal{B})\,; \text{ and} \tag{3.13}$$

$$\mathcal{B} = (\mathcal{B} \setminus \mathcal{A}) \cup (\mathcal{A} \cap \mathcal{B})\,, \tag{3.14}$$

where the sets $\mathcal{A} \setminus \mathcal{B}$, $\mathcal{A} \cap \mathcal{B}$ and $\mathcal{B} \setminus \mathcal{A}$ are mutually disjoint. Therefore,

$$\nu(\mathcal{A}) = \nu(\mathcal{A} \setminus \mathcal{B}) + \nu(\mathcal{A} \cap \mathcal{B}), \tag{3.15a}$$

$$\nu(\mathcal{B}) = \nu(\mathcal{B} \setminus \mathcal{A}) + \nu(\mathcal{A} \cap \mathcal{B}), \text{ and} \tag{3.15b}$$

$$\nu(\mathcal{A} \cup \mathcal{B}) = \nu(\mathcal{A} \cap \mathcal{B}) + \nu(\mathcal{B} \setminus \mathcal{A}) + \nu(\mathcal{A} \setminus \mathcal{B}). \tag{3.15c}$$

Adding the equations in (3.15) yields,

$$\nu(\mathcal{A}) + \nu(\mathcal{B}) = \nu(\mathcal{A} \cup \mathcal{B}) + \nu(\mathcal{A} \cap \mathcal{B}). \tag{3.16}$$

The proof of (*ii*) is based on the fact that $\mathcal{A}$ and $\mathcal{B} \setminus \mathcal{A}$ are disjoint sets and thus,

$$\nu(\mathcal{A}) + \nu(\mathcal{B} \setminus \mathcal{A}) = \nu(\mathcal{A} \cup (\mathcal{B} \setminus \mathcal{A})) = \nu(\mathcal{A} \cup \mathcal{B}) = \nu(\mathcal{B}). \tag{3.17}$$

where the last equality follows from the fact that $\mathcal{A} \subseteq \mathcal{B}$. This completes the proof. □

THEOREM 3.17. *Let* $(\mathcal{O}, \mathscr{F}, \nu)$ *be a measure space. Consider also an infinite sequence of subsets* $\mathcal{A}_1, \mathcal{A}_2, \ldots,$ *in* $\mathscr{F}$. *Then,*

(*i*) *if* $\mathcal{A}_n \uparrow \mathcal{A}$, $\lim_{n\to\infty} \nu(\mathcal{A}_n) = \nu(\mathcal{A})$; *and*
(*ii*) *if* $\mathcal{A}_n \downarrow \mathcal{A}$ *and* $\nu(\mathcal{O}) < \infty$, $\lim_{n\to\infty} \nu(\mathcal{A}_n) = \nu(\mathcal{A})$.

Proof Let $\mathcal{B}_1 \triangleq \mathcal{A}_1$ and for all $n \in \mathbb{N} \setminus \{1\}$, let $\mathcal{B}_n \triangleq \mathcal{A}_n \setminus \mathcal{A}_{n-1}$. Note that $\mathcal{B}_1$, $\mathcal{B}_2$, ... are mutually disjoint sets and $\bigcup_{i\in\mathbb{N}} \mathcal{B}_i = \bigcup_{i\in\mathbb{N}} \mathcal{A}_i = \mathcal{A}$ (Theorem 1.16), which yields:

$$\nu\left(\mathcal{A}\right) = \nu\left(\bigcup_{i\in\mathbb{N}} \mathcal{B}_i\right) = \sum_{i\in\mathbb{N}} \nu(\mathcal{B}_i). \tag{3.18}$$

Note also that for all $n \in \mathbb{N} \setminus \{1\}$, it holds that $\mathcal{A}_{n-1} \subseteq \mathcal{A}_n$. Hence, from Theorem 3.16, it follows that

$$\nu(\mathcal{A}_n) = \nu(\mathcal{A}_{n-1}) + \nu(\mathcal{A}_n \setminus \mathcal{A}_{n-1}) = \nu(\mathcal{A}_{n-1}) + \nu(\mathcal{B}_n), \tag{3.19}$$

and thus, $\nu(\mathcal{B}_n) = \nu(\mathcal{A}_n) - \nu(\mathcal{A}_{n-1})$; and $\nu(\mathcal{B}_1) = \nu(\mathcal{A}_1)$. Using these elements, the following holds:

$$\begin{aligned}
\nu(\mathcal{A}) &= \nu\left(\bigcup_{i\in\mathbb{N}} \mathcal{A}_i\right) && (3.20)\\
&= \nu\left(\bigcup_{i\in\mathbb{N}} \mathcal{B}_i\right) && (3.21)\\
&= \sum_{i\in\mathbb{N}} \nu(\mathcal{B}_i) && (3.22)\\
&= \nu(\mathcal{B}_1) + \sum_{i\geq 2} \nu(\mathcal{B}_i) && (3.23)\\
&= \nu(\mathcal{A}_1) + \sum_{i\geq 2} \left(\nu(\mathcal{A}_i) - \nu(\mathcal{A}_{i-1})\right) && (3.24)\\
&= \nu(\mathcal{A}_1) + \lim_{n\to\infty} \sum_{i=2}^{n} \left(\nu(\mathcal{A}_i) - \nu(\mathcal{A}_{i-1})\right) && (3.25)\\
&= \nu(\mathcal{A}_1) + \lim_{n\to\infty} \left(\nu(\mathcal{A}_n) - \nu(\mathcal{A}_1)\right) && (3.26)\\
&= \lim_{n\to\infty} \nu(\mathcal{A}_n), && (3.27)
\end{aligned}$$

which completes the proof of statement (*i*).

The proof of statement (*ii*) is as follows. For all $n \in \mathbb{N} \setminus \{1\}$, let $\mathcal{B}_n = \mathcal{A}_n^{\mathsf{c}}$. This implies that $\mathcal{B}_1 \subseteq \mathcal{B}_2 \subseteq \ldots$. Hence, from Theorem 1.12, it holds that,

$$\left(\bigcup_{i\in\mathbb{N}} \mathcal{B}_i\right)^{\mathsf{c}} = \bigcap_{i\in\mathbb{N}} \mathcal{B}_i^{\mathsf{c}} = \bigcap_{i\in\mathbb{N}} \mathcal{A}_i = \mathcal{A} \tag{3.28}$$

Therefore, $\mathcal{B}_n \uparrow \mathcal{A}^{\mathsf{c}}$ and thus, from (3.27) it follows that $\lim_{n\to\infty} \nu(\mathcal{B}_n) = \nu(\mathcal{A}^c)$. Given that ν is a finite measure, i.e., $\nu(\mathcal{O}) < \infty$, and $\mathcal{A} \subset \mathcal{O}$, it holds from

Theorem 3.16 that

$$\nu(\mathcal{A}) = \nu(\mathcal{O}) - \nu(\mathcal{A}^c) = \nu(\mathcal{O}) - \lim_{n\to\infty} \nu(\mathcal{B}_n) \tag{3.29}$$

$$= \nu(\mathcal{O}) - \lim_{n\to\infty} \nu\left((\mathcal{O}) - \nu(\mathcal{A}_n)\right) \tag{3.30}$$

$$= \nu(\mathcal{O}) - (\nu(\mathcal{O}) - \lim \nu(\mathcal{A}_n)) \tag{3.31}$$

$$= \lim \nu(\mathcal{A}_n), \tag{3.32}$$

which completes the proof. □

3.3 General Integration

Given a measure space $(\mathcal{O}, \mathscr{F}, \nu)$ and a Borel measurable function f relative to $(\mathcal{O}, \mathscr{F})$, the integral of the function f with respect to ν, often referred to as *Lebesgue integral*, is denoted by

$$\int_{\mathcal{O}} f \mathrm{d}\nu, \text{ or } \int_{\mathcal{O}} f(x)\nu(\mathrm{d}x), \text{ or } \int_{\mathcal{O}} f(x)\mathrm{d}\nu(x), \text{ or } \nu f, \text{ or } \nu(f). \tag{3.33}$$

Nonetheless, the notation used in the following would be $\int_{\mathcal{O}} f \mathrm{d}\nu$.

Definition 3.18 (Borel Measurable Simple Functions). Consider a mesurable space $(\mathcal{O}, \mathscr{F})$. Then, a function $f : \mathcal{O} \to \mathbb{R}$ is said to be a Borel measurable simple function if it is Borel measurable relative to $(\mathcal{O}, \mathscr{F})$ and it takes finitely many different values.

Every Borel measurable simple function $f : \mathcal{O} \to \mathbb{R}$ relative to $(\mathcal{O}, \mathscr{F})$ can be written as follows:

$$f(x) = \sum_{t=1}^{m} a_t \mathbb{1}_{\{x \in \mathcal{A}_t\}}, \tag{3.34}$$

where $m \in \mathbb{N}$ is finite, $(a_1, a_2, \ldots, a_m) \in \mathbb{R}^m$ and $\mathcal{A}_1, \mathcal{A}_2, \ldots \mathcal{A}_m$ are disjoint sets in $\mathscr{F}$. Note that a Borel measurable simple function is a Lebesgue measurable simple function. Nonetheless, the converse is not necessarily true.

Definition 3.19 (Increasing and Decreasing Sequences of Functions). Consider a sequence of Borel measurable functions relative to $(\mathcal{O}, \mathscr{F})$, denoted by f_1, f_2, f_3, ... The sequence is said to be increasing if for all (m, n) with $m < n$, it holds that for all $x \in \mathcal{O}$,

$$f_m(x) < f_n(x). \tag{3.35}$$

Alternatively, the sequence is said to be decreasing if for all (m, n) with

$m < n$, it holds that for all $x \in \mathcal{O}$,

$$f_m(x) > f_n(x). \tag{3.36}$$

The following is a fundamental property of Borel measurable functions in terms of increasing sequences of simple functions.

THEOREM 3.20. *Given a mesurable space $(\mathcal{O}, \mathscr{F})$, any non-negative Borel measurable function f relative to $(\mathcal{O}, \mathscr{F})$ is the limit of an increasing sequence of non-negative, finite Borel measurable simple functions.*

Proof The proof is by construction. For all $n \in \mathbb{N}$, consider the functions $f_n : \mathcal{O} \to \mathbb{R}$ defined as follows:

$$f_n(x) = \begin{cases} \frac{k-1}{2^n} & \text{if} \quad \frac{k-1}{2^n} \leqslant f(x) < \frac{k}{2^n} \text{ for some } k \in \{1, 2, \ldots, n2^n\} \\ n & \text{if} \quad f(x) \geqslant n. \end{cases} \tag{3.37}$$

Note that, for all $n \in \mathbb{N}$, f_n is a non-negative quantizer of the function f with resolution $\frac{1}{2^n}$ and span n. Thus, it is a Borel measurable simple function. Moreover, for all $x \in \mathcal{O}$, $\lim_{n \to \infty} f_n(x) = f(x)$. □

Theorem 3.20 leads to the following more general result.

THEOREM 3.21. *Given a mesurable space $(\mathcal{O}, \mathscr{F})$, any arbitrary Borel measurable function f relative to $(\mathcal{O}, \mathscr{F})$ is the limit of a sequence of finite Borel measurable simple functions f_1, f_2, …, such that for all $n \in \mathbb{N}$ and for all $x \in \mathcal{O}$, $|f_n(x)| < |f(x)|$.*

Using these notations, given a measure space $(\mathcal{O}, \mathscr{F}, \nu)$ and a Borel measurable function f relative to $(\mathcal{O}, \mathscr{F})$, the Lebesgue integral of the function f with respect to ν is defined hereunder.

DEFINITION 3.22 (Lebesgue Integral). Given a measurable space $(\mathcal{O}, \mathscr{F}, \nu)$ and a Borel measurable function $f : \mathcal{O} \to \mathbb{R}$ relative to $(\mathcal{O}, \mathscr{F})$, the integral of the function f with respect to ν is defined as follows:

- when f is a non-negative Borel measurable simple function (BMSF), that is, for all $x \in \mathcal{O}$, $f(x) = \sum_{t=1}^{m} a_t \mathbb{1}_{\{x \in \mathcal{A}_t\}}$, for some finite $m \in \mathbb{N}$, $(a_1, a_2, \ldots, a_m) \in [0, +\infty]^m$ and $\mathcal{A}_1, \mathcal{A}_2, \ldots \mathcal{A}_m$ are sets in $\mathscr{F}$, then

$$\int_{\mathcal{O}} f \mathrm{d}\nu \triangleq \sum_{t=1}^{m} a_t \nu\left(\mathcal{A}_t\right); \tag{3.38}$$

- when f is non-negative Borel measurable relative to $(\mathcal{O}, \mathscr{F})$, then,

$$\int_{\mathcal{O}} f \mathrm{d}\nu \triangleq \sup \left\{ \int_{\mathcal{O}} g \mathrm{d}\nu : g \text{ is a BMSF relative to } (\mathcal{O}, \mathscr{F}) \text{ and} \right.$$
$$\left. \forall x \in \mathcal{O}, 0 \leqslant g(x) \leqslant f(x) \right\}; \text{and} \tag{3.39}$$

- when f is an arbitrary Borel measurable function relative to $(\mathcal{O}, \mathscr{F})$, then,

$$\int_{\mathcal{O}} f \mathrm{d}\nu \triangleq \int_{\mathcal{O}} f^{+} \mathrm{d}\nu - \int_{\mathcal{O}} f^{-} \mathrm{d}\nu. \tag{3.40}$$

The Lebesgue integral $\int_{\mathcal{O}} f \mathrm{d}\nu$ of an arbitrary Borel measurable function f relative to $(\mathcal{O}, \mathscr{F})$ is said to exist if the indetermination $+\infty + -\infty$ does not appear in the sum in (3.40). This indetermination is always avoided in the case in which the function f is either non-negative or non-positive. Therefore, the Lebesgue integral of a non-negative function or a non-positive function always exists. Nonetheless, the former might be $+\infty$, whereas the latter might be $-\infty$.

When the Lebesgue integral $\int_{\mathcal{O}} f \mathrm{d}\nu$ is finite, the function f is said to be Lebesgue integrable with respect to the measure space $(\mathcal{O}, \mathscr{F}, \nu)$.

A condition that ensures the finiteness of the Lebesgue integral is the absolute integrability.

Definition 3.23 (Absolute Integrability). Consider a measure space $(\mathcal{O}, \mathscr{F}, \nu)$ and a Borel measurable function f relative to $(\mathcal{O}, \mathscr{F})$. The function f is said to be absolutely integrable if

$$\int_{\mathcal{O}} |f| \, \mathrm{d}\nu < \infty. \tag{3.41}$$

When the integral of f with respect to the measure ν is over a particular set $\mathcal{A} \in \mathscr{F}$ other than $\mathcal{O}$, i.e., $\int_{\mathcal{A}} f \mathrm{d}\nu$, it follows that:

$$\int_{\mathcal{A}} f \mathrm{d}\nu = \int_{\mathcal{O}} f(x) \mathbb{1}_{\{x \in \mathcal{A}\}} \mathrm{d}\nu(x). \tag{3.42}$$

The following theorem compares the integrals $\int_{\mathcal{A}} f \mathrm{d}\nu$ and $\int_{\mathcal{O}} f \mathrm{d}\nu$.

Theorem 3.24. *Let f be a non-negative Borel measurable function relative to $(\mathcal{O}, \mathscr{F})$ and ν a measure on $(\mathcal{O}, \mathscr{F})$. Assume also that* $\int_{\mathcal{O}} f \mathrm{d}\nu < +\infty$.

Then, for all $\mathcal{A} \subset \mathcal{O}$, it holds that $\int_{\mathcal{A}} f \mathrm{d}\nu < +\infty$ and

$$\int_{\mathcal{A}} f \mathrm{d}\nu \leqslant \int_{\mathcal{O}} f \mathrm{d}\nu. \tag{3.43}$$

Proof Assume that f is a non-negative simple function with the form in (3.38). Then, let $g : \mathcal{O} \to \mathbb{R}$ be for all $x \in \mathcal{O}$,

$$g(x) = f(x)\mathbb{1}_{\{x\in\mathcal{A}\}} = \sum_{t=1}^{m} a_t \mathbb{1}_{\{x\in\mathcal{A}_t\}}\mathbb{1}_{\{x\in\mathcal{A}\}} = \sum_{t=1}^{m} a_t \mathbb{1}_{\{x\in\mathcal{A}_t\cap\mathcal{A}\}}, \tag{3.44}$$

which is also a simple function. Hence,

$$\int_{\mathcal{A}} f \mathrm{d}\nu = \int_{\mathcal{A}} g \mathrm{d}\nu = \sum_{t=1}^{m} a_t \nu(\mathcal{A}_t \cap \mathcal{A}) \leqslant \sum_{t=1}^{m} a_t \nu(\mathcal{A}_t) = \int_{\mathcal{O}} f \mathrm{d}\nu < +\infty. \tag{3.45}$$

The proof continues with the analysis of non-negative functions (other than simple functions) using the same argument. □

Definition 3.25. Given measurable space $(\mathcal{O}, \mathscr{F})$ a set $\mathcal{A} \in \mathscr{F}$ and an arbitrary Borel measurable function f : relative to $(\mathcal{O}, \mathscr{F})$, the integral

$$\int_{\mathcal{A}} f \mathrm{d}\nu \tag{3.46}$$

is referred to as the indefinite integral of f with respect to ν on $\mathcal{A}$.

The denomination of indefinite integral stems from the fact that if $\mathcal{O} = \mathbb{R}$, $\mathscr{F} = \mathscr{B}(\mathbb{R})$ and ν is the Lebesgue measure (Definition 2.56), then given an interval $\mathcal{A} = [a, x]$, it follows that if f is Riemman integrable,

$$\int_{\mathcal{A}} f \mathrm{d}\nu = \int_{a}^{x} f(t)\mathrm{d}t, \tag{3.47}$$

where the integral on the right hand side of (3.47) is the Riemman indefinite integral.

Lebesgue integrals exhibit several properties that are reminiscent to those of Riemman integrals. The following theorem highlights one of those properties.

Theorem 3.26 (Integration of a weighted function). *Let f be a Borel measurable function relative to $(\mathcal{O}, \mathscr{F})$ and ν a measure on $(\mathcal{O}, \mathscr{F})$. Then, if $\int_{\mathcal{O}} f \mathrm{d}\nu$ exists, it holds that for all $c \in \mathbb{R}$, $\int_{\mathcal{O}} c f \mathrm{d}\nu$ exists and*

$$\int_{\mathcal{O}} c f \mathrm{d}\nu = c \int_{\mathcal{O}} f \mathrm{d}\nu. \tag{3.48}$$

Proof TBW □

THEOREM 3.27. *Let f and g be two Borel measurable functions relative to $(\mathcal{O}, \mathscr{F})$ and ν a measure on $(\mathcal{O}, \mathscr{F})$. Then, if for all $x \in \mathcal{O}$, $f(x) \geqslant g(x)$, it follows that $\int_{\mathcal{O}} f \mathrm{d}\nu \geqslant \int_{\mathcal{O}} g \mathrm{d}\nu$, given that both integrals exist.*

Proof TBW □

THEOREM 3.28. *Let f be a Borel measurable function with respect to $(\mathcal{O}, \mathscr{F})$ and ν a measure on $(\mathcal{O}, \mathscr{F})$. Then, if $\int_{\mathcal{O}} f \mathrm{d}\nu$ exists, it holds that*

$$\left| \int_{\mathcal{O}} f \mathrm{d}\nu \right| \leqslant \int_{\mathcal{O}} |f| \,\mathrm{d}\nu. \tag{3.49}$$

Proof TBW □

THEOREM 3.29 (Additivity of Integrals). *Let f and g be two Borel measurable functions with respect to $(\mathcal{O}, \mathscr{F})$ and ν a measure on $(\mathcal{O}, \mathscr{F})$. Then, when the integrals $\int_{\mathcal{O}} f \mathrm{d}\nu$ and $\int_{\mathcal{O}} g \mathrm{d}\nu$ exist, and $\int_{\mathcal{O}} f \mathrm{d}\nu + \int_{\mathcal{O}} f \mathrm{d}\nu$ is not of the form $+\infty + -\infty$ or $-\infty + \infty$, it holds that*

$$\int_{\mathcal{O}} f + g \,\mathrm{d}\nu = \int_{\mathcal{O}} f \mathrm{d}\nu + \int_{\mathcal{O}} g \mathrm{d}\nu. \tag{3.50}$$

Proof TBW □

3.4 Monotone Convergence

Using the notion of increasing sequences of functions (Definition 3.19), the monotone convergence theorem can be stated as follows.

THEOREM 3.30 (Monotone Convergence). *Let $(\mathcal{O}, \mathscr{F}, \nu)$ be a measure space and f be a non-negative Borel measurable function relative to $(\mathcal{O}, \mathscr{F})$. Let also f_1, f_2, f_3, ... be an increasing sequence of non-negative Borel measurable functions relative to $(\mathcal{O}, \mathscr{F})$. Assume that for all $x \in \mathcal{O}$,*

$$\lim_{t \to \infty} f_t(x) = f(x). \tag{3.51}$$

Then, if follows that

$$\lim_{t \to \infty} \int_{\mathcal{O}} f_t(x) \mathrm{d}\nu = \int_{\mathcal{O}} f(x) \mathrm{d}\nu. \tag{3.52}$$

Proof TBW □

THEOREM 3.31 (Fatou's Lemma). *Let* $(\mathcal{O}, \mathscr{F}, \nu)$ *be a measure space and* f *and* f_1, f_2, f_3, ... *be non-negative Borel measurable functions relative to* $(\mathcal{O}, \mathscr{F})$. *Then,*

- *when for all* $n \in \mathbb{N}$ *and for all* $x \in \mathcal{O}$, $f_n(x) \geqslant f(x)$ *and* $\int_{\mathcal{O}} f \mathrm{d}\nu > -\infty$, *it holds that*

$$\liminf_{n\to\infty} \int_{\mathcal{O}} f_n \mathrm{d}\nu \geqslant \int_{\mathcal{O}} \left(\liminf_{n\to\infty} f_n\right) \mathrm{d}\nu \tag{3.53}$$

- *when for all* $n \in \mathbb{N}$ *and for all* $x \in \mathcal{O}$, $f_n(x) \leqslant f(x)$ *and* $\int_{\mathcal{O}} f \mathrm{d}\nu < \infty$, *it holds that*

$$\limsup_{n\to\infty} \int_{\mathcal{O}} f_n \mathrm{d}\nu \leqslant \int_{\mathcal{O}} \left(\limsup_{n\to\infty} f_n\right) \mathrm{d}\nu \tag{3.54}$$

Proof TBW □

3.5 Dominated Convergence

THEOREM 3.32 (Dominated Convergence). *Let* $(\mathcal{O}, \mathscr{F}, \nu)$ *be a measure space and* f, g *and* f_1, f_2, f_3, ... *be Borel measurable functions relative to* $(\mathcal{O}, \mathscr{F})$. *Assume that for all* $n \in \mathbb{N}$ *and for all* $x \in \mathcal{O}$, $|f_n(x)| \leqslant g(x)$ *and* $\int_{\mathcal{O}} |g| \, \mathrm{d}\nu < +\infty$ *and* $\lim_{n\to\infty} f_n(x) = f(x)$ *almost everywhere with respect to* ν. *Then,* $\int_{\mathcal{O}} |f| \, \mathrm{d}\nu < +\infty$ *and*

$$\lim_{n\to\infty} \int_{\mathcal{O}} f_n \mathrm{d}\nu = \int_{\mathcal{O}} f \mathrm{d}\nu. \tag{3.55}$$

Proof TBW □

3.6 Radon-Nikodym Derivative

In order to introduce the notion of Radon-Nikodym derivative, the notion of absolute continuity of a measure with respect to another is introduced.

DEFINITION 3.33 (Absolute Continuity). Given two measures ν and λ on a measurable space $(\mathcal{O}, \mathscr{F})$, λ is said to be absolutely continuous with respect to ν, if for all $\mathcal{A} \in \mathscr{F}$ for which $\nu(\mathcal{A}) = 0$, it holds that $\lambda(\mathcal{A}) = 0$.

THEOREM 3.34. *Given a measure space* $(\mathcal{O}, \mathscr{F}, \nu)$ *and a non-negative Borel measurable function* $f : \mathcal{O} \to \mathbb{R}$ *relative to* $(\mathcal{O}, \mathscr{F})$ *such that* $\displaystyle\int_{\mathcal{O}} f \mathrm{d}\nu < +\infty$, *let $\lambda : \mathscr{F} \to [0, +\infty]$ be such that for all the $\mathcal{A} \in \mathscr{F}$,*

$$\lambda(\mathcal{A}) = \int_{\mathcal{A}} f \mathrm{d}\nu. \tag{3.56}$$

Then, λ is a measure on $(\mathcal{O}, \mathscr{F})$.

Proof To prove the conditions imposed by Definition 3.11, it suffices to verify that λ is a non-negative function, $\lambda(\emptyset) = 0$, and

$$\lambda\left(\bigcup_{i\in\mathbb{N}} \mathcal{A}_i\right) = \sum_{i\in\mathbb{N}} \lambda(\mathcal{A}_i). \tag{3.57}$$

Note that f is a non-negative function and thus, from Theorem 3.27, it follows that λ is non-negative. Let $\mathcal{A}_1$, $\mathcal{A}_2$, ... be mutually disjoint sets in $\mathscr{F}$. Hence, the following holds:

$$\lambda\left(\bigcup_{i\in\mathbb{N}} \mathcal{A}_i\right) = \int_{\bigcup_{i\in\mathbb{N}} \mathcal{A}_i} f(x)\mathrm{d}\nu(x) = \int_{\mathcal{O}} f(x)\mathbb{1}_{\left\{x\in \bigcup_{i\in\mathbb{N}} \mathcal{A}_i\right\}}\mathrm{d}\nu(x). \tag{3.58}$$

From the assumption that $\mathcal{A}_1$, $\mathcal{A}_2$, ... are mutually disjoint sets, it holds that $\mathbb{1}_{\left\{x\in \bigcup_{i\in\mathbb{N}} \mathcal{A}_i\right\}} = \sum_{i\in\mathbb{N}} \mathbb{1}_{\{x\in\mathcal{A}_i\}}$. Thus, from (3.58), it holds that

$$\lambda\left(\bigcup_{i\in\mathbb{N}} \mathcal{A}_i\right) = \int_{\mathcal{O}} f(x) \sum_{i\in\mathbb{N}} \mathbb{1}_{\{x\in\mathcal{A}_i\}}\mathrm{d}\nu(x) \tag{3.59}$$

$$= \int_{\mathcal{O}} \left(\sum_{i\in\mathbb{N}} f(x)\mathbb{1}_{\{x\in\mathcal{A}_i\}}\right)\mathrm{d}\nu(x). \tag{3.60}$$

For all $n \in \mathbb{N}$, let the function $g_n : \mathcal{O} \to \mathbb{R}$ be such that for all $x \in \mathcal{O}$, it holds that $g_n(x) = \sum_{i=1}^{n} f(x)\mathbb{1}_{\{x\in\mathcal{A}_i\}}$, which is a sum of non-negative terms, and thus, $g_1, g_2, \ldots$ form an increasing sequence of nonnegative functions. Hence, from the monotone convergence theorem (Theorem 3.30), it follows that

$$\int_{\mathcal{O}} \lim_{n\to\infty} g_n(x)\mathrm{d}\nu(x) = \lim_{n\to\infty} \int_{\mathcal{O}} g_n(x)\mathrm{d}\nu(x), \tag{3.61}$$

which implies the following,

$$\int_{\mathcal{O}} \left(\sum_{i\in\mathbb{N}} f(x)\mathbb{1}_{\{x\in\mathcal{A}_i\}}\right)\mathrm{d}\nu(x) = \int_{\mathcal{O}} \lim_{n\to\infty} g_n(x)\mathrm{d}\nu(x) \tag{3.62}$$

$$= \lim_{n\to\infty} \int_{\mathcal{O}} g_n(x)\mathrm{d}\nu(x) \tag{3.63}$$

$$= \lim_{n\to\infty} \int_{\mathcal{O}} \sum_{i=1}^{n} f(x)\mathbb{1}_{\{x\in\mathcal{A}_i\}}\mathrm{d}\nu(x). \tag{3.64}$$

From the assumptions of the theorem, it follows that the integral $\int_{\mathcal{O}} f\mathrm{d}\nu$ exists, which implies that the integral $\int_{\mathcal{A}} f\mathrm{d}\nu$ exists for all $\mathcal{A} \in \mathscr{F}$ (Theorem 3.24). Hence, from (3.59) and Theorem 3.29, the following holds

$$\lambda\left(\bigcup_{i\in\mathbb{N}} \mathcal{A}_i\right) = \lim_{n\to\infty} \int_{\mathcal{O}} \sum_{i=1}^{n} f(x)\mathbb{1}_{\{x\in\mathcal{A}_i\}}\mathrm{d}\nu(x) \tag{3.65}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \int_{\mathcal{O}} f(x)\mathbb{1}_{\{x\in\mathcal{A}_i\}}\mathrm{d}\nu(x) \tag{3.66}$$

$$= \sum_{i=1}^{\infty} \int_{\mathcal{O}} f(x)\mathbb{1}_{\{x\in\mathcal{A}_i\}}\mathrm{d}\nu(x) \tag{3.67}$$

$$= \sum_{i=1}^{\infty} \int_{\mathcal{A}_i} f(x)\mathrm{d}\nu(x) \tag{3.68}$$

$$= \sum_{i\in\mathbb{N}} \lambda(\mathcal{A}_i), \tag{3.69}$$

which proves the second condition imposed by Definition 3.11.

To prove the first condition imposed by Definition 3.11, note that from the assumption that $\int_{\mathcal{O}} f\mathrm{d}\nu < \infty$, it holds that $\lambda(\mathcal{O}) < +\infty$. Then, $\lambda(\mathcal{O}) = \lambda(\mathcal{O} \cup \emptyset) = \lambda(\mathcal{O}) + \lambda(\emptyset) < +\infty$. This implies that $\lambda(\emptyset) = 0$, which proves the first condition imposed by Definition 3.11, and completes the proof. □

The measure λ in Theorem 3.34 is the only measure that can be generated from ν through the function f up to negligible sets with respect to ν. The following theorem formalizes this intuition.

Theorem 3.35. *Consider a measure space $(\mathcal{O}, \mathscr{F}, \nu)$, with ν a σ-finite measure, and consider also two non-negative Borel measurable functions f and g relative to $(\mathcal{O}, \mathscr{F})$ such that $\int_{\mathcal{O}} f\mathrm{d}\nu$ and $\int_{\mathcal{O}} g\mathrm{d}\nu$ exist. Let $\lambda_1 : \mathscr{F} \to [0, +\infty]$ and $\lambda_2 : \mathscr{F} \to [0, +\infty]$ be two measures on $(\mathcal{O}, \mathscr{F})$ such that for all the $\mathcal{A} \in \mathscr{F}$,*

$$\lambda_1(\mathcal{A}) = \int_{\mathcal{A}} f\mathrm{d}\nu, \text{ and} \tag{3.70}$$

$$\lambda_2(\mathcal{A}) = \int_{\mathcal{A}} g\mathrm{d}\nu. \tag{3.71}$$

Then, λ_1 and λ_2 are identical if and only if for almost every $x \in \mathcal{O}$ with respect to ν, $f(x) = g(x)$.

Proof See Homework 3. □

EXERCISE 3.36. Show via an example that Theorem 3.35 fails if the assumption that ν is σ-finite is neglected.

Note that in Theorem 3.34, for all $\mathcal{A}$ for which $\nu(\mathcal{A}) = 0$, it follows that $\lambda(\mathcal{A}) = 0$. That is, the measure λ generated from ν through the function f is absolutely continuous with respect to ν. The following theorem states the converse: if λ is absolutely continuous with respect to ν, then λ is obtained as the indefinite integral of f with respect to a measure ν, with f being a unique non-negative Boreal measurable function relative to $(\mathcal{O}, \mathscr{F})$.

THEOREM 3.37 (Radon-Nikodym Theorem). *Let λ and ν be two measures on a given measurable space $(\mathcal{O}, \mathscr{F})$, such that ν is σ-finite and λ is absolutely continuous with respect to ν. Then, there exists a Borel measurable function $f : \mathcal{O} \to \bar{\mathbb{R}}$ such that for all $\mathcal{A} \in \mathscr{F}$,*

$$\lambda(\mathcal{A}) = \int_{\mathcal{A}} f \mathrm{d}\nu. \tag{3.72}$$

Moreover, the function g is unique almost everywhere with respect to λ.

Proof See Homework 3. □

The function f in (3.72) is often referred to as the **density** of λ with respect to ν, the **likelihood ratio** of λ with respect to ν, or the **Radon-Nikodym derivative** of λ with respect to ν. To emphasize this denomination, it is often denoted by $\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}$.

The following results are immediate extensions of Theorem 3.37.

COROLLARY 3.38. *Let λ and ν be two measures on a given measurable space $(\mathcal{O}, \mathscr{F})$. Then, ν is absolutely continuous with respect to λ if and only if there exists a Borel measurable function $f : \mathcal{O} \to \mathbb{R}$ such that for all $\mathcal{A} \in \mathscr{F}$, the equality in* (3.72) *holds.*

Proof See Homework 3. □

Another immediate result from Theorem 3.37 is the following.

COROLLARY 3.39. *Let $(\mathcal{O}, \mathscr{F}, \nu)$ be a measure space with $\mathcal{O}$ a countable set and ν a counting measure. That is, for all $\mathcal{A} \in \mathscr{F}$,*

$$\nu(\mathcal{A}) = |A|. \tag{3.73}$$

Let also λ be a measure on $(\mathcal{O}, \mathscr{F})$. Then, λ is absolutely continuous with respect to ν.

Proof See Homework 3. □

Exercise 3.40. Consider the measure space $([0,+\infty[, \mathscr{B}([0,+\infty[,), \mu)$, with μ the Lebesgue measure. Let ν be absolutely continuous with respect to μ and assume that the Radon-Nikodym derivative $\frac{\mathrm{d}\nu}{\mathrm{d}\mu}$ is a continuous function. Show that for all $x \in [0,+\infty[$, the following holds:

$$\frac{\mathrm{d}}{\mathrm{d}x}\nu([0,x]) = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x). \tag{3.74}$$

The following theorem describes some of the properties of the Randon-Nikodym derivative.

Theorem 3.41. *Let λ and ν be two measures on a given measurable space $(\mathcal{O}, \mathscr{F})$ with ν being absolutely continuous with respect to λ and λ being σ-finite. Then,*
(i) The function $\frac{\mathrm{d}\lambda}{\mathrm{d}\lambda}$ is constant equal to one almost everywhere with respect to λ;
(ii) if $f : \mathcal{O} \to \mathbb{R}_+$ is a non-negative Borel measurable function with respect to $(\mathcal{O}, \mathscr{F})$, it holds that for all $\mathcal{A} \in \mathscr{F}$,

$$\int_{\mathcal{A}} f \mathrm{d}\nu = \int_{\mathcal{A}} f \frac{\mathrm{d}\nu}{\mathrm{d}\lambda}\mathrm{d}\lambda, \tag{3.75}$$

if the integrals exist;
(iii) if γ is a σ-finite measure on $(\mathcal{O}, \mathscr{F})$, λ is absolutely continuous with respect to γ, it holds that

$$\frac{\mathrm{d}\nu}{\mathrm{d}\gamma} = \frac{\mathrm{d}\nu}{\mathrm{d}\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}\gamma} \text{ almost everywhere w.r.t. } \gamma\text{; and} \tag{3.76}$$

(iv) if λ is absolutely continuous with respect to ν, and ν is σ-finite, it holds that the product of functions

$$\frac{\mathrm{d}\nu}{\mathrm{d}\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}\nu} = 1 \text{ almost everywhere w.r.t. } \lambda \text{ and } \nu. \tag{3.77}$$

Proof *Proof of (i):* First, consider that the σ-finite measure λ is absolutely continuous with respect to itself. Hence, from Theorem 3.37, it follows that for all $\mathcal{A} \in \mathscr{F}$,

$$\lambda(\mathcal{A}) = \int_{\mathcal{A}} \mathrm{d}\lambda = \int_{\mathcal{A}} \frac{\mathrm{d}\lambda}{\mathrm{d}\lambda}\, \mathrm{d}\lambda, \tag{3.78}$$

where the function $\frac{\mathrm{d}\lambda}{\mathrm{d}\lambda}$ is unique up to sets of measure zero with respect to λ. Hence, the function $\frac{\mathrm{d}\lambda}{\mathrm{d}\lambda}$ is equal to one almost everywhere with respect to λ.

Proof of (ii): From Theorem 3.37, it holds that for all $\mathcal{A} \in \mathscr{F}$,

$$\nu(\mathcal{A}) = \int_{\mathcal{A}} \frac{\mathrm{d}\nu}{\mathrm{d}\lambda}\mathrm{d}\lambda. \tag{3.79}$$

Let $\mathcal{A}$ be an arbitrary set in $\mathscr{F}$ and assume that f is a simple function of the form $f(x) = \sum_{i=1}^{m} a_i \mathbb{1}_{\{x \in \mathcal{A}_i\}}$, for some finite $m \in \mathbb{N}$, some subsets $\mathcal{A}_1$, $\mathcal{A}_2$ ...,

$\mathcal{A}_m$ forming a partition of $\mathcal{O}$ and a_1, a_2, …, a_m being non-negative reals. For all $i \in \{1, 2, \ldots, m\}$, let $\mathcal{B}_i = \mathcal{A} \cap \mathcal{A}_i$ be a subset of $\mathcal{F}$. Hence,

$$\int_{\mathcal{A}} f \frac{\mathrm{d}\nu}{\mathrm{d}\lambda} \mathrm{d}\lambda = \int_{\mathcal{A}} \frac{\mathrm{d}\nu}{\mathrm{d}\lambda}(x) \sum_{i=1}^{n} a_i \mathbb{1}_{\{x \in \mathcal{A}_i\}} \mathrm{d}\lambda(x) \tag{3.80}$$

$$= \int_{\mathcal{A}} \sum_{i=1}^{n} a_i \frac{\mathrm{d}\nu}{\mathrm{d}\lambda}(x) \mathbb{1}_{\{x \in \mathcal{A}_i\}} \mathrm{d}\lambda(x) \tag{3.81}$$

$$= \sum_{i=1}^{n} a_i \int_{\mathcal{A}} \frac{\mathrm{d}\nu}{\mathrm{d}\lambda}(x) \mathbb{1}_{\{x \in \mathcal{A}_i\}} \mathrm{d}\lambda(x) \quad \text{(from Theorem 3.29)} \tag{3.82}$$

$$= \sum_{i=1}^{n} a_i \int_{\mathcal{B}_i} \frac{\mathrm{d}\nu}{\mathrm{d}\lambda} \mathrm{d}\lambda \tag{3.83}$$

$$= \sum_{i=1}^{n} a_i \nu(\mathcal{B}_i) \quad \text{(from (3.79))}. \tag{3.84}$$

On the other hand,

$$\int_{\mathcal{A}} f d\nu = \int_{\mathcal{A}} \sum_{i=1}^{n} a_i \mathbb{1}_{\{x \in \mathcal{A}_i\}} \mathrm{d}\nu(x) \tag{3.85}$$

$$= \sum_{i=1}^{n} a_i \int_{\mathcal{A}} \mathbb{1}_{\{x \in \mathcal{A}_i\}} \mathrm{d}\nu(x) \quad \text{(from Theorem 3.29)} \tag{3.86}$$

$$= \sum_{i=1}^{n} a_i \int_{\mathcal{B}_i} \mathrm{d}\nu \tag{3.87}$$

$$= \sum_{i=1}^{n} a_i \nu(\mathcal{B}_i), \tag{3.88}$$

which proves the equaility (3.75) in the case of simple functions.

For the case in which f is an arbitrary non-negative Borel measurable function, it follows from Theorem 3.20 that there always exists an increasing sequence of non-negative finite simple functions h_1, h_2, … that converge point-wise to f. Hence, from Theorem 3.30 and the previous result for simple functions, it holds that:

$$\int_{\mathcal{A}} f d\nu = \lim_{n \to \infty} \int_{\mathcal{A}} h_n d\nu = \lim_{n \to \infty} \int_{\mathcal{A}} h_n \frac{\mathrm{d}\lambda}{\mathrm{d}\nu} \mathrm{d}\nu = \int_{\mathcal{A}} f \frac{\mathrm{d}\lambda}{\mathrm{d}\nu} \mathrm{d}\nu, \tag{3.89}$$

which completes the proof of (ii).

Proof of (iii)*:* From Theorem 3.37, it holds that for all $\mathcal{A} \in \mathscr{F}$,

$$\nu(\mathcal{A}) = \int_{\mathcal{A}} \mathrm{d}\nu = \int_{\mathcal{A}} \frac{\mathrm{d}\nu}{\mathrm{d}\gamma} \mathrm{d}\gamma. \tag{3.90}$$

Again, from Theorem 3.37, it holds that for all $\mathcal{A} \in \mathscr{F}$,

$$\nu(\mathcal{A}) = \int_{\mathcal{A}} \mathrm{d}\nu = \int_{\mathcal{A}} \frac{\mathrm{d}\nu}{\mathrm{d}\lambda} \mathrm{d}\lambda = \int_{\mathcal{A}} \frac{\mathrm{d}\nu}{\mathrm{d}\lambda} \frac{d\lambda}{d\gamma} d\gamma, \tag{3.91}$$

and thus, given that the Radon-Nikodym derivative is unique up to sets of measure zero, it holds that (3.76) follows sfrom (3.90) and (3.91).

Proof of (*iv*)*:* From the proof of (*iii*), it follows that

$$\frac{d\nu}{d\mu}.\frac{d\mu}{d\nu} = \frac{d\mu}{d\mu}, \tag{3.92}$$

and from the proof of (*i*), it follows that

$$\frac{d\nu}{d\mu}.\frac{d\mu}{d\nu} = \frac{d\mu}{d\mu} = 1, \tag{3.93}$$

almost everywhere, which completes the proof of (*iv*). □

Theorem 3.42. *Let λ be a σ-finite measure on $(\mathcal{O}, \mathscr{F})$ and $\nu_1, \nu_2, \ldots, \nu_n$ be finite measures on $(\mathcal{O}, \mathscr{F})$ such that for all $k \in \{1, 2, \ldots, n\}$, ν_k is absolutely continuous with λ. Then, the following holds,*

$$\frac{d\sum_{t=1}^{n} \nu_t}{d\lambda} = \sum_{t=1}^{n} \frac{d\nu_t}{d\lambda} \text{ almost everywhere w.r.t. } \lambda \tag{3.94}$$

Moreover, if ν is a measure on $(\mathcal{O}, \mathscr{F})$ such that for all $\mathcal{A} \in \mathscr{F}$, $\nu(\mathcal{A}) = \lim_{n\to\infty} \sum_{t=1}^{n} \nu_t(\mathcal{A})$, then ν is absolutely continuous with λ and

$$\lim_{n\to\infty} \frac{d\sum_{t=1}^{n} \nu_t}{d\lambda} = \frac{d\nu}{d\lambda} \text{ almost everywhere w.r.t. } \lambda. \tag{3.95}$$

Proof For all $t \in \{1, 2, \ldots, n\}$, let the measure $\gamma_t : \mathscr{F} \to [0, +\infty]$ be such that for all $\mathcal{A} \in \mathscr{F}$,

$$\gamma_t(\mathcal{A}) = \sum_{i=1}^{t} \nu_i(\mathcal{A}). \tag{3.96}$$

Then, the following holds for all $\mathcal{A} \in \mathscr{F}$,

$$\gamma_n(\mathcal{A}) = \int_{\mathcal{A}} d\sum_{t=1}^{n} \nu_t = \int_{\mathcal{A}} \frac{d\sum_{t=1}^{n} \nu_t}{d\lambda} d\lambda. \tag{3.97}$$

On the other hand, from Theorem 3.29, it follows that

$$\gamma_n(\mathcal{A}) = \sum_{i=1}^{n} \nu_i(\mathcal{A}) = \sum_{i=1}^{n} \int_{\mathcal{A}} d\nu_i = \sum_{i=1}^{n} \int_{\mathcal{A}} \frac{d\nu_i}{d\lambda} d\lambda = \int_{\mathcal{A}} \sum_{i=1}^{n} \frac{d\nu_i}{d\lambda} d\lambda. \tag{3.98}$$

Hence, the proof of (3.95) follows from (3.97) and (3.98).

The proof of (3.95) follows by noticing that

$$\frac{\mathrm{d}\nu_1}{\mathrm{d}\lambda}, \frac{\mathrm{d}(\nu_1+\nu_2)}{\mathrm{d}\lambda}, \ldots \frac{\mathrm{d}\sum_{t=1}^{n}\nu_t}{\mathrm{d}\lambda}, \ldots \tag{3.99}$$

form an increasing sequence of non-negative measurable functions. Hence, for all $\mathcal{A} \in \mathscr{F}$, it holds that

$$\nu(\mathcal{A}) = \lim_{n\to\infty}\sum_{i=1}^{n}\nu_i(\mathcal{A}) \tag{3.100}$$

$$= \lim_{n\to\infty}\int_{\mathcal{A}}\frac{\mathrm{d}\sum_{t=1}^{n}\nu_t}{\mathrm{d}\lambda}\mathrm{d}\lambda \quad \text{(from (3.97))} \tag{3.101}$$

$$= \int_{\mathcal{A}}\lim_{n\to\infty}\frac{\mathrm{d}\sum_{t=1}^{n}\nu_t}{\mathrm{d}\lambda}\mathrm{d}\lambda \quad \text{(from Theorem 3.30).} \tag{3.102}$$

On the other hand, given that for all $\mathcal{A} \in \mathscr{F}$, it holds that

$$\nu(\mathcal{A}) = \int_{\mathcal{A}}\frac{\mathrm{d}\nu}{\mathrm{d}\lambda}d\lambda. \tag{3.103}$$

The equalities in (3.102) and (3.103) imply (3.95), which completes the proof. □

THEOREM 3.43. *For all $i \in \{1,2\}$, let $(\mathcal{O},\mathscr{F},\nu_i)$ and $(\mathcal{O},\mathscr{F},\lambda_i)$ be two measure spaces with ν_i and λ_i two σ-finite measures; and λ_i absolutely continuous with respect to ν_i. Let also ν and λ be two measures on the measurable space $\left(\mathcal{O}^2,\sigma\left(\mathscr{F}^2\right)\right)$ such that for all $(\mathcal{A},\mathcal{B}) \in \sigma\left(\mathscr{F}^2\right)$,*

$$\nu(\mathcal{A},\mathcal{B}) = \nu_1(\mathcal{A})\,\nu_2(\mathcal{B}), \text{ and} \tag{3.104}$$

$$\lambda(\mathcal{A},\mathcal{B}) = \lambda_1(\mathcal{A})\,\lambda_2(\mathcal{B}). \tag{3.105}$$

Then, λ is absolutely continuous with respect to ν; and for all $(x,y) \in \mathcal{O}^2$,

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\nu}(x,y) = \frac{\mathrm{d}\lambda_1}{\mathrm{d}\nu_1}(x)\frac{\mathrm{d}\lambda_2}{\mathrm{d}\nu_2}(y). \tag{3.106}$$

Proof See Homework 3. □

3.7 Distances and Pseudo-Distances between Measures

3.7.1 Total Variation

3.7.2 Levy-Prokhorov Distance

3.7.3 Kullback-Liebler Divergence

3.8 Inequalities

3.8.1 Markov Inequality

THEOREM 3.44. *Let $f : \mathbb{R}^n \to [0, \infty]$, with $n \in \mathbb{N}$, be a non-negative Borel measurable function. Then, for all $\gamma \in]0, +\infty[$, it holds that*

$$\mu\left(\{\boldsymbol{x} \in \mathbb{R}^n : f(\boldsymbol{x}) \geqslant \gamma\}\right) \leqslant \frac{1}{\gamma}\int_{\mathbb{R}^n} f \mathrm{d}\mu, \tag{3.107}$$

where μ is the Lebesgue measure.

Proof See Homework 3. □

3.8.2 Jensen Inequality

THEOREM 3.45. *Consider a probability space $(\mathcal{O}, \mathscr{F}, P)$ and let g be a Borel measurable function relative to $(\mathcal{O}, \mathscr{F})$ such that $-\infty < \int_{\mathcal{O}} g \mathrm{d}P < +\infty$. Let also $f : \mathbb{R} \to \mathbb{R}$ be a convex Borel measurable function. Then,*

$$f\left(\int_{\mathcal{O}} g \mathrm{d}P\right) \leqslant \int_{\mathcal{O}} (f \circ g)\, \mathrm{d}P. \tag{3.108}$$

Proof See Homework 3. □

EXERCISE 3.46. State a result of the form of Theorem 3.45 when the function f is assumed to be concave.

4 Probability Theory

4.1 Independence

Two events are independent if the occurrence of one does not influence the occurrence of the other. More specifically, given a probability space $(\mathcal{O}, \mathscr{F}, P)$ and two sets (events) $\mathcal{A}$ and $\mathcal{B}$ in $\mathscr{F}$, they are said to be independent, with respect to P, if and only if: $P(\mathcal{A} \cap \mathcal{B}) = P(\mathcal{A})\, P(\mathcal{B})$. Note also that

$$\begin{aligned} P(\mathcal{A}^{\mathsf{c}} \cap \mathcal{B}) &= P\left((\mathcal{A} \cup \mathcal{B}^{\mathsf{c}})^{\mathsf{c}}\right) \\ &= 1 - P(\mathcal{A} \cup \mathcal{B}^{\mathsf{c}}) \\ &= 1 - P(\mathcal{B}^{\mathsf{c}} \cup (\mathcal{B} \cap \mathcal{A})) \\ &= 1 - P(\mathcal{B}^{\mathsf{c}}) - P(\mathcal{B} \cap \mathcal{A}), \end{aligned} \tag{4.1}$$

and in the case in which $\mathcal{A}$ and $\mathcal{B}$ are independent, the equality in (4.1) can be written as follows:

$$\begin{aligned} P(\mathcal{A}^{\mathsf{c}} \cap \mathcal{B}) &= 1 - P(\mathcal{B}^{\mathsf{c}}) - P(\mathcal{A})\, P(\mathcal{B}) \\ &= 1 - P(\mathcal{B}^{\mathsf{c}}) - P(\mathcal{A}) \left(1 - P(\mathcal{B})^{\mathsf{c}}\right) && (4.2) \\ &= (1 - P(\mathcal{A})) \left(1 - P(\mathcal{B})^{\mathsf{c}}\right) && (4.3) \\ &= P(\mathcal{A}^{\mathsf{c}})\, P(\mathcal{B}), && (4.4) \end{aligned}$$

which implies the independence of $\mathcal{A}^{\mathsf{c}}$ and $\mathcal{B}$ with respect to the probability measure P. Following the same argument, if $\mathcal{A}$ and $\mathcal{B}$ are independent with respect to P, so are $\mathcal{A}$ and $\mathcal{B}^{\mathsf{c}}$. That is, $P(\mathcal{A} \cap \mathcal{B}^{\mathsf{c}}) = P(\mathcal{A})\, P(\mathcal{B}^{\mathsf{c}})$.

The notion of independence of a pair of events can be extended to any finite sequence of events. A sequence of n events is said to be formed by mutually independent events if and only if any subset of events is also formed by mutually independent events. The following definition formalizes this extension.

Definition 4.1 (Mutual Independent Events). Consider a probability space $(\mathcal{O}, \mathscr{F}, P)$ and a sequence of n sets denoted by $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$, such that for all $i \in \mathcal{I} \triangleq \{1, 2, \ldots, n\}$, $\mathcal{A}_i \in \mathscr{F}$. Then, the sets $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ are said to be mutually independent if and only if for all finite subsets $\mathcal{J} \subseteq \mathcal{I}$, it holds that

$$P\left(\bigcap_{j\in\mathcal{J}} \mathcal{A}_j\right) = \prod_{i\in\mathcal{J}} P(\mathcal{A}_i). \tag{4.5}$$

In order to determined whether a sequence of n sets are mutually independent, it is required to verify whether $\sum_{t=2}^{n}\binom{n}{t} = 2^n - n - 1$ subsets of sets are mutually independent. When, one of the $2^n - n - 1$ verifications fails, the sequence is said to be **dependent**. When all the verifications fail, the sequence is said to be **totally dependent**. If there exists a $k \in \mathbb{N}$ for which any sequence of k sets out of the n sets is mutually independent, the sequence is said to be **k-wise independent**. In particular, if $k = 2$, the sequence is said to be **pair-wise independent**, whereas if $k = n$, the sequence is said to be **mutually independent**.

From Definition 4.1, it follows that for all $2 \leqslant k < n$, k-wise independence does not imply mutual independence.

4.2 Conditional Probability

When two events are dependent, a natural question is to determine the probability of one of the events given that the other has been observed. The answer to this question is the notion of **conditional probability**.

Definition 4.2 (Conditional Probability). Given a probability space $(\mathcal{O}, \mathscr{F}, P)$ and two events (sets) $\mathcal{A}$ and $\mathcal{B}$ in $\mathscr{F}$, the probability of $\mathcal{A}$ conditioning on $\mathcal{B}$ is denoted by $P(\mathcal{A}|\mathcal{B})$ and it is defined as the ratio:

$$P(\mathcal{A}|\mathcal{B}) \triangleq \frac{P(\mathcal{A}\cap\mathcal{B})}{P(\mathcal{B})}, \tag{4.6}$$

when $P(\mathcal{B}) > 0$.

The definition of conditional probability with respect to negligible sets, e.g., $P(\mathcal{A}|\mathcal{B})$ when $P(\mathcal{B}) = 0$ is left for a latter discussion.

The notion of conditional probability allows for a reformulation of the notion of independence. The following theorem highlights this observation.

Theorem 4.3 (Independence and Conditional Probability). *Consider a probability space $(\mathcal{O}, \mathscr{F}, P)$ and two sets (events) $\mathcal{A}$ and $\mathcal{B}$ in $\mathscr{F}$, with $P(\mathcal{A}) > 0$. Then, $\mathcal{A}$ and $\mathcal{B}$ are independent if and only if $P(\mathcal{B}|\mathcal{A}) = P(\mathcal{B})$. Alternatively, if $P(\mathcal{B}) > 0$, $\mathcal{A}$ and $\mathcal{B}$ are independent if and only if $P(\mathcal{A}|\mathcal{B}) = P(\mathcal{A})$.*

Proof The proof is immediate from Definition 4.1 and Definition 4.2. □

Theorem 4.4 (Total Probability). *Consider a probability space* $(\mathcal{O}, \mathscr{F}, P)$ *and a set* $\mathcal{A} \in \mathscr{F}$. *Let the sets* $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n$ *be an exact partition of* $\mathcal{O}$. *Then, the following holds:*

$$P(\mathcal{A}) = \sum_{t=1}^{n} P(\mathcal{A} \cap \mathcal{B}_t) \text{ and} \tag{4.7}$$

$$P(\mathcal{A}) = \sum_{\{t:P(\mathcal{B}_t)>0\}} P(\mathcal{A}|\mathcal{B}_t)\, P(\mathcal{B}_t). \tag{4.8}$$

Proof The proof of (4.7) follows from the equalities hereunder

$$P(\mathcal{A}) = P(\mathcal{A} \cap \mathcal{O}) \tag{4.9}$$

$$= P\left(\mathcal{A} \cap \left(\bigcup_{t=1}^{n} \mathcal{B}_t\right)\right) \tag{4.10}$$

$$= P\left(\bigcup_{t=1}^{n} (\mathcal{A} \cap \mathcal{B}_t)\right) \tag{4.11}$$

$$= \sum_{t=1}^{n} P(\mathcal{A} \cap \mathcal{B}_t), \tag{4.12}$$

where (4.10) follows from the assumption that $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n$ form a partition of $\mathcal{O}$ (Definition 1.52); (4.11) follows from the distributive property of unions and intersections (Theorem 1.6); and (4.12) follows from the fact that the sets $\mathcal{A} \cap \mathcal{B}_1$, $\mathcal{A} \cap \mathcal{B}_2$, ..., $\mathcal{A} \cap \mathcal{B}_n$ are disjoint (Definition 1.15). The proof of (4.8) follows from (4.7) and Definition (4.2). □

4.3 Random Variables

A random variable is essentially a Borel measurable function defined over a probability space. Assume for instance a cubic dice that is rolled once. The outcomes of the experiment are $\mathcal{O} \triangleq \{1, 2, \ldots, 6\}$. Consider a σ-field $\mathscr{F}$ consisting in the largest σ-field induced by $\mathcal{O}$ and let $P : \mathscr{F} \to [0, 1]$ be for all $\mathcal{A} \in \mathscr{F}$,

$$P(\mathcal{A}) = \frac{|\mathcal{A}|}{6}. \tag{4.13}$$

Hence, given the probability space $(\mathcal{O}, \mathscr{F}, P)$, some random variables can be defined. Let the random variable X be 1 if the outcome is an even number or 0 otherwise. That is, $X(1) = X(3) = X(5) = 0$ and $X(2) = X(4) = X(6) = 1$. Let the random variable Y be 1 if the outcome is bigger than three or 0 otherwise. That is, $Y(1) = Y(2) = Y(3) = 0$ and $Y(4) = Y(5) = Y(6) = 1$. Let also the random variable Z be simply the outcome of the experiment, i.e., for all $i \in \{1, 2, \ldots, 6\}$, $Z(i) = i$.

In general, a random variable defined in $(\mathcal{O}, \mathscr{F}, P)$ maps each of the elements of $\mathcal{O}$ into the reals. That is, it associates a real number to each of the outcomes of the experiment. This rises an interest in the probability of events (sets) of the form $\{x : X(x) \in \mathcal{B}\}$, where $\mathcal{B} =]a, b]$ is a Borel set with $a \leqslant b$. This implies that for all $(a, b) \in \mathbb{R}^2$, with $a \leqslant b$, the sets of the form $X^{-1}(]a, b])$ must be measurable with respect to $(\mathcal{O}, \mathscr{F})$ and $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$. That is, $X^{-1}(]a, b]) \in \mathscr{F}$. This highlights the fact that random variables are Borel measurable functions with respect to the corresponding measurable space. This becomes clearer in the following definition.

Definition 4.5 (Random Variables). Given a probability space $(\mathcal{O}, \mathscr{F}, P)$, a function $X : \mathcal{O} \to \mathbb{R}$ Borel measurable with respect to $(\mathcal{O}, \mathscr{F})$ is said to be a random variable.

Let $X : \mathcal{O} \to \mathbb{R}$ be a random variable defined on $(\mathcal{O}, \mathscr{F}, P)$. Note that X induces a measure in $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$, which is denoted by P_X. More specifically, for all $\mathcal{B} \in \mathscr{B}(\mathbb{R})$, it holds that

$$P_X(\mathcal{B}) = P(\{x \in \mathcal{O} : X(x) \in \mathcal{B}\}). \tag{4.14}$$

Note that if the value $P_X(\mathcal{B})$ is known for all $\mathcal{B} \in \mathscr{B}(\mathbb{R})$, the random variable X is completely characterized. Nonetheless, this characterization might be long and tedious. Random variables might be characterized by their distribution function.

Definition 4.6 (Distribution Functions). Given a random variable X defined on a given probability space $(\mathcal{O}, \mathscr{F}, P)$, the distribution function of X, denoted by $F_X : \mathbb{R} \to [0, 1]$ is

$$F_X(a) = P(\{x \in \mathcal{O} : X(x) \leqslant a\}), \tag{4.15}$$

and satisfies

$$\lim_{x \to +\infty} F(x) = 1; \text{ and } \lim_{x \to 0} F(x) = 0. \tag{4.16}$$

The distribution function F_X determines the probability measure P_X.

Note that once a distribution function is associated to random variable, it is no longer needed to specify the probability space in which it has been defined. Nonetheless, it is implicit that all random variables are defined in a probability space. More importantly, it can be verified that for all random variables there always exists a probability space in which they can be formally defined. Let for instance, X be a random variable with probability distribution function F_X. Then, it can be defined as a Borel measurable function $X : \mathbb{R} \to \mathbb{R}$ on the probability space $(\mathbb{R}, \mathscr{B}(\mathbb{R}), Q)$ in such a way that for all $r \in \mathbb{R}$, $X(r) = r$, where for all $\mathcal{B} =]a, b] \in \mathscr{B}(\mathbb{R})$ with $a \leqslant b$,

$$Q(\mathcal{B}) = F_X(b) - F_X(a). \tag{4.17}$$

Radom variables can be broadly classified among **discrete random vari-**

ables, **absolutely continuous random variables** and **continuous random variables**.

4.3.1 Discrete Random Variables

Discrete random variables take essentially finite or infinitely countable values. A particular class of discrete random variables is that of simple random variables.

> Definition 4.7 (Simple Random Variables). A random variable is said to be simple if and only if it takes finitely many different values.

Note that Definition 4.7 is reminiscent of that of simple functions (Definition 3.18).

> Definition 4.8 (Discrete Random Variables). A random variable is said to be discrete if and only if its image is a countable set.

Consider a random variable X defined in the probability space $(\mathcal{O}, \mathscr{F}, P)$ and assume it is discrete. Then, its image can be described by a set $\mathcal{X} \triangleq \{x_1, x_2, \ldots, x_n\}$, with $n \leqslant \infty$ of isolated points. In this case, the random variable X can be fully characterized by the **probability mass function**, denoted by $p_X : \mathbb{R} \to [0, 1]$. This function is zero everywhere in $\mathbb{R}$ except in $\{x_1, x_2, \ldots, x_n\}$. More specifically, for all $i \in \{1, 2, \ldots, n\}$,

$$p_X(x_i) = P\left(\{x \in \mathcal{O} : X(x) = x_i\}\right), \tag{4.18}$$

and

$$\sum_{i=1}^{n} p_X(x_i) = 1, \tag{4.19}$$

which implies that for all $a \in \mathbb{R}$, the distribution function of the random variable X is

$$F_X(a) = \sum_{j=1}^{n} p_X(x_j) \mathbb{1}_{\{x_j \leqslant a\}}. \tag{4.20}$$

Assume that there exists a set $\{i_1, i_2, \ldots, i_n\} \subseteq \{1, 2, \ldots, n\}$ of indices such that $\{x_1, x_2, \ldots, x_n\}$ can be ordered as follows:

$$x_{i_1} < x_{i_2} < \ldots < x_{i_n}. \tag{4.21}$$

Hence, the distribution function F_X of X is a function with discontinuities at each value $x_{i_1}, x_{i_2}, \ldots, x_{i_n}$. More specifically, for all pairs $(a, b) \in [x_t, x_{t+1}[$ with $a < b$ and $t \in \{1, 2, \ldots, n-1\}$, it holds that F_X is constant, i.e.,

$$F_X(b) - F_X(a) = 0; \text{ and} \tag{4.22}$$

for all pairs $(a, b) \in]x_{t-1}, x_{t+1}[$ with $a < x_t \leqslant b$ and $t \in \{2, 3, \dots, n-1\}$,

$$F_X(b) - F_X(a) = p_X(x_t). \tag{4.23}$$

The random variable X induces the probability measure P_X on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$, which satisfies for all $\mathcal{B} \in \mathscr{B}(\mathbb{R})$,

$$P_X(\mathcal{B}) = P(\{x \in \mathcal{O} : X(x) \in \mathcal{B}\}) \tag{4.24}$$

$$= \int_{\mathcal{B}} p_X \mathrm{d}\nu \tag{4.25}$$

$$= \sum_{i \in \{j : x_j \in \mathcal{B}\}} p_X(x_i), \tag{4.26}$$

where ν is a counting measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$. That is, for all $\mathcal{A} \in \mathscr{B}(\mathbb{R})$,

$$\nu(\mathcal{A}) = |A|. \tag{4.27}$$

From Theorem 3.37, it holds that the probability mass function p_X is the Radon-Nikodym derivative of P_X with respect to a counting measure ν in (4.27).

In particular, for all $\mathcal{B} =]a, b] \in \mathscr{B}(\mathbb{R})$, with $a < b$,

$$P_X(\mathcal{B}) = \sum_{i \in \{j : x_j \in \mathcal{B}\}} p_X(x_i) = F_X(b) - F_X(a). \tag{4.28}$$

Note that every simple random variable is discrete. Nonetheless, the converse is not necessarily true.

Note also that the random variable X is fully described by the set $\mathcal{X}$ and probability mass function (PMF) p_X.

4.3.2 Absolutely Continuous and Continuous Random Variables

DEFINITION 4.9 (Absolutely Continuous Random Variables). Consider a random variable X described by a distribution function $F_X : \mathbb{R} \to [0, 1]$. The random variable X is said to be absolutely continuous if and only if there exists a non-negative Borel measurable function $f_X : \mathbb{R} \to \mathbb{R}$ such that

$$F_X(x) = \int_{-\infty}^{x} f_X(t) \mathrm{d}t. \tag{4.29}$$

The function f_X in (4.29) is referred to as the **probability density function** of the random variable X. The distribution function F_X is non-negative, increasing and bounded by one. Hence, from (4.29), it follows that

$$\int_{-\infty}^{+\infty} f_X(t) \mathrm{d}t = \int_{\mathbb{R}} f_X \mathrm{d}\mu = 1, \tag{4.30}$$

where the integral on the left is in the sense of Riemann, whereas the one on the right is in the sense of Lebesgue with u the Lebesgue measure (Definition 2.56).

Exercise 4.10. Given an absolutely continuous random variable X that induces the probability measure P_X on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$, show that P_X is absolutely continuous with respect to the Lebesgue measure. This justifies the denomination *absolutely continuous random variable.*

The probability measure P_X induced by the random variable X satisfies for all $\mathcal{B} =]a, b] \in \mathscr{B}(\mathbb{R})$, with $a < b$,

$$P_X(\mathcal{B}) = \int_{\mathcal{B}} f_X \, \mathrm{d}\mu = F_X(b) - F_X(a), \tag{4.31}$$

which is reminiscent to (4.28) in the case of discrete random variables.

Definition 4.11 (Continuous Random Variables). Consider a random variable X described by a distribution function $F_X : \mathbb{R} \to [0, 1]$. The random variable X is said to be continuous if and only if it can be verified that F_X is continuous everywhere on $\mathbb{R}$

Exercise 4.12. Prove that absolute continuity implies continuity and show via an example that the converse is not true.

Exercise 4.13. Given a random variable X that induces the probability measure P_X on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$, show that X is absolutely continuous if and only if for all $x \in \mathbb{R}$, $P_X(\{x\}) = 0$.

4.4 Random Vectors

A random vector is essentially a (Borel measurable) vector-valued function defined over a probability space.

Definition 4.14 (Random Vector). Given a probability space $(\mathcal{O}, \mathscr{F}, P)$, any measurable function with respect to $(\mathcal{O}, \mathscr{F})$ and $(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n))$ is said to be an n-dimensional random variable.

Consider a probability space $(\mathcal{O}, \mathscr{F}, P)$ and let $\boldsymbol{X} : \mathcal{O} \to \mathbb{R}^n$ be an n-dimensional random vector. For all $i \in \{1, 2, \ldots, n\}$, let $X_i : \mathcal{O} \to \mathbb{R}$ be such that for all $w \in \mathcal{O}$, $\boldsymbol{X}(w) = (X_1(w), X_2(w), \ldots, X_n(w))$. That is, X_i is the projection of $\boldsymbol{X}$ into the i-th coordinate space. Hence, given that $\boldsymbol{X}$ is Borel measurable with respect to $(\mathcal{O}, \mathscr{F})$, so is each of the functions $X_1, X_2, \ldots, X_{n-1}$, and X_n. Therefore, the random vector $\boldsymbol{X}$ can be understood as an n-tuple of random variables $(X_1, X_2, \ldots, X_n)$.

The probability measure induced by the random vector $\boldsymbol{X}$ is denoted by $P_{\boldsymbol{X}}$ and for all $\mathcal{B} \in \mathscr{B}(\mathbb{R}^n)$,

$$P_{\boldsymbol{X}}(\mathcal{B}) = P(\{w \in \mathcal{O} : \boldsymbol{X}(w) \in \mathcal{B}\}). \tag{4.32}$$

The distribution function $F_{\boldsymbol{X}} : \mathbb{R}^n \to [0,1]$ associated with the random variable $\boldsymbol{X}$ is for all $\boldsymbol{w} = (w_1, w_2, \ldots, w_n) \in \mathbb{R}^n$,

$$F_{\boldsymbol{X}}(\boldsymbol{w}) = P(\{r \in \mathcal{O} : \forall i \in \{1, 2, \ldots, n\}, X_i(r) < w_i\}). \tag{4.33}$$

Often, the distribution function $F_{\boldsymbol{X}}$ is referred to as the joint probability distribution of $X_1, X_2, \ldots, X_{n-1}$ and X_n. The function $F_{\boldsymbol{X}}$ is increasing and right-continuous on $\mathbb{R}$.

4.5 Independent Random Variables

The notion of independent random variables is reminiscent to the notion of independent events in Section 4.1.

Definition 4.15 (Independent Random Variables). Let $X_1, X_2, \ldots, X_{n-1}$ and X_n be random variables on a given probability space $(\mathcal{O}, \mathscr{F}, P)$. Then, X_1, X_2, …, X_{n-1} and X_n are said to be independent if and only if for all sets $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n$, with $\mathcal{B}_i \in \mathscr{B}(\mathbb{R})$ for all $i \in \{1, 2, \ldots, n\}$,

$$P(X_1 \in \mathcal{B}_1, X_2 \in \mathcal{B}_2, \ldots, X_n \in \mathcal{B}_n) = \prod_{t=1}^{n} P(X_t \in \mathcal{B}_t). \tag{4.34}$$

The following theorems describe two essential properties of independent random variables.

Theorem 4.16. *Let $X_1, X_2, \ldots, X_{n-1}$ and X_n be random variables on a given probability space $(\mathcal{O}, \mathscr{F}, P)$ with joint probability distribution function $F_{X_1, X_2, \ldots, X_n}$. For all $i \in \{1, 2, \ldots, n\}$, let F_{X_i} be the probability distribution function of the random variable X_i. Then, $X_1, X_2, \ldots, X_{n-1}$ and X_n are independent if and only if for all $\boldsymbol{w} = (w_1, w_2, \ldots, w_n) \in \mathbb{R}^n$,*

$$F_{\boldsymbol{X}}(w_1, w_2, \ldots, w_n) = \prod_{t=1}^{n} F_{X_t}(w_t). \tag{4.35}$$

Theorem 4.17. *Let $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ be random vector with a probability mass function $p_{\boldsymbol{X}}$. For all $i \in \{1, 2, \ldots, n\}$, let p_{X_i} be the probability mass function of the random variable X_i. Then, $X_1, X_2, \ldots, X_{n-1}$ and X_n*

are independent if and only if for all $\boldsymbol{w} = (w_1, w_2, \ldots, w_n) \in \mathbb{R}^n$,

$$p_{\boldsymbol{X}}(w_1, w_2, \ldots, w_n) = \prod_{t=1}^{n} p_{X_t}(w_t). \tag{4.36}$$

THEOREM 4.18. *Let* $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ *be random vector with a probability density function* $f_{\boldsymbol{X}}$*. For all* $i \in \{1, 2, \ldots, n\}$*, let* f_{X_i} *be the probability density function of the random variable* X_i*. Then,* $X_1, X_2, \ldots, X_{n-1}$ *and* X_n *are independent if and only if for all* $\boldsymbol{w} = (w_1, w_2, \ldots, w_n) \in \mathbb{R}^n$,

$$f_{\boldsymbol{X}}(w_1, w_2, \ldots, w_n) = \prod_{t=1}^{n} f_{X_t}(w_t). \tag{4.37}$$

4.6 Expectation

The expectation of a random variable is, intuitively, the average of a large set of its realizations.

DEFINITION 4.19 (Expectation). Let X be a random variable defined on $(\mathcal{O}, \mathscr{F}, P)$. Then, the expectation of X (with respect to P) is denoted by $\mathbb{E}[X]$ and

$$\mathbb{E}[X] \triangleq \int_{\mathcal{O}} X \mathrm{d}P. \tag{4.38}$$

The expectation is said to exists if and only if the integral in (4.38) exists as well. In the case of discrete random variables, the expectation simplifies to a well-known expression.

THEOREM 4.20 (Expectation of Discrete Random Variables). *Let* X *be a discrete random variable defined on* $(\mathcal{O}, \mathscr{F}, P)$ *with probability mass function* $p_X : \mathbb{R} \to [0, 1]$*, with* $\operatorname{supp} p_X = \{x_1, x_2, \ldots, x_n\}$ *and* $n \leqslant \infty$*. Then,*

$$\mathbb{E}[X] = \sum_{i=1}^{n} x_i p_X(x_i). \tag{4.39}$$

Proof Let $\mathcal{O}_1$, $\mathcal{O}_2$, …, $\mathcal{O}_{n-1}$ and $\mathcal{O}_n$ form a partition of the set $\mathcal{O}$ such that for all $i \in \{1, 2, \ldots, n\}$ and for all $v \in \mathcal{O}_i$, it holds that $X(v) = x_i$. Hence,

$$\mathbb{E}[X] \triangleq \int_{\mathcal{O}} X \mathrm{d}P \tag{4.40}$$
$$= \sum_{i=1}^{n} x_i P(\mathcal{O}_i) \tag{4.41}$$
$$= \sum_{i=1}^{n} x_i P(\{v \in \mathcal{O} : X(v) = x_i\}) \tag{4.42}$$
$$= \sum_{x \in \mathcal{X}} x p_X(x), \tag{4.43}$$

which completes the proof. □

The equality in 4.43 implies that the expectation of X, in the discrete case, can be calculated using the probability mass function p_X instead of the measure P. This observation highlights the fact once the probability mass function p_X is known, it is not needed to specify the probability space $(\mathcal{O}, \mathscr{F}, P)$ to fully describe the random variable X.

The case of absolutely continuous random variables is described by the following theorem.

Theorem 4.21 (Expectation of Absolutely Continuous Random Variables). *Let X be an absolutely continuous random variable with probability density function f_X. Then,*

$$\mathbb{E}[X] = \int_{\mathbb{R}} x f(x) \mathrm{d}x. \tag{4.44}$$

Proof Assume, without any loss of generality that the random variable X is defined as a Borel measurable function $X : \mathbb{R} \to \mathbb{R}$ on the probability space $(\mathbb{R}, \mathscr{B}(\mathbb{R}), P)$ in such a way that for all $r \in \mathbb{R}$, $X(r) = r$, where for all $\mathcal{B} =]a, b] \in \mathscr{B}(\mathbb{R})$ with $a < b$, it holds that $P(\mathcal{B}) = F_X(b) - F_X(a)$. Assume also that $X \geqslant 0$. From Theorem 3.20, it follows that there always exists an increasing sequence of simple finite functions $X_1, X_2, X_3, \ldots$ Borel measurable with respect to $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ with probability density functions f_{X_1}, f_{X_2}, …, such that $\lim_{i \to \infty} X_i = X$. Note also that for all $i \in \mathbb{N} \setminus \{0\}$, there exists a partition of $\mathbb{R}_+$ of the form $\mathcal{B}_{i,t} =]t\,\epsilon_i, (t+1)\epsilon_i]$, with $t \in \mathbb{N}$ such that for all $x \in \mathcal{B}_{i,t}$, $X_i(x) = X(t\,\epsilon_i) = t\,\epsilon_i$, with $\epsilon_1 > \epsilon_2 > \ldots > 0$ arbitrarily small. Hence, for all

$i \in \mathbb{N} \setminus \{0\}$,

$$\mathbb{E}[X_i] \triangleq \int_{\mathcal{O}} X_i \mathrm{d}P_i \tag{4.45}$$

$$= \sum_{t=0}^{\infty} t\,\epsilon_i P(\mathcal{B}_{i,t}) \tag{4.46}$$

$$= \sum_{t=0}^{\infty} t\,\epsilon_i \int_{t\,\epsilon_i}^{(t+1)\,\epsilon_i} f_{X_i}(u)\mathrm{d}u. \tag{4.47}$$

Note that the integration variable $u \in \mathbb{R}_+$ satisfies $t\,\epsilon_i < u \leqslant (t+1)\,\epsilon_i$. Hence, there always exists a $\delta_i > 0$ such that $t\,\epsilon_i = u + \delta_i$, which leads to the following:

$$\mathbb{E}[X_i] = \sum_{t=0}^{\infty} \int_{t\,\epsilon_i}^{(t+1)\,\epsilon_i} u f_{X_i}(u)\mathrm{d}u + \sum_{s=0}^{\infty} \delta_i \int_{s\,\epsilon_i}^{(s+1)\,\epsilon_i} f_{X_i}(u)\mathrm{d}u \tag{4.48}$$

$$= \int_0^{\infty} u f_{X_i}(u)\mathrm{d}u + \delta_i \int_0^{\infty} f_{X_i}(u)\mathrm{d}u \tag{4.49}$$

$$= \int_0^{\infty} u f_{X_i}(u)\mathrm{d}u + \delta_i, \tag{4.50}$$

where, for all $i \in \mathbb{N}$, $\delta_i > \delta_{i+1} > 0$. Note that X_1, X_2, ..., is an increasing sequence of functions such that $\lim_{i\to\infty} X_i = X$. Hence, from the monotone convergence theorem (Theorem 3.30), it holds that

$$\lim_{i\to\infty} \mathbb{E}[X_i] = \lim_{i\to\infty} \int_0^{\infty} u f_{X_i}(u)\mathrm{d}u + \lim_{i\to\infty} \delta_i \tag{4.51}$$

$$= \int_0^{\infty} u f_X(u)\mathrm{d}u, \tag{4.52}$$

$$= \mathbb{E}[X], \tag{4.53}$$

which completes the part of the proof for non-negative random variables. The proof continues by noting that given an arbitrary absolutely continuous random variable X, the same analysis holds for its positive and negative parts, i.e., X^+ and X^-. Hence, for an arbitrary random variable, it holds that

$$\mathbb{E}[X] = \int_{\mathbb{R}} x f_X(x)\mathrm{d}x, \tag{4.54}$$

which completes the proof. □

The equality in 4.54 implies that the expectation of X, in the continuous case, can be calculated using the distribution function f_X instead of P. This observation highlights the fact once the probability density function f_X is known, it is not needed to specify the probability space $(\mathcal{O}, \mathscr{F}, P)$ to fully describe the random variable X.

In general, the expectation of a random variable is its integral with respect to the measure of the corresponding probability space. Hence, all the existing results for integration hold.

Theorem 4.22 (Expectation and Probability). *Let X be a discrete random variable with induced probability measure P_X. Then,*

$$\mathbb{E}\left[\mathbb{1}_{\{X\in\mathcal{A}\}}\right] = P_X\left(\mathcal{A}\right). \tag{4.55}$$

Proof TBW □

Theorem 4.23 (Properties of Expectations). *Let X and Y be two random variables defined on $(\mathcal{O}, \mathscr{F}, P)$ and assume that both $\mathbb{E}[X]$ and $\mathbb{E}[Y]$ exist. Then,*

- *if $X \geqslant 0$, then $\mathbb{E}[X] \geqslant 0$;*
- *if $X \geqslant 0$, then $\mathbb{E}[X] = 0$ if and only if $X = 0$;*
- *if $X \leqslant Y$, $\mathbb{E}[X] \leqslant \mathbb{E}[Y]$; and*
- *for all $(a,b) \in \mathbb{R}^2$, $\mathbb{E}[aX + bY] = a\mathbb{E}[X] + b\mathbb{E}[Y]$.*

Proof TBW □

4.7 Moments

4.8 Markov Chains

5 Information Measures

5.1 Information

Definition 5.1 (Information). Given a discrete random variable X with probability mass function $p_X : \mathbb{R} \to [0,1]$, for all $x \in \operatorname{supp} p_X$, the information provided by x is

$$\iota_X(x) = -\log\left(p_X(x)\right). \tag{5.1}$$

Information is measured in bits or nats depending on whether the base of the logarithm is either two or the natural base.

Definition 5.2 (Information Spectrum). Given a discrete random variable X with probability mass function $p_X : \mathbb{R} \to [0,1]$, the information spectrum of X, denoted by $S_X : \mathbb{R} \to [0,1]$, is the cumulative distribution function of the random variable $\iota_X(X)$. That is, for all $a \in \mathbb{R}$,

$$S_X(a) = \sum_{x \in \operatorname{supp} p_X} p_X(x) \mathbb{1}_{\{\iota_X(x) \leqslant a\}}. \tag{5.2}$$

Definition 5.3 (Joint Information). Given two discrete random variables X and Y with joint probability mass function $p_{XY} : \mathbb{R}^2 \to [0,1]$, for all $(x,y) \in \operatorname{supp} p_X \times \operatorname{supp} p_Y$, the information provided by (x,y) is

$$\iota_{XY}(x,y) = -\log\left(p_{XY}(x,y)\right). \tag{5.3}$$

Definition 5.4 (Conditional Information). Consider two discrete random variables X and Y with joint probability mass function $p_{XY} : \mathbb{R}^2 \to [0,1]$ and marginal probability mass functions $p_X : \mathbb{R} \to [0,1]$ and $p_Y : \mathbb{R} \to [0,1]$. For all $(x,y) \in \operatorname{supp} p_X \times \operatorname{supp} p_Y$, the information provided by the event $Y = y$ conditioning on the event $X = x$, is denoted by $\iota_{Y|X=x}(y)$ or

$\iota_{Y|X}(y|x)$, and

$$\iota_{Y|X=x}(y) = -\log\left(\frac{p_{XY}(x,y)}{p_X(y)}\right) = \iota_{XY}(x,y) - \iota_X(x). \tag{5.4}$$

5.2 Entropy – Case of Discrete Random Variables

The entropy of a given discrete random variable is defined as follows.

DEFINITION 5.5 (Entropy). Let X be a discrete random variable and assume it induces the probability measure P_X on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Let also $p_X : \mathbb{R} \to [0,1]$ be the probability mass function of X. Then, the entropy of X, denoted by $H(X)$, $H(p_X)$ or $H(P_X)$, is:

$$H(X) \triangleq \int_{\mathbb{R}} \iota_X \mathrm{d}P_X = -\sum_{x \in \operatorname{supp} p_X} p_X(x) \log p_X(x). \tag{5.5}$$

The entropy is measured in bits or nats depending on whether the base of the logarithm is either two or the natural base.

Let $\operatorname{supp} X = \{x_1, x_2, \ldots, x_n\}$ be the support of the random variable X, with $n \in \mathbb{N}$. Note that $H(X)$ in (5.5) depends upon the values x_1, x_2, …, x_{n-1} and x_n only trough the values $p_X(x_1)$, $p_X(x_2)$, …, $p_X(x_{n-1})$ and $p_X(x_n)$. That is, the entropy $H(X)$ in (5.5) does not depend on the values that the random variable X might take, but the probability with which it takes such values. This property of entropy is inherited from the information function (Definition 5.1).

An alternative way for calculating the entropy $H(X)$ in (5.5) is using the properties of expectation with respect to the probability measure P_X:

THEOREM 5.6 (Entropy and Information Spectrum). *Let X be a discrete random variable and assume it induces the probability measure P_X on $\big(\mathbb{R}, \mathcal{B}(\mathbb{R})\big)$. Let also $p_X : \mathbb{R} \to [0,1]$ be the probability mass function of X. Then,*

$$H(X) = \mathbb{E}_X\left[\iota_X(X)\right] = \int_0^{\infty} (1 - S_X(x))\mathrm{d}x, \tag{5.6}$$

where S_X is the information spectrum of X (Definition 5.2).

Proof See Homework 3. □

From Theorem 5.6, it follows that the entropy of a given random variable is the expectation of the amount of information that can be obtained from such random variable.

The following Theorem provides a closed-form expression of the entropy of a binary random variable.

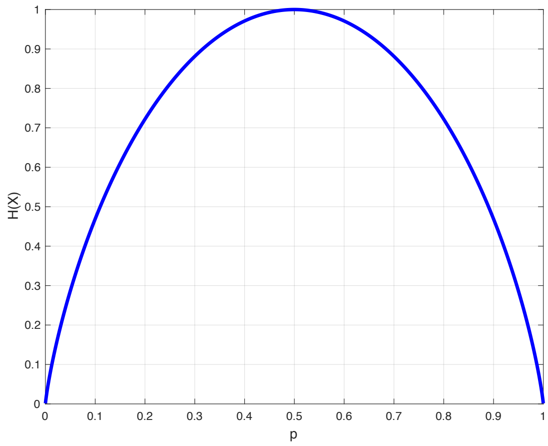

Figure 5.1 Entropy $H(X)$ of a binary random variable X with probability mass function $p_X(0) = 1 - p_X(1) = p$, with $p \in [0, 1]$.

THEOREM 5.7. *Let X be a Bernoulli random variable with probability mass function p_X, such that $p_X(0) = 1 - p_X(1) = p$, with $p \in [0, 1]$. Then,*

$$H(X) = \begin{cases} 0 & \text{if } p \in \{0, 1\} \\ -p \log p - (1-p) \log(1-p) & \text{otherwise.} \end{cases} \tag{5.7}$$

Often, the entropy $H(X)$ of a binary random variable X with probability mass function $p_X(0) = 1 - p_X(1) = p$ is denoted by $H(p)$, with $p \in [0, 1]$. Figure 5.1 shows that $0 \leqslant H(p) \leqslant 1$. That is, the entropy $H(p)$ is a non-negative bounded function of p. The maximum is achieved when $p_X(0) = 1 - p_X(1) = \frac{1}{2}$ (uniform distribution). Alternatively, the lower bound is achieved when X is an ill distribution, i.e., $p_X(0) = 1 - p_X(1) \in \{0, 1\}$. The following Theorem generalizes these observations for random variables with countable supports.

THEOREM 5.8. *Let X be a discrete random variable with probability mass function $p_X : \mathbb{R} \to [0, 1]$. Then,*

$$0 \leqslant H(X) \leqslant \log |\mathcal{X}|. \tag{5.8}$$

Proof The lower-bound follows from Theorem 4.23 and the fact that for all $x \in \mathcal{X}$, $\iota_X(x) \geqslant 0$. The upper-bound follows from (5.6), and the following inequalities:

$$H(X) = \mathbb{E}_X \left[\log \frac{1}{p_X(X)} \right] \tag{5.9a}$$
$$\leqslant \log \mathbb{E}_X \left[\frac{1}{p_X(X)} \right] \tag{5.9b}$$
$$= \log \sum_{x \in \operatorname{supp} p_X} 1 \tag{5.9c}$$
$$= \log |\operatorname{supp} p_X| \tag{5.9d}$$
$$\leqslant \log |\mathcal{X}|, \tag{5.9e}$$

where, (5.9b) follows from Jensen's inequality (Theorem 3.45). Thus, the maximum value of the entropy of a random variable X is obtained when it is uniformly distributed, i.e., $|\operatorname{supp} p_X| = |\mathcal{X}|$ and $p_X(x) = \frac{1}{|\mathcal{X}|}$ for all $x \in \mathcal{X}$. This completes the proof of Theorem 5.8. □

5.2.1 Joint Entropy

The joint entropy of two discrete random variables is defined as follows.

DEFINITION 5.9 (Joint Entropy). Let X and Y be two discrete random variables and assume they induce the probability measure P_{XY} on $\left(\mathbb{R}^2, \mathcal{B}\left(\mathbb{R}^2\right)\right)$. Let also $p_{XY} : \mathbb{R}^2 \to [0,1]$ be their joint probability mass function. Then, the joint entropy of X and Y, denoted by $H(X,Y)$, $H\left(p_{XY}\right)$, or $H\left(P_{XY}\right)$, is:

$$H(X,Y) = \int_{\mathbb{R}^2} \iota_{XY} \mathrm{d}P_{XY} = -\sum_{(x,y)\in \operatorname{supp}(p_{XY})} p_{XY}(x,y) \log p_{XY}(x,y). \tag{5.10}$$

The joint entropy of the random variables X and Y can also be written as follows:

$$H(X,Y) = \mathbb{E}_{XY}\left[\iota_{XY}(X,Y)\right]. \tag{5.11}$$

The joint entropy of two or more discrete random variables exhibit the following property.

THEOREM 5.10. *Let X and Y be two discrete random variables with joint probability mass function $p_{XY} : \mathbb{R}^2 \to [0,1]$, with $p_X : \mathbb{R} \to [0,1]$ and $p_Y : \mathbb{R} \to [0,1]$ the marginal probability mass functions. Then,*

$$H(X,Y) \leqslant H(X) + H(Y), \tag{5.12}$$

with equality if and only if the random variables X and Y are independent.

Proof From (5.11), the following holds:

$$H(X,Y) = -\mathbb{E}_{XY}\left[\log\left(\frac{p_X(X)p_Y(Y)p_{XY}(X,Y)}{p_X(X)p_Y(Y)}\right)\right] \tag{5.13a}$$
$$= -\mathbb{E}_X\left[\log p_X(X)\right] - \mathbb{E}_Y\left[\log p_Y(Y)\right]$$
$$-\mathbb{E}_{XY}\left[\log\left(\frac{p_{XY}(X,Y)}{p_X(X)p_Y(Y)}\right)\right] \tag{5.13b}$$
$$= H(X) + H(Y) + \mathbb{E}_{XY}\left[\log\left(\frac{p_X(X)p_Y(Y)}{p_{X,Y}(XY)}\right)\right] \tag{5.13c}$$
$$\leqslant H(X) + H(Y) + \log\left(\mathbb{E}_{p_{XY}}\left[\left(\frac{p_X(X)p_Y(Y)}{p_{XY}(X,Y)}\right)\right]\right) \tag{5.13d}$$
$$= H(X) + H(Y) + \log\left(\sum_{(x,y)\in\operatorname{supp}(p_{XY})} p_X(X)p_Y(Y)\right) \tag{5.13e}$$
$$= H(X) + H(Y), \tag{5.13f}$$

where (5.13d) follows from Jensen's inequality (Theorem 3.45).

Note that when the random variables X and Y are independent, it holds from (5.13c) that:

$$H(X,Y) = H(X) + H(Y) + \mathbb{E}_{XY}\left[\log\left(\frac{p_X(X)p_Y(Y)}{p_X(X)p_Y(Y)}\right)\right] \tag{5.14a}$$
$$= H(X) + H(Y), \tag{5.14b}$$

and this completes the proof of Theorem 5.10. □

Definition 5.11 provides a definition of joint entropy with respect to an arbitrary number of discrete random variables.

Definition 5.11. Let $\boldsymbol{X} = \left(X_1, X_2, \ldots, X_n\right)^\mathsf{T}$ be an n-dimensional discrete random vector and assume that it induces the probability measure $P_{\boldsymbol{X}}$ on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. Let $p_{\boldsymbol{X}} : \mathbb{R}^n \to [0,1]$ be the probability mass function. Then, the joint entropy of $\boldsymbol{X}$, denoted by $H(\boldsymbol{X})$, $H(p_{\boldsymbol{X}})$ or $H(P_{\boldsymbol{X}})$, is:

$$H(\boldsymbol{X}) = \int_{\mathbb{R}^n} \iota_{\boldsymbol{X}} \, \mathrm{d}P_{\boldsymbol{X}} = -\sum_{\boldsymbol{x}\in\operatorname{supp}(p_{\boldsymbol{X}})} p_{\boldsymbol{X}}(\boldsymbol{x}) \log p_{\boldsymbol{X}}(\boldsymbol{x}). \tag{5.15}$$

The joint entropy of a vector of discrete random variables $\boldsymbol{X}$ can also be written as follows:

$$H(\boldsymbol{X}) = \mathbb{E}_{\boldsymbol{X}}\left[\iota_{\boldsymbol{X}}(\boldsymbol{X})\right]. \tag{5.16}$$

From Theorem 5.10, it follows that when $X_1, X_2, \ldots, X_n$ are mutually independent, the following holds:

$$H(\boldsymbol{X}) = \sum_{t=1}^{n} H(X_t). \tag{5.17}$$

5.2.2 Conditional Entropy

The conditional entropy of a discrete random variable Y conditioning on the random variable X is defined hereunder.

DEFINITION 5.12 (Conditional Entropy). Let X and Y be two discrete random variables and assume they induce the probability measure P_{XY} on $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$. Let $p_{XY} : \mathbb{R}^2 \to [0,1]$ be the joint probability mass function. Then, the entropy of Y conditioning on X, denoted by $H(Y|X)$ or $H(p_{Y|X})$, is:

$$H(Y|X) = \int_{\mathbb{R}^2} \iota_{Y|X} \mathrm{d}P_{XY} = -\sum_{(x,y)\in\mathrm{supp}(p_{XY})} p_{XY}(x,y) \log \frac{p_{YX}(y,x)}{p_X(x)}. \tag{5.18}$$

The entropy of the random variable Y conditioning on the random variable X can also be written as follows:

$$H(Y|X) = \mathbb{E}_{XY}\left[\iota_{Y|X}(Y|X)\right]. \tag{5.19}$$

Note also that the conditional entropy in (5.18) can be written as follows:

$$\begin{aligned} H(Y|X) &= \sum_{x\in\mathrm{supp}\, p_X} p_X(x) \left[-\sum_{y\in\mathrm{supp}\, p_{Y|X=x}} p_{Y|X}(y|x) \log p_{Y|X}(y|x) \right] \\ &= \sum_{x\in\mathrm{supp}\, p_X} p_X(x) H(Y|X=x), \end{aligned} \tag{5.20}$$

where, $H(Y|X=x) \triangleq -\sum_{y\in\mathrm{supp}\, p_{Y|X}} p_{Y|X}(y|x) \log p_{Y|X}(y|x)$ is the entropy of Y conditioning on the event $X = x$.

The following Theorem presents an important property of the conditional entropy.

THEOREM 5.13 (Chain rule for entropy and conditional entropy). *Let* $\boldsymbol{X} = (X_0, X_1, X_2, \ldots, X_n)^{\mathsf{T}}$ *be an* $(n+1)$*-dimensional discrete random vector with joint probability mass function* $p_{\boldsymbol{X}} : \mathbb{R}^{n+1} \to [0,1]$. *Then,*

$$\begin{aligned} H(X_1,\ldots,X_n) &= H(X_1) + H(X_2|X_1) \\ &\quad + \sum_{n=3}^{n} H(X_n|X_1,\ldots,X_{n-1}), \text{ and} \end{aligned} \tag{5.21}$$

$$\begin{aligned} H(X_1,\ldots,X_n|X_0) &= H(X_1|X_0) + H(X_2|X_0,X_1) \\ &\quad + \sum_{n=3}^{n} H(X_n|X_0,X_1,\ldots,X_{n-1}). \end{aligned} \tag{5.22}$$

Proof **Proof of** (5.21): From (5.16), the following holds:

$$\begin{aligned} H(\boldsymbol{X}_{1:n}) &= -\mathbb{E}_{\boldsymbol{X}_{1:n}}\Big[\log\big(p_{X1}(X_1)p_{X_2|X_1}(X_2|X_1)\ldots \\ &\quad p_{X_n|X_1X_2\ldots X_{n-1}}(X_n|X_1,X_2,\ldots,X_{n-1})\big)\Big] && (5.23a)\\ &= -\mathbb{E}_{X_1}\left[\log p_{X1}(X_1)\right] - \mathbb{E}_{X_1X_2}\left[\log p_{X_2|X_1}(X_2|X_1)\right] - \ldots \\ &\quad -\mathbb{E}_{\boldsymbol{X}_{1:n}}\left[\log p_{X_n|X_1,X_2,\ldots,X_{n-1}}\right] && (5.23b)\\ &= H(X_1) + H(X_2|X_1) + \ldots + H(X_n|X_1,X_2,\ldots,X_{n-1}), && (5.23c) \end{aligned}$$

and this completes the proof of (5.21).

Proof of (5.22): From (5.19), the following holds:

$$\begin{aligned} H(\boldsymbol{X}_{1:n}|X_0) &= -\mathbb{E}_{\boldsymbol{X}}\left[\log p_{\boldsymbol{X}_{1:n}|X_0}(\boldsymbol{X}_{1:n}|X_0)\right] && (5.24a)\\ &= -\mathbb{E}_{\boldsymbol{X}}\Big[\log\Big(p_{X_1|X_0}(X_1|X_0)p_{X_2|X_0X_1}(X_2|X_0,X_1)\ldots \\ &\quad p_{X_n|X_0X_1X_2\ldots X_{n-1}}(X_n|X_0,X_1,X_2,\ldots,X_{n-1})\Big)\Big] && (5.24b)\\ &= -\mathbb{E}_{X_0X_1}\left[\log p_{X_1|X_0}(X_1|X_0)\right] \\ &\quad -\mathbb{E}_{X_0X_1X_2}\left[\log p_{X_2|X_0X_1}(X_2|X_0,X_1)\right] - \ldots && (5.24c)\\ &\quad -\mathbb{E}_{\boldsymbol{X}}\left[\log p_{X_n|\boldsymbol{X}_{0:n-1}}(X_n|X_0,X_1,\ldots,X_{n-1})\right] && (5.24d)\\ &= H(X_1|X_0) + H(X_2|X_0,X_1) + \ldots \\ &\quad + H(X_n|X_0,X_1,X_2,\ldots,X_{n-1}), && (5.24e) \end{aligned}$$

and this completes the proof of (5.22). □

Conditioning a random variable on another one does not increase, in expectation, the information it provides. This observation is formalized by the following result.

THEOREM 5.14 (Conditioning does not increase entropy). *Let X and Y be two discrete random variables with joint probability mass function $p_{XY} : \mathbb{R}^2 \to [0,1]$. Then,*

$$H(Y|X) \leqslant H(Y), \tag{5.25}$$

with equality if and only if the random variables X and Y are independent.

Proof From (5.19), the following holds:

$$H(Y|X) = -\mathbb{E}_{XY}\left[\log\left(\frac{p_Y(Y)p_{X|Y}(X|Y)}{p_X(X)}\right)\right] \tag{5.26a}$$
$$= -\mathbb{E}_Y\left[\log p_Y(Y)\right] - \mathbb{E}_{XY}\left[\log\left(\frac{p_{X|Y}(X|Y)}{p_X(X)}\right)\right] \tag{5.26b}$$
$$= H(Y) - \mathbb{E}_{XY}\left[\log\left(\frac{p_{XY}(X,Y)}{p_X(X)p_Y(Y)}\right)\right] \tag{5.26c}$$
$$= H(Y) + \mathbb{E}_{XY}\left[\log\left(\frac{p_X(X)p_Y(Y)}{p_{XY}(X,Y)}\right)\right] \tag{5.26d}$$
$$\leqslant H(Y) + \log\left(\mathbb{E}_{XY}\left[\left(\frac{p_X(X)p_Y(Y)}{p_{XY}(X,Y)}\right)\right]\right) \tag{5.26e}$$
$$= H(Y) + \log\left(\sum_{(x,y)\in\mathrm{supp}(p_{XY})} p_X(x)p_Y(y)\right) \tag{5.26f}$$
$$= H(Y), \tag{5.26g}$$

where (5.26e) follows from Jensen's inequality (Theorem 3.45) and holds with equality, only if the random variables X and Y are independent. In this case, it holds from (5.26d) that,

$$H(Y|X) = H(Y) + \mathbb{E}_{XY}\left[\log\left(\frac{p_X(X)p_Y(Y)}{p_X(X)p_Y(Y)}\right)\right] \tag{5.27a}$$
$$= H(Y). \tag{5.27b}$$

Finally, if the random variables X and Y are independent, it holds that for all $(x, y) \in \mathrm{supp}\, p_X \times \mathrm{supp}\, p_Y$, $p_{XY}(x, y) = p_X(x)p_Y(y)$ (Theorem 4.17), and thus, from Definition 5.12, it holds that $H(Y|X) = H(Y)$. This completes the proof of Theorem 5.14. □

The joint entropy of a collection of random variables is less than or equal to the sum of the entropy of each random variable. This statement is formalized by the following result.

THEOREM 5.15. *Let* $\boldsymbol{X} = \left(X_1, X_2, \ldots, X_n\right)^{\mathsf{T}}$ *be an* n*-dimensional discrete random vector with joint probability mass function* $p_{\boldsymbol{X}} : \mathbb{R}^n \to [0, 1]$*. Then,*

$$H(X_1, \ldots, X_N) \leqslant \sum_{t=1}^{n} H(X_t), \tag{5.28}$$

with equality if and only if the random variables $X_1, X_2, \ldots, X_n$ *are mutually independent.*

Proof The proof of Theorem 5.15 follows by combining Theorem 5.13 and Theorem 5.14. □

The entropy of a Borel measurable function of the random variable X is smaller than or equal to the entropy of the random variable X, with equality only when the function is an injective function. This observation is formalized by the following Theorem.

Theorem 5.16 (Entropy of a function). *Let X be a discrete random variable with probability mass function $p_X : \mathbb{R} \to [0,1]$. Let also $f : \mathbb{R} \to \mathbb{R}$ be a Borel measurable function. Then,*

$$H(X) \geqslant H(f(X)). \tag{5.29}$$

Proof Denote by $\mathcal{Y} \subseteq \mathbb{R}$ the image of the support of p_X through the function f. That is,

$$\mathcal{Y} \triangleq f^{-1}(\mathrm{supp} p_X). \tag{5.30}$$

Consider the random variable $Y = f(X)$, which satisfies $Y \in \mathcal{Y}$, and let p_Y be the corresponding probability mass function. That is, for all $y \in \mathcal{Y}$,

$$p_Y(y) = \sum_{x \in \{a \in \mathrm{supp} p_X : y = f(a)\}} p_X(x). \tag{5.31}$$

Hence, for all $y \in \mathcal{Y}$, assume that $\mathcal{I}_y \triangleq \{a \in \mathrm{supp}\, P_X : y = f(a)\}$. From (5.6), the following holds:

$$\begin{aligned}
H(Y) &= -\sum_{y \in \mathcal{Y}} p_Y(y) \log p_Y(y) && \text{(5.32a)}\\
&= -\sum_{y \in \mathcal{Y}} \left(\sum_{x \in \mathcal{I}_y} p_X(x) \right) \log \left(\sum_{v \in \mathcal{I}_y} p_X(v) \right) && \text{(5.32b)}\\
&\leqslant -\sum_{y \in \mathcal{Y}} \sum_{x \in \mathcal{I}_y} p_X(x) \log p_X(x) && \text{(5.32c)}\\
&= -\sum_{x \in \mathrm{supp}\, p_X} p_X(x) \log p_X(x) && \text{(5.32d)}\\
&= H(X), && \text{(5.32e)}
\end{aligned}$$

If f is an injective function, it holds that for all $y \in \mathcal{Y}$, $|\mathcal{I}_y| = 1$, and thus (5.32c) holds with equality. This completes the proof of Theorem 5.16. □

5.3 Entropy – Case of Absolutely Continuous Random Variables

The entropy of an absolutely continuous random variable X is often referred to as differential entropy, and it is denoted by $h(X)$ to distinguish from the discrete case.

Definition 5.17 (Differential Entropy). Let X be an absolutely continuous random variable with probability density function $f_X : \mathbb{R} \to [0, \infty)$. Then, the differential entropy of X, denoted by $h(X)$ or $h(f_X)$, is:

$$h(X) = -\int_{-\infty}^{\infty} f_X(x) \log f_X(x) \, \mathrm{d}x. \tag{5.33}$$

The differential entropy of a random variable X can also be written as follows:

$$h(X) = -\mathbb{E}_X \left[\log f_X(X)\right]. \tag{5.34}$$

The following two theorems provide closed-form expressions for some absolutely continuous random variables.

Theorem 5.18. *Let X be an absolutely continuous random variable uniformly distributed on $[0, a]$. Then,*

$$h(X) = \ln a. \tag{5.35}$$

Proof See Homework 3. □

Note that $h(X)$ in (5.35) is negative if $a < 1$. This is in sharp contrast with the entropy of discrete random variables, which is always non-negative.

Theorem 5.19. *Let X be a Gaussian random variable with zero mean and variance σ^2. Then,*

$$h(X) = \frac{1}{2} \ln \left(2\pi e \sigma^2\right) \text{ in nats,} \tag{5.36}$$

where e is Néper's constant.

Proof The probability density function of X is for all $x \in \mathbb{R}$,

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp -\frac{x^2}{2\sigma^2}. \tag{5.37}$$

Hence, the following holds:

$$h(X) = -\int_{-\infty}^{\infty} f_X(x) \ln f_X(x) \, \mathrm{d}x \tag{5.38a}$$

$$= -\int_{-\infty}^{\infty} f_X(x) \left(-\frac{x^2}{2\sigma^2} - \ln \sqrt{2\pi\sigma^2} \right) \mathrm{d}x \tag{5.38b}$$

$$= \frac{1}{2\sigma^2} \int_{-\infty}^{\infty} x^2 f_X(x) \, \mathrm{d}x + \ln \sqrt{2\pi\sigma^2} \int_{-\infty}^{\infty} f_X(x) \, \mathrm{d}x \tag{5.38c}$$

$$= \frac{\mathbb{E}_X\left[X^2\right]}{2\sigma^2} + \ln \sqrt{2\pi\sigma^2} \tag{5.38d}$$

$$= \frac{1}{2} + \frac{1}{2} \ln\left(2\pi\sigma^2\right) \tag{5.38e}$$

$$= \frac{1}{2} \ln e + \frac{1}{2} \ln\left(2\pi\sigma^2\right) \tag{5.38f}$$

$$= \frac{1}{2} \ln\left(2\pi e\sigma^2\right) \text{ in nats,} \tag{5.38g}$$

which completes the proof of Theorem 5.19. □

Note that $h(X)$ in (5.36) is negative when the variance of X is such that $\sigma^2 < \frac{1}{2\pi e}$.

The following theorem provides an upper bound on the differential entropy.

THEOREM 5.20. *Let X be an absolutely continuous random variable with probability density function $f_X : \mathbb{R} \to [0, \infty)$, zero mean, and variance σ^2. Then,*

$$h(X) \leqslant \frac{1}{2} \log\left(2\pi e\sigma^2\right), \tag{5.39}$$

and equality holds if and only if f_X satisfies (5.37).

Proof *Proof* See Homework 3. □

□

5.3.1 Joint Entropy

The joint differential entropy of two absolutely continuous random variables is defined as follows.

DEFINITION 5.21 (Joint Differential Entropy). Let X and Y be two absolutely continuous random variables with joint probability density function $f_{XY} : \mathbb{R}^2 \to [0, \infty)$. Then, the joint differential entropy of the random variables X and Y, denoted by $h(X, Y)$ or $h(f_{XY})$, is:

$$h(X, Y) = -\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x, y) \log f_{XY}(x, y) \, \mathrm{d}x \, \mathrm{d}y. \tag{5.40}$$

The joint differential entropy of the random variables X and Y can also be written as follows:

$$h(X,Y) = -\mathbb{E}_{XY}\left[\log f_{XY}(X,Y)\right]. \tag{5.41}$$

The following theorem provides the differential entropy of two correlated Gaussian random variables.

THEOREM 5.22 (Differential Entropy of a Bivariate Gaussian Distribution). *Let X and Y be two random variables with joint probability function f_{XY} such that for all $(x,y) \in \mathbb{R}^2$,*

$$f_{XY}(x,y) = \frac{1}{\sqrt{(2\pi)^2 \det \mathbb{K}}} \exp\left(-\frac{1}{2}\left([x\,y]\,\mathbb{K}^{-1}\begin{bmatrix} x \\ y \end{bmatrix}\right)\right), \tag{5.42}$$

where,

$$\mathbb{K} \triangleq \mathbb{E}_{XY}\left[\begin{bmatrix} X \\ Y \end{bmatrix}\begin{bmatrix} X & Y \end{bmatrix}\right] = \begin{bmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{bmatrix}, \tag{5.43}$$

with σ_X^2 and σ_Y^2 are the variances of X and Y, respectively; and $\rho \triangleq \frac{\mathbb{E}_{XY}[XY]}{\sigma_X\sigma_Y}$ is the Pearson correlation coefficient. Then,

$$h(X,Y) = \frac{1}{2}\log\left((2\pi e)^2 \det \mathbb{K}\right), \tag{5.44}$$

where e is the Néper's constant.

Proof From (5.41), the following holds:

$$h(X,Y) = -\mathbb{E}_{XY}\left[\log f_{XY}(X,Y)\right]. \tag{5.45a}$$

Moreover, the determinant of the covariance matrix $\mathbb{K}$ is

$$\det \mathbb{K} = \sigma_X^2\sigma_Y^2\left(1-\rho^2\right), \tag{5.45b}$$

and the inverse of the covariance matrix $\mathbb{K}$ is

$$\mathbb{K}^{-1} = \frac{1}{\det \mathbb{K}}\begin{bmatrix} \sigma_Y^2 & -\rho\sigma_X\sigma_Y \\ -\rho\sigma_X\sigma_Y & \sigma_X^2 \end{bmatrix}. \tag{5.45c}$$

Plugging (5.45c) into (5.42) the following holds:

$$f_{XY}(x,y) = \frac{1}{\sqrt{(2\pi)^2 \det \mathbb{K}}} \exp\left(-\frac{[x\,y]\begin{bmatrix} \sigma_Y^2 & -\rho\sigma_X\sigma_Y \\ -\rho\sigma_X\sigma_Y & \sigma_X^2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}}{2\det \mathbb{K}}\right) \tag{5.45d}$$

$$= \frac{1}{\sqrt{(2\pi)^2 \det \mathbb{K}}} \exp\left(-\frac{x^2\sigma_Y^2 - 2xy\rho\sigma_X\sigma_Y + y^2\sigma_X^2}{2\det \mathbb{K}}\right). \tag{5.45e}$$

Plugging (5.45e) into (5.45a) and taking the logarithm yields:

$$h(X,Y) = -\mathbb{E}_{XY}\left[\ln\left(\frac{1}{\sqrt{(2\pi)^2 \det \mathbb{K}}} \exp\left(-\frac{X^2\sigma_Y^2 - 2\,X\,Y\rho\,\sigma_X\sigma_Y + Y^2\sigma_X^2}{2\det\mathbb{K}}\right)\right)\right] \tag{5.45f}$$

$$= \ln\left(\sqrt{(2\pi)^2 \det \mathbb{K}}\right) + \frac{\sigma_Y^2\mathbb{E}_X\left[X^2\right] - 2\rho\sigma_X\sigma_Y\mathbb{E}_{XY}\left[XY\right] + \sigma_X^2\mathbb{E}_Y\left[Y^2\right]}{2\det\mathbb{K}} \tag{5.45g}$$

$$= \ln\left(\sqrt{(2\pi)^2 \det \mathbb{K}}\right) + \frac{\sigma_Y^2\sigma_X^2 - 2\rho^2\sigma_X^2\sigma_Y^2 + \sigma_X^2\sigma_Y^2}{2\det\mathbb{K}} \tag{5.45h}$$

$$= \ln\left(\sqrt{(2\pi)^2 \det \mathbb{K}}\right) + \frac{\sigma_X^2\sigma_Y^2\left(1-\rho^2\right)}{\det\mathbb{K}} \tag{5.45i}$$

$$= \ln\left(\sqrt{(2\pi)^2 \det \mathbb{K}}\right) + 1 \tag{5.45j}$$

$$= \frac{1}{2}\ln\left((2\pi e)^2 \det\mathbb{K}\right), \tag{5.45k}$$

where (5.45j) follows from (5.45b), which completes the proof of Theorem 5.22. □

Theorem 5.22 can be generalized as follows:

THEOREM 5.23. *Let* $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)^\mathsf{T} \in \mathbb{R}^n$ *be an* n*-dimensional absolutely continuous random vector with joint probability density function* $f_{\boldsymbol{X}}$ *such that for all* $\boldsymbol{x} \in \mathbb{R}^n$

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \frac{1}{\sqrt{(2\pi)^n \det\mathbb{K}}} \exp\left(-\frac{\boldsymbol{x}^\mathsf{T}\mathbb{K}^{-1}\boldsymbol{x}}{2}\right), \tag{5.46}$$

where $\mathbb{K} \triangleq \mathbb{E}_{f_{XY}}\left[\boldsymbol{X}\boldsymbol{X}^\mathsf{T}\right]$ *is the covariance matrix of* $\boldsymbol{X}$*. Then, the joint differential entropy of* $\boldsymbol{X}$ *is:*

$$h(\boldsymbol{X}) = \frac{1}{2}\log\left((2\pi e)^n \det\mathbb{K}\right). \tag{5.47}$$

Proof See Homework 3 □

5.3.2 Conditional Entropy

The conditional entropy of an absolutely continuous random variable Y conditioning on the random variable X is defined hereunder.

Definition 5.24 (Conditional Differential Entropy). Let X and Y be two absolutely continuous random variables with joint probability density function $f_{XY} : \mathbb{R}^2 \to [0, \infty)$. Then, the differential entropy of Y conditioning on X, denoted by $h(Y|X)$ or $h(f_{Y|X})$, is:

$$h(Y|X) = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x, y) \log f_{Y|X}(y|x)\, \mathrm{d}x\, \mathrm{d}y. \tag{5.48}$$

The differential entropy of the random variable Y conditioning on the random variable X can be written as follows:

$$h(Y|X) = -\mathbb{E}_{XY}\left[\log f_{Y|X}(Y|X)\right], \tag{5.49}$$

and alternatively,

$$\begin{aligned} h(Y|X) &= \int_{-\infty}^{\infty} f_X(x)\left(-\int_{-\infty}^{\infty} f_{Y|X}(y|x) \log f_{Y|X}(y|x)\mathrm{d}y\right)\, \mathrm{d}x \\ &= \int_{-\infty}^{\infty} f_X(x) h(Y|X = x)\, \mathrm{d}x, \end{aligned} \tag{5.50}$$

where $h(Y|X = x) \triangleq -\int_{-\infty}^{\infty} f_{Y|X}(y|x) \log f_{Y|X}(y|x)\, \mathrm{d}y$, the differental entropy of Y conditioning on the event $X = x$.

The following theorems highlight some of the properties of the differential entropy. These properties are reminiscent to those of the entropy of discrete random variables.

Theorem 5.25 (Chain rule for differential entropy). *Let* $\boldsymbol{X} = (X_0, X_1, X_2, \ldots, X_n)$ *be a vector formed by* $n+1$ *absolutely continuous random variables, with joint probability density function* $f_{\boldsymbol{X}} : \mathbb{R}^{n+1} \to [0, +\infty[$. *Then,*

$$h(X_1, \ldots, X_n) = h(X_1) + h(X_2|X_1) + \sum_{t=3}^{n} h(X_t|X_1, \ldots, X_{t-1}); \tag{5.51}$$

and

$$h(X_1, \ldots, X_n|X_0) = h(X_1|X_0) + h(X_2|X_0, X_1) + \sum_{t=3}^{n} h(X_t|X_0, X_1, \ldots, X_{t-1}). \tag{5.52}$$

Proof The proof of Theorem 5.25 follows along the same lines as the proof of Theorem 5.13. □

Theorem 5.26 (Conditioning reduces differential entropy). *Let* X *and* Y *be two absolutely continuous random variables. Then,*

$$h(Y|X) \leqslant h(Y), \tag{5.53}$$

with equality if and only if the random variables X and Y are mutually independent.

Proof See Homework 3 □

THEOREM 5.27. *Let X_1, X_2, ..., X_n be n absolutely continuous random variables. Then,*

$$h(X_1, \ldots, X_n) \leqslant \sum_{t=1}^{n} h(X_n), \tag{5.54}$$

with equality if and only if the random variables $X_1, X_2, \ldots, X_n$ are mutually independent.

Proof See Homework 3 □

5.4 Mutual Information – Case of Discrete Random Variables

The mutual information between two discrete random variables X and Y is defined as follows.

DEFINITION 5.28 (Mutual Information). Let X and Y be two discrete random variables and assume that they respectively induce the probability measures P_X and P_Y on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$. Jointly, X and Y induce the probability measure P_{XY} on $\left(\mathbb{R}^2, \mathscr{B}\left(\mathbb{R}^2\right)\right)$. Let also $p_{XY} : \mathbb{R}^2 \to [0,1]$ be the joint probability mass function. Then, the mutual information between X and Y, denoted by $I(X;Y)$, $I(p_{XY})$ or $I(P_{XY})$, is:

$$I(X;Y) \triangleq \int_{\mathbb{R}^2} (\iota_{P_X P_Y} - \iota_{P_{XY}}) \, \mathrm{d}P_{XY} \tag{5.55}$$

$$= \sum_{(x,y) \in \operatorname{supp} p_{XY}} p_{XY}(x,y) \log\left(\frac{p_{XY}(x,y)}{p_X(x) p_Y(y)}\right), \tag{5.56}$$

with p_X and p_Y the marginal probability mass functions obtained from p_{XY}.

Note that in Definition 5.28, the information functions $\iota_{P_X P_Y}$ and $\iota_{P_{XY}}$ specify the probability measure that must be used. This notation might appear overburden but it is useful to avoid confusion between $-\log(p_{XY}(x,y))$ and $-\log(p_X(x)p_Y(y))$, which can both be denoted by $\iota_{XY}(x,y)$, for all pairs $(x,y) \in \mathbb{R}^2$.

The mutual information between the random variables X and Y can be written in a variety of ways. In order to avoid confusion in the following, the expectation indicates the probability measures that must be used instead of the random variables.

THEOREM 5.29 (Mutual Information). *Let X and Y be two discrete random variables and assume that they respectively induce the probability measures P_X and P_Y on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$. Jointly, X and Y induce the probability measure P_{XY} on $\left(\mathbb{R}^2, \mathscr{B}\left(\mathbb{R}^2\right)\right)$. Let also $p_{XY} : \mathbb{R}^2 \to [0,1]$ be the joint probability mass function. Then, the following holds:*

$$
\begin{aligned}
I(X;Y) &= \mathbb{E}_{P_{XY}}\left[\log\left(\frac{p_{XY}(X,Y)}{p_X(X)p_Y(Y)}\right)\right] && (5.57)\\
&= \mathbb{E}_{P_{XY}}\left[\log\left(\frac{\mathrm{d}P_{XY}}{\mathrm{d}P_X P_Y}(X,Y)\right)\right] && (5.58)\\
&= \mathbb{E}_{P_{XY}}\left[\log\left(\frac{p_{Y|X}(Y|X)}{p_Y(Y)}\right)\right] && (5.59)\\
&= \mathbb{E}_{P_{XY}}\left[\log\left(\frac{\mathrm{d}P_{Y|X}}{\mathrm{d}P_Y}(X,Y)\right)\right] && (5.60)\\
&= \mathbb{E}_{P_{XY}}\left[\log\left(\frac{p_{X|Y}(X|Y)}{p_X(X)}\right)\right] && (5.61)\\
&= \mathbb{E}_{P_{XY}}\left[\log\left(\frac{\mathrm{d}P_{X|Y}}{\mathrm{d}P_X}(X,Y)\right)\right], && (5.62)
\end{aligned}
$$

where $\frac{\mathrm{d}P_{XY}}{\mathrm{d}P_X P_Y}$, $\frac{\mathrm{d}P_{Y|X}}{\mathrm{d}P_Y}$, and $\frac{\mathrm{d}P_{X|Y}}{\mathrm{d}P_X}$ are Radon-Nikodym derivatives; and p_X and p_Y are the marginal probability mass functions obtained from p_{XY}.

Proof See Homework 3 □

The following Theorem presents some useful properties of the mutual information.

THEOREM 5.30. *Given three discrete random variables X, Y, and Z, the following holds:*

$$
\begin{aligned}
&I(X;Y) = I(Y;X), && (5.63)\\
&I(X;Y) = H(X) - H(X|Y), && (5.64)\\
&I(X;Y) = H(Y) - H(Y|X), && (5.65)\\
&I(X;Y) \geqslant 0, && (5.66)\\
&I(X;Y) = H(X) + H(Y) - H(X,Y), && (5.67)\\
&I(X;X) = H(X). && (5.68)
\end{aligned}
$$

Proof Let P_X and P_Y be the probability measures on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ induced by X and Y. Let also P_{XY} on $\left(\mathbb{R}^2, \mathscr{B}\left(\mathbb{R}^2\right)\right)$ be the probability measure jointly induced by X and Y. Let also $p_{XY} : \mathbb{R}^2 \to [0,1]$ be the joint probability mass function.

Proof of (5.63): This follows directly from Definition 5.28.

Proof of (5.64): From (5.61), the following holds:

$$I(X;Y) = -\mathbb{E}_{P_X}\left[\log p_X(X)\right] + \mathbb{E}_{P_{XY}}\left[\log p_{X|Y}(X|Y)\right] \tag{5.69a}$$
$$= H(X) - H(X|Y), \tag{5.69b}$$

and this completes the proof of (5.64).

Proof of (5.65): From (5.59), the following holds:

$$I(X;Y) = \mathbb{E}_{P_{XY}}\left[\log\left(\frac{p_{Y|X}(Y|X)}{p_Y(Y)}\right)\right] \tag{5.70a}$$
$$= -\mathbb{E}_{P_Y}\left[\log p_Y(Y)\right] + \mathbb{E}_{P_{XY}}\left[\log p_{Y|X}(Y|X)\right] \tag{5.70b}$$
$$= H(Y) - H(Y|X), \tag{5.70c}$$

and this completes the proof of (5.65).

Proof of (5.66): From (5.64) and (5.65), the following holds:

$$I(X;Y) \geqslant H(X) - H(X) \tag{5.71a}$$
$$= 0, \tag{5.71b}$$

where, (5.71a) follows from Theorem 5.14. This completes the proof of (5.66).

Proof of (5.67): From (5.57), the following holds:

$$I(X;Y) = -\mathbb{E}_{P_X}\left[\log p_X(X)\right] - \mathbb{E}_{P_Y}\left[\log p_Y(Y)\right] + \mathbb{E}_{P_{XY}}\left[\log p_{XY}(X,Y)\right] \tag{5.72a}$$
$$= H(X) + H(Y) - H(X,Y), \tag{5.72b}$$

and this completes the proof of (5.67).

Proof of (5.68): Let Y be a random variable identical to the random variable X, i.e., $Y = X$. From (5.57), the following holds:

$$I(X;X) = \mathbb{E}_{P_{XY}}\left[\log\left(\frac{p_{XY}(X,Y)}{p_X(X)p_Y(Y)}\right)\right] \tag{5.73a}$$
$$= \mathbb{E}_{P_X}\left[\log\left(\frac{p_X(X)}{p_X(X)p_X(X)}\right)\right] \tag{5.73b}$$
$$= \mathbb{E}_{P_X}\left[\log\left(\frac{1}{p_X(X)}\right)\right] \tag{5.73c}$$
$$= -\mathbb{E}_{P_X}\left[\log p_X(X)\right] \tag{5.73d}$$
$$= H(X), \tag{5.73e}$$

and this completes the proof of (5.68) and the proof of Theorem 5.30. □

The occurrence of one random variable does not provide any information about the occurrence of another random variable, when the random variables are independent. Hence, the mutual information between two independent random variables must be equal to zero. This is proved hereunder.

THEOREM 5.31 (Mutual information of independent random variables). *Let X and Y be two independent discrete random variables. Then,*

$$I(X;Y) = 0. \tag{5.74}$$

Proof Let P_X and P_Y be the probability measures on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ induced by X and Y, respectively. Let also P_{XY} be the probability measure jointly induced by X and Y on $\left(\mathbb{R}^2, \mathscr{B}\left(\mathbb{R}^2\right)\right)$. Let $p_X : \mathbb{R} \to [0,1]$ and $p_Y : \mathbb{R} \to [0,1]$ be the probability mass functions of X and Y, respectively. From the assumption of the theorem, it follows that $p_{XY}(x,y) = p_X(x)p_Y(y)$ for all $(x,y) \in \mathbb{R}^2$. Hence, from (5.57) the following holds:

$$I(X;Y) = \mathbb{E}_{P_{XY}}\left[\log\left(\frac{p_X(X)p_Y(Y)}{p_X(X)p_Y(Y)}\right)\right] \tag{5.75a}$$

$$= \mathbb{E}_{P_{XY}}[\log 1] \tag{5.75b}$$

$$= 0. \tag{5.75c}$$

This completes the proof. □

Given three discrete random variables X, Y, and Z, the mutual information between X and both Y and Z is bigger than or equal to the mutual information between X and one of the random variables Y or Z. This observation is formalized hereunder.

THEOREM 5.32. *Let X, Y, and Z be three discrete random variables. Then,*

$$I(X;Y,Z) \geqslant I(X;Y), \tag{5.76}$$

with equality if and only if $X \to Y \to Z$.

Proof From (5.64), the following holds:

$$I(X;Y,Z) = H(Y,Z) - H(Y,Z|X) \tag{5.77a}$$

$$= H(Y) + H(Z|Y) - H(Y|X) - H(Z|X,Y) \tag{5.77b}$$

$$= I(X;Y) + H(Z|Y) - H(Z|X,Y) \tag{5.77c}$$

$$\geqslant I(X;Y), \tag{5.77d}$$

where, (5.77d) follows from the fact the fact that $H(Z|Y) - H(Z|X,Y) \geqslant 0$ given that conditioning does not increase entropy (Theorem 5.14). Note that the equality holds if $H(Z|Y) - H(Z|X,Y) = H(Z|Y) - H(Z|Y) = 0$. This means that the random variables X and Z are independent conditioning on the random variable Y, i.e., $X \to Y \to Z$. This completes the proof of Theorem 5.32. □

5.4.1 Conditional Mutual Information

DEFINITION 5.33 (Conditional Mutual Information). Given three discrete random variables X, Y, and Z, the mutual information between X and Y conditioning on Z, denoted by $I(X;Y|Z)$, is:

$$I(X;Y|Z) = -\sum_{(x,y,z)\in\text{supp}(p_{XYZ})} p_{XYZ}(x,y,z)\log\left(\frac{p_{XY|Z}(x,y|z)}{p_{X|Z}(x|z)p_{Y|Z}(y|z)}\right). \quad (5.78)$$

Let X, Y and Z be three discrete random variables and assume that they respectively induce the probability measures P_X, P_Y and P_Z on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$. Jointly, X, Y and Z induce the probability measure P_{XYZ} on $\left(\mathbb{R}^3, \mathscr{B}\left(\mathbb{R}^3\right)\right)$. Let also $p_{XYZ} : \mathbb{R}^2 \to [0,1]$ be the joint probability mass function. Then, the mutual information between the random variables X and Y conditioning on the random variable Z can also be written as follows:

$$\begin{aligned} I(X;Y|Z) &= \mathbb{E}_{P_{XYZ}}\left[\log\left(\frac{p_{XY|Z}(X,Y|Z)}{p_{X|Z}(X|Z)p_{Y|Z}(Y|Z)}\right)\right] && (5.79a)\\ &= \mathbb{E}_{P_{XYZ}}\left[\log\left(\frac{p_{Y|XZ}(Y|X,Z)}{p_{Y|Z}(Y|Z)}\right)\right] && (5.79b)\\ &= \mathbb{E}_{P_{XYZ}}\left[\log\left(\frac{p_{X|YZ}(X|Y,Z)}{p_{X|Z}(X|Z)}\right)\right]. && (5.79c)\end{aligned}$$

Note also that the conditional mutual information in (5.78) can be written as follows:

$$\begin{aligned} I(X;Y|Z) &= \sum_{z\in\text{supp}(p_Z)} p_Z(z)\left(-\sum_{(x,y)\in\text{supp}\left(p_{XY|Z=z}\right)} p_{X,Y|Z}(x,y|z)\log\left(\frac{p_{XY|Z}(x,y|z)}{p_{X|Z}(x|z)p_{Y|Z}(y|z)}\right)\right)\\ &= \sum_{z\in\text{supp}(p_Z)} p_Z(z) I(X;Y|Z=z), && (5.80)\end{aligned}$$

where, $I(X;Y|Z=z) = -\sum_{(x,y)\in\text{supp}\left(p_{XY|Z=z}\right)} p_{XY|Z}(x,y|z)\log\left(\frac{p_{XY|Z}(x,y|z)}{p_{X|Z}(x|z)p_{Y|Z}(y|z)}\right)$ is the mutual information between X and Y conditioning on the event $Z = z$.

The following Theorem presents some useful properties of the mutual information and conditional mutual information.

THEOREM 5.34. *Given three discrete random variables X, Y and Z, the following holds:*

$$\begin{aligned} I(X;Y|Z) &= H(Y|Z) - H(Y|X,Z) && (5.81)\\ &= H(X|Z) - H(X|Y,Z) \text{ and} && (5.82)\\ I(X,Y;Z) &= I(X;Z) + I(Y;Z|X) && (5.83)\\ &= I(Y;Z) + I(X;Z|Y). && (5.84)\end{aligned}$$

Proof Let P_X, P_Y and P_Z be the probability measures on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ induced by X, Y and Z, respectively. Jointly, X, Y and Z induce the probability measure P_{XYZ} on $\left(\mathbb{R}^3, \mathscr{B}\left(\mathbb{R}^3\right)\right)$. Let also $p_{XYZ} : \mathbb{R}^2 \to [0,1]$ be the joint probability mass function.

Proof of (5.81): From (5.79b), the following holds:

$$I(X;Y|Z) = \mathbb{E}_{P_{XYZ}}\left[\log p_{Y|XZ}(Y|X,Z)\right] - \mathbb{E}_{P_{YZ}}\left[\log p_{Y|Z}(Y|Z)\right] \quad (5.85a)$$
$$= H(Y|Z) - H(Y|X,Z), \quad (5.85b)$$

and this completes the proof of (5.81).

Proof of (5.82): From (5.79c), the following holds:

$$I(X;Y|Z) = \mathbb{E}_{P_{XYZ}}\left[\log p_{X|YZ}(X|Y,Z)\right] - \mathbb{E}_{P_{XZ}}\left[\log p_{X|Z}(X|Z)\right] \quad (5.86a)$$
$$= H(X|Z) - H(X|Y,Z), \quad (5.86b)$$

and this completes the proof of (5.82).

Proof of (5.83): From (5.64), the following holds:

$$I(X,Y;Z) = H(X,Y) - H(X,Y|Z) \quad (5.87a)$$
$$= H(X) + H(Y|X) - H(X|Z) - H(Y|X,Z) \quad (5.87b)$$
$$= I(X;Z) + I(Y;Z|X), \quad (5.87c)$$

and this completes the proof of (5.83).

Proof of (5.84): From (5.64), the following holds:

$$I(X,Y;Z) = H(X,Y) - H(X,Y|Z) \quad (5.88a)$$
$$= H(Y) + H(X|Y) - H(Y|Z) - H(X|Y,Z) \quad (5.88b)$$
$$= I(Y;Z) + I(X;Z|Y), \quad (5.88c)$$

and this completes the proof of (5.84). This completes the proof of Theorem 5.34.

□

The mutual information between the random variables X and Y conditioning on the random variable Z is equal to zero if X and Y are independent conditioning on Z, i.e., $X \to Z \to Y$.

Theorem 5.35. *Let X, Y and Z be three discrete random variables such that $X \to Z \to Y$. Then,*

$$I(X;Y|Z) = 0. \quad (5.89)$$

Proof Let P_{XYZ} be the probability measure jointly induced by X, Y and Z on $\left(\mathbb{R}^3, \mathscr{B}\left(\mathbb{R}^3\right)\right)$. Let also $p_{XYZ} : \mathbb{R}^2 \to [0,1]$ be the joint probability mass function. From (5.79c), the following holds:

$$I(X;Y|Z) = \mathbb{E}_{P_{XYZ}}\left[\log\left(\frac{p_{X|Z}(x|z)}{p_{X|Z}(x|z)}\right)\right] \tag{5.90a}$$

$$= \mathbb{E}_{P_{XYZ}}[\log 1] \tag{5.90b}$$

$$= 0. \tag{5.90c}$$

where, (5.90a) follows from the fact that the random variables X and Y are mutually independent conditioning on the random variable Z, i.e., $X \to Z \to Y$. This completes the proof of Theorem 5.35. □

The following Theorem presents some additional useful properties of the mutual information and conditional mutual information.

THEOREM 5.36 (Chain rules). *Let $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)^\mathsf{T}$ be an n-dimensional discrete random vector and let also Y and Z be two discrete random variables. Then,*

$$I(X_1, X_2, \ldots, X_n; Y) = I(X_1; Y) + I(X_2; Y|X_1) + \sum_{n=3}^{n} I(X_n; Y|X_1, X_2, \ldots, X_{n-1}), \tag{5.91a}$$

$$I(X_1, X_2, \ldots, X_n; Y) \geqslant 0, \text{ and} \tag{5.91b}$$

$$I(X_1, X_2, \ldots, X_n; Y|Z) = I(X_1; Y|Z) + I(X_2; Y|Z, X_1) + \sum_{n=3}^{n} I(X_n; Y|Z, X_1, X_2, \ldots, X_{n-1}). \tag{5.91c}$$

Proof **Proof of** (5.91a): Let $P_{\boldsymbol{X}Y}$ be the probability measure jointly induced by $\boldsymbol{X}$, and Y on $\left(\mathbb{R}^{n+1}, \mathscr{B}\left(\mathbb{R}^{n+1}\right)\right)$. Let also $p_{\boldsymbol{X}Y} : \mathbb{R}^{n+1} \to [0,1]$ be the joint probability mass function. Hence, from (5.57), the following holds:

$$I(\boldsymbol{X};Y) = \mathbb{E}_{P_{\boldsymbol{X}Y}}\left[\log\left(\frac{p_{\boldsymbol{X}Y}(\boldsymbol{X},Y)}{p_{\boldsymbol{X}}(\boldsymbol{X})p_Y(Y)}\right)\right] \tag{5.92a}$$

$$\begin{aligned}
&= \mathbb{E}_{P_{X_1Y}}\left[\log\frac{p_{X_1Y}(X_1,Y)}{p_{X_1}(X_1)p_Y(Y)}\right] \\
&\quad + \mathbb{E}_{P_{X_1X_2Y}}\left[\log\frac{p_{X_2|X_1Y}(X_2|X_1,Y)}{p_{X_2|X_1}(X_2|X_1)}\right] \\
&\quad + \mathbb{E}_{P_{X_1X_2X_3Y}}\left[\log\frac{p_{X_3|X_1X_2Y}(X_3|X_1,X_2,Y)}{p_{X_3|X_1X_2}(X_3|X_1,X_2)}\right] + \ldots \\
&\quad + \mathbb{E}_{P_{\boldsymbol{X}Y}}\left[\log\frac{p_{X_n|X_1X_2\ldots X_{n-1}Y}(X_n|X_1,X_2,\ldots,X_{n-1},Y)}{p_{X_n|X_1X_2\ldots X_{n-1}}(X_3|X_1,X_2,\ldots,X_n-1)}\right]
\end{aligned} \tag{5.92b}$$

$$\begin{aligned}
&= I(X_1;Y) + I(X_2;Y|X_1) + I(X_3;Y|X_1,X_2) + \ldots \\
&\quad + I(X_n;Y|X_1,X_2,\ldots,X_{n-1}),
\end{aligned} \tag{5.92c}$$

where, (5.92c) follows from (5.57) and (5.79b). This completes the proof of (5.91a).

Proof of (5.91b): From (5.91a), the following holds:

$$\begin{aligned}
I(X_1, X_2, \dots, X_n; Y) &= I(X_1; Y) + I(X_2; Y|X_1) \\
&\quad + \sum_{n=3}^{n} I(X_n; Y|X_1, X_2, \dots, X_{n-1}) && \text{(5.93a)} \\
&= H(Y) - H(Y|X_1) + H(Y|X_1) - H(Y|X_1, X_2) \\
&\quad + \sum_{n=3}^{n} H(Y|X_1, X_2, \dots, X_{n-1}) \\
&\quad - H(Y|X_1, X_2, \dots, X_{n-1}, X_n) && \text{(5.93b)} \\
&\geqslant 0, && \text{(5.93c)}
\end{aligned}$$

where (5.93c) follows from Theorem 5.14 and the fact that $H(Y) \geqslant H(Y|X_1)$, $H(Y|X_1) \geqslant H(Y|X_1, X_2)$, ..., and for all $n > 2$, $H(Y|X_1, X_2, \dots, X_{n-1}) \geqslant H(Y|X_1, X_2, \dots, X_{n-1}, X_n)$. This completes the proof of of (5.91b).

Proof of (5.91c): Let $P_{\boldsymbol{X}YZ}$ be the probability measure jointly induced by $\boldsymbol{X}$, Y and Z on $\left(\mathbb{R}^{n+2}, \mathscr{B}\left(\mathbb{R}^{n+2}\right)\right)$. Let also $p_{\boldsymbol{X}YZ} : \mathbb{R}^{n+2} \to [0,1]$ be the joint probability mass function. Hence, from (5.79c), the following holds:

$$\begin{aligned}
I(\boldsymbol{X}; Y|Z) &= \mathbb{E}_{P_{\boldsymbol{X}YZ}}\left[\log\left(\frac{p_{\boldsymbol{X}|YZ}(\boldsymbol{X}|Y, Z)}{p_{\boldsymbol{X}|Z}(\boldsymbol{X})}\right)\right] && \text{(5.94a)} \\
&= \mathbb{E}_{P_{X_1YZ}}\left[\log\frac{p_{X_1|YZ}(X_1|Y, Z)}{p_{X_1|Z}(X_1|Z)}\right] \\
&\quad + \mathbb{E}_{P_{X_1X_2YZ}}\left[\log\frac{p_{X_2|X_1YZ}(X_2|X_1, Y, Z)}{p_{X_2|X_1Z}(X_2|X_1, Z)}\right] \\
&\quad + \mathbb{E}_{P_{X_1X_2X_3YZ}}\left[\log\frac{p_{X_3|X_1X_2YZ}(X_3|X_1, X_2, Y, Z)}{p_{X_3|X_1X_2Z}(X_3|X_1, X_2, Z)}\right] + \dots \\
&\quad + \mathbb{E}_{P_{\boldsymbol{X}YZ}}\left[\log\frac{p_{X_n|X_1X_2\dots X_{n-1}YZ}(X_n|X_1, X_2, \dots, X_{n-1}, Y, Z)}{p_{X_n|X_1X_2\dots X_{n-1}Z}(X_3|X_1, X_2, \dots, X_n - 1, Z)}\right] \\
&= I(X_1; Y|Z) + I(X_2; Y|X_1, Z) + I(X_3; Y|X_1, X_2, Z) + \dots \\
&\quad + I(X_n; Y|X_1, X_2, \dots, X_{n-1}, Z), && \text{(5.94b)}
\end{aligned}$$

where (5.94b) follows from (5.79c). This completes the proof of (5.91c). This completes the proof of Theorem 5.36.

□

The following Theorems state some properties of the mutual information between the discrete random variables X, Y and Z when they form a Markov chain, i.e., $X \to Y \to Z$.

THEOREM 5.37 (Data Processing Inequality). *Let X, Y, and Z be three* *discrete random variables forming the Markov chain $X \to Y \to Z$. Then,*

$$I(X;Z) \leqslant I(X;Y) \text{ and} \tag{5.95a}$$

$$I(X;Z) \leqslant I(Y;Z), \tag{5.95b}$$

and when $Z = g(Y)$, for a given Borel measurable function $g : \mathbb{R} \to \mathbb{R}$, then

$$I(X;g(Y)) \leqslant I(X;Y). \tag{5.95c}$$

Proof **Proof of** (5.95a): From (5.91a), the following holds:

$$\begin{aligned} I(X;Y,Z) &= I(X;Z) + I(X;Y|Z) && (5.96a)\\ &\geqslant I(X;Z) && (5.96b) \end{aligned}$$

and

$$\begin{aligned} I(X;Y,Z) &= I(X;Y) + I(X;Z|Y) && (5.96c)\\ &= I(X;Y), && (5.96d) \end{aligned}$$

where (5.96d) follows from the fact that the random variables X and Z are mutually independent conditioning on the random variable Y, i.e., $X \to Y \to Z$. From (5.96b) and (5.96d), the following holds:

$$I(X;Z) \leqslant I(X;Y), \tag{5.96e}$$

and this completes the proof of (5.95a).

Proof of (5.95b): From (5.91a), the following holds:

$$\begin{aligned} I(X,Y;Z) &= I(Y;Z) + I(X;Z|Y) && (5.97a)\\ &= I(Y;Z) && (5.97b) \end{aligned}$$

and

$$\begin{aligned} I(X,Y;Z) &= I(X;Z) + I(Y;Z|X) && (5.97c)\\ &\geqslant I(X;Z), && (5.97d) \end{aligned}$$

where (5.97b) follows from the fact that the random variables X and Z are mutually independent conditioning on the random variable Y, i.e., $X \to Y \to Z$. From (5.97b) and (5.97d), the following holds:

$$I(X;Z) \leqslant I(Y;Z), \tag{5.97e}$$

and this completes the proof of (5.95b).

Proof of (5.95c): Plugging $Z = g(Y)$ into (5.96e), yields:

$$I\left(X;g(Y)\right) \leqslant I\left(X;Y\right), \tag{5.98}$$

and this completes the proof of (5.95c). □

Theorem 5.38. *Let X, Y and Z be three discrete random variables such* *that* $X \to Y \to Z$. *Then,*

$$I(X;Y|Z) \leqslant I(X;Y) \text{ and} \tag{5.99a}$$

$$I(Y;Z|X) \leqslant I(Y;Z). \tag{5.99b}$$

Proof **Proof of** (5.99a): From (5.91a), the following holds:

$$I(X;Y,Z) = I(X;Z) + I(X;Y|Z) \tag{5.100a}$$

$$\geqslant I(X;Y|Z). \tag{5.100b}$$

From (5.96d) and (5.100b), the following holds:

$$I(X;Y|Z) \leqslant I(X;Y), \tag{5.100c}$$

and this completes the proof of (5.99a).

Proof of (5.99b): From (5.91a), the following holds:

$$I(X,Y;Z) = I(X;Z) + I(Y;Z|X)$$

$$\geqslant I(Y;Z|X). \tag{5.101a}$$

From (5.97b) and (5.101a), the following holds:

$$I(Y;Z|X) \leqslant I(Y;Z), \tag{5.101b}$$

and this completes the proof of (5.99b). This completes the proof of Theorem 5.38. □

THEOREM 5.39. *Let* X, Y *and* Z *be three discrete random variables with joint probability mass function* $p_{XYZ} : \mathbb{R}^3 \to [0,1]$ *such that for* $(x,y,z) \in \mathbb{R}^3$, $p_{XYZ}(x,y,z) = p_X(x)p_Y(y)p_{Z|XY}(z|x,y)$. *Then,*

$$I(X;Y|Z) \geqslant I(X;Y). \tag{5.102}$$

Proof From the assumption of the theorem, X and Y are two independent random variables, then $I(X;Y) = 0$. Hence, the inequality follows from the non-negativity of mutual information. □

The following two theorems play central roles in the analysis of the fundamental limits of data transmission.

THEOREM 5.40. *Let* $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)^\mathsf{T}$ *and* $\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_n)^\mathsf{T}$ *be two* n*-dimensional discrete random vectors with joint probability mass function* $p_{\boldsymbol{XY}} : \mathbb{R}^{2n} \to [0,1]$. *Assume that for all* $(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^{2n}$,

$$p_{\boldsymbol{XY}}(\boldsymbol{x}, \boldsymbol{y}) = p_{\boldsymbol{Y}|\boldsymbol{X}}(\boldsymbol{y}|\boldsymbol{x}) \prod_{t=1}^{n} p_X(x_t), \tag{5.103}$$

for some probability mass function $p_X : \mathbb{R} \to [0,1]$. *Then,*

$$I(\boldsymbol{X};\boldsymbol{Y}) \geqslant \sum_{n=1}^{n} I(X_n;Y_n). \tag{5.104}$$

Proof Let $P_{\boldsymbol{XY}}$ be the probability measure jointly induced by $\boldsymbol{X}$ and $\boldsymbol{Y}$ on $\left(\mathbb{R}^{2n}, \mathscr{B}\left(\mathbb{R}^{2n}\right)\right)$. Similarly, for all $t \in \{1,2,\ldots,n\}$, let $P_{X_tY_t}$ be the probability measure jointly induced by X_t and Y_t on $\left(\mathbb{R}^2, \mathscr{B}\left(\mathbb{R}^2\right)\right)$. Hence, from (5.61), the following holds:

$$\begin{aligned} I(\boldsymbol{X};\boldsymbol{Y}) &= \mathbb{E}_{P_{\boldsymbol{XY}}}\left[\log\left(\frac{p_{\boldsymbol{X}|\boldsymbol{Y}}(\boldsymbol{X}|\boldsymbol{Y})}{p_{\boldsymbol{X}}(\boldsymbol{X})}\right)\right] && (5.105\text{a}) \\ &= \mathbb{E}_{P_{\boldsymbol{XY}}}\left[\log\left(\frac{p_{\boldsymbol{X}|\boldsymbol{Y}}(\boldsymbol{X}|\boldsymbol{Y})}{p_{X_1}(X_1)p_{X_2}(X_2)\ldots p_{X_n}(X_n)}\right)\right], && (5.105\text{b}) \end{aligned}$$

where, (5.105b) follows from the fact that $X_1, X_2, \ldots, X_n$ are mutually independent. On the other hand,

$$\begin{aligned} \sum_{t=1}^{n} I(X_t;Y_t) &= \sum_{t=1}^{n} \mathbb{E}_{P_{X_tY_t}}\left[\log\left(\frac{p_{X_t|Y_t}(X_t|Y_t)}{p_{X_t}(X_t)}\right)\right] \\ &= \mathbb{E}_{P_{\boldsymbol{XY}}}\left[\log\left(\frac{p_{X_1|Y_1}(X_1|Y_1)p_{X_2|Y_2}(X_2|Y_2)\ldots p_{X_n|Y_n}(X_n|Y_n)}{p_{X_1}(X_1)p_{X_2}(X_2)\ldots p_{X_n}(X_n)}\right)\right]. \end{aligned}$$

Hence,

$$\begin{aligned} &\sum_{t=1}^{n} I\ (X_t;Y_t) - I(\boldsymbol{X};\boldsymbol{Y}) \\ &\quad= \mathbb{E}_{P_{\boldsymbol{XY}}}\left[\log\left(\frac{p_{X_1|Y_1}(X_1|Y_1)p_{X_2|Y_2}(X_2|Y_2)\ldots p_{X_n|Y_n}(X_n|Y_n)}{p_{\boldsymbol{X}|\boldsymbol{Y}}(\boldsymbol{X}|\boldsymbol{Y})}\right)\right] \\ &\quad\leqslant \log\left(\mathbb{E}_{P_{\boldsymbol{XY}}}\left[\left(\frac{p_{X_1|Y_1}(X_1|Y_1)p_{X_2|Y_2}(X_2|Y_2)\ldots p_{X_n|Y_n}(X_n|Y_n)}{p_{\boldsymbol{X}|\boldsymbol{Y}}(\boldsymbol{X}|\boldsymbol{Y})}\right)\right]\right) && (5.105\text{c}) \\ &\quad= \log\left(\sum_{\boldsymbol{y}\in\operatorname{supp} p_{\boldsymbol{Y}}} p_{\boldsymbol{Y}}(\boldsymbol{y}) \sum_{\boldsymbol{x}\in\operatorname{supp} p_{\boldsymbol{X}}} \left(p_{X_1|Y_1}(x_1|y_1)p_{X_2|Y_2}(x_2|y_2)\ldots p_{X_n|Y_n}(x_n|y_n)\right)\right) \\ &\quad= \log\left(\sum_{\boldsymbol{y}\in\operatorname{supp} p_{\boldsymbol{Y}}} p_{\boldsymbol{Y}}(\boldsymbol{y})\right) \\ &\quad= \log 1 \\ &\quad= 0, && (5.105\text{d}) \end{aligned}$$

where (5.105c) follows from Jensen's inequality (Theorem 3.45). Then,

$$I(\boldsymbol{X};\boldsymbol{Y}) \geqslant \sum_{t=1}^{n} I(X_t;Y_t), \tag{5.105e}$$

and this completes the proof of 5.40. □

THEOREM 5.41. *Let $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ be $2n$ discrete random variables with joint probability mass function $p_{\boldsymbol{XY}} : \mathbb{R}^{2n} \to [0,1]$, such that for all $(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^{2n}$ it holds that*

$$P_{\boldsymbol{XY}}(\boldsymbol{x}, \boldsymbol{y}) = \prod_{t=1}^{n} P_{Y|X}(y_t|x_t) P_X(x_t), \tag{5.106}$$

for some given probability mass functions $p_{Y|X}$ and p_X. Then,

$$I(\boldsymbol{X}; \boldsymbol{Y}) \leqslant \sum_{t=1}^{n} I(X_t; Y_t). \tag{5.107}$$

Proof Let $P_{\boldsymbol{XY}}$ be the probability measure jointly induced by $\boldsymbol{X}$ and $\boldsymbol{Y}$ on $\left(\mathbb{R}^{2n}, \mathscr{B}\left(\mathbb{R}^{2n}\right)\right)$. Similarly, for all $t \in \{1, 2, \ldots, n\}$, let $P_{X_t Y_t}$ be the probability measure jointly induced by X_t and Y_t on $\left(\mathbb{R}^2, \mathscr{B}\left(\mathbb{R}^2\right)\right)$. Hence, from (5.59), the following holds:

$$\begin{aligned} I(\boldsymbol{X}; \boldsymbol{Y}) &= \mathbb{E}_{P_{\boldsymbol{XY}}}\left[\log\left(\frac{p_{\boldsymbol{Y}|\boldsymbol{X}}(\boldsymbol{Y}|\boldsymbol{X})}{p_{\boldsymbol{Y}}(\boldsymbol{Y})}\right)\right] \\ &= \mathbb{E}_{P_{\boldsymbol{XY}}}\left[\log\left(\frac{p_{Y_1|X_1}(Y_1|X_1) p_{Y_2|X_2}(Y_2|X_2) \ldots p_{Y_n|X_n}(Y_n|X_n)}{p_{\boldsymbol{Y}}(\boldsymbol{Y})}\right)\right]. \end{aligned}$$

On the other hand,

$$\begin{aligned} \sum_{t=1}^{n} (X_t; Y_t) &= \sum_{t=1}^{n} \mathbb{E}_{P_{X_t Y_t}}\left[\log\left(\frac{p_{Y_t|X_t}(Y_t|X_t)}{p_{Y_t}(Y_t)}\right)\right] \\ &= \mathbb{E}_{P_{\boldsymbol{XY}}}\left[\log\left(\frac{p_{Y_1|X_1}(Y_1|X_1) p_{Y_2|X_2}(Y_2|X_2) \ldots p_{Y_n|X_n}(Y_n|X_n)}{p_{Y_1}(Y_1) p_{Y_2}(Y_2) \ldots p_{Y_n}(Y_n)}\right)\right]. \end{aligned}$$

Hence,

$$\begin{aligned} I(\boldsymbol{X}; \boldsymbol{Y}) - \sum_{t=1}^{n} I(X_t; Y_t) &= \mathbb{E}_{\boldsymbol{Y}}\left[\log\left(\frac{p_{Y_1}(Y_1) p_{Y_2}(Y_2) \ldots p_{Y_t}(Y_t)}{p_{\boldsymbol{Y}}(\boldsymbol{Y})}\right)\right] \\ &\leqslant \log\left(\mathbb{E}_{\boldsymbol{Y}}\left[\left(\frac{p_{Y_1}(Y_1) p_{Y_2}(Y_2) \ldots p_{Y_n}(Y_n)}{p_{\boldsymbol{Y}}(\boldsymbol{Y})}\right)\right]\right) \\ &= \log\left(\sum_{\boldsymbol{y} \in \operatorname{supp} p_{\boldsymbol{Y}}} \left(p_{Y_1}(y_1) p_{Y_2}(y_2) \ldots p_{Y_n}(y_n)\right)\right) \\ &= \log 1 \\ &= 0, \end{aligned} \tag{5.108a}$$

where the inequality follows from Jensen's inequality (Theorem 3.45). Then,

$$I(\boldsymbol{X}; \boldsymbol{Y}) \leqslant \sum_{t=1}^{n} I(X_t; Y_t), \tag{5.108b}$$

which completes the proof of Theorem 5.41. □

5.5 Mutual Information – Case of Absolutely Continuous Random Variables

The mutual information between two absolutely continuous random variables is defined as follows.

Definition 5.42 (Mutual Information). Let X and Y be two absolutely continuous random variables with joint probability density function $f_{XY} : \mathbb{R}^2 \to [0, \infty)$. Then, the mutual information between X and Y, denoted by $I(X;Y)$, is:

$$I(X;Y) = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y) \log\left(\frac{f_{XY}(x,y)}{f_X(x) f_Y(y)}\right) \mathrm{d}x\, \mathrm{d}y. \tag{5.109}$$

The mutual information between the real-valued random variables X and Y can also be written as follows:

$$I(X;Y) = \mathbb{E}_{XY}\left[\log\left(\frac{f_{XY}(X,Y)}{f_X(X) f_Y(Y)}\right)\right] \tag{5.110a}$$

$$= \mathbb{E}_{XY}\left[\log\left(\frac{f_{Y|X}(Y|X)}{f_Y(Y)}\right)\right] \tag{5.110b}$$

$$= \mathbb{E}_{XY}\left[\log\left(\frac{f_{X|Y}(X|Y)}{f_X(X)}\right)\right]. \tag{5.110c}$$

Theorems 5.30-5.32 can be extended to real-valued random variables.

Theorem 5.43 (Mutual information between two Gaussian random variables). *Let X and Y be two random variables with joint probability function f_{XY} such that for all $(x, y) \in \mathbb{R}^2$,*

$$f_{XY}(x,y) = \frac{1}{\sqrt{(2\pi)^2 \det \mathbb{K}}} \exp\left(-\frac{1}{2}\,[x\, y]\, \mathbb{K}^{-1} \begin{bmatrix} x \\ y \end{bmatrix}\right), \tag{5.111}$$

where,

$$\mathbb{K} \triangleq \mathbb{E}_{f_{XY}}\left[\begin{bmatrix} X \\ Y \end{bmatrix} \begin{bmatrix} X & Y \end{bmatrix}\right] = \begin{bmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{bmatrix}, \tag{5.112}$$

with σ_X^2 and σ_Y^2 are the variances of X and Y, respectively; and $\rho \triangleq \frac{\mathbb{E}_{XY}[XY]}{\sigma_X\sigma_Y}$ is the Pearson correlation coefficient. Then,

$$I(X;Y) = -\frac{1}{2}\log\left(1-\rho^2\right). \tag{5.113}$$

Proof From Theorem 5.67, the following holds:

$$I(X;Y) = h(X) + h(Y) - h(X,Y). \tag{5.114a}$$

Plugging (5.36) and (5.44) into (5.114a), the following holds:

$$I(X;Y) = \frac{1}{2}\log\left(2\pi e\sigma_X^2\right) + \frac{1}{2}\log\left(2\pi e\sigma_Y^2\right) - \frac{1}{2}\log\left((2\pi e)^2 \det \mathbb{K}\right) \tag{5.114b}$$

$$= \frac{1}{2}\log\left(\frac{\sigma_X^2\sigma_Y^2}{\det \mathbb{K}}\right) \tag{5.114c}$$

$$= -\frac{1}{2}\log\left(1-\rho^2\right), \tag{5.114d}$$

where (5.114d) follows from the fact that $\det \mathbb{K} = \sigma_X^2\sigma_Y^2\left(1-\rho^2\right)$. This completes the proof. □

Note that if $\rho = \pm 1$ (perfect correlation) then $I(X;Y)$ is infinite.

5.5.1 Conditional Mutual Information

Definition 5.44 (Conditional Mutual Information). Let X, Y, and Z be three absolutely continuous random variables with joint probability density function $f_{XYZ} : \mathbb{R}^3 \to [0,\infty)$. Then, the mutual information between X and Y conditioning on Z, denoted by $I(X;Y|Z)$, is:

$$I(X;Y|Z) = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XYZ}(x,y,z)\log\left(\frac{f_{XY|Z}(x,y|z)}{f_{X|Z}(x|z)f_{Y|Z}(y|z)}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \tag{5.115}$$

The mutual information between the real-valued random variables X and Y conditioning on the real-valued random variable Z can also be written as follows:

$$I(X;Y|Z) = \mathbb{E}_{XYZ}\left[\log\left(\frac{f_{XY|Z}(X,Y|Z)}{f_{X|Z}(X|Z)f_{Y|Z}(Y|Z)}\right)\right] \tag{5.116}$$

$$= \mathbb{E}_{XYZ}\left[\log\left(\frac{f_{Y|XZ}(Y|X,Z)}{f_{Y|Z}(Y|Z)}\right)\right] \tag{5.117}$$

$$= \mathbb{E}_{XYZ}\left[\log\left(\frac{f_{X|YZ}(X|YZ)}{f_{X|Z}(X|)}\right)\right]. \tag{5.118}$$

Theorems 5.34-5.41 can be extended to real-valued random variables.

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