Interpolation via Barycentric Coordinates

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Some material from D. Anisimov

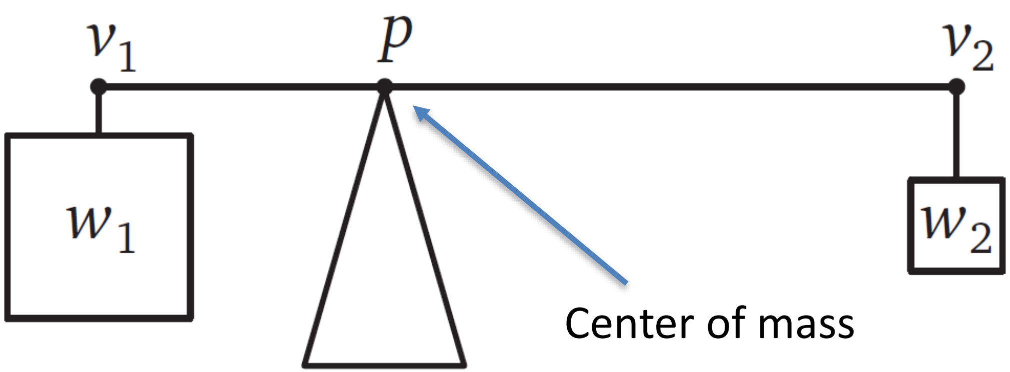
Outline

- Barycenter
- Convexity
- Barycentric coordinates
  - For Simplices
  - For point sets
    - Inverse distance (Shepard)
    - Delaunay / Voronoi
    - Natural neighbors (Sibson)
  - For convex polyhedra
  - Generalized barycentric coordinates
  - Applications

BARYCENTER

Barycenter

- Law of lever: $w_1 l_1 = w_2 l_2$



$$l_1 = p - v_1$$
$$l_2 = v_2 - p$$

Barycentric coordinates

$$w_1(v_1 - p) + w_2(v_2 - p) = 0$$

$$w_1 v_1 + w_2 v_2 = W p$$

$$b_1 v_1 + b_2 v_2 = p$$

$$w_1 + w_2$$

Opposite problem?

$$b_1 v_1 + b_2 v_2 = p$$

?

Opposite problem

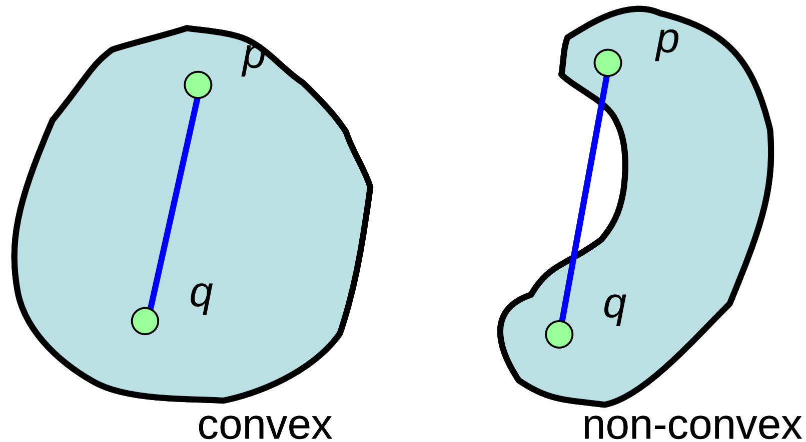
$$b_1 v_1 + b_2 v_2 = p$$

l_2/l $\quad$ l_1/l $\quad$ unique!

CONVEXITY

Convexity

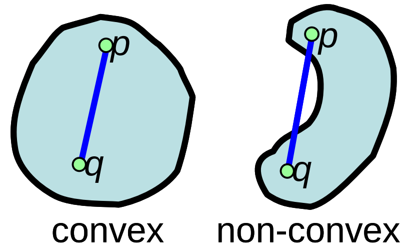
A set S is *convex* if any pair of points $p,q \in S$ satisfy $pq \subseteq S$.

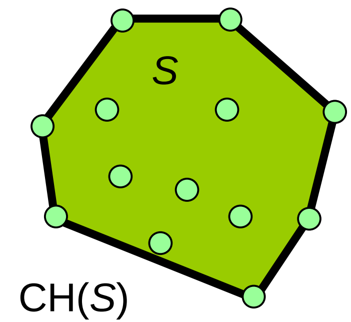


Convex Hull

- The *convex hull* of a set S is:
  - The minimal convex set that contains S, i.e. any convex set C such that $S \subseteq C$ satisfies CH(S) $\subseteq C$.
  - The intersection of all convex sets that contain S.
  - The set of all convex combinations of $p_i \in S$, i.e. all points of the form:

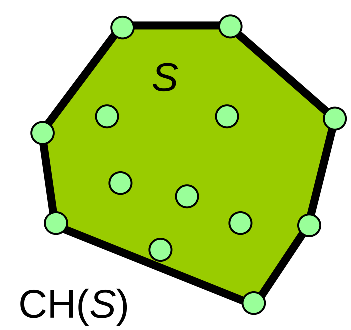
$$\sum_{i=1}^{n} \alpha_i p_i \ , \qquad \alpha_i \geq 0, \ \sum_{i=1}^{n} \alpha_i = 1$$





Convex Hulls

- The convex hull of a set is unique, up to collinearities.
- The boundary of the convex hull of a point set is a polygon on a subset of the points.

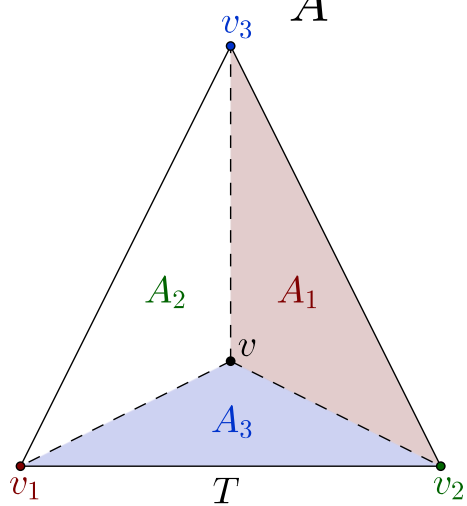


BARYCENTRIC COORDINATES FOR SIMPLICES

Triangle barycentric coordinates

$$b_3 = \frac{A_3}{A}$$

$$b_1 = \frac{A_1}{A} \qquad b_2 = \frac{A_2}{A}$$

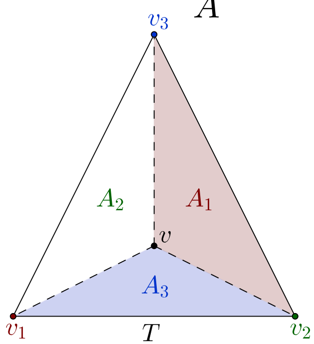


$$A = A_1 + A_2 + A_3$$

A. F. Möbius
[1790–1868]

Triangle barycentric coordinates

$$b_3 = \frac{A_3}{A}$$



$$b_1 = \frac{A_1}{A} \qquad b_2 = \frac{A_2}{A}$$

$$A = A_1 + A_2 + A_3$$

$$\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

Properties:

Constant precision: $b_1 + b_2 + b_3 = 1$

Linear precision: $b_1 v_1 + b_2 v_2 + b_3 v_3 = v$

The Lagrange property: $b_i(v_j) = \delta_{ij}$

Non-negativity: $b_i \geq 0$ for $i = 1, 2, 3$

Linearity along edges: $b_i = av + b$ for $v \in \partial T$

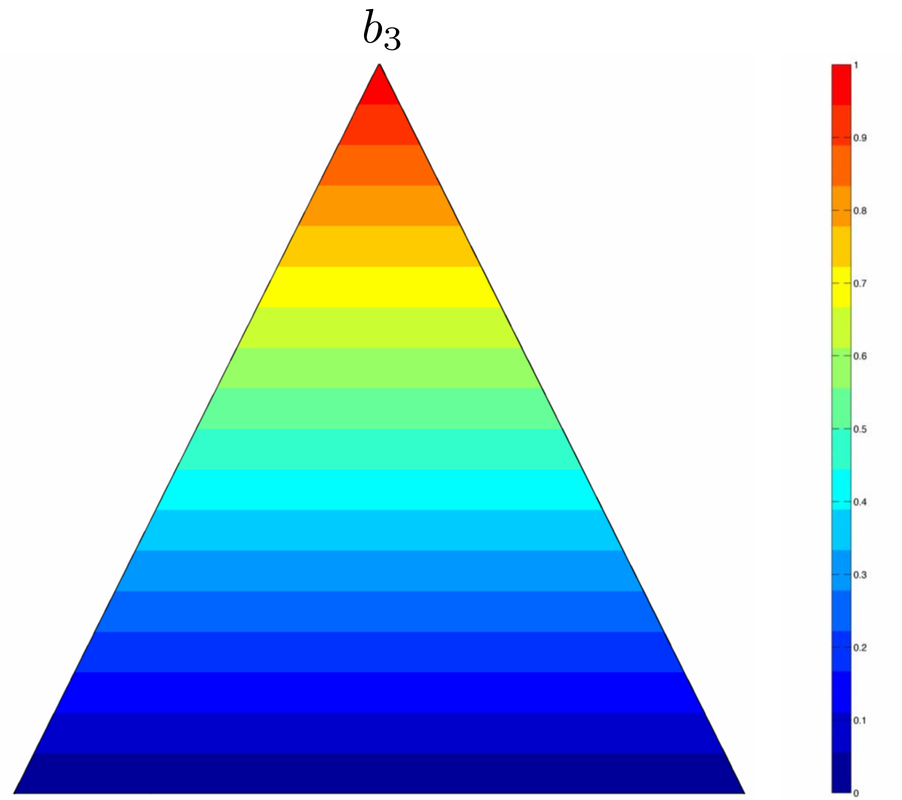
Smoothness: $b_i \in C^k,\ k > 0$

Closed-form: b_i are expressed in analytic form

Triangle barycentric coordinates

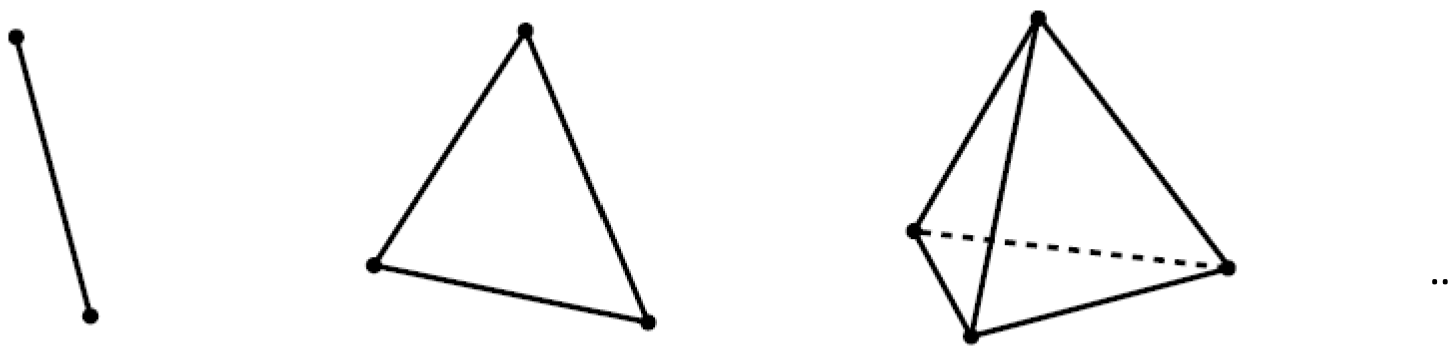
$$\sum_{i=1}^{d+1} b_i(\boldsymbol{p})\boldsymbol{v}_i$$

Data interpolation



Linear function

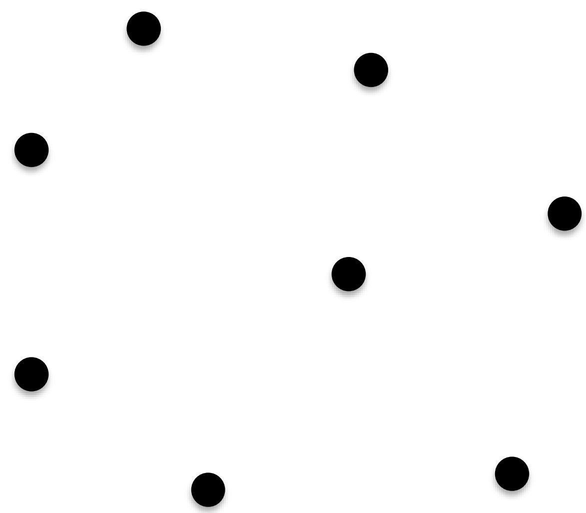
Barycentric Coordinates for Simplices



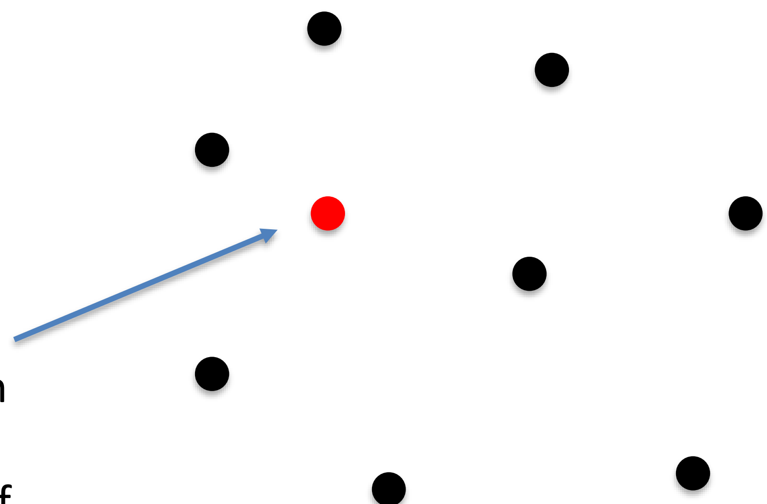
$$\sum_{i=1}^{d+1} b_i(\boldsymbol{p})\boldsymbol{v}_i$$

BARYCENTRIC COORDINATES FOR POINT SETS

Point Set



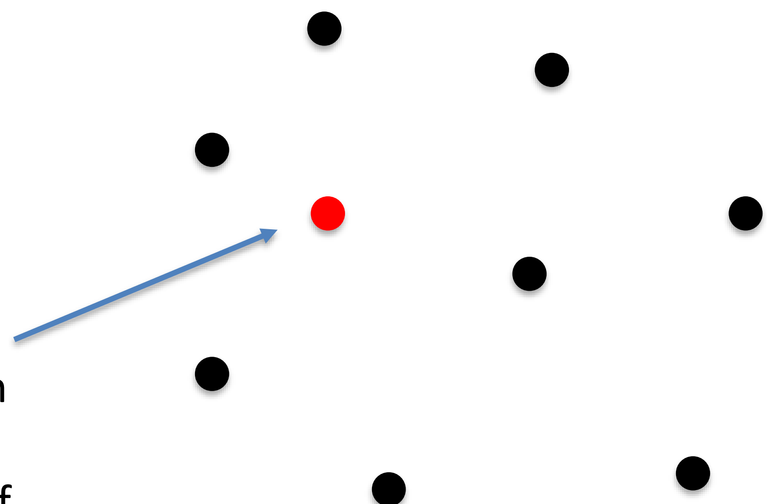
Barycentric Coordinates?



Weights such that it is a barycenter of the point set?

Lagrange property: $b_i(v_j) = \delta_{ij}$

Barycentric Coordinates?



Weights such that it is a barycenter of the point set?

A. F. Möbius
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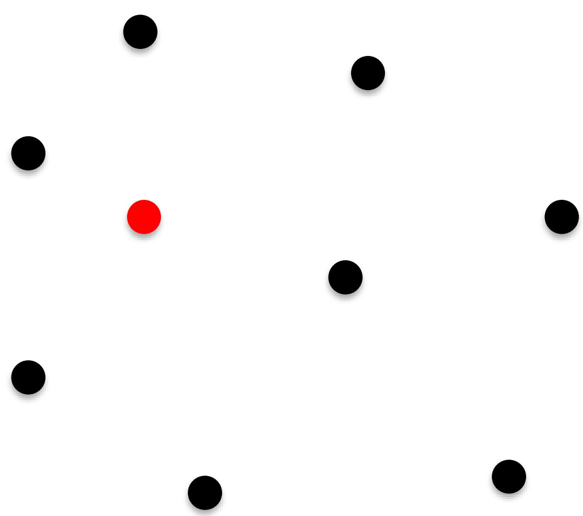
Weights always exist if #points >= dimension

Lagrange property: $b_i(v_j) = \delta_{ij}$

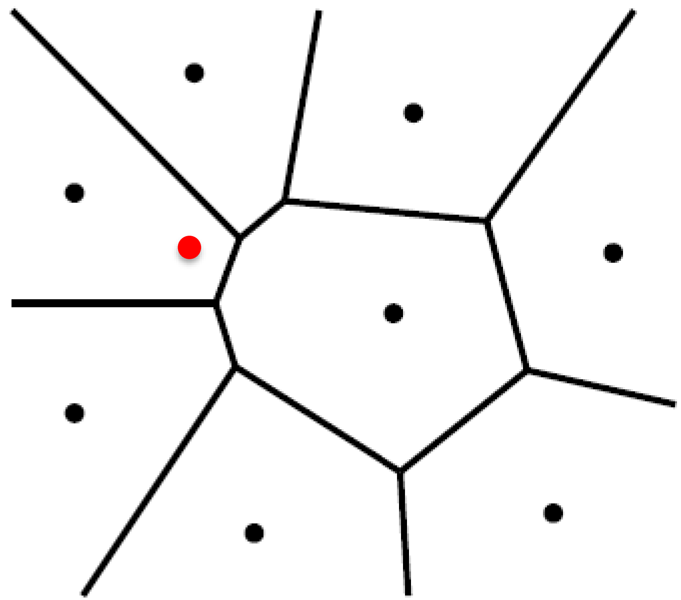
Inverse Distance [Shepard]



$$u(\mathbf{x}) = \begin{cases} \dfrac{\sum_{i=1}^{N} w_i(\mathbf{x}) u_i}{\sum_{i=1}^{N} w_i(\mathbf{x})}, & \text{if } d(\mathbf{x}, \mathbf{x}_i) \neq 0 \text{ for all } i \\ u_i, & \text{if } d(\mathbf{x}, \mathbf{x}_i) = 0 \text{ for some } i \end{cases}$$

$$w_i(\mathbf{x}) = \frac{1}{d(\mathbf{x}, \mathbf{x}_i)^p}$$

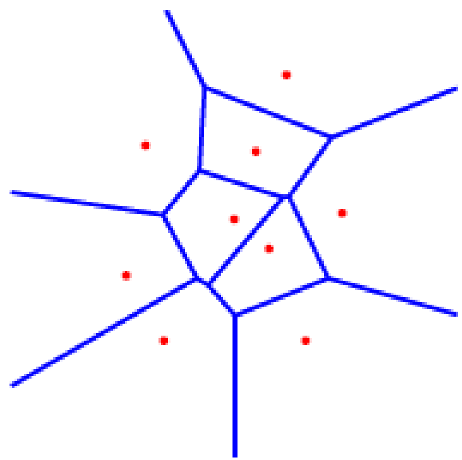
Natural Neighbor Coordinates [Sibson]



Voronoi Diagram

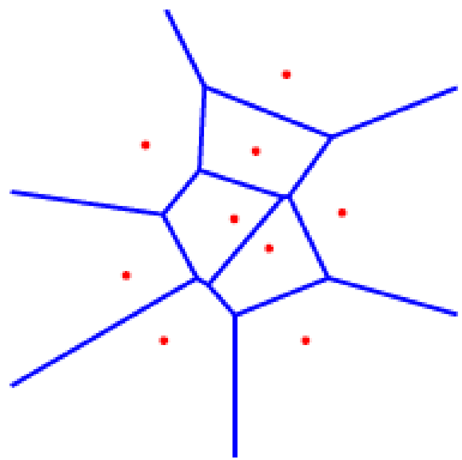
Let $\mathcal{E} = \{\mathbf{p_1}, \ldots, \mathbf{p_n}\}$ be a set of points (so-called sites) in $\mathbb{R}^d$. We associate to each site $\mathbf{p_i}$ its Voronoi region $V(\mathbf{p_i})$ such that:

$$V(\mathbf{p_i}) = \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x} - \mathbf{p_i}\| \leq \|\mathbf{x} - \mathbf{p_j}\|, \forall j \leq n\}.$$



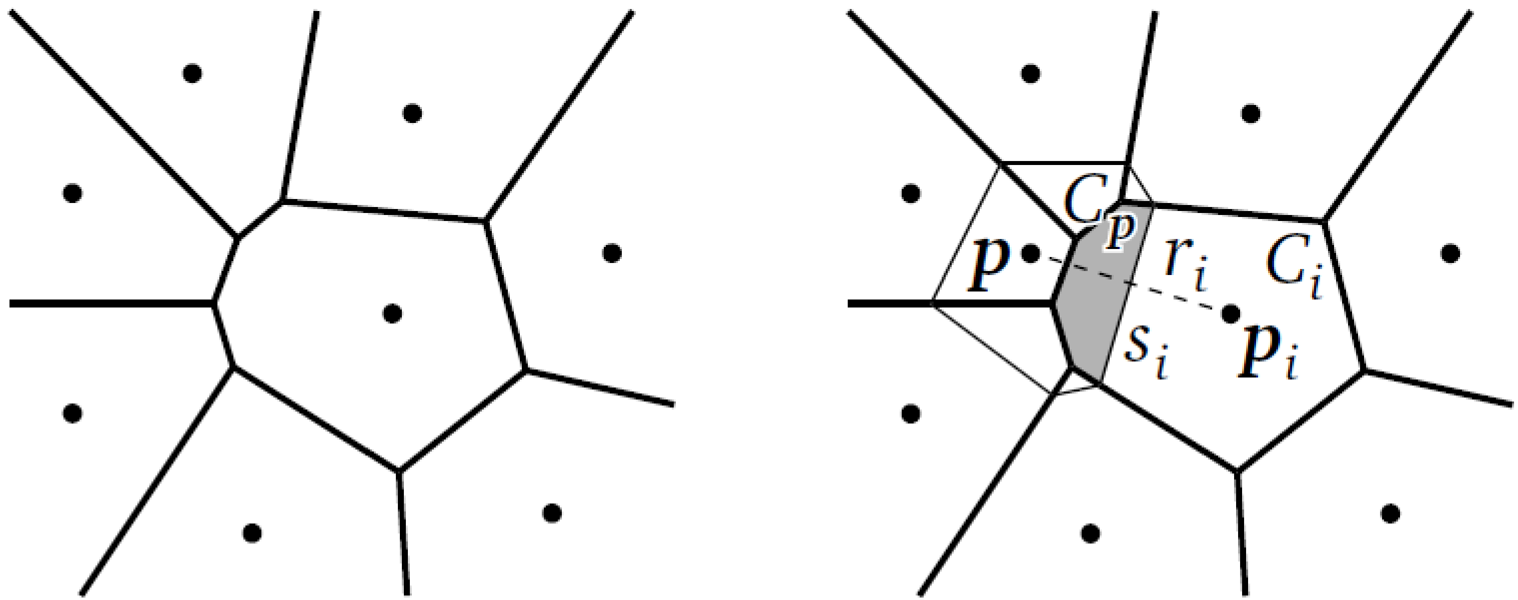
Voronoi Diagram

- The collection of the non-empty Voronoi regions and their faces, together with their incidence relations, constitute a cell complex called the **Voronoi diagram** of E.
- The locus of points which are equidistant to two sites pi and pj is called a **bisector**, all bisectors being affine subspaces of IR^d (lines in 2D).



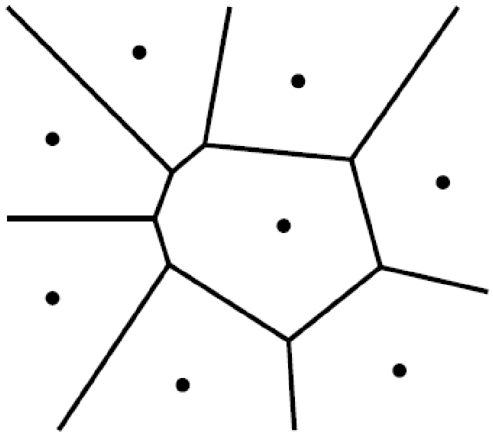
demo

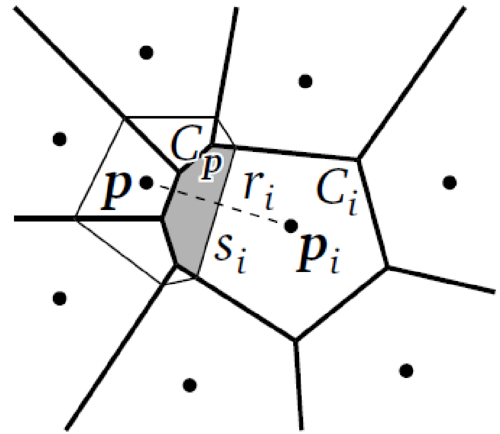
Sibson Coordinates



$$w_i = \text{Area}[C_i \cap C_p], \qquad i = 1, \ldots, n.$$

Sibson Coordinates

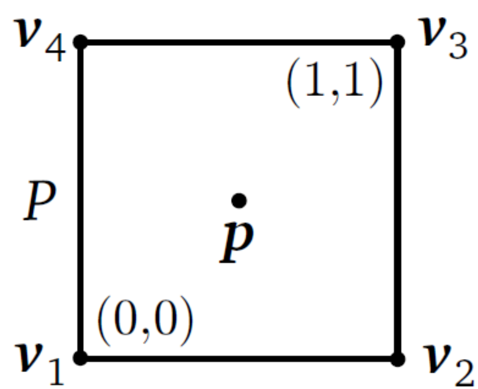




- Well define over convex hull
- Local support
- Satisfy Lagrange property
- C^1 continuity, except at points p_i (only C^0)

GENERALIZED BARYCENTRIC COORDINATES

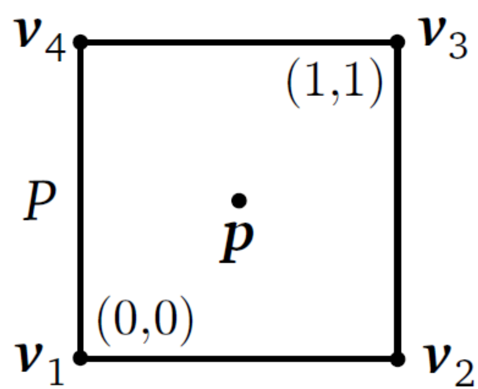
Quadrilateral?



$$b_1(\boldsymbol{p}) = (1-s)(1-t), \quad b_2(\boldsymbol{p}) = s(1-t), \quad b_3(\boldsymbol{p}) = st, \quad b_4(\boldsymbol{p}) = (1-s)t$$

Bilinear interpolation

Quadrilateral



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Bilinear map on unit square

Quadrilateral

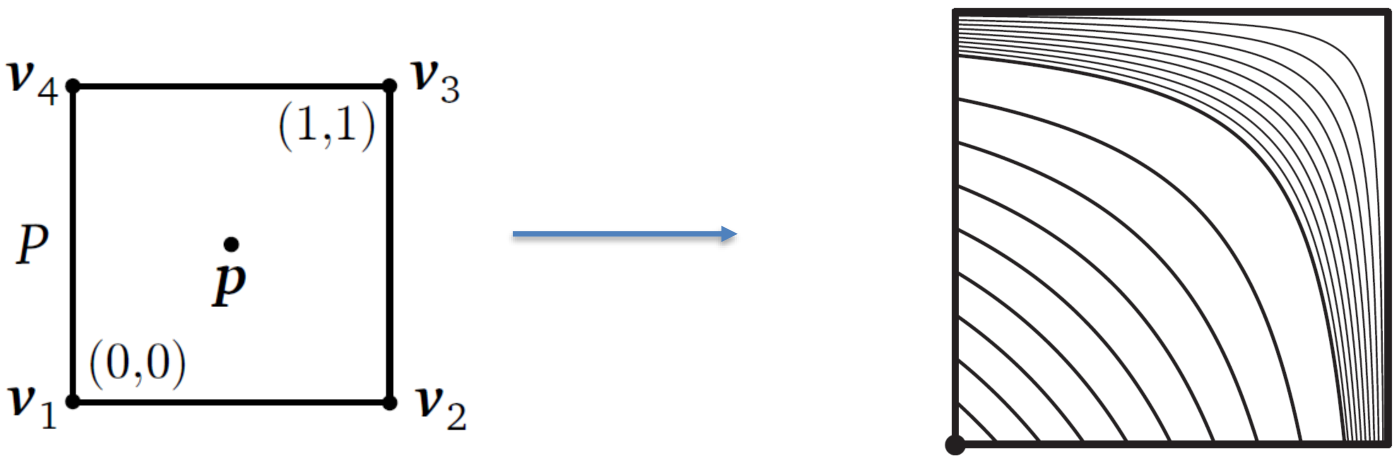

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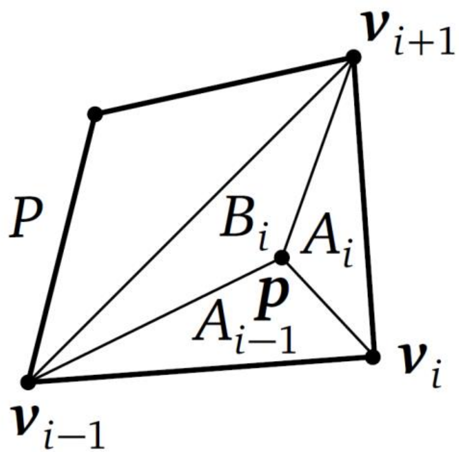
Image of bilinear map on unit square

Unified Formula! [Floater]

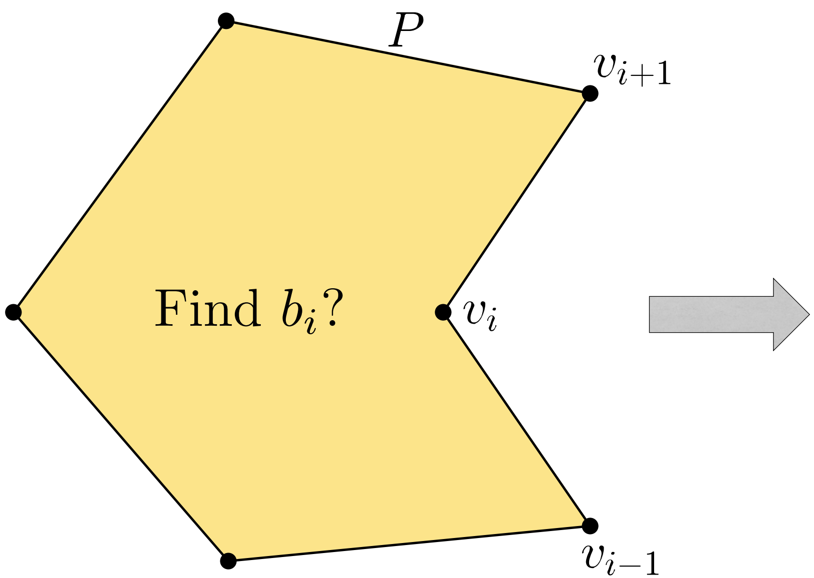


$$b_i(\boldsymbol{p}) = \frac{4A_{i+1}(\boldsymbol{p})A_{i+2}(\boldsymbol{p})}{G_{i+1}(\boldsymbol{p})G_{i+2}(\boldsymbol{p})}, \qquad i = 1, \ldots, 4$$

$$G_i = 2A_i - B_i - B_{i+1} + \sqrt{B_1^2 + B_2^2 + 2A_1A_3 + 2A_2A_4},$$

signed area

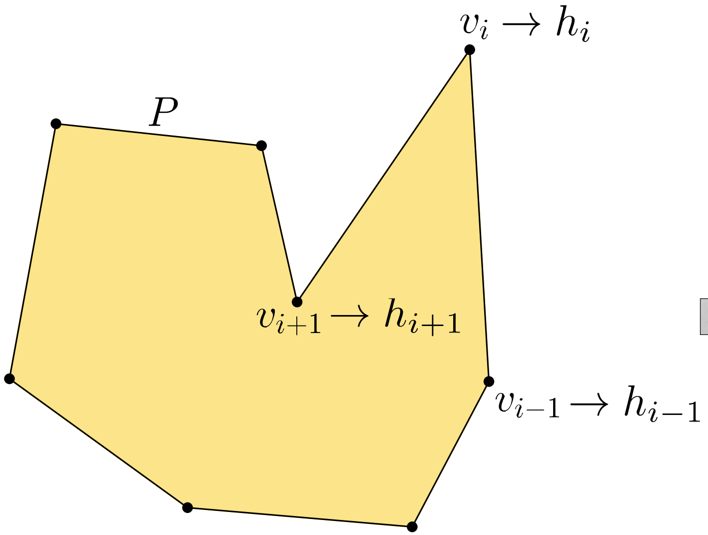
Generalized barycentric coordinates



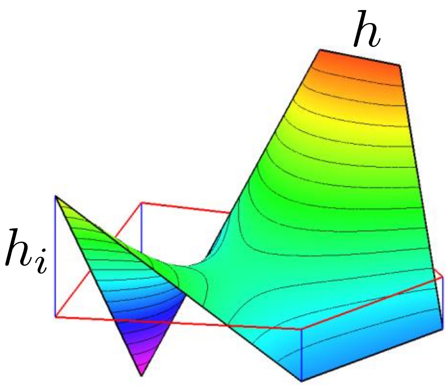
Properties:

- Constant precision
- Linear precision
- The Lagrange property
- Non-negativity
- Linearity along edges
- Smoothness
- Closed-form

Interpolation



$$\sum_{i=1}^{n} b_i h_i = h$$



Generalized barycentric coordinates

$$b_i = \frac{w_i}{W}, \text{ where } W = \sum_{j=1}^{n} w_j$$

Different weights -> different coordinate functions

Generalized barycentric coordinates

- Wachspress coordinates [Wachspress, 1975]
- Discrete harmonic coordinates [Pinkall and Polthier, 1993]
- Mean value coordinates [Floater, 2003]
- Metric coordinates [Malsch et al., 2005]
- Harmonic coordinates [Joshi et al., 2007]
- Maximum entropy coordinates [Hormann and Sukumar, 2008]
- Complex barycentric coordinates [Weber et al., 2009]
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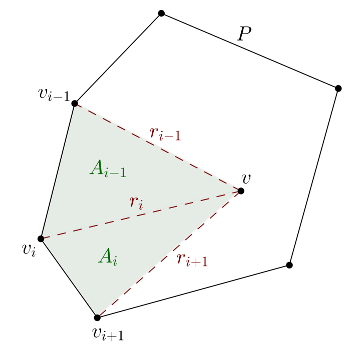
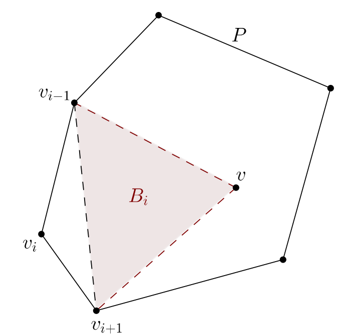


- Three-point coordinates [Floater et al., 2006]

$$p \in \mathbb{R}, \qquad w_i = \frac{r_{i-1}^p A_i - r_i^p B_i + r_{i+1}^p A_{i-1}}{A_{i-1} A_i}$$



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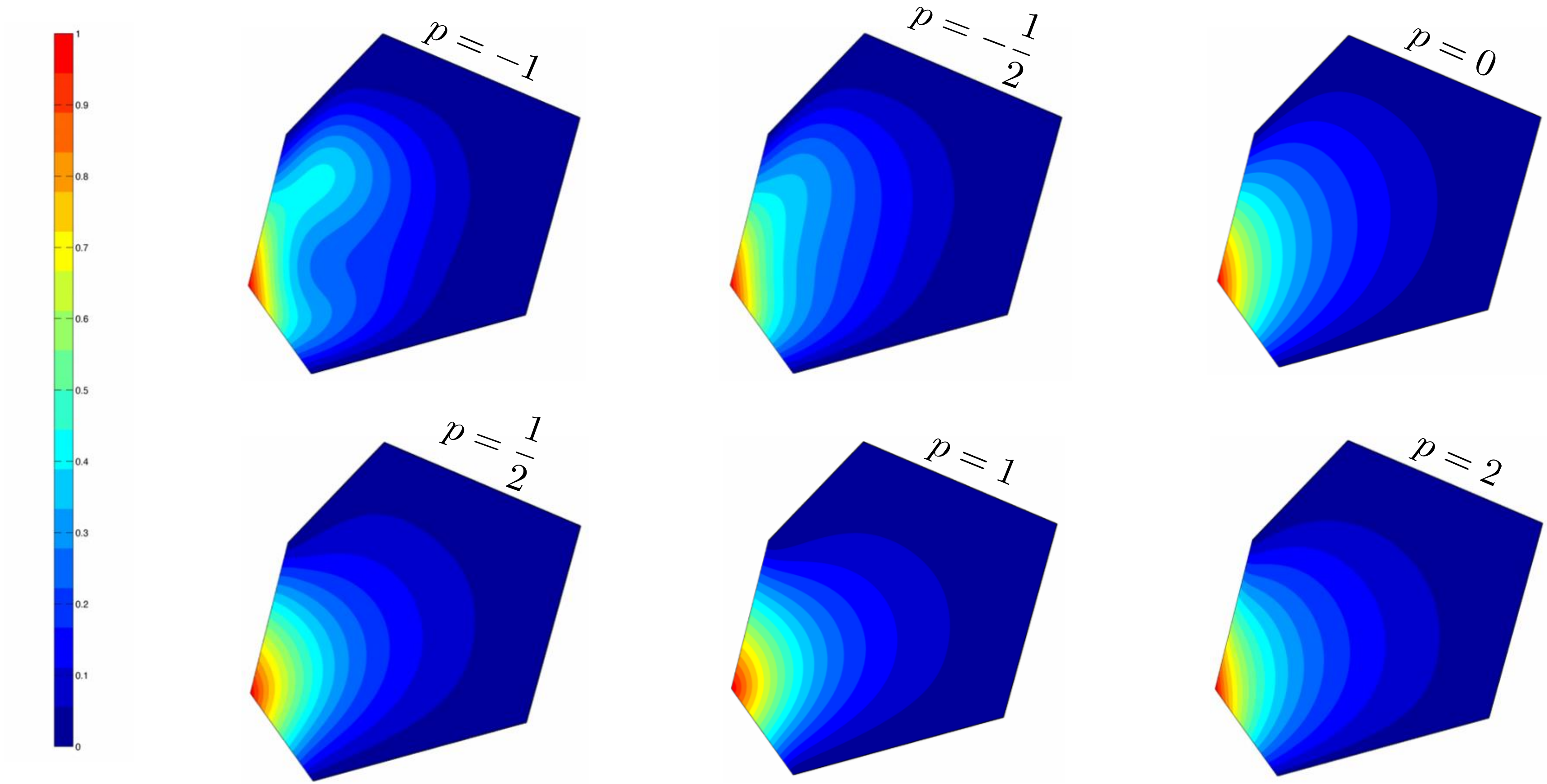

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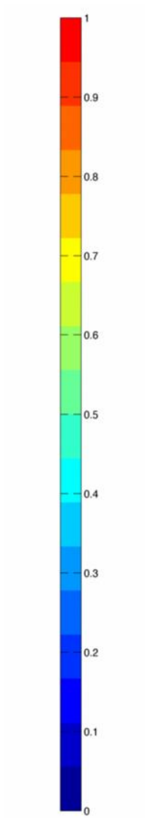
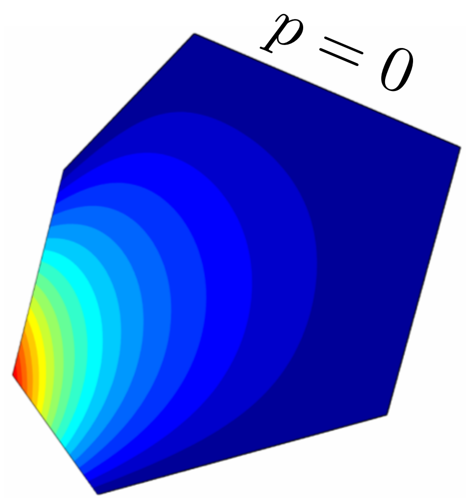
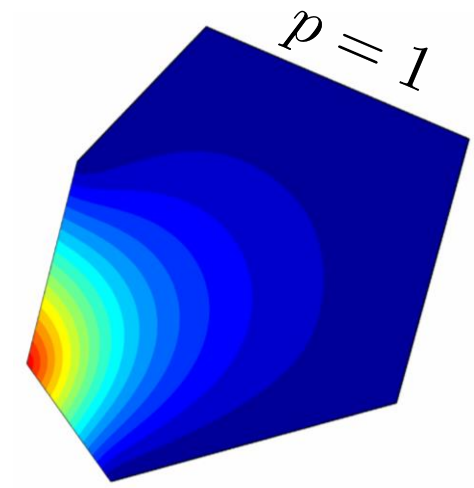
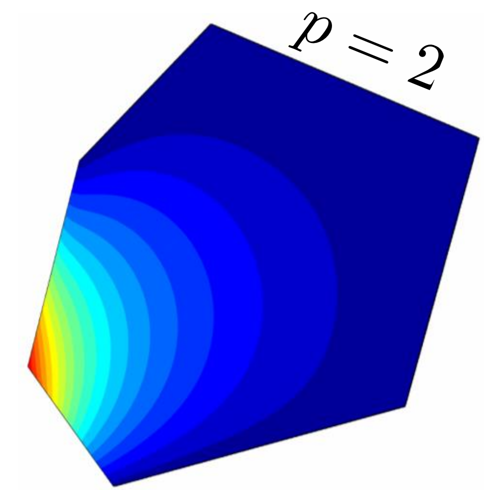




Wachspress

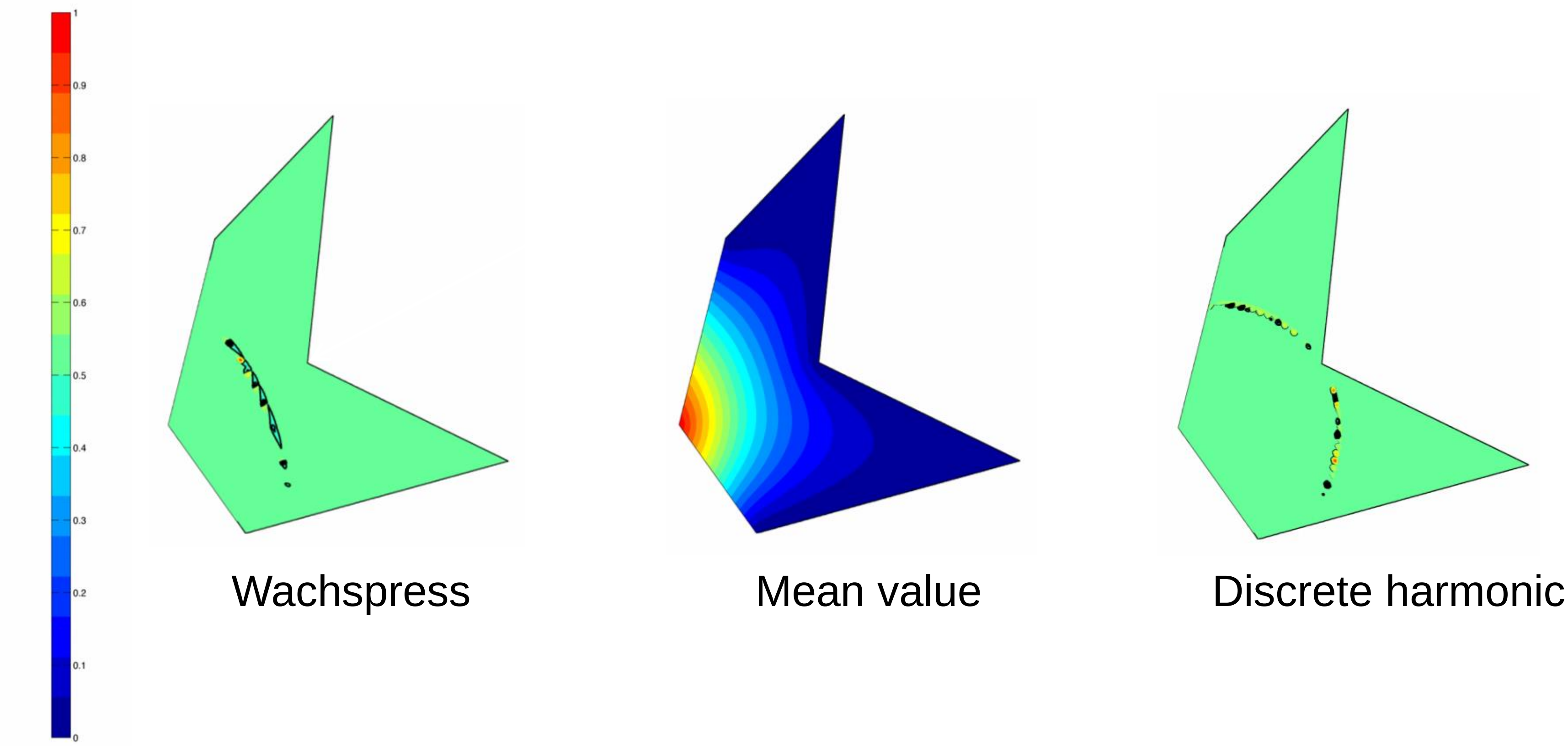


Mean value



Discrete harmonic

Three-point coordinates



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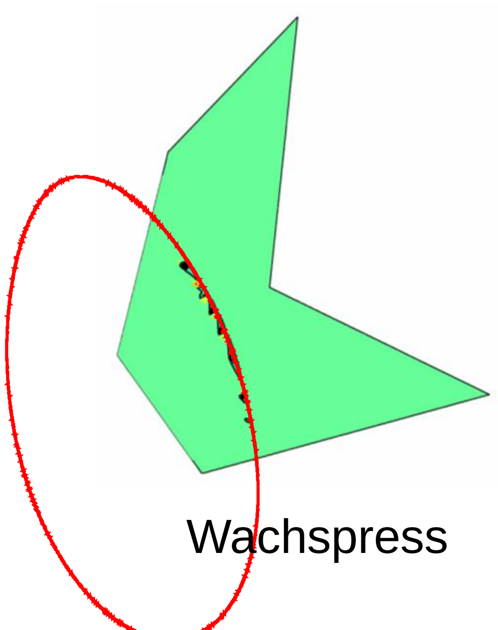
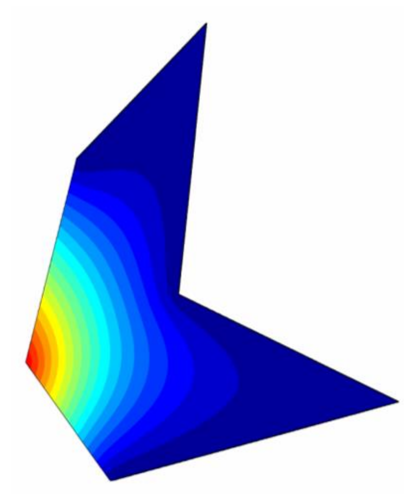


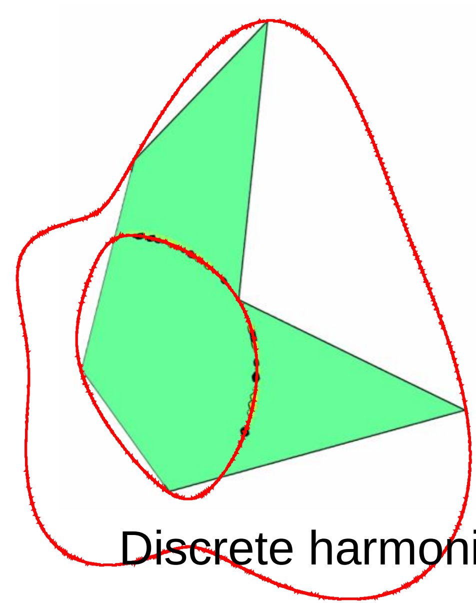
Wachspress

$W = 0$



Mean value

$$b_i = \frac{w_i}{W}$$



Discrete harmonic

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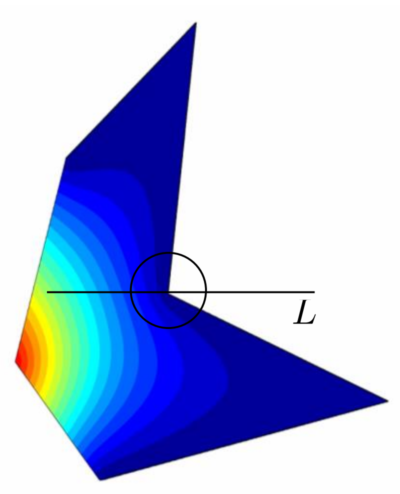
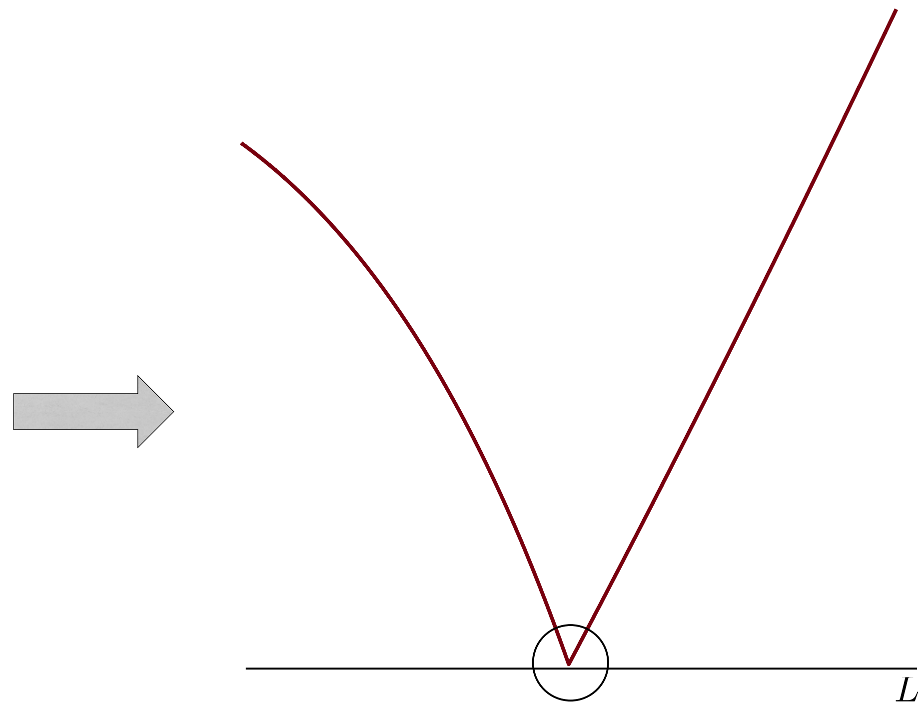
Mean value coordinates



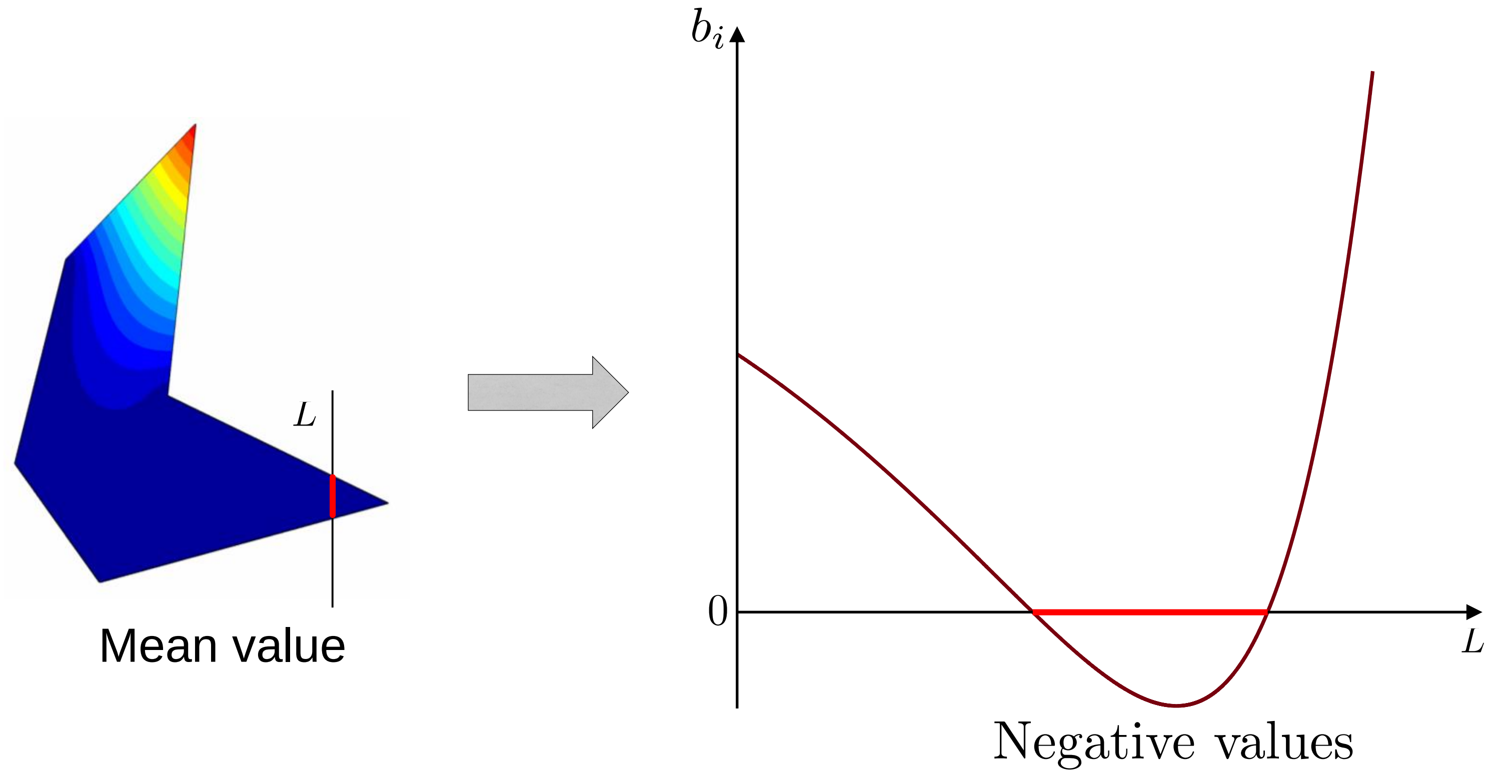
Mean value



C^0 continuity

Mean value coordinates

Positive mean value coordinates [Lipman et al., 2007]



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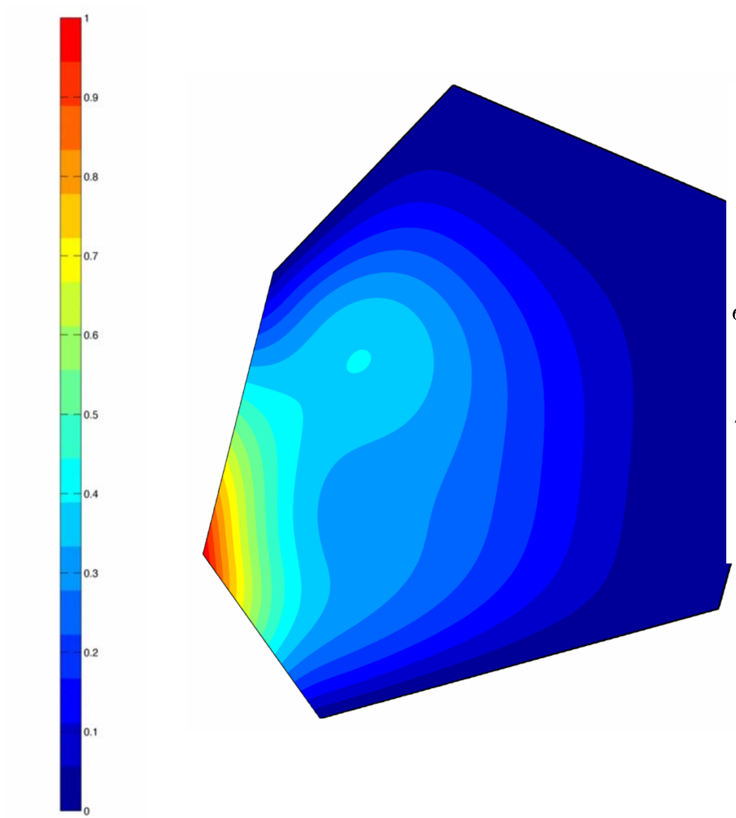
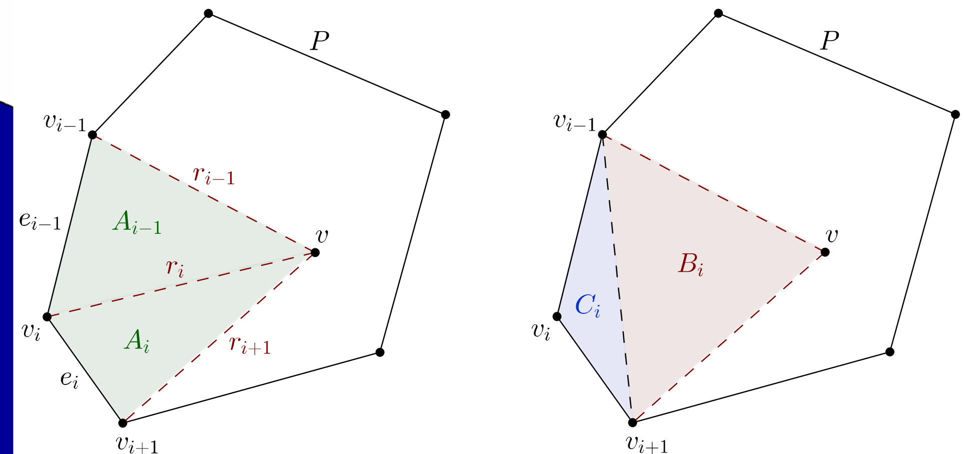
Metric coordinates

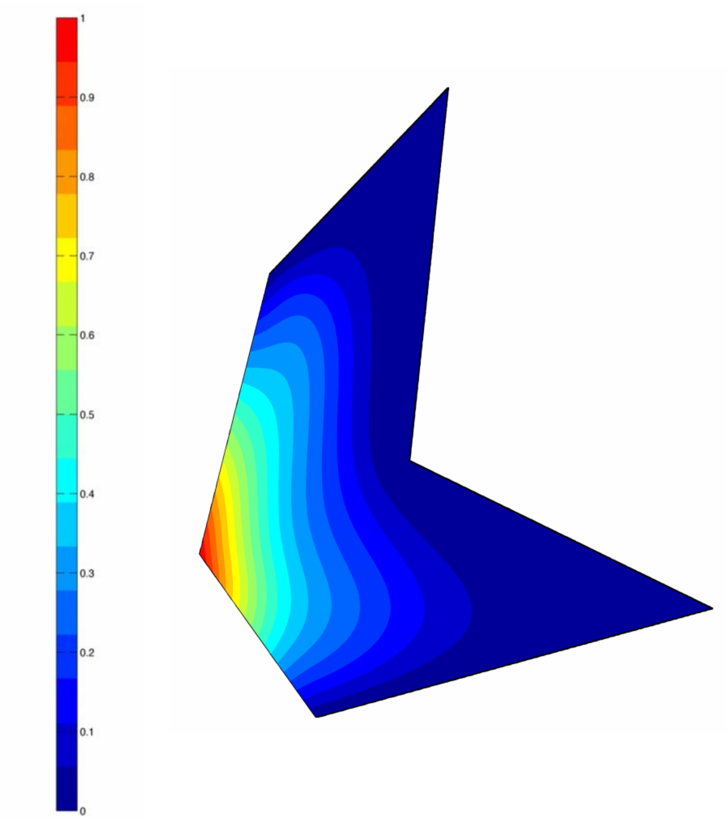


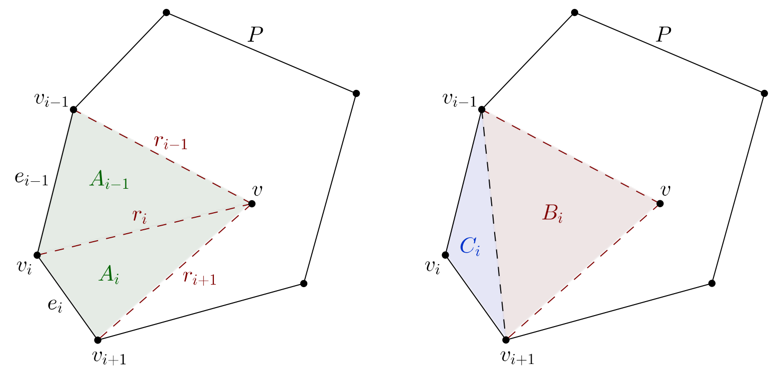


$$q_i = r_i + r_{i+1} - e_i$$

$$w_i = \frac{A_{i-2}}{C_{i-1} q_{i-2} q_{i-1}} - \frac{B_i}{C_i q_{i-1} q_i} + \frac{A_{i+1}}{C_{i+1} q_i q_{i+1}}$$

Metric coordinates





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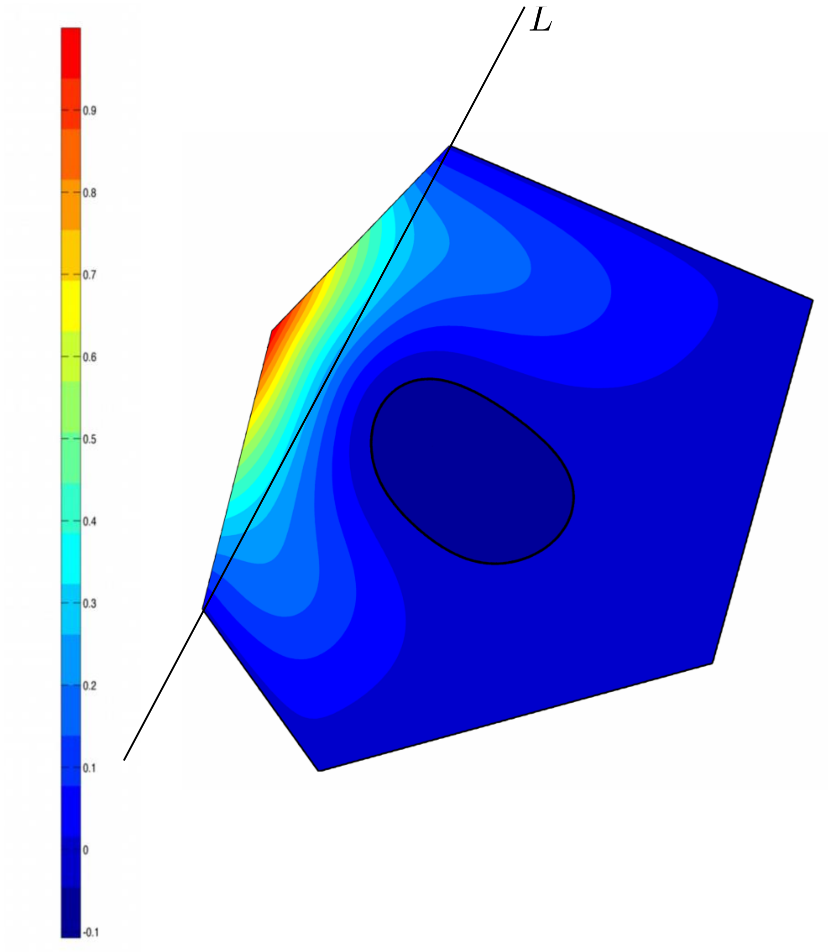
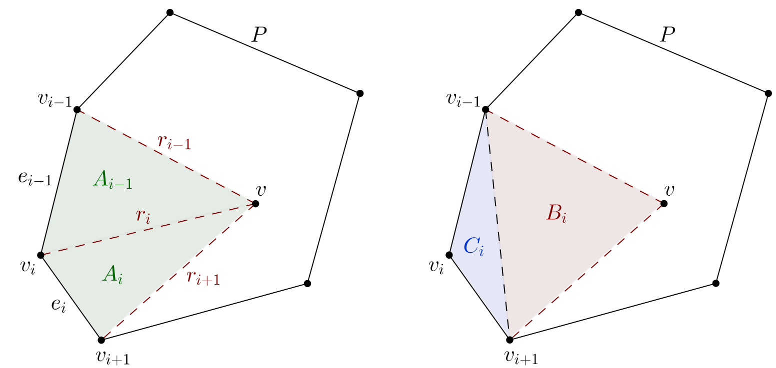
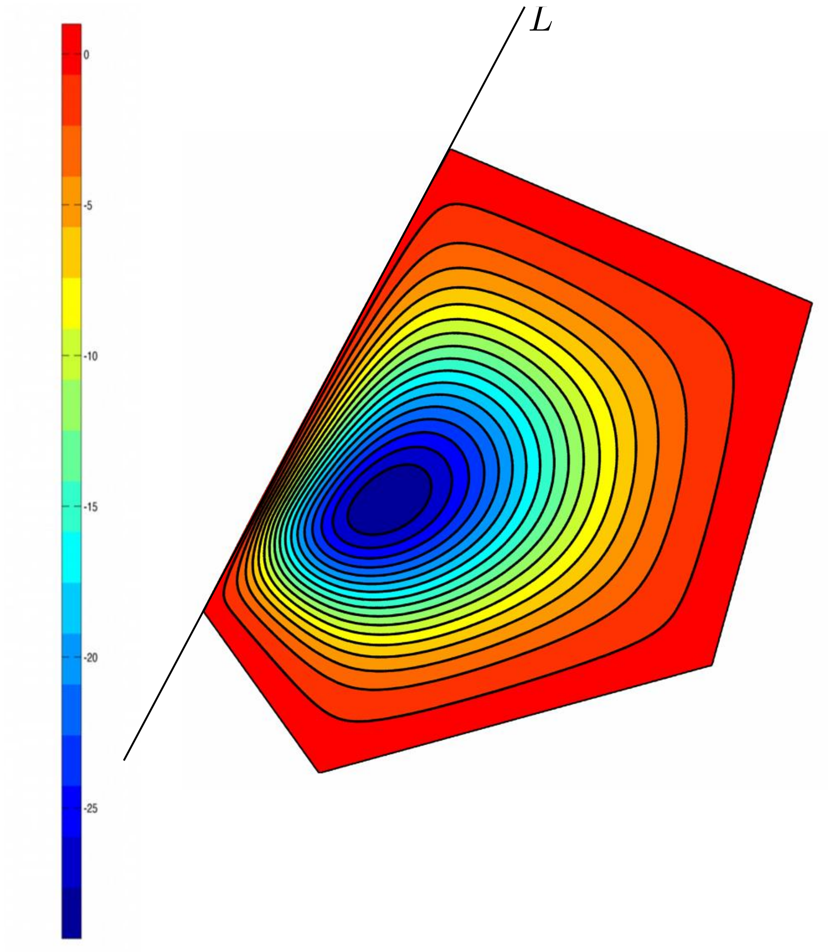


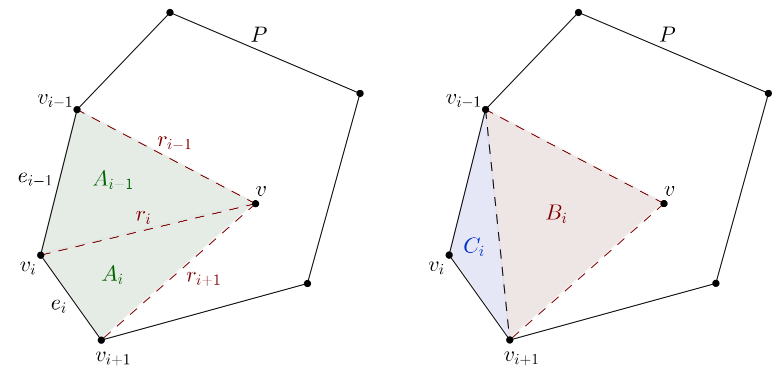


$$q_i = r_i + r_{i+1} - e_i$$

$$w_i = \frac{A_{i-2}}{C_{i-1}q_{i-2}q_{i-1}} - \frac{B_i}{C_i q_{i-1} q_i} + \frac{A_{i+1}}{C_{i+1} q_i q_{i+1}}$$

Metric coordinates





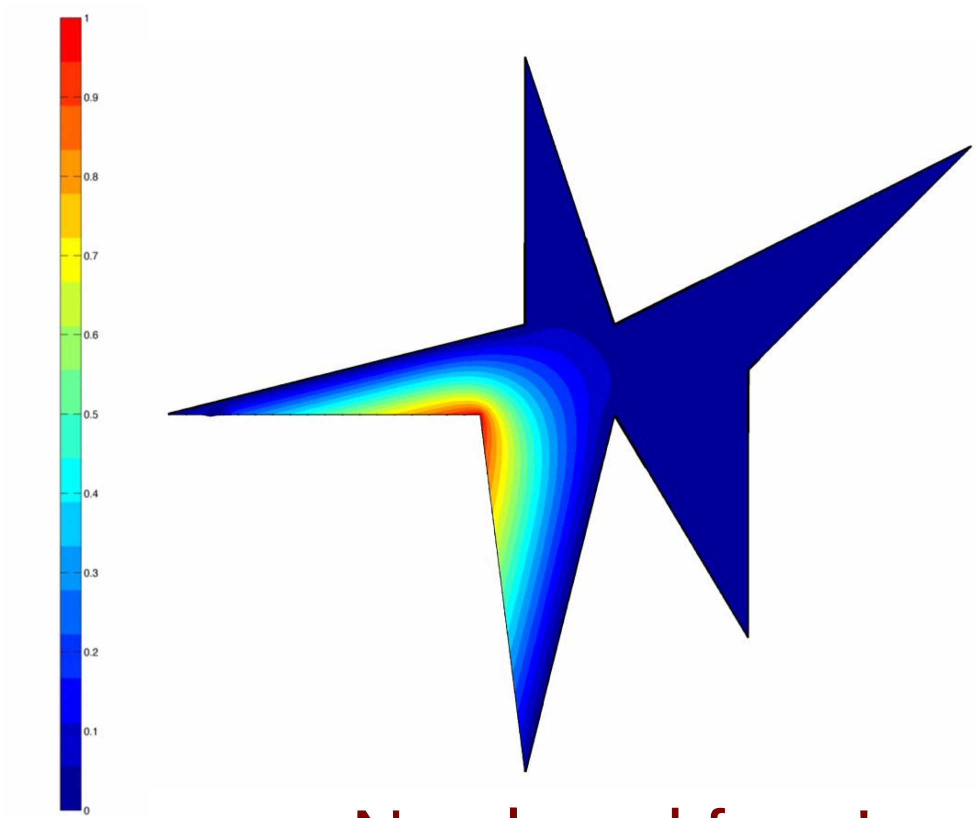
$$q_i = r_i + r_{i+1} - e_i$$

$$w_i = \frac{A_{i-2}}{C_{i-1}q_{i-2}q_{i-1}} - \frac{B_i}{C_i q_{i-1} q_i} + \frac{A_{i+1}}{C_{i+1} q_i q_{i+1}}$$

Generalized barycentric coordinates

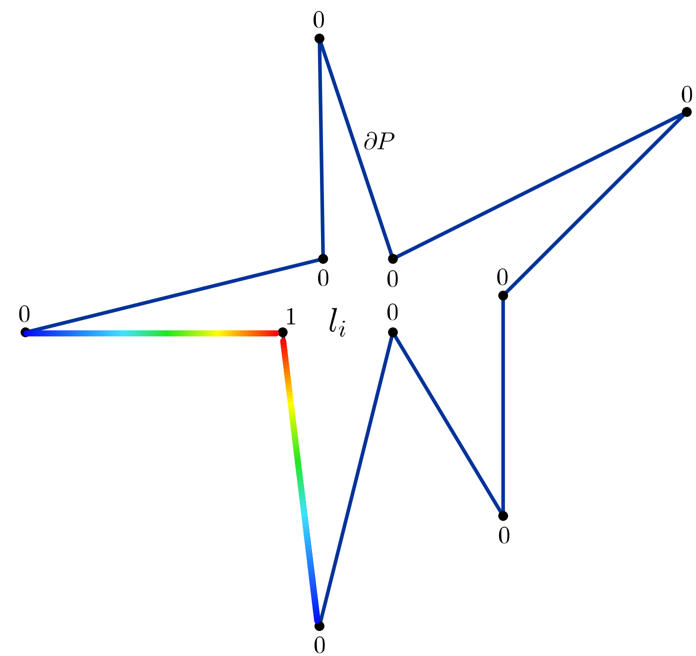
- Wachspress coordinates [Wachspress, 1975]
- Discrete harmonic coordinates [Pinkall and Polthier, 1993]
- Mean value coordinates [Floater, 2003]
- Metric coordinates [Malsch et al., 2005]
- Harmonic coordinates [Joshi et al., 2007]
- Maximum entropy coordinates [Hormann and Sukumar, 2008]
- Complex barycentric coordinates [Weber et al., 2009]
- Moving least squares coordinates [Manson and Schaefer, 2010]
- Cubic mean value coordinates [Li and Hu, 2013]
- Poisson coordinates [Li et al., 2013]

Harmonic coordinates



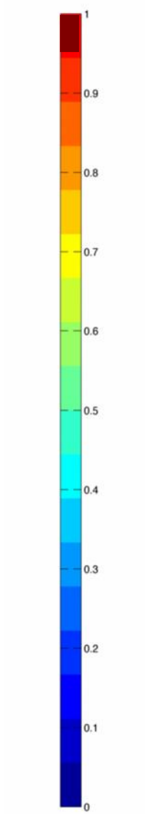
Solve the Laplacian equation:

$$\Delta b_i = 0 \text{ s.t. } b_i|_{\partial P} = l_i$$

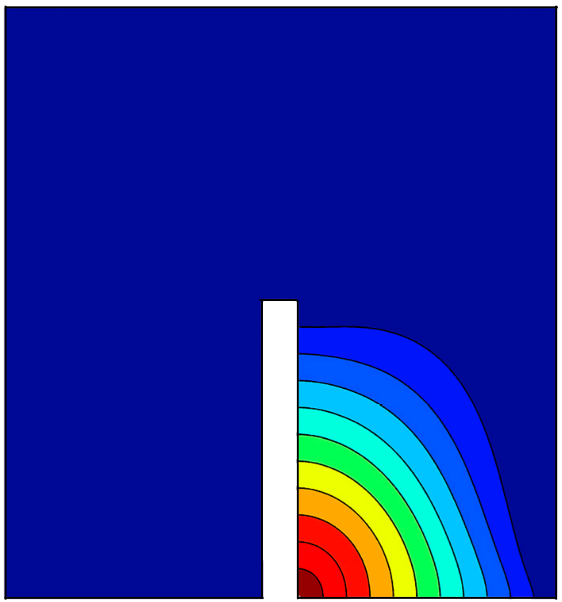


No closed form!

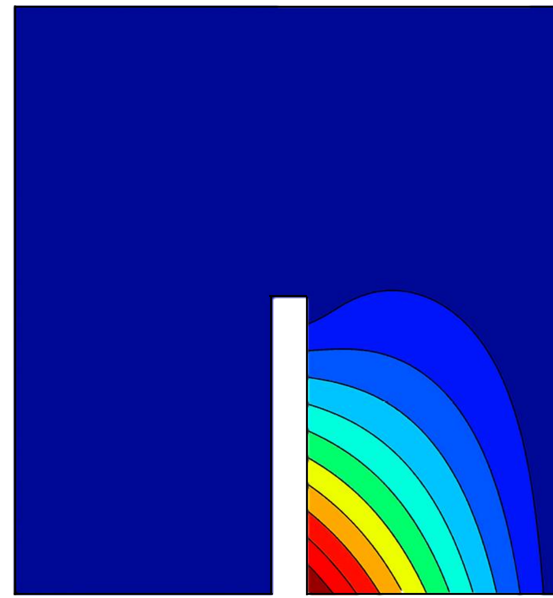
Biharmonic vs Harmonic



[Weber et al., 2012]



Biharmonic



Harmonic

Generalized barycentric coordinates

- Wachspress coordinates [Wachspress, 1975]
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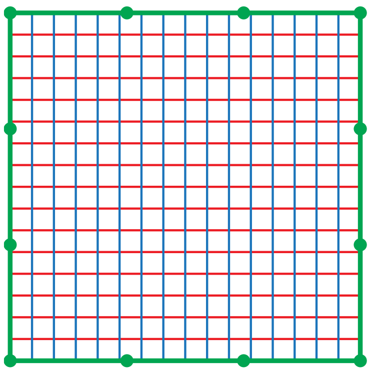
Barycentric mapping

Source polygon



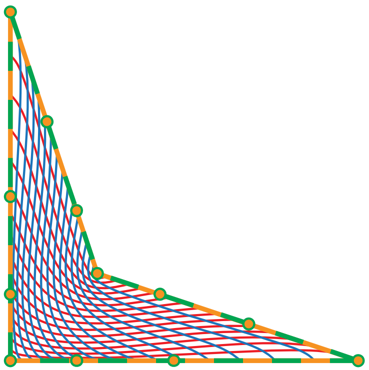
$$P = [v_1, \ldots, v_n]$$

$$v_i \in \mathbb{R}^2$$

For $v \in P$:

$$f(v) = \sum_{i=1}^{n} b_i(v)\hat{v}_i$$

Target polygon

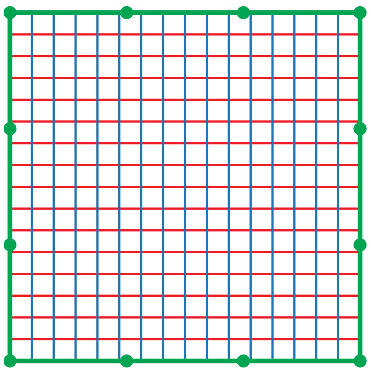


$$\hat{P} = [\hat{v}_1, \ldots, \hat{v}_n]$$

$$\hat{v}_i \in \mathbb{R}^2$$

Complex barycentric mapping

Source polygon



$P = [z_1, \ldots, z_n]$

$z_i \in \mathbb{C}$

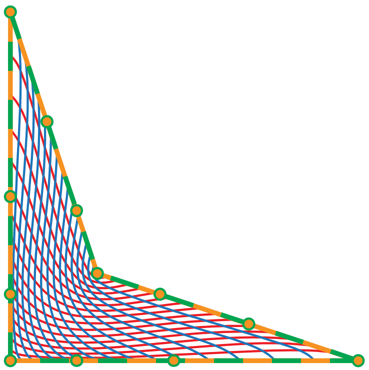
For $z \in P$:

$$g(z) = \sum_{i=1}^{n} c_i(z) \hat{z}_i$$

with complex barycentric coordinates

$c_i : P \to \mathbb{C}$

Target polygon



$\hat{P} = [\hat{z}_1, \ldots, \hat{z}_n]$

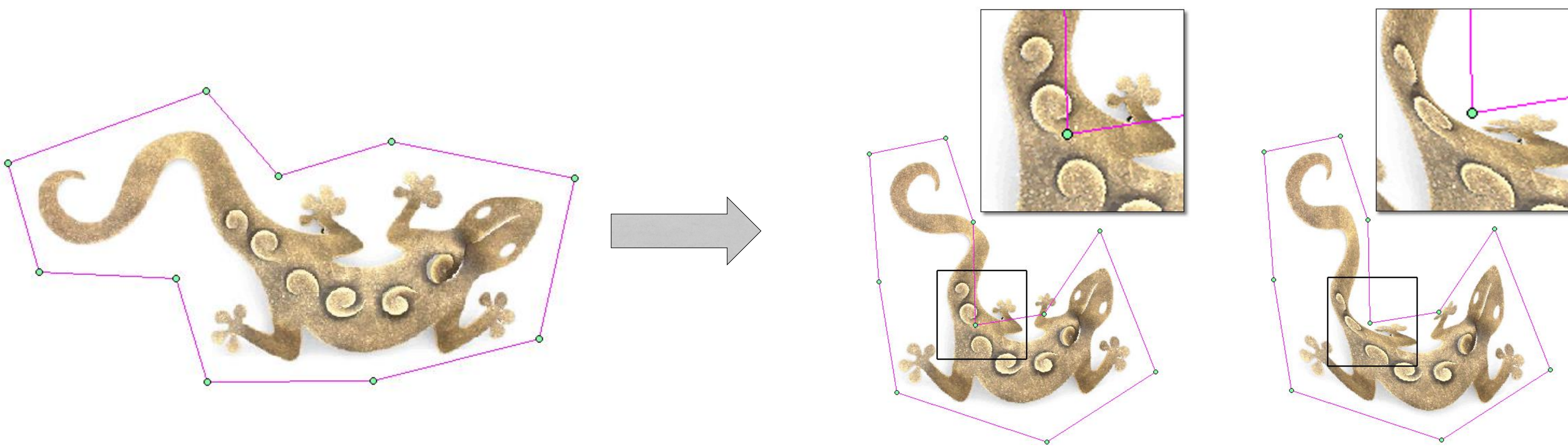
$\hat{z}_i \in \mathbb{C}$

Complex barycentric coordinates

Three-point coordinates are generalized to complex three-point coordinates

Green coordinates are members of complex three-point coordinates
[Lipman et al., 2008]

Induce conformal mappings



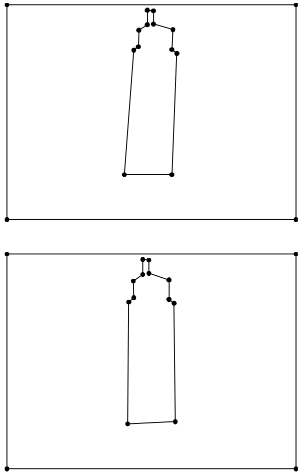
Green Harmonic

APPLICATIONS

Editing

Image warping

Original image



Mask

Warped image

Principal Component Analysis

Linear Least Squares Approximation

Pierre Alliez
Inria Sophia Antipolis

Definition (point set case)

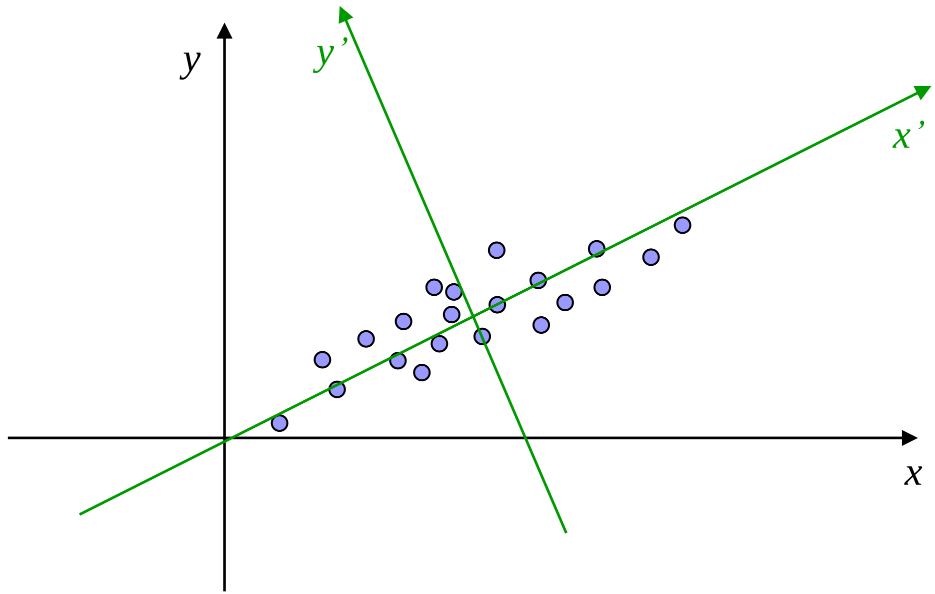
- Given a point set $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \in R^d$, linear least squares fitting amounts to find the linear sub-space of R^d which **minimizes the sum of squared distances** from the points to their projection onto this linear sub-space.

Definition (point set case)

- This problem is equivalent to search for the linear sub-space which **maximizes the variance of projected points**, the latter being obtained by eigen decomposition of the covariance (scatter) matrix.
- Eigenvectors corresponding to large eigenvalues are the directions in which the data has **strong component**, or equivalently large variance. If eigenvalues are the same there is no preferable sub-space.

PCA – the general idea

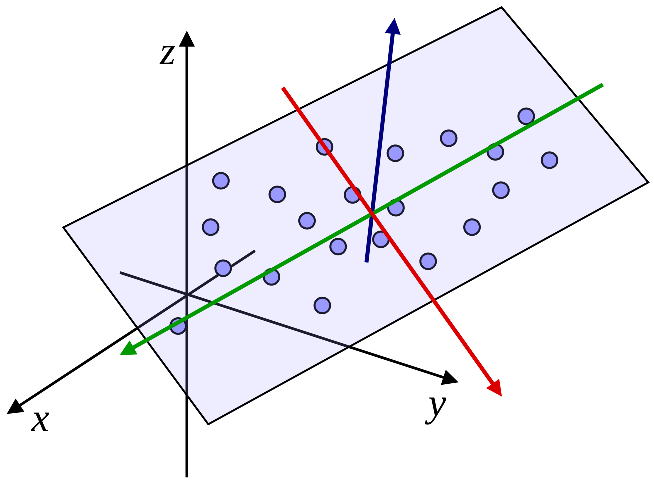
- PCA finds an orthogonal basis that best represents given data set.



- The sum of distances^2 from the x' axis is minimized.

PCA – the general idea

- PCA finds an orthogonal basis that best represents given data set.



3D point set

- PCA finds a best approximating plane (in terms of $\Sigma distances^2$)

Notations

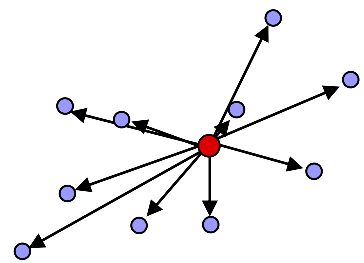
- Denote our data points by $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \in R^d$

$$\mathbf{x_1} = \begin{pmatrix} x_1^1 \\ x_1^2 \\ \vdots \\ x_1^d \end{pmatrix}, \; \mathbf{x_2} = \begin{pmatrix} x_2^1 \\ x_2^2 \\ \vdots \\ x_2^d \end{pmatrix}, \ldots, \; \mathbf{x_d} = \begin{pmatrix} x_n^1 \\ x_n^2 \\ \vdots \\ x_n^d \end{pmatrix}$$

The origin of the new axes

- The origin is zero-order approximation of our data set (a point)
- It will be the center of mass:

$$\mathbf{m} = \frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i$$



- It can be shown that:

$$\mathbf{m} = \underset{\mathbf{x}}{\operatorname{argmin}} \sum_{i=1}^{n} \left\| \mathbf{x}_i - \mathbf{x} \right\|^2$$

Scatter matrix

- Denote $\mathbf{y}_i = \mathbf{x}_i - \mathbf{m}, \quad i = 1, 2, \ldots, n$

$$\boxed{S = YY^T}$$

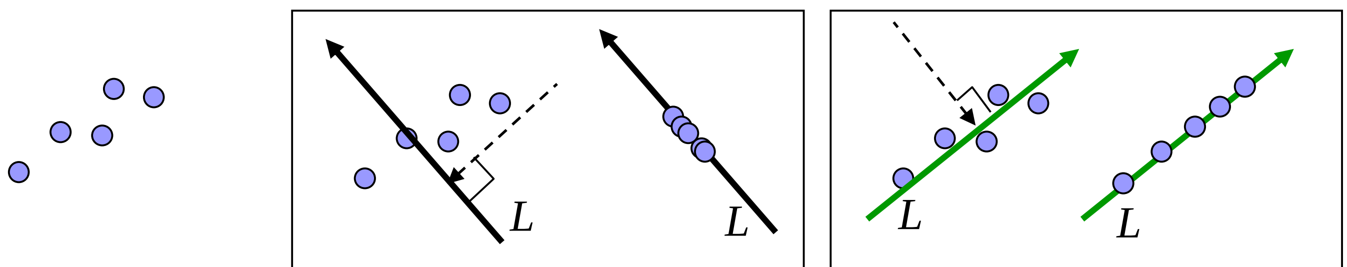
where Y is $d \times n$ matrix with $\mathbf{y}_k$ as columns ($k = 1, 2, \ldots, n$)

$$S = \underbrace{\begin{pmatrix} y_1^1 & y_2^1 & \cdots & y_n^1 \\ y_1^2 & y_2^2 & & y_n^2 \\ \vdots & \vdots & & \vdots \\ y_1^d & y_2^d & \cdots & y_n^d \end{pmatrix}}_{\mathrm{Y}} \underbrace{\begin{pmatrix} y_1^1 & y_1^2 & \cdots & y_1^d \\ y_2^1 & y_2^2 & \cdots & y_2^d \\ \vdots & & & \vdots \\ y_n^1 & y_n^2 & \cdots & y_n^d \end{pmatrix}}_{\mathrm{Y}^T}$$

Variance of projected points

- In a way, S measures variance (= scatterness) of the data in different directions.
- Let's look at a line L through the center of mass $\mathbf{m}$, and project our points $\mathbf{x}_i$ onto it. The variance of the projected points $\mathbf{x}'_i$ is:

$$\mathrm{var}(L) = \frac{1}{n}\sum_{i=1}^{n} \| \mathbf{x}'_i - \mathbf{m} \|^2$$

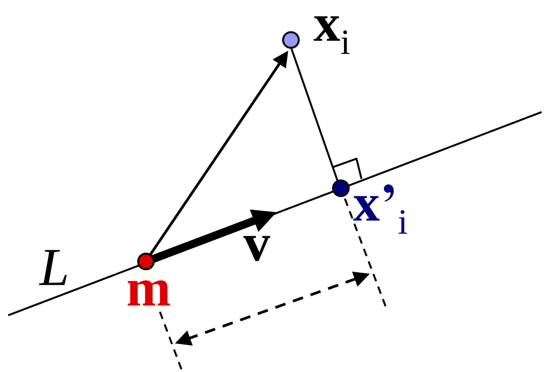


Original set | Small variance | Large variance

Variance of projected points

- Given a direction $\mathbf{v}$, $\|\mathbf{v}\| = 1$, the projection of $\mathbf{x}_\mathrm{i}$ onto $L = \mathbf{m} + \mathbf{v}t$ is:

$$\| \mathbf{x}'_i - \mathbf{m} \| = <\mathbf{v},\ \mathbf{x}_\mathbf{i} - \mathbf{m}> \| \mathbf{v} \| = <\mathbf{v},\ \mathbf{y}_\mathbf{i}> = \mathbf{v}^T \mathbf{y}_\mathbf{i}$$



Variance of projected points

- So,

$$\mathrm{var}(L) = \tfrac{1}{n}\sum_{i=1}^{n} \| \mathbf{x}'_{\mathbf{i}} - \mathbf{m} \|^2 = \tfrac{1}{n}\sum_{i=1}^{n} (\mathbf{v}^{\mathbf{T}}\mathbf{y}_{\mathbf{i}})^2 = \tfrac{1}{n} \| \mathbf{v}^{\mathbf{T}} Y \|^2 =$$

$$= \tfrac{1}{n} \| Y^T \mathbf{v} \|^2 = \tfrac{1}{n} < Y^T\mathbf{v},\ Y^{\mathbf{T}}\mathbf{v} > = \tfrac{1}{n}\mathbf{v}^{\mathbf{T}} Y Y^T \mathbf{v} = \tfrac{1}{n}\mathbf{v}^{\mathbf{T}} S\mathbf{v} = \tfrac{1}{n} < S\mathbf{v},\ \mathbf{v} >$$

$$\sum_{i=1}^{n} (\mathbf{v}^T \mathbf{y}_{\mathbf{i}})^2 = \sum_{i=1}^{n} \left(\begin{pmatrix} v^1 & v^2 & \cdots & v^d \end{pmatrix} \begin{pmatrix} y_i^1 \\ y_i^2 \\ \vdots \\ y_i^d \end{pmatrix} \right)^2 = \left\| \begin{pmatrix} v^1 & v^2 & \cdots & v^d \end{pmatrix} \begin{pmatrix} y_1^1 & y_2^1 & \cdots & y_n^1 \\ y_1^2 & y_2^2 & & y_n^2 \\ \vdots & \vdots & & \vdots \\ y_1^d & y_2^d & \cdots & y_n^d \end{pmatrix} \right\|^2 = \| \mathbf{v}^{\mathbf{T}} Y \|^2$$

Directions of maximal variance

- So, we have: $\text{var}(L) = <S\mathbf{v}, \mathbf{v}>$
- <u>Theorem</u>:

  Let $f : \{\mathbf{v} \in R^d \mid ||\mathbf{v}|| = 1\} \rightarrow R,$

  $f(\mathbf{v}) = <S\mathbf{v}, \mathbf{v}>$ (and S is a symmetric matrix).

  Then, the extrema of f are attained at the eigenvectors of S.

- So, eigenvectors of S are directions of maximal/minimal variance.

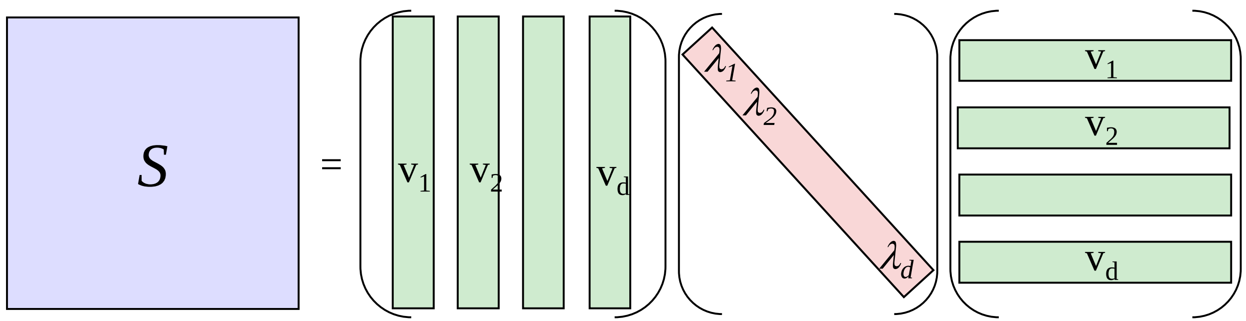
Summary so far

- We take the centered data points $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_\mathrm{n} \in R^d$
- Construct the scatter matrix $S = YY^T$
- S measures the variance of the data points
- Eigenvectors of S are directions of max/min variance.

Scatter matrix - eigendecomposition

- $\boldsymbol{S}$ is symmetric

$\Rightarrow$ $\boldsymbol{S}$ has eigendecomposition: $\boldsymbol{S} = V\Lambda V^T$

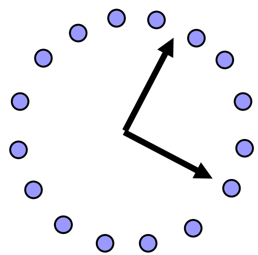


The eigenvectors form orthogonal basis

Principal components

- Eigenvectors that correspond to big eigenvalues are the directions in which the data has strong components (= large variance).
- If the eigenvalues are more or less the same – there is no preferable direction.

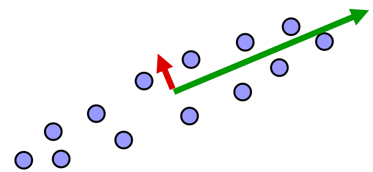
Principal components



- There's no preferable direction
- $\boldsymbol{S}$ looks like this:

$$\begin{pmatrix} \lambda & \\ & \lambda \end{pmatrix}$$

- Any vector is an eigenvector

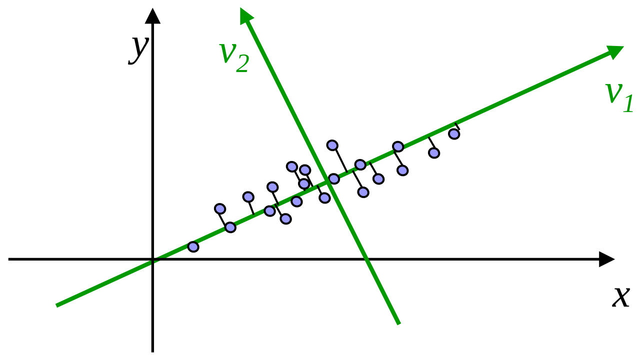


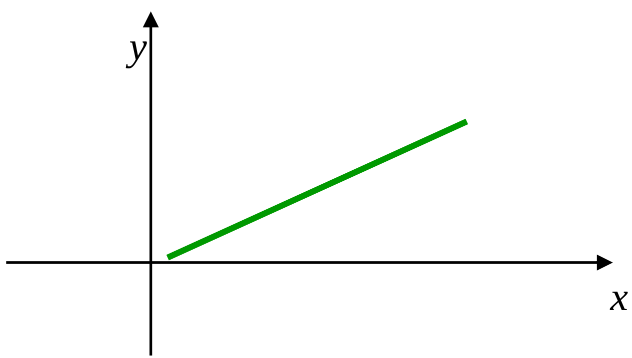
- There is a clear preferable direction
- $\boldsymbol{S}$ looks like this:

$$V\begin{pmatrix} \lambda & \\ & \mu \end{pmatrix}V^T$$

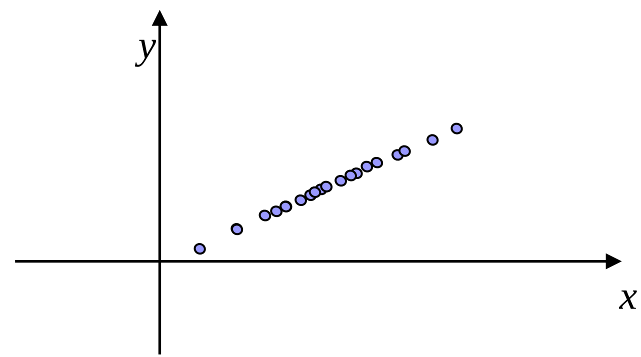
- μ is close to zero, much smaller than λ.

For approximation





This line segment approximates the original data set



The projected data set approximates the original data set

For approximation

- In general dimension d, the eigenvalues are sorted in descending order:

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_d$$

- The eigenvectors are sorted accordingly.
- To get an approximation of dimension $d' < d$, we take the d' first eigenvectors and look at the subspace they span ($d' = 1$ is a line, $d' = 2$ is a plane…)

For approximation

- To get an approximating set, we project the original data points onto the chosen subspace:

$$\mathbf{x}_\mathrm{i} = \mathbf{m} + \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_{d'}\mathbf{v}_{d'} + \ldots + \alpha_d\mathbf{v}_d$$

Projection:

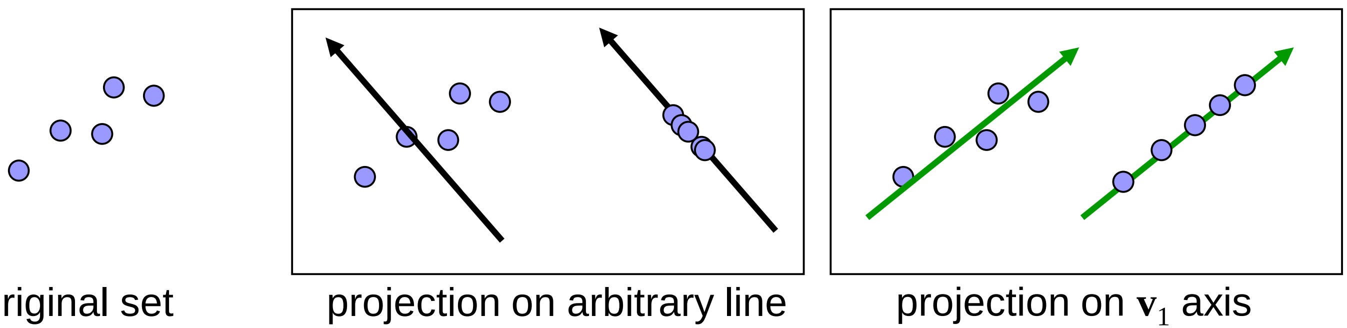
$$\mathbf{x}_\mathrm{i}' = \mathbf{m} + \underbrace{\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_{d'}\mathbf{v}_{d'}} + 0\cdot\mathbf{v}_{d'+1} + \ldots + 0\cdot\mathbf{v}_d$$

Optimality of approximation

- The approximation is optimal in least-squares sense. It gives the minimal of:

$$\sum_{k=1}^{n} \left\| \mathbf{x}_k - \mathbf{x}'_k \right\|^2$$

- The projected points have maximal variance.

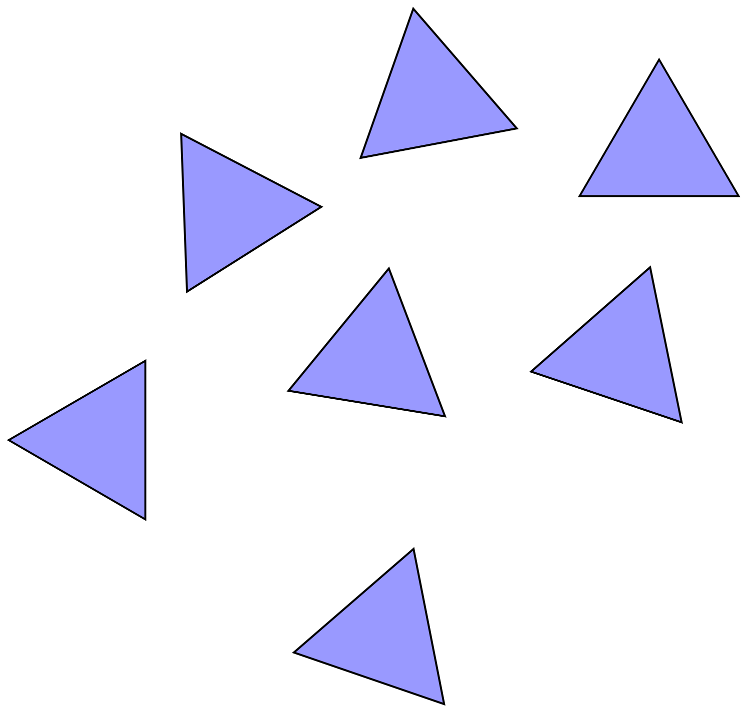


PCA on Point Sets

$$S = \begin{pmatrix} y_1^1 & y_2^1 & \cdots & y_n^1 \\ y_1^2 & y_2^2 & & y_n^2 \\ \vdots & \vdots & & \vdots \\ y_1^d & y_2^d & \cdots & y_n^d \end{pmatrix} \begin{pmatrix} y_1^1 & y_1^2 & \cdots & y_1^d \\ y_2^1 & y_2^2 & \cdots & y_2^d \\ \vdots & & & \vdots \\ y_n^1 & y_n^2 & \cdots & y_n^d \end{pmatrix}$$

demo

PCA on Geometric Primitives?



$$C_i = \int_{s_i} x x^T dx$$

Coordinate relative to center of mass