BOUNDS FOR A CLASS OF STOCHASTIC RECURSIVE EQUATIONS

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Abstract

In this note we develop a framework for computing upper and lower bounds of an exponential form for a class of stochastic recursive equations with uniformly recurrent Markov modulated inputs. These bounds generalize Kingman's bounds for queues with renewal inputs.

QUEUES; EXPONENTIAL BOUNDS; GENERAL STATE SPACE MARKOV CHAIN

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1 Introduction

Let $(\Omega, \mathcal{F})$ be a measurable space large enough to carry a $\mathbb{R}_+$-valued random variable (r.v.) X_0, a sequence of $\mathbb{E}$-valued r.v.'s $\{Y_n,\, n = 0, 1, \ldots\}$ and a sequence of $\mathbb{R}$-valued r.v.'s $\{\xi_n,\, n = 0, 1, \ldots\}$. The set $\mathbb{R}$ (resp. $\mathbb{R}_+$) denotes the set of all real numbers (resp. the set of all nonnegative real numbers) endowed with the σ-algebra $\mathcal{B}$ (resp. $\mathcal{B}_+$). We will assume that $\mathbb{E}$ is a general space endowed with the σ-algebra $\mathcal{E}$.

Let $\{F(x; \cdot),\, x \in \mathbb{E}\}$ be a family of probability measures on $(\mathbb{E} \times \mathbb{R}, \mathcal{E} \times \mathcal{B})$ and let η be a probability measure on $(\mathbb{R}_+ \times \mathbb{E}, \mathcal{B}_+ \times \mathcal{E})$. Let $\mu(A) \equiv \eta(\mathbb{R}_+ \times A)$, $A \in \mathcal{E}$, be a probability measure on $(\mathbb{E}, \mathcal{E})$.

We postulate the existence of a probability measure P on $(\Omega, \mathcal{F})$, with expectation operator E, such that for all $A \in \mathcal{E}$, $B \in \mathcal{B}$, $C \in \mathcal{B}_+$,

$$P\left((X_0, Y_0) \in C \times A\right) = \eta(C \times A) \tag{1.1}$$

and

$$P\left((Y_{n+1}, \xi_n) \in A \times B \,|\, X_0, Y_0, \ldots, Y_n, \xi_0, \ldots, \xi_{n-1}\right) = F(Y_n; A \times B) \tag{1.2}$$

for all $n = 0, 1, \ldots$. The definition (1.2) implies, in particular, that $\mathbf{Y} = \{Y_n,\, n = 0, 1, \ldots\}$ is a time-homogeneous Markov chain with state-space $\mathbb{E}$, transition kernel given by $Q(x, A) \equiv F(x; A \times \mathbb{R})$ for all $x \in \mathbb{E}$, $A \in \mathcal{E}$, and initial probability distribution μ.

We will assume that the Markov chain $\mathbf{Y}$ is aperiodic, positive Harris recurrent (see [21, Theorem 13.0.1]) and will denote by π its invariant probability measure. Let $\{\pi_n,\, n = 0, 1, \ldots\}$ be a family of probability measures on $\mathbb{E}$ recursively defined by $\pi_0 \equiv \mu$ and $\pi_n(A) = \int_{\mathbb{E}} Q(x, A)\, \pi_{n-1}(dx)$ for $A \in \mathcal{E}$, $n = 1, 2, \ldots$. In other words, π_n is the probability distribution of Y_n given that the probability distribution of Y_0 is μ.

On $(\Omega, \mathcal{F})$ we define the new sequence $\{X_n,\, n = 0, 1, \ldots\}$ of $\mathbb{R}_+$-valued r.v.'s by

$$X_{n+1} = \max\left(0,\, X_n + \xi_n\right), \quad n = 0, 1, \ldots. \tag{1.3}$$

In the queueing literature the recursion (1.3) is called the *Lindley's equation*. For instance, X_n may represent the waiting time of the n-th customer in a first-in-first-out G/G/1 queue, where ξ_n is the difference between the service time of the n-th customer and the interarrival time between the n-th and the $(n+1)$-st customer.

The aim of this note is to compute exponential upper and lower bounds for the tail distribution of X_n, both for every $n = 1, 2, \ldots$ and for the stationary regime X of X_n (when it exists), namely, to find real numbers $a, a_n \geq 0$, $b, b_n, \theta > 0$ such that

$$a_n\, e^{-\theta s} \leq P(X_n > s) \leq b_n\, e^{-\theta s} \tag{1.4}$$

$$a\, e^{-\theta s} \leq P(X > s) \leq b\, e^{-\theta s} \tag{1.5}$$

for all $s \geq 0$, $n = 1, 2, \ldots$.

When $\mathbb{E}$ is a singleton – which implies that $\{\xi_n,\ n=0,1,\ldots\}$ is a renewal sequence with common p.d.f. denoted as H – the first upper bound for $P(X>s)$ was obtained by Kingman [17] via martingale theory. Successive refinements by Kingman [18] and Ross [22] have finally given the following bounds (see also Borovkov [3, p. 139]):

$$\left(\inf_{s>0} g(s,\theta^\star)\right) e^{-\theta^\star s} \le P(X>s) \le \left(\sup_{s>0} g(s,\theta)\right) e^{-\theta s}, \quad s\ge 0 \tag{1.6}$$

for all $0\le\theta\le\theta^* = \sup\{\alpha>0 \,:\, E[e^{\alpha\xi_0}]\le 1\}$, where $g(s,\theta) := (1-H(s))/\int_s^\infty e^{\theta(u-s)}dH(u)$. These bounds hold under the stability condition $E[\xi_0]<0$.

When $\mathbb{E}$ is a finite set, and if the stability condition $E_\pi[\xi_0]<0$ holds, then Liu, Nain and Towsley [20] have shown that the upper bound in (1.5) holds for every $0\le\theta\le\theta^*=\sup\{\alpha>0,\ \rho(\alpha)\le 1\}$ with

$$b = \sup_{(s,j)\in\mathcal{A}} \frac{\sum_{k\in\mathbf{E}} \pi(\{k\})\, F(k;\{j\}\times(s,\infty))}{\sum_{k\in\mathbf{E}} l(k;\theta) \int_s^\infty e^{\theta(u-s)}\, F(k;\{j\}\times du)} \tag{1.7}$$

where $(l(k;\theta),\ k\in\mathbb{E})$ is the unique positive left eigenvector of the matrix $\left[\int_{\mathbf{R}} e^{\theta u}\, F(k;\{j\}\times du)\right]_{k,j}$ associated with its largest eigenvalue $\rho(\theta)$ such that $\sum_{k\in\mathbf{E}} l(k;\theta)=1$. In (1.7) the set $\mathcal{A}$ is defined as $\mathcal{A}=\{(s,j)\in\mathbb{R}_+\times\mathbb{E} \,:\, F(k;\{j\}\times(s,\infty))>0 \text{ for some } k\in\mathbb{E}\}$. The lower bound (1.5) is obtained for $\theta=\theta^\star$ and with the coefficient a defined as the r.h.s. of (1.7) after substituting "sup" for "inf" in this expression. Transient bounds of the form (1.4) are also reported in [20].

Still for finite state-space $\mathbb{E}$, steady-state bounds of the form (1.5) have been obtained by Asmussen and Rolski [2] in the context of risk theory. It is not easy to compare analytically bounds in [2] to those in [20]. Numerical experiments conducted in [20] indicate that, in general, bounds in [20] are better than those in [2].

Upper bounds of the form (1.5) have already been derived by Duffield [9] for arbitrary set $\mathbb{E}$. Duffield's approach, based on the maximal inequality for positive super-martingales, does not yield very tight bounds in general (see [20, Section 3.4]) and does not seem to easily yield a lower bound. A more general setting than the one in the present note is studied by Chang and Chen in [6]. In [6] the increment process $\{\xi_n, n=0,1,\ldots\}$ in (1.3) is in the form $\xi_n=a_n-c$ for all $n=0,1,\ldots$ with c a nonnegative constant. Under the assumption that the process $\{a_n,\ n=0,1,\ldots\}$ satisfies a sample criterion, Chang and Chen derive lower and upper bounds for $P(X>s)$; bounds for intree routing networks are also reported in [6]. As acknowledged by the authors, their bounds are not as tight as the bounds in [2] and in [20] when specialized to the case when $\{a_n, n=0,1,\ldots\}$ is a Markov arrival process.

The results in this note generalize the bounds in (1.6) and the bounds in [20] to the case when $(\mathbb{E},\mathcal{E})$ is a general measurable space. Upper and lower bounds for $P(X_n>s)$ and $P(X>s)$ are obtained through a unified and simple approach.

Besides the theoretical interest of obtaining bounds like those in (1.4)-(1.5), bounds on the tail distribution of quantities such as buffer occupancy and response times can be used in the design of high speed networks. In addition, bounds can also be used to develop policies for controlling the admission of new applications or sessions to the network. The interested reader is referred to [1, 5, 7, 8, 10, 11, 15, 16, 19, 20, 23] where these issues have been lately addressed.

We conclude this section by introducing some further notation and by stating some preliminary results.

A first step toward the generalization to general state-space $\mathbb{E}$ for the Markov chain $\mathbf{Y}$ is to note that the process $\{(Y_{n+1}, \sum_{i=0}^{n} \xi_i),\ n = 0, 1, \ldots\}$ is a discrete-time *Markov-Additive* process [14] with MA-kernel given by $F(x; A \times B)$. This observation will allow us to borrow several results from [14].

To this end, define the transform

$$\hat{Q}(x, A; \theta) = \int_{-\infty}^{\infty} e^{\theta u}\, F(x; A \times du) \tag{1.8}$$

for all $x \in \mathbb{E}$, $A \in \mathcal{E}$, $\theta \in \mathbb{R}$.

Let $F^{(1)}(x; A \times B) = F(x; A \times B)$ and $F^{(i)}(x; A \times B) = \int_{\mathbf{E}} \int_{-\infty}^{\infty} F^{(i-1)}(x; dy \times du)\, F(y; A \times (B - u))$, $i \geq 2$, and define $\hat{Q}^{(i)}(x, A; \theta) = \int_{-\infty}^{\infty} e^{\theta u} F^{(i)}(x; A \times du)$ for all $i \geq 1$.

From now on we will assume that there exist a probability measure m on $(\mathbb{E} \times \mathbb{R}, \mathcal{E} \times \mathcal{B})$, an integer i, and real numbers $0 < a_1 \leq a_2 < \infty$ such that

$$a_1\, m(A \times B) \leq F^{(i)}(x; A \times B) \leq a_2\, m(A \times B), \quad \forall x \in \mathbb{E},\ A \in \mathcal{E},\ B \in \mathcal{B}. \tag{1.9}$$

The above condition is the "recurrence hypothesis" (3.1) in [14]. Condition (1.9) holds automatically when $\mathbb{E}$ is finite and when the Markov chain $\mathbf{Y}$ is irreducible and aperiodic. The interested reader is referred to [14, Section 7] for a discussion on cases where condition (1.9) holds.

Define $\hat{m}(A; \theta) = \int_{\mathbf{R}} e^{\theta u}\, m(A \times du)$ for all $A \in \mathcal{E}$, and let $\mathcal{D} = \{\theta \in \mathbb{R} : \hat{m}(\mathbb{E}; \theta) < \infty\}$.

By applying Lemma 3.1 in [14] we deduce that for each $\theta \in \mathcal{D}$, $\hat{Q}(\cdot, \cdot; \theta)$ has a maximal simple eigenvalue $\rho(\theta) > 0$ with uniformly positive and bounded associated left eigenmeasure (see [13, Theorem III.10.1]) $l(\theta) = \{l(A; \theta),\ A \in \mathcal{E}\}$. Recall that the left eigenmeasure $l(\theta)$ satisfy the relationship $\rho(\theta)\, l(A; \theta) = \int_{\mathbf{E}} \hat{Q}(x, A; \theta)\, l(dx; \theta)$, for all $A \in \mathcal{E}$. We will assume without loss of generality that the left eigenmeasure is chosen so that

$$l(\mathbb{E}; \theta) = 1, \quad \forall \theta \in \mathcal{D}. \tag{1.10}$$

2 Exponential Bounds

The approach used in this paper generalized Kingman's in [18].

Let $\{\gamma_n(s,\cdot),\ s \geq 0,\ n = 0,1,\ldots\}$, be a collection of measures on $(\mathbb{E},\mathcal{E})$ such that

$$\int_{\mathbf{E}} \int_{-\infty}^{s} \gamma_n(s-u,dx)\, F(x;A\times du) + F(x;A\times(s,\infty))\,\pi_n(dx) \leq \gamma_{n+1}(s,A) \tag{2.1}$$

for all $s \geq 0,\ A \in \mathcal{E},\ n = 0,1,\ldots$.

The following technical lemma holds:

Lemma 2.1 *Let $\mathcal{P}_n$ denote the property that*

$$P(X_n > s, Y_n \in A) \leq \gamma_n(s,A) \tag{2.2}$$

for all $s \geq 0,\ A \in \mathcal{E},\ n = 0,1,\ldots$.

If $\mathcal{P}_0$ is true, then $\mathcal{P}_n$ is true for all $n \geq 0$.

Proof. We will use an induction argument on n. Assume that $\mathcal{P}_m$ is true for $m = 0,1,\ldots,n$ and let us show that $\mathcal{P}_{n+1}$ is true.

We have for all $s \geq 0,\ A \in \mathcal{E}$,

$$\begin{aligned}
& P(X_{n+1} > s,\ Y_{n+1} \in A) \\
&= \int_{\mathbf{E}} \int_{-\infty}^{\infty} P(X_n > s-u,\ Y_{n+1} \in A,\ \xi_n \in du \,|\, Y_n = x)\,\pi_n(dx) \\
&= \int_{\mathbf{E}} \int_{-\infty}^{s} P(X_n > s-u,\ Y_{n+1} \in A,\ \xi_n \in du \,|\, Y_n = x)\,\pi_n(dx) \\
&\quad + \int_{\mathbf{E}} \int_{s}^{\infty} P(Y_{n+1} \in A,\ \xi_n \in du \,|\, Y_n = x)\,\pi_n(dx) \\
&= \int_{\mathbf{E}} \int_{-\infty}^{s} P(Y_{n+1} \in A,\ \xi_n \in du \,|\, X_n > s-u, Y_n = x)\, P(X_n > s-u, Y_n \in dx) \\
&\quad + \int_{\mathbf{E}} F(x;A\times(s,\infty))\,\pi_n(dx) \\
&= \int_{\mathbf{E}} \left[\int_{-\infty}^{s} F(x;A\times du)\, P(X_n > s-u, Y_n \in dx) + F(x;A\times(s,\infty))\,\pi_n(dx) \right] && (2.3) \\
&\leq \int_{\mathbf{E}} \left[\int_{-\infty}^{s} \gamma_n(s-u,dx)\, F(x;A\times du) + F(x;A\times(s,\infty))\,\pi_n(dx) \right] && (2.4) \\
&\leq \gamma_{n+1}(s,A) && (2.5)
\end{aligned}$$

where (2.3), (2.4) and (2.5) follow from (1.2), the induction hypothesis and the definition (2.1), respectively.

■

Introduce the set $\mathcal{G} = \{\alpha \geq 0 \,:\, \rho(\alpha) \leq 1\} \cap \mathcal{D}$. Observe that $\mathcal{G}$ is nonempty since $\rho(0) = 1$ and $\hat{m}(\mathbb{E};0) = 1$ which implies that $0 \in \mathcal{G}$.

We are now in position to prove the main result of this paper.

Proposition 2.1 *Let $\theta \in \mathcal{G}$. For $n = 0, 1, \ldots$, define*

$$b_n(\theta) = \sup_{\substack{(s,A)\in\mathcal{K} \\ 0\le m\le n}} \frac{\int_{\mathbf{E}} F(x; A\times(s,\infty))\,\pi_m(dx)}{\int_{\mathbf{E}}\int_s^{\infty} e^{\theta(u-s)}\,F(x; A\times du)\,l(dx;\theta)} < \infty \tag{2.6}$$

with $\mathcal{K} = \{(s,A) \in \mathbb{R}_+ \times \mathcal{E} \,:\, F(x; A\times(s,\infty)) > 0$ for some $x \in \mathbf{E}\}$.

If

$$\eta((s,\infty)\times A) \le b_0(\theta)\, l(A;\theta)\, e^{-\theta s}, \quad \forall s \ge 0,\ A \in \mathcal{E} \tag{2.7}$$

then

$$P(X_n > s,\, Y_n \in A) \le b_n(\theta)\, l(A;\theta)\, e^{-\theta s}, \quad \forall s \ge 0,\ A\in\mathcal{E},\ n = 0,1,\ldots. \tag{2.8}$$

In particular,

$$P(X_n > s) \le b_n(\theta)\, e^{-\theta s}, \qquad \forall s\ge 0,\ n = 0,1,\ldots. \tag{2.9}$$

Proof. Fix $\theta \in \mathcal{G}$. Define $\gamma_n(s,A) = b_n(\theta)\, e^{-\theta s}\, l(A;\theta)$ for all $s \ge 0$, $A \in \mathcal{E}$, $n = 0,1,\ldots$.

Thanks to Lemma 2.1, the derivation of (2.8) will follow if we can show that the mappings $\gamma_n(s,A)$ defined above satisfy (2.1).

We have

$$\begin{aligned}
&\int_{\mathbf{E}}\int_{-\infty}^{s} \gamma_n(s-u, dx)\, F(x; A\times du) + F(x; A\times(s,\infty))\,\pi_n(dx) \\
&= b_n(\theta)\, e^{-\theta s}\int_{\mathbf{E}}\int_{-\infty}^{\infty} e^{\theta u}\, F(x; A\times du)\, l(dx;\theta) - b_n(\theta)\int_{\mathbf{E}}\int_s^{\infty} e^{\theta(u-s)}\, F(x; A\times du) l(dx;\theta) \\
&\quad + \int_{\mathbf{E}} F(x; A\times(s,\infty))\,\pi_n(dx) \\
&\le b_n(\theta)\, e^{-\theta s}\int_{\mathbf{E}}\int_{-\infty}^{\infty} e^{\theta u}\, F(x; A\times du)\, l(dx;\theta), \qquad \text{from the definition of } b_n(\theta) \\
&= b_n(\theta)\, e^{-\theta s}\int_{\mathbf{E}} \hat{Q}(x, A;\theta)\, l(dx;\theta) \\
&= b_n(\theta)\, e^{-\theta s}\,\rho(\theta)\, l(A;\theta) \\
&\le \gamma_{n+1}(\theta)
\end{aligned}$$

where the last inequality follows from $\rho(\theta) \le 1$ and $b_n(\theta) \le b_{n+1}(\theta)$. This proves (2.8).

For the proof of (2.9) simply observe that

$$\begin{aligned}
P(X_n > s) &= P(X_n > s, Y_n \in \mathbf{E}) \\
&\le b_n(\theta)\, e^{-\theta s}\, l(\mathbf{E};\theta) \qquad \text{from (2.8)} \\
&= b_n(\theta)\, e^{-\theta s}, \qquad \forall s\ge 0,\ n = 0,1,\ldots
\end{aligned}$$

by using the normalizing condition (1.10).

We conclude this proof by showing that the constants $b_0(\theta), b_1(\theta), \ldots$ are all finite for all $\theta \in \mathcal{G}$. This result follows from the inequalities

$$b_n(\theta) \leq \sup_{\substack{(s,A)\in\mathcal{K} \\ 0\leq m\leq n}} \frac{\int_{\mathbf{E}} F(x; A\times(s,\infty))\,\pi_m(dx)}{\int_{\mathbf{E}} F(x; A\times(s,\infty))\,l(dx;\theta)} \leq \sup_{\substack{0\leq m\leq n \\ A\in\mathcal{E}}} \frac{\pi_m(A)}{l(A;\theta)} < \infty \tag{2.10}$$

where the last inequality follows from the positiveness of the left eigenmeasure $l(\theta)$ (see Section 1).

■

From now on we will assume that there exists $0 < B < \infty$ such that $m(\mathbb{E}, (B,\infty)) > 0$. If this assumption does not hold then it can be shown from (1.9) and (1.3) that $X_n \to_n 0$ almost surely and the system becomes trivial.

Define $\theta^\star = \sup\{\theta \in \mathcal{G}\}$. It is shown in [9] that when the set $\mathcal{D}$ is open then $\theta^\star > 0$ if the *stability condition* (see [4]) $E_\pi[\xi_0] < 0$ is satisfied, where E_π stands for the expectation operator associated with a stationary Markov chain $\mathbf{Y}$ (i.e. $\mu = \pi$). In that case, $\theta^\star$ is the *largest exponential decay rate* among all positive decay rates such that $\rho(\theta) \leq 1$. Actually, $\theta^\star$ is seen to be the largest exponential decay rate among *all* $\theta > 0$ whenever (i) the set $\mathcal{D}$ is open, (ii) condition (1.9) holds and (iii) $E_\pi[\xi_0] < 0$. This result is a direct consequence of the identity $\lim_{x\to\infty}(1/x)\log P(X > x) = -\theta^\star$ that holds under the aforementioned conditions (i)-(iii) (see Duffield [9], Glynn and Whitt [12, Theorem 1]).

We now establish the transient lower bound.

Proposition 2.2 *For $n = 0, 1, \ldots$, define*

$$a_n = \inf_{\substack{(s,A)\in\mathcal{K} \\ 0\leq m\leq n}} \frac{\int_{\mathbf{E}} F(x; A\times(s,\infty))\,\pi_m(dx)}{\int_{\mathbf{E}}\int_s^\infty e^{\theta^\star(u-s)}\,F(x; A\times du))\,l(dx;\theta^\star)}. \tag{2.11}$$

Assume that $E_\pi[\xi_0] < 0$, $\mathcal{D}$ is an open set and $\rho(\theta^\star) = 1$ (see Remark 2.1 below).

If

$$\eta((s,\infty)\times A) \geq a_0\, l(A;\theta)\, e^{-\theta^\star s}, \quad \forall s \geq 0,\ A\in\mathcal{E} \tag{2.12}$$

then

$$P(X_n > s,\, Y_n \in A) \geq a_n\, l(A;\theta)\, e^{-\theta^\star s}, \quad \forall s\geq 0,\ A\in\mathcal{E},\ n = 0,1,\ldots. \tag{2.13}$$

In particular,

$$P(X_n > s) \geq a_n\, e^{-\theta^\star s}, \qquad \forall s \geq 0,\ n = 0,1,\ldots. \tag{2.14}$$

Proof. Let $\{\delta_n(s,\cdot),\ s\geq 0,\ n=0,1,\ldots\}$ be a collection of measures on $(\mathbb{E},\mathcal{E})$ such that

$$\int_{\mathbf{E}}\int_{-\infty}^{s} \delta_n(s-u,dx)\,F(x;A\times du)+F(x;A\times(s,\infty))\,\pi_n(dx)\geq \delta_{n+1}(s,A) \tag{2.15}$$

for all $s\geq 0,\ A\in\mathcal{E},\ n=0,1,\ldots$. As in Lemma 2.1 we can easily show that if the property

$$P(X_n>s,\,Y_n\in A)\geq \delta_n(s,A),\qquad \forall s>0,\ A\in\mathcal{E} \tag{2.16}$$

holds for $n=0$ then it holds for all $n=0,1,\ldots$.

Define $\delta_n(s,A)=a_n\,e^{-\theta^\star s}\,l(A;\theta^\star)$ for all $s\geq 0,\ A\in\mathcal{E},\ n=0,1,\ldots$.

If the mappings $\{\delta_n(s,A),\ s\geq 0,\ A\in\mathcal{E},\ n=0,1,\ldots\}$ satisfy (2.15) then, according to (2.16),

$$P(X_n>s,\,Y_n\in A)\geq a_n\,e^{-\theta^\star s}\,l(A;\theta^\star)$$

for all $s>0,\ A\in\mathcal{E},\ n=0,1,\ldots$, which in turn implies with (1.10) that

$$P(X_n>s)=P(X_n>s,\,Y_n\in\mathbb{E})\geq a_n\,e^{-\theta^\star s}.$$

It remains to check that $\{\delta_n(s,A),\ s\geq 0,\ A\in\mathcal{E},\ n=0,1,\ldots\}$ satisfy (2.15). This can be done by mimicking the proof of Proposition 2.1 and by using the assumption $\rho(\theta^\star)=1$. The proof is omitted. ∎

Remark 2.1 The equation $\rho(\theta)=1$ has always one and only one solution $\theta=\theta^\star$ in $\mathcal{D}\cap(0,\infty)$ when the set $\mathcal{D}$ is open. This follows from the strict convexity of $\rho(\theta)$ on $\mathcal{D}$ (which itself is a consequence of the strict convexity of $\log\rho(\theta)$ [14, Lemma 3.4(i)]), of $\lim_{\theta\to\partial\mathcal{D}}\rho(\theta)=\infty$ [14, Corollary 3.1], of $\rho(0)=1$, and of $\rho'(0)=E_\pi[\xi_0]<0$.

We now turn to the derivation of steady-state bounds. As already mentioned, we know that there exists a proper r.v. X such that $P(X_n\leq s)\to_n P(X\leq s)$ for all $s\geq 0$ independently of the initial distribution η of (X_0,Y_0) whenever $E_\pi[\xi_0]<0$ [9]. This result, combined with Propositions 2.1-2.2, yields the following:

Corollary 2.1 *Assume that $E_\pi[\xi_0]<0$. Then, $\forall\theta\in\mathcal{G}$,*

$$P(X>s)\leq b(\theta)\,e^{-\theta s},\quad \forall s\geq 0 \tag{2.17}$$

where

$$b(\theta)=\sup_{(s,A)\in\mathcal{K}}\frac{\displaystyle\int_{\mathbf{E}}F(x;A\times(s,\infty))\,\pi(dx)}{\displaystyle\int_{\mathbf{E}}\int_{s}^{\infty}e^{\theta(u-s)}\,F(x;A\times du)\,l(dx;\theta)}<\infty. \tag{2.18}$$

If we further assume that $\mathcal{D}$ is an open set, then

$$P(X > s) \geq a\, e^{-\theta^\star s}, \quad \forall s \geq 0 \tag{2.19}$$

where

$$a = \inf_{(s,A)\in\mathcal{K}} \frac{\displaystyle\int_{\mathbf{E}} F(x; A \times (s,\infty))\, \pi(dx)}{\displaystyle\int_{\mathbf{E}} \int_s^\infty e^{\theta^\star (u-s)}\, F(x; A \times du)\, l(dx; \theta^\star)}. \tag{2.20}$$

As already noticed by Kingman [17] it is simple to construct examples where the coefficient a in the lower bound is equal to 0. Instances where $a > 0$ are discussed in [20].

References

[1] D. Artiges and P. Nain, "Upper and Lower Bounds for the Multiplexing of Multiclass Markovian On/Off Sources", *Perf. Evaluation*, **27**&**28**, pp. 673–698, Oct. 1996.

[2] S. Asmussen and T. Rolski, "Risk Theory in a Periodic Environment: the Cramer-Lundberg Approximation and Lundberg's Inequality", *Mathematics of Operations Research*, **2**, 2, pp. 410-433, May 1994.

[3] A. A. Borovkov, *Stochastic Processes in Queueing Theory.* Springer-Verlag, New York, 1976.

[4] A. A. Borovkov and S. G. Foss, "Stochastically Recursive Sequences and Their Generalizations," *Siberian Advances in Mathematics*, **2**, pp. 16-81, 1992.

[5] C.-S. Chang, "Stability, Queue Length and Delay of Deterministic and Stochastic Queueing Networks," *IEEE Trans. Aut. Contr.*, **39**, No. 5, pp. 913-931, May 1994.

[6] C.-S. Chang and J. Cheng, "Computable exponential bounds for intree networks with routing," *Proc. of the Infocom'96 Conf.*, Boston, MA, Apr.4-6, 1995, Vol. 1, pp. 197-204.

[7] R. L. Cruz, "A Calculus for Network Delay, Part I: Network Elements in Isolation," *IEEE Trans. Inf. Theory*, **37**, No. 1, pp. 114-131, Jan. 1991.

[8] R. L. Cruz, "A Calculus for Network Delay, Part II: Network Analysis," *IEEE Trans. Inf. Theory*, **37**, No. 1, pp. 132-141, Jan. 1991.

[9] N. G. Duffield, "Exponential Bounds for Queues with Markovian Arrivals," *QUESTA*, **17**, pp. 413-430, 1994.

[10] A. I. Elwalid and D. Mitra, "Effective Bandwidth of General Markovian Traffic Sources and Admission Control of High Speed Networks," *IEEE/ACM Trans. on Networking*, **1**, No. 3, pp. 329-343, Jun. 1993.

[11] R. J. Gibbens and P. J. Hunt, "Effective Bandwidths for the Multi-Type UAS Channel," *QUESTA*, **9**, pp. 17-28, 1991.

[12] P. W. Glynn and W. Whitt, "Logarithmic Asymptotics for Steady-State Tail Probabilities in a Single-Server Queue," *Studies in Appl. Prob.*, J. Galambos and J. Gani eds, pp. 131–156, 1994.

[13] T. E. Harris, *The Theory of Branching Processes.* Dover Publications Inc., New York, 1989.

[14] I. Iscoe, P. Ney and E. Nummelin, "Large Deviations of Uniformly Recurrent Markov Additive Processes," *Adv. in Appl. Math.*, **6**, pp. 373-412, 1985.

[15] F. P. Kelly, "Effective Bandwidths at Multi-Class Queues," *QUESTA*, **9**, pp. 5-16, 1991.

[16] G. Kesidis, J. Walrand and C.-S. Chang, "Effective Bandwidths for Multiclass Markov Fluids and Other ATM Sources," *IEEE/ACM Trans. Networking*, **1**, No. 4, pp. 424-428, Aug. 1993.

[17] J. F. C. Kingman, "A Martingales Inequality in the Theory of Queues," *Proc. Camb. Phil. Soc.*, **59**, pp. 359-361, 1964.

[18] J. F. C. Kingman, "Inequalities in the Theory of Queues," *J. Roy. Stat. Soc.*, Series B, **32**, pp. 102-110, 1970.

[19] J. F. Kurose, "On Computing per-Session Performance Bounds in High-Speed Multi-Hop Computer Networks", *Proc. of the joint ACM Sigmetrics and Performance'92 Conf.*, Newport, RI, pp. 128-139, Jun. 1992.

[20] Z. Liu, P. Nain and D. Towsley, "Exponential Bounds with Applications to Call Admission", *Journal of the ACM*, Vol. 44, No. 3, pp. 366-394, May 1997.

[21] S. P. Meyn and R. L. Tweedie, *Markov Chains and Stochastic Stability.* Springer Verlag, London, 1993.

[22] S. M. Ross, "Bounds on the Delay Distribution in GI/GI/1 Queues," *J. Appl. Prob.*, **11**, 417-421, 1974.

[23] O. Yaron and M. Sidi, "Performance and Stability of Communication Networks Via Robust Exponential Bounds," *IEEE/ACM Trans. Networking*, **1**, No. 3, pp. 372-385, Jun. 1993.