Mechanism and Machine Theory 81 (2014) 21–35

Contents lists available at ScienceDirect

Mechanism and Machine Theory

journal homepage: www.elsevier.com/locate/mechmt

Dynamic modeling of parallel robots with flexible platforms

Philip Long *, Wisama Khalil, Philippe Martinet

Institut de Recherche en Communications et Cybernétique de Nantes, UMR CNRS n° 6597 1 rue de la Noë, École Centrale de Nantes, 44321, Nantes, France

ARTICLE INFO

Article history:
Received 12 December 2013
Received in revised form 11 June 2014
Accepted 12 June 2014
Available online 5 July 2014

Keywords:
Parallel robots
Flexible manipulators
Dynamic modeling

ABSTRACT

This paper presents a method for calculating the direct and inverse dynamic models of a parallel robot with a flexible platform. The system considered in this study is a Gough–Stewart 6-DOF parallel robot however the method is general and can be used for other structures. The platform of the parallel manipulator is considered as a flexible body and modeled using distributed flexibility while the links of the legs are considered as rigid. The direct dynamic model gives the elastic and Cartesian accelerations in terms of the input torques and the current state of the system i.e. the position and velocities of both the rigid and elastic variables. The inverse dynamic model calculates the elastic accelerations and the actuator torques from the current state variables and the desired acceleration of the platform.

1. Introduction

The dynamic modeling of Gough–Stewart robot with rigid elements has attracted many works with different algorithms. For instance, the Lagrange–Euler formalism has been used in the works of Lee and Shah [1], Geng et al. [2] and Lebret et al. [3], Ait-Ahmed [4], Bhattacharya et al. [5,6] and Liu et al. [7]. The principle of virtual work has been used by Tsai [8], Codourey [9] and Staicu [10,11]. On the other hand, Newton–Euler equations have been used in the work of Sugimoto [12], Reboulet et al. [13], Ji [14], Gosselin [15] and Dasgupta et al. [16,17]. However, recently, Carricato and Gosselin [18], Afroun et al. [19], Fu et al. [20] and Vakil et al. [21], have pointed out common errors in many methods related to parameterization and instantaneous kinematic behavior of the legs. These errors may cause kinematic and dynamic miscalculations. The correct dynamic modeling of the rigid Gough–Stewart robot, which avoids these errors, has been demonstrated using different formalisms. For example using screw theory in Gallardo et al. [22], the Newton–Euler approach in Khalil and Guegan [23], Khalil and Ouarda [24] and by Lagrange methods in Abdellatif and Heimann [25].

The aim of this paper is to extend the dynamic method in [23,24] to parallel robots with flexible platforms. There are two possible applications for this work. The first application is for robots with large platforms, where flexibility can no longer be neglected. The platform's flexibility can be taken into account in the design of the controller, thanks to this model. The second application is for robots that carry out high speed machining tasks, during which large vibrations are induced. Generally to counteract this, the platform's mass is increased until the effects of vibration are negligible. This solution leads to manipulators with high mass and greater energy consumption.

To give an idea of the dimension involved, consider CMW's 6-DOF parallel robot the hexapode. The platform of this robot has a mass of over 200 kg with a diameter of 600 mm. The total mass of the system is 900 kg. The maximum speed is just over 0.8 m/s. If the flexibility is modeled, these manipulators can be designed with low weight platforms, thereby reducing the total mass and permitting the use of high acceleration trajectories.

In the literature the main approaches to modeling flexibility in parallel robots are concerned with limb flexibility, this is because the limb's flexibility can be approximated using beam elements. For instance, for the Gough–Stewart robot the effects of leg flexibility

* Corresponding author. Tel.: +33 2 40 37 69 27; fax: +33 2 40 37 69 30.
E-mail address: philip.long@irccyn.ec-nantes.fr (P. Long).

http://dx.doi.org/10.1016/j.mechmachtheory.2014.06.009
0094-114X/

Nomenclature

$\mathbf{q}_i$	Joint positions of leg i
$\dot{\mathbf{q}}_i$	Joint velocities of leg i
$\ddot{\mathbf{q}}_i$	Joint accelerations of leg i
Γ_i	Joint accelerations of leg i
$\mathbf{q}_e$	Generalized elastic position variables of platform
$\dot{\mathbf{q}}_e$	Generalized elastic velocity variables of platform
$\ddot{\mathbf{q}}_e$	Generalized elastic acceleration variables of platform
$\Phi_{dk}(\mathrm{i})$	Displacement shape function of mode k at point i
$\Phi_{rk}(\mathrm{i})$	Rotation shape function of mode k at point i
$\mathbf{V}_i$	Kinematic twist at point i 30
$\mathbf{v}_i$	Linear velocity at point i
ω_i	Angular velocity at point i
$\mathbf{r}_i$	Vector from platform origin to point i
$\mathbf{F}_i$	Wrench at point i
$\mathbf{f}_i$	Force at point i
$\mathbf{n}_i$	Moment at point i
$\mathbf{Q}_p$	Elastic generalized forces of the platform

are examined in [26,27]. The optimum choice of flexibility representation is investigated in [28]. In [29,30] lumped spring mass approximations have been used. By using distributed flexibility in [31], a solution for the dynamics calculation of parallel robots is proposed in the case of flexible legs but with a rigid platform.

The parallel robot treated in this paper is the well known Gough–Stewart platform, which is considered as a good representation of parallel robot's characteristics, however the proposed methods are general and can be applied to other parallel robots. The paper is organized in the following way. In Section 2, the overall procedure is outlined, as well as the prescribed solution. In Section 3 the geometric, kinematic and dynamic models of the manipulator legs are presented. In Section 4, the generalized Newton–Euler model of a flexible platform is given. Furthermore the geometric and flexible parameters for the target platform are described. Section 5 describes how the inverse dynamic model and direct dynamic model of the flexible robot are derived. In Section 6, a numerical simulation validating the proposed model is given. Finally in Section 7 the conclusions are drawn and future areas of research are described.

2. Problem statement

The objective of this work is to calculate the dynamic models of the Gough–Stewart robot with flexible platform. The inverse dynamic model obtains the joint torques and forces for a desired acceleration of the platform using the state variables of the robot (the positions and velocities). The direct dynamic model gives the elastic and rigid accelerations of the system's variables in terms of the input torques and the state of the system.

In order to proceed, the system is decomposed into two subsystems, one is flexible and the other is rigid. The decomposition is performed by opening (virtually) the spherical joints representing the connection points between the legs and the platform. The flexible subsystem represents the platform, that is described using distributed flexibility [32,33] and modeled using Cartesian coordinates and the Newton–Euler formulation. The rigid elements of the robot, which consist of the legs and the fixed base of the legs, are described as a tree structure robot using the Modified Denavit Hartenberg Parameters [34] and modeled using joint variables. The two subsystems are connected by calculating the reaction forces at the connection points between the platform and the legs.

3. Leg system description and modeling

3.1. Geometric parameters

The studied system is a Gough–Stewart structure, as shown in Fig. 1. The platform has 6-DOF and is connected to the fixed base by six legs. Each leg is connected to the base with a 2-DOF universal joint (U-joint) and to the platform with a 3-DOF spherical joint (S-joint). Each leg has a variable length by means of an actuated prismatic joint (P-joint).

The base frame and the platform frame are denoted by Σ_o and Σ_p, respectively. The connection points between the base and the U-Joints are denoted as $\mathbf{b}_i$ and are arranged according to the convention established in [23]. The connection points between the platform origin and the legs are denoted as $\mathbf{p}_i$, for $i = 1 \dots 6$.

After opening virtually the spherical joints, each leg i is composed of three joints and three links. The geometric parameters of the links, $j = 1 \dots 3$, for each leg i are given in Table 1, for $i = 1 \dots 6$.

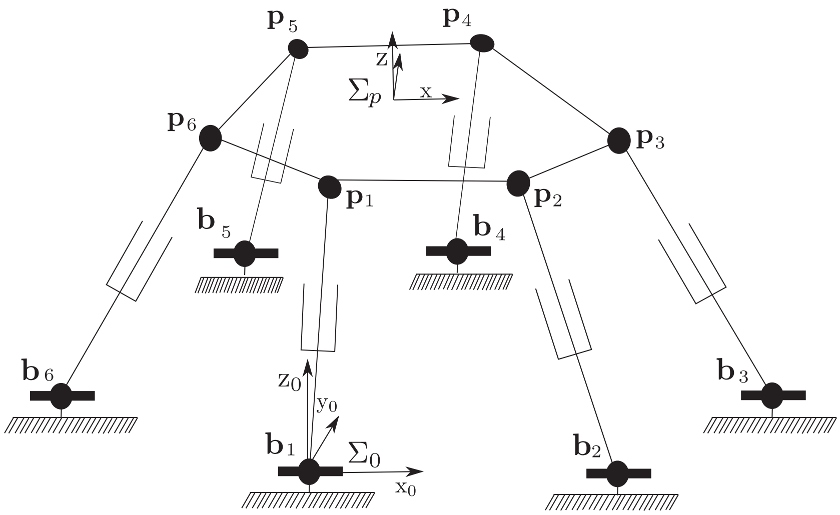

Fig. 1. Gough–Stewart manipulator.

$\mu_{ji} = 1$ for an actuated joint or $\mu_{ji} = 0$ for a passive joint. $\sigma_{ji} = 1$ indicates the joint is prismatic and $\sigma_{ji} = 0$ indicates that the joint is revolute. The parameters γ_{ji}, b_{ji}, α_{ji}, d_{ji}, $þeta_{ji}$ and r_{ji} define the location of frame j of leg i, defined as Σ_{ji}, with respect to its antecedent frame.

3.2. Kinematic models of the legs

The 3×1 vectors denoting joint position, velocity and acceleration for leg i are denoted as $\mathbf{q}_i$, $\dot{\mathbf{q}}_i$ and $\ddot{\mathbf{q}}_i$ respectively. The actuated joint of leg i is denoted as q_{3i} and the vector of all actuated joints of the system is given as $\mathbf{q}_a = [q_{31} \quad q_{32} \quad q_{33} \quad q_{34} \quad q_{35} \quad q_{36}]^T$ where q_{3i} is the distance between $\mathbf{p}_i$ and $\mathbf{b}_i$. The velocity of the connection point, $\mathbf{p}_i$, is a linear velocity $\mathbf{v}_i$ which can be obtained from the joint velocity of the corresponding leg using the kinematic Jacobian matrix of the leg:

$$\mathbf{v}_i = \mathbf{J}_i \dot{\mathbf{q}}_i. \tag{1}$$

The inverse kinematic model is written as:

$$\dot{\mathbf{q}}_i = \mathbf{J}_i^{-1} \mathbf{v}_i. \tag{2}$$

The Jacobian matrix of leg i represented in the frame of Σ_{0i} is given as:

$$^{0i}\mathbf{J}_i = \begin{bmatrix} -q_{3i}sin(q_{1i})sin(q_{2i}) & q_{3i}cos(q_{1i})cos(q_{2i}) & cos(q_{1i})sin(q_{2i}) \\ 0 & q_{3i}sin(q_{2i}) & -cos(q_{2i})(3) \\ -q_{3i}cos(q_{1i})sin(q_{2i}) & -q_{3i}cos(q_{2i})sin(q_{1i}) & -sin(q_{1i})sin(q_{2i}) \end{bmatrix}. \tag{3}$$

The inverse Jacobian matrix of leg i represented in the frame of Σ_{0i} is given as:

$$^{0i}\mathbf{J}_i^{-1} = \begin{bmatrix} -\dfrac{sin(q_{1i})}{(q_{3i}sin(q_{2i}))} & 0 & -\dfrac{cos(q_{1i})}{(q_{3i}sin(q_{2i}))} \\ \dfrac{(cos(q_{1i})cos(q_{2i}))}{q_{3i}} & \dfrac{sin(q_{2i})}{q_{3i}} & -\dfrac{(cos(q_{2i})sin(q_{1i}))}{q_{3i}} \\ cos(q_{1i})sin(q_{2i}) & -cos(q_{2i}) & -sin(q_{1i})sin(q_{2i}) \end{bmatrix}. \tag{4}$$

Table 1
Geometric parameters for leg i.

j_i	μ_{ji}	σ_{ji}	γ_{ji}	b_{ji}	α_{ji}	dji	θ_{ji}	r_{ji}
1_i	0	0	γ_{1i}	b_{1i}	$-\frac{\pi}{2}$	d_{1i}	q_{1i}	0
2_i	0	0	0	0	$\frac{\pi}{2}$	0	q_{2i}	0
3_i	1	1	0	0	$\frac{\pi}{2}$	0	0	q_{3i}

It should be noted that the third row of the Jacobian matrix corresponds to the unit vector along the axis of the prismatic joint of the serial leg i, $\mathbf{a}_{3i}^T$. This leads to an expression of the actuated joint velocity of leg i in terms of $\mathbf{v}_i$ as:

$$\dot{q}_{3i} = \mathbf{a}_{3i}^T \mathbf{v}_i. \tag{5}$$

The second order inverse kinematic model of the leg is given by:

$$\ddot{\mathbf{q}}_i = \mathbf{J}_i^{-1}(\dot{\mathbf{v}}_i - \dot{\mathbf{J}}_i \dot{\mathbf{q}}_i). \tag{6}$$

$\mathbf{v}_i$ and $\dot{\mathbf{v}}_i$ are obtained from $\mathbf{V}_p$ and $\dot{\mathbf{V}}_p$, the velocity and acceleration of the platform respectively, using the elastic equations given in Section 4.

3.3. Inverse dynamic model of the legs

The inverse dynamic model of leg i is obtained by considering the tree structure sub-system of the legs (after separating the platform).

Let the dynamic model of the 3 DOF system be given as $\tau_i = \mathbf{A}_i(\mathbf{q}_i)\ddot{\mathbf{q}}_i + \mathbf{c}_i(\mathbf{q}_i, \dot{\mathbf{q}}_i)$ where τ_i represents the joint torques if the leg is not connected to the platform. The positions, velocities and accelerations of the joints are obtained from the position, velocity and acceleration of point $\mathbf{p}_i$ using the inverse kinematic model. $\mathbf{A}_i$ is the inertia matrix of leg i whereas $\mathbf{c}_i$ is the vector of Coriolis, centrifugal and gravity torques.

Γ_i is the torque of the closed loop structure of leg i, composed of the dynamic of open loop $\mathbf{A}_i(\mathbf{q}_i)\ddot{\mathbf{q}}_i + \mathbf{c}_i(\mathbf{q}_i, \dot{\mathbf{q}}_i)$ and the effect of the forces generated by the moving platform on the legs. It can be written as:

$$\Gamma_i = \mathbf{A}_i \ddot{\mathbf{q}}_i + \mathbf{c}_i + \mathbf{J}_i^T \mathbf{f}_i \tag{7}$$

with

$$\Gamma_i = [0 \;\; 0 \;\; \Gamma_{3i}]^T. \tag{8}$$

The first two components of Γ_i are zero as they correspond to the torques of the passive joints. This equation permits the calculation of the reaction forces of the leg on the platform in terms of the actuated joint torques of the manipulator:

$$\mathbf{f}_i = \mathbf{a}_{3i}\Gamma_{3i} - \mathbf{J}_i^{-T}(\mathbf{A}_i \ddot{\mathbf{q}}_i + \mathbf{c}_i). \tag{9}$$

Eqs. (4) and (8), (9) can be manipulated to leave and expression in terms of the torque of the actuated joint:

$$\mathbf{f}_i = \mathbf{a}_{3i}\Gamma_{3i} - \mathbf{J}_i^{-T}(\mathbf{A}_i \ddot{\mathbf{q}}_i + \mathbf{c}_i). \tag{10}$$

4. Modeling of flexible Gough–Stewart platform

4.1. Platform description

Fig. 2 shows the flexible platform. The flexibility is represented by a series of shape functions and is modeled using the generalized Newton–Euler model [32,35]. The platform contains N flexible DOF, i.e. the total number of shape functions characterizing the flexible behavior.

For the generalized Newton–Euler model, the Cartesian variables are used to describe the rigid body motions while the Lagrangian variables are used to describe its elastic motions. The main hypothesis in this formalism is the manipulator undergoes small deformations that can be described using modal shape functions. Therefore the motion of the flexible body can be approximated by the sum of the rigid body motion and the flexible body deformation. The frame Σ_p is fixed with the platform, and its origin is located at an operational point of the platform, for example the geometric center. The location of the Σ_p in the world frame is defined by ${}^0\mathbf{T}_p$, the 4×4 homogeneous transformation matrix.

The most common boundary conditions for representing the flexibility are clamped-free, free–free, pinned–pinned and clamped–clamped. The boundary conditions not only define the deformation characteristics of the object but also the reaction force transmitted by the joints. In this case the platform has six boundary conditions, located at the connection points of the platform i.e. the S-joints. Each joint allows the platform to freely rotate but not translate, corresponding to a pinned–pinned boundary condition. However in this paper, we outline a more general approach and model the platform using free–free boundary conditions, where the shape functions can be obtained using finite element software, for instance MSC Nastran©.

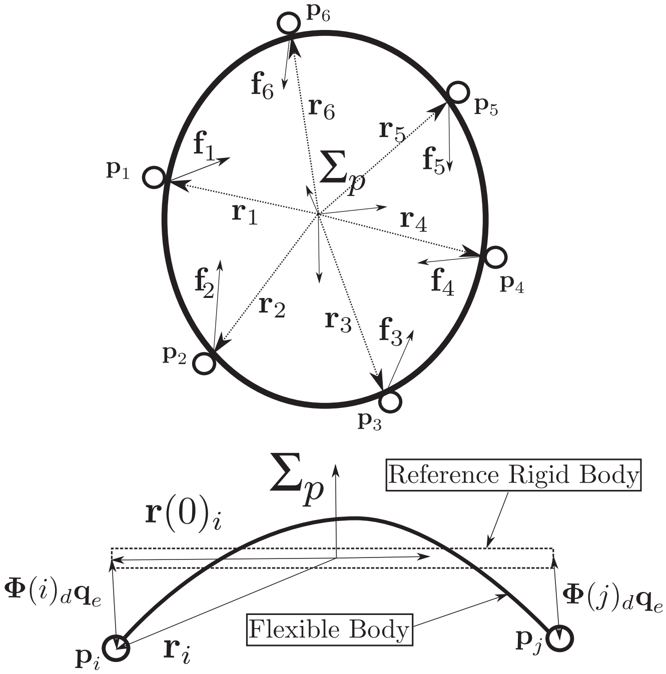

Fig. 2. Flexible platform, (TOP) flexible platform forces and attachment point vectors (bottom) representation of free–free boundary conditions.

4.2. Platform kinematics

The position of the connection point i, denoted as $\mathbf{p}_i$ for $i = 1 \ldots 6$, can be calculated from the position of the platform and the position vector from the origin to this connection point as:

$$\mathbf{p}_i = \mathbf{p}_p + \mathbf{r}_i \tag{11}$$

where $\mathbf{r}_i$ is defined as the position vector from the origin of the platform frame, denoted as $\mathbf{p}_p$, to $\mathbf{p}_i$. The vector $\mathbf{r}_i$ is a function of the flexible parameters of the platform. It is obtained by the summation of the rigid body position, $\mathbf{r}_i(0)$, and the deformation due to flexibility using Eq. (12), as shown in Fig. 2.

$$\mathbf{r}_i = \mathbf{r}_i(0) + \sum_{k=1}^{N} \mathbf{\Phi}_{dk}(i)\dot{q}_{ek} \tag{12}$$

$\mathbf{q}_e = (q_{e1} \ldots q_{ek} \ldots q_{eN})$, is the N × 1 vector of generalized elastic coordinates. The derivative of Eq. (12), leads to

$$\dot{\mathbf{r}}_i = \omega_p \times \mathbf{r}_i + \sum_{k=1}^{N} \mathbf{\Phi}_{dk}(i)\,\dot{q}_{ek} \tag{13}$$

$\mathbf{\Phi}_{dk}(i)$ and $\mathbf{\Phi}_{rk}(i)$ are the *kth* displacement and rotation shape functions at point *i*. $\mathbf{\Phi}_{dk}(i)$ and $\mathbf{\Phi}_{rk}(i)$ are not independent, rather they are linked by the *curl operator*. Since the shape functions are defined with respect to the platform frame Σ_p, all other variables in the following are represented in this frame unless otherwise stated. ω_p is defined as the vector of angular velocity of the moving platform. $\dot{\mathbf{q}}_e$ and $\ddot{\mathbf{q}}_e$ are the vectors of velocity and acceleration of the generalized elastic coordinates.

The velocity screw at the platform origin is defined as $\mathbf{V}_t$ which is composed of $\mathbf{v}_t$ and ω_t the total (including the effects of flexibility) linear and angular velocity of the platform respectively. The total velocity screw can be obtained as the sum of the rigid body velocity screw evaluated at that point and the velocity due to the effects of the flexibility.

$$\mathbf{V}_t = \mathbf{V}_p + \begin{bmatrix} \Phi_d(p) \\ \Phi_r(p) \end{bmatrix} \dot{\mathbf{q}}_e \tag{14}$$

$$\mathbf{\Phi}_d(p) = [\,\Phi_{d1}(p) \quad \Phi_{d2}(p) \quad \ldots \quad \Phi_{dN}(p)\,] \tag{15}$$

$$\mathbf{\Phi}_r(p) = [\,\Phi_{r1}(p) \quad \Phi_{r2}(p) \quad \ldots \quad \Phi_{rN}(p)\,] \tag{16}$$

$$\mathbf{V}_p = \begin{bmatrix} \mathbf{v}_p \\ \omega_p \end{bmatrix} \qquad \mathbf{V}_t = \begin{bmatrix} \mathbf{v}_t \\ \omega_t \end{bmatrix}. \tag{17}$$

$\mathbf{v}_p$ is defined as the component of rigid velocity of the platform, while $\dot{\mathbf{v}}_p$ and $\dot{\omega}_p$ denote the linear and angular acceleration.

The linear velocity and acceleration at connection point are defined as $\mathbf{v}_i$ and $\dot{\mathbf{v}}_i$ respectively. The linear velocity can be obtained from the effect of the rigid platform velocity, plus the effects of flexibility at point i:

$$\mathbf{v}_i = \mathbf{v}_p + \omega_p \times \mathbf{r}_i + \Phi_d(i)\dot{\mathbf{q}}_e. \tag{18}$$

The joint variables can be obtained by using Eq. (2). Therefore, substituting Eq. (5) into Eq. (18):

$$\dot{\mathbf{q}}_a = \begin{bmatrix} \mathbf{a}_{31}^T & (\hat{\mathbf{r}}_1\mathbf{a}_{31})^T & \mathbf{a}_{31}^T\Phi_d(1) \\ \vdots & \vdots & \vdots \\ \mathbf{a}_{36}^T & (\hat{\mathbf{r}}_6\mathbf{a}_{36})^T & \mathbf{a}_{36}^T\Phi_d(6) \end{bmatrix} \begin{bmatrix} \mathbf{v}_p \\ \omega_p \\ \dot{\mathbf{q}}_e \end{bmatrix}. \tag{19}$$

Eq. (19) is rewritten as

$$\dot{\mathbf{q}}_a = \begin{bmatrix} \mathbf{J}_p^{-1} & \mathbf{J}_e^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{V}_p \\ \dot{\mathbf{q}}_e \end{bmatrix} \tag{20}$$

with

$$\mathbf{J}_p^{-1} = \begin{bmatrix} \mathbf{a}_{31}^T & (\hat{\mathbf{r}}_1\mathbf{a}_{31})^T \\ \vdots & \vdots \\ \mathbf{a}_{36}^T & (\hat{\mathbf{r}}_6\mathbf{a}_{36})^T \end{bmatrix} \qquad \mathbf{J}_e^{-1} = \begin{bmatrix} \mathbf{a}_{31}^T\Phi_d(1) \\ \vdots \\ \mathbf{a}_{36}^T\Phi_d(6) \end{bmatrix} \tag{21}$$

where $\hat{\mathbf{x}}$ designates the 3×3 skew symmetric matrix associated with a vector x, such that $\hat{\mathbf{x}}\mathbf{y} = \mathbf{x} \times \mathbf{y}$. It should be noted that $\mathbf{J}_p^{-1}$ and $\mathbf{J}_e^{-1}$ are defined directly and are not obtained from inverting any matrices.

By differentiation of Eq. (18), the acceleration of the connection point i can be obtained:

$$\dot{\mathbf{v}}_i = \dot{\mathbf{v}}_p + \dot{\omega}_p \times \mathbf{r}_i + \omega_p \times \left(\omega_p \times \mathbf{r}_i + \Phi_d(i)\dot{\mathbf{q}}_e\right) + \Phi_d(i)\ddot{\mathbf{q}}_e + \omega_p \times \Phi_d(i)\dot{\mathbf{q}}_e \tag{22}$$

which can be rearranged as:

$$\dot{\mathbf{v}}_i = \begin{bmatrix} \mathbf{1}_3 & -\hat{\mathbf{r}}_i\Phi_d(i) \end{bmatrix} \begin{bmatrix} \dot{\mathbf{v}}_p \\ \dot{\omega}_p \\ \ddot{\mathbf{q}}_e \end{bmatrix} + \mathbf{h}_i \tag{23}$$

with

$$\mathbf{h}_i = \omega_p \times \omega_p \times \mathbf{r}_i + 2\omega_p \times \Phi_d(i)\dot{\mathbf{q}}_e. \tag{24}$$

Finally, taking into account all legs of the system, $\mathbf{h}$ is defined as:

$$\mathbf{h} = \begin{bmatrix} \mathbf{h}_1^T & \mathbf{h}_2^T & \mathbf{h}_3^T & \mathbf{h}_4^T & \mathbf{h}_5^T & \mathbf{h}_6^T \end{bmatrix}^T. \tag{25}$$

4.3. Platform forces

Since the connection points of the platform constitute S-joints, only pure forces can be transmitted to the platform from the robot legs. The force transmitted by connection point i is denoted as $\mathbf{f}_i$. The contribution of this force at the platform origin, $\mathbf{f}_{pi}$, $\mathbf{n}_{pi}$ and $\mathbf{Q}_{pi}$, is given by:

$$\begin{bmatrix} \mathbf{f}_{pi} \\ \mathbf{n}_{pi} \\ \mathbf{Q}_{pi} \end{bmatrix} = \begin{bmatrix} \mathbf{1}_3 \\ \hat{\mathbf{r}}_i \\ \Phi_d^T(i) \end{bmatrix} \mathbf{f}_i. \tag{26}$$

Taking into account all six connection points of the flexible platform, the platform forces, moments and elastic generalized forces denoted as f_p n_p and $\mathbf{Q}_p$ respectively can be obtained as:

$$\begin{bmatrix} \mathbf{f}_p \\ \mathbf{n}_p \\ \mathbf{Q}_p \end{bmatrix} = \begin{bmatrix} 1_3 & \ldots & 1_3 \\ \hat{\mathbf{r}}_1 & \ldots & \mathbf{r}_6 \\ \Phi_d^T(1) & \ldots & \Phi_d^T(6) \end{bmatrix} \begin{bmatrix} \mathbf{f}_1 \\ \vdots \\ \mathbf{f}_6 \end{bmatrix}. \tag{27}$$

Eq. (27) can be rewritten as:

$$\begin{bmatrix} \mathbf{F}_p \\ \mathbf{Q}_p \end{bmatrix} = \mathbf{W} \begin{bmatrix} \mathbf{f}_1 \\ \vdots \\ \mathbf{f}_6 \end{bmatrix} \tag{28}$$

where $\mathbf{F}p$ contains the forces and moments at frame $\sum_p$,

$$\mathbf{F}_p = \begin{bmatrix} \mathbf{f}_p \\ \mathbf{n}_p \end{bmatrix}. \tag{29}$$

The matrix $\mathbf{W}$, given in Eq. (28), is a $(6 + N) \times 18$ matrix, whose transpose relates the velocities of the connection points to the platform velocity as shown in Eq. (18). Likewise it transforms forces applied by the legs of the parallel robot to the total platform forces. $\mathbf{W}$ can be decomposed into $\mathbf{W_p}$ a 6×18 matrix and $\mathbf{W}_e$ a $N \times 18$ matrix as shown in Eq. (30). $\mathbf{W}_p$ and $\mathbf{W}_e$ relate the leg forces to the platform's rigid and flexible variables respectively.

$$\mathbf{W} = \begin{bmatrix} 1_3 & \ldots & 1_3 \\ \hat{\mathbf{r}}_1 \ldots \hat{\mathbf{r}}_6 & & \\ \Phi_d^T(1) & \ldots & \Phi_d^T \end{bmatrix} = \begin{bmatrix} \mathbf{W}_p \\ \mathbf{W}_e \end{bmatrix}. \tag{30}$$

Each column of $\mathbf{W}$ can also be decomposed into its rigid and elastic parts:

$$\mathbf{W}_i = \begin{bmatrix} \mathbf{W}_{pi} \\ \mathbf{W}_{ei} \end{bmatrix} = \begin{bmatrix} 1_3 \\ \hat{\mathbf{r}}_i \\ \Phi_d^T(i) \end{bmatrix}. \tag{31}$$

Comparing Eq. (31) with Eq. (18), it can be seen that

$$\mathbf{W}_{pi} = \left(\frac{\partial \mathbf{v}_i}{\partial \mathbf{V}_p}\right)^T \qquad \mathbf{W}_{ei} = \left(\frac{\partial \mathbf{v}_i}{\partial \dot{\mathbf{q}}_e}\right)^T. \tag{32}$$

From Eq. (32), it is obvious that $\mathbf{W}$ relates the kineostatic variables at the platform frame to the variables at connection point i. These relations are used in the derivation of the dynamic model given in Section 5. However, since the objective is to relate the joint torques to the Cartesian acceleration, an expression must be obtained that links the platform and the leg variables. Therefore using Eq. (21) and Eq. (32), it can be seen that:

$$\mathbf{J}_{pi}^{-T} = \mathbf{W}_{pi}\mathbf{a}_i = \left(\frac{\partial \dot{q}_{3i}}{\partial \mathbf{V}_p}\right)^T \qquad \mathbf{J}_{ei}^{-T} = \mathbf{W}_{ei}\mathbf{a}_i = \left(\frac{\partial \dot{q}_{3i}}{\partial \dot{\mathbf{q}}_e}\right)^T. \tag{33}$$

4.4. Platform dynamics

In order to find the dynamics of a flexible body moving in space the principle of virtual powers is used. The principle of virtual power is analogous to the principle of virtual work, the difference being the use of a virtual velocity instead of a virtual displacement. It states that the virtual power due to the acceleration of the body is equal to the sum of the virtual power due to internal forces and the virtual power due to external forces. The generalized Newton Euler model for the platform taking into account the flexibility is given as [32,35]:

$$\begin{bmatrix} \mathbf{f}_p \\ \mathbf{n}_p \\ \mathbf{Q}_p) \end{bmatrix} = \begin{bmatrix} m1_3 & \mathbf{M}\hat{\mathbf{S}}_r^T & \mathbf{MS}_{de} \\ \mathbf{M}\hat{\mathbf{S}}_r & \mathbf{I}_{0p} & \mathbf{MS}_{re} \\ \mathbf{MS}_{de}^T & \mathbf{MS}_{re}^T & \mathbf{m}_{ee} \end{bmatrix} \left[\dot{\mathbf{v}}_p - \mathbf{g}\dot{\omega}_p \ddot{\mathbf{q}}_e \right] + \begin{bmatrix} \mathbf{f}_c \\ \mathbf{n}_c \\ \mathbf{Q}_c \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \mathbf{K}_{ee}\mathbf{q}_e \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \mathbf{D}_{ee}\dot{\mathbf{q}}_e \end{bmatrix} \tag{34}$$

where $m = \int_{\Sigma_{P0}} dm$ is the mass of the body. $\mathbf{M}\hat{\mathbf{S}}_r = \mathbf{M}\hat{\mathbf{S}}_{r0} + \sum_{k=1}^{N} \hat{\mathbf{s}}_k q_{ek}$ is the anti-symmetric tensor of first moments of inertia of the platform where $\mathbf{MS}_{r0}$ is the vector of rigid first moments of inertia and $\int_{\Sigma_{P0}} \Phi_{dk} dm = \mathbf{s}_k$ is the kth elastic counterpart. $\mathbf{MS}_{de} = (\mathbf{s}_1 \ldots \mathbf{s}_n)$

is the displacement first elastic moments whereas $\mathbf{MS}_{re} = (\beta_1 + \sum_{k=1}^{N} \lambda_{k1} q_{ek} + \ldots + \beta_N + \sum_{k=1}^{N} \lambda_{kN} q_{ek})$ is the rotational first elastic moments. $\mathbf{I}_{0p}$ is the 3×3 total rigid inertia matrix of the platform. $\mathbf{m}_{ee} = \mathrm{diag}_{i,j=1\ldots N} \int_{\Sigma_{P0}} \left(\Phi_{dk}^2\right) dm$ is the matrix of generalized mass. $\mathbf{f}_c$, $\mathbf{n}_c$ and $\mathbf{Q}_c$ are the vectors of centrifugal and Coriolis forces and moments. $\mathbf{K}_{ee} = \mathrm{diag}_{i,j=1\ \ldots\ N} k_{ij}$ is the matrix of generalized stiffness. $\mathbf{D}_{ee} = \mathrm{diag}_{i,j=1sN} 2d_{ij}\sqrt{k_{ij}}$ is the matrix of generalized damping. Finally, Σ_{P0} is defined as the initial, undeformed state of the flexible platform, $\lambda_{ik} = \int_{\Sigma_{P0}} (\Phi_{dk}\Phi_{di}) dm$ and $\beta_k = \int_{\Sigma_{P0}} \mathbf{r}_i \times \Phi_{dk} dm$ while $\mathbf{g}$ is the gravity vector.

For the numerical simulation given in Section 6, to calculate the above variables, the platform is discretized in a series of elements joined together at nodes. The modal analysis gives the nodal mass (dm), the generalized stiffness for each mode, the shape functions of each mode evaluated at every node and the distance from each node to the platform origin.

Rewriting Eq. (34) the dynamic equation of the flexible platform of the Gough Stewart platform is obtained as:

$$\begin{bmatrix} \mathbf{F}_p \\ \mathbf{Q}_p \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{rr} & \mathbf{A}_{re} \\ \mathbf{A}_{re}^T & \mathbf{m}_{ee} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{V}}_p \ddot{\mathbf{q}}_e \end{bmatrix} + \begin{bmatrix} \mathbf{c}_r \\ \mathbf{c}_e \end{bmatrix} \tag{35}$$

where $\mathbf{A}_{rr}$ is the spatial 6×6 inertia matrix of the platform.

5. Dynamic model of the Gough Stewart robot

In the following, the steps taken to calculate the dynamic model of the system are outlined. From the dynamic relations, the inverse dynamic problem, outlined in Section 5.1, and the direct dynamic problem, outlined in Section 5.2 can be solved.

The first step is to transform the dynamics of leg i to attachment point i. By using Eq. (6), the dynamic equation of leg i, given in Eq. (10), can be rewritten in terms of the acceleration of the connection point:

$$\mathbf{f}_i = \mathbf{a}_{3i}\Gamma_{3i} - \mathbf{J}_i^{-T}\mathbf{A}_i\mathbf{J}_i^{-1}(\dot{\mathbf{v}}_i - \dot{\mathbf{J}}_i\dot{\mathbf{q}}_i) - \mathbf{J}_i^{-T}\mathbf{c}_i \tag{36}$$

which becomes:

$$\mathbf{f}_i = \mathbf{a}_{3i}\Gamma_{3i} - \mathbf{A}_{xi}\dot{\mathbf{v}}_i - \mathbf{c}_{xi}. \tag{37}$$

$\mathbf{A}_{xi}$ is the inertia matrix of leg i transformed into the Cartesian space at the connection point $\mathbf{p}_i$, such that $\mathbf{A}_{xi} = \mathbf{J}_i^{-T}\mathbf{A}_i\mathbf{J}_i^{-1}$. Since $\mathbf{c}_i$, A_i and $\dot{\mathbf{J}}_i\dot{\mathbf{q}}_i$ are functions of the state variables of leg i, i.e. the joint positions and velocities of leg i, a new variable $\mathbf{c}_{xi}$ is defined that groups these terms together, where $\mathbf{c}_{xi} = \mathbf{J}_i^{-T}\mathbf{c}_i - \mathbf{A}_{xi}\dot{\mathbf{J}}_i\dot{\mathbf{q}}$.

The second step is to eliminate the acceleration at the connection points by expressing it terms of platform acceleration. Rewriting Eq. (23) as:

$$\begin{bmatrix} \dot{\mathbf{v}}_1 \\ \vdots \\ \dot{\mathbf{v}}_6 \end{bmatrix} = \mathbf{W}_p^T\dot{\mathbf{V}}_p + \mathbf{W}_e^T\ddot{\mathbf{q}}_e + \mathbf{h}. \tag{38}$$

Therefore $\mathbf{f}_i$ can be written in terms of Γ_{3i} such that:

$$\mathbf{f}_i = \mathbf{a}_{3i}\Gamma_{3i} - \mathbf{H}_{xi} \tag{39}$$

where from Eq. (38), $\mathbf{H}_{xi}$ is defined as:

$$\mathbf{H}_{xi} = \mathbf{A}_{xi}\mathbf{W}_{pi}^T\dot{\mathbf{V}}_p + \mathbf{A}_{xi}\mathbf{W}_{ei}^T\ddot{\mathbf{q}}_e + \mathbf{A}_{xi}\mathbf{h}_i + \mathbf{c}_{xi}. \tag{40}$$

Table 2
Modal properties of flexible plate.

Mode no.	Natural frequency (rad/s)	Generalized stiffness	Damping ratio
7	188.7473	35625.55	0.4
8	188.7475	35625.63	0.4
9	315.7169	99677.16	0.4
10	401.0187	160816	0.4
11	474.7468	225384.5	0.4
12	709.5586	503473.4	0.4
13	709.5814	503505.8	0.4
14	828.2868	686059.1	0.4
15	828.3062	686091.1	0.4
16	1310.581	1717622	0.4

Table 3
Base inertial parameters of legs.

Link	1	2	3
XX (kg m^2)	0	4.823	1.068
YY (kg m^2)	0	0.945	1.068
ZZ (kg m^2)	5.11	5.11	0.0234
XY (kg m^2)	0	0.215	0
XZ (kg m^2)	0	0	0
YZ (kg m^2)	0	0	0
MX (kg m)	0	4.29	0
MY (kg m)	0	−12.628	0
MZ (kg m)	0	0	−3.018
M (kg)	3.176	30.176	5.390
IA (kg m^2)	0	0	45.89

The force at attachment point i can be transformed to the platform frame using Eq. (26). Thus, by gathering all six legs and making use of Eq. (33), the following expressions are obtained:

$$\mathbf{W}_p \begin{bmatrix} \mathbf{f}_1 \\ \vdots \\ \mathbf{f}_6 \end{bmatrix} = \mathbf{J}_p^{-T}\Gamma - \mathbf{W}_p\left(\mathbf{A}_x\mathbf{W}_p^T\dot{\mathbf{V}} + \mathbf{A}_x\mathbf{W}_e^T\ddot{\mathbf{q}}_e + \mathbf{A}_x\mathbf{h} + \mathbf{c}_x\right) \tag{41}$$

$$\mathbf{W}_e \begin{bmatrix} \mathbf{f}_1 \\ \vdots \\ \mathbf{f}_6 \end{bmatrix} = \mathbf{J}_e^{-T}\Gamma - \mathbf{W}_e\left(\mathbf{A}_x\mathbf{W}_p^T\dot{\mathbf{V}}_p + \mathbf{A}_x\mathbf{W}_e^T\ddot{\mathbf{q}}_e + \mathbf{A}_x\mathbf{h} + \mathbf{c}_x\right) \tag{42}$$

where $\mathbf{A}_x$ is an 18×18 block diagonal matrix whose 3×3 diagonal components are equal to $\mathbf{A}_{x1} \ldots \mathbf{A}_{x6}$. $\mathbf{c}_x$ is an 18×1 vector such that $\mathbf{c}_x = [\mathbf{c}_{x1} \ldots \mathbf{c}_{x6}]^T$.

Eqs. (41) and (42) constitute the dynamics of the system in terms of the platform acceleration, the generalized elastic accelerations, the joint torques and the forces at the connection points.

The next step is to introduce the platform dynamics into the above expressions. By using Eq. (28), the Newton–Euler equation of the flexible platform can be rewritten in terms of the forces at the connection points.

$$\begin{bmatrix} \mathbf{W}_p \\ \mathbf{W}_e \end{bmatrix} \begin{bmatrix} \mathbf{f}_1 \\ \vdots \\ \mathbf{f}_6 \end{bmatrix} = \begin{bmatrix} \mathbf{A}_r & \mathbf{A}_{re} \\ \mathbf{A}_{re}^T & \mathbf{m}_{ee} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{V}}_p \ddot{\mathbf{q}}_e \end{bmatrix} + \begin{bmatrix} \mathbf{c}_r \\ \mathbf{c}_e \end{bmatrix}. \tag{43}$$

Therefore by equating the elastic part of Eq. (43) with Eq. (42), an expression for $\ddot{\mathbf{q}}_e$ is obtained in terms of the joint torque and platform accelerations:

$$\ddot{\mathbf{q}}_e = \mathbf{A}_{ee}^{-1}\left(\mathbf{J}_e^{-T}\Gamma - \left(\mathbf{A}_{re}^T + \mathbf{W}_e\mathbf{A}_x\mathbf{W}_p^T\right)\dot{\mathbf{V}}_p - \mathbf{W}_e\mathbf{A}_x\mathbf{h} - \mathbf{W}_e\mathbf{c}_x - \mathbf{c}_e\right) \tag{44}$$

where for convenience:

$$\mathbf{A}_{ee}^{-1} = \left(\mathbf{m}_{ee} + \mathbf{W}_e\mathbf{A}_x\mathbf{W}_e^T\right)^{-1}. \tag{45}$$

This equation can be integrated to obtain $\dot{\mathbf{q}}_e$ and $\mathbf{q}_e$. It should be noted that $\mathbf{A}_{ee}$ is a definite positive square matrix of dimension $N \times N$.

Table 4
Coordinates of legs in the world frame.

leg	1	2	3	4	5	6
$\mathbf{b}_j(x)(m)$	0.1	0.3	1.3	1.0	−0.2	−0.7
$\mathbf{b}_j(y)(m)$	−0.2	−0.2	0.4	1.0	0.9	0.5
$\mathbf{q}_{1j}(rad)$	−1.671	−1.373	−2.016	−1.958	−1.656	−1.936
$\mathbf{q}_{2j}(rad)$	1.767	1.764	1.814	1.859	1.389	1.345
$\mathbf{q}_{3j}(m)$	1.025	1.039	1.142	1.126	1.0203	1.099

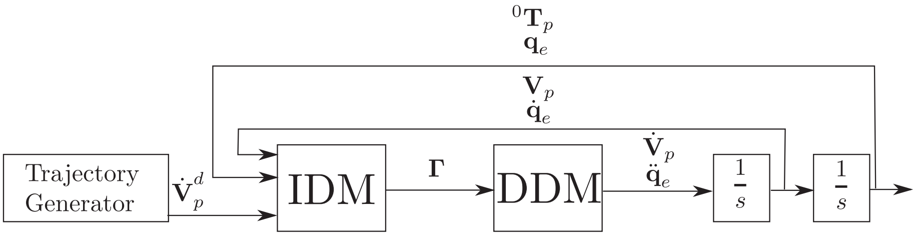

Fig. 3. Simulation schematic.

Finally, by equating the rigid part of Eq. (43) with Eq. (41) and replacing the acceleration of the generalized elastic variables with Eq. (44), the dynamic model is written as:

$$\mathbf{A}\dot{\mathbf{V}}_p + \mathbf{c} = \mathbf{J}_{sys}^{-T}\Gamma. \tag{46}$$

Eq. (46) constitutes the closed form equation of the Cartesian dynamic model of the system. Both **A** and $\mathbf{J}_{sys}$ are square matrices which, outside special configurations, are invertible. Thus, both the inverse dynamic problem and the direct dynamic problem can be solved using this expression, after which the generalized elastic accelerations of the platform $\ddot{\mathbf{q}}_e$ are obtained using Eq. (44).

The 6 × 6 matrix **A** is the equivalent total inertia matrix of the legs and the flexible platform is written as:

$$\begin{aligned}\mathbf{A} = {} & \mathbf{A}_{rr} + \mathbf{W}_p\mathbf{A}_x\mathbf{W}_p^T \\ & -\mathbf{A}_{re}\mathbf{A}_{ee}^{-1}\left(\mathbf{A}_{re}^T + \mathbf{W}_e\mathbf{A}_x\mathbf{W}_p^T\right) \\ & -\mathbf{W}_p\mathbf{A}_x\mathbf{W}_e^T\mathbf{A}_{ee}^{-1}\left(\mathbf{A}_{re}^T + \mathbf{W}_e\mathbf{A}_x\mathbf{W}_p^T\right)\end{aligned} \tag{47}$$

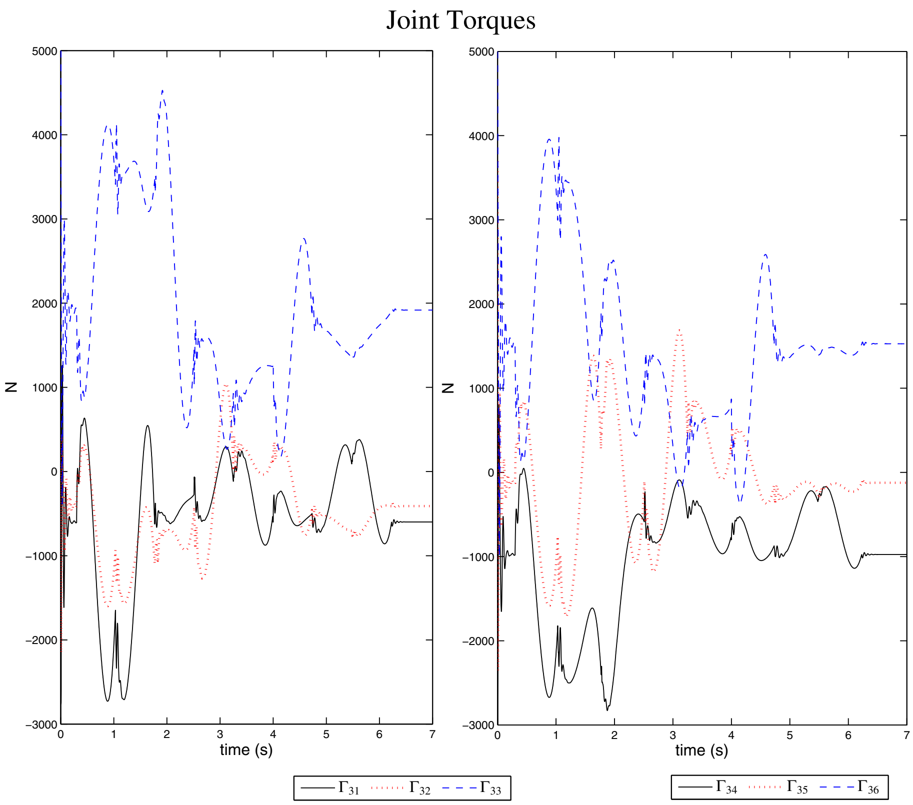

Fig. 4. Actuated prismatic joint forces.

where the 6 × 6 system Jacobian matrix is given by

$$\mathbf{J}_{sys}^{-T} = \mathbf{J}_p^{-T} - \left(\mathbf{A}_{re} + \mathbf{W}_p\mathbf{A}_x\mathbf{W}_e^T\right)\mathbf{A}_{ee}^{-1}\mathbf{J}_e^{-T}. \tag{48}$$

The 6 × 1 vector $\mathbf{c}$, the total Coriolis, centrifugal and gravity torques of the legs and the flexible platform, are given as:

$$\begin{aligned}\mathbf{c} = {} & \mathbf{c}_r + \mathbf{W}_p\mathbf{A}_x\mathbf{h} + \mathbf{W}_p\mathbf{c}_x \\ & -\left(\mathbf{A}_{re} + \mathbf{W}_p\mathbf{A}_x\mathbf{W}_e^T\right)\mathbf{A}_{ee}^{-1}(\mathbf{W}_e\mathbf{A}_x\mathbf{h} + \mathbf{W}_e\mathbf{c}_x + \mathbf{c}_e)\end{aligned} \tag{49}$$

where $\mathbf{c}_x$ is an 18 × 1 vector whose such that $\mathbf{c}_x = [c_{x1} \ldots c_{x6}]^T$.

5.1. Inverse dynamic problem

The inverse dynamic model of a parallel robot gives the actuated joint torques as a function of the desired trajectory of the platform frame and the current state of the robot. The main objective of this model is in non-linear control strategies, for instance the computed torque algorithm.

Inputs:
$\dot{\mathbf{V}}_p$: The desired rigid body Cartesian velocity of the platform.
$(\mathbf{q},\dot{\mathbf{q}}\ \mathbf{q}_e,\dot{\mathbf{q}}_e)$: The state of the robot i.e. the position and velocity of the joint and the generalized elastic variables respectively.
The joint positions and velocities are obtained from the platform variables using Eq. (11) and Eq. (18), followed by the inverse geometric and kinematic model of each leg, respectively. The generalized elastic variables are obtained by integration.

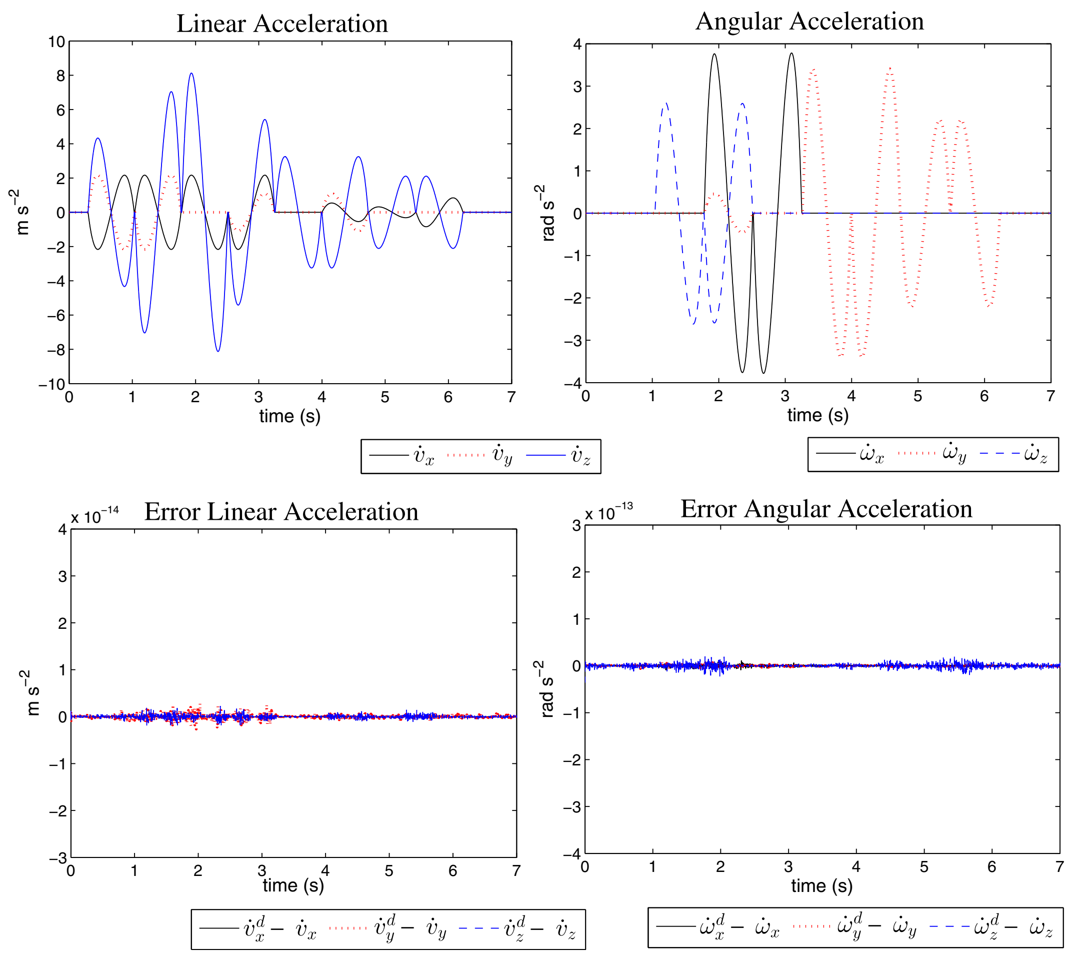

Fig. 5. Rigid body acceleration as calculated by direct dynamic model (top), difference between desired acceleration and current acceleration (bottom).

Outputs:
Γ: The vector of motor torques is obtained by solving Eq. (46), by using the Jacobian matrix of the system, $\mathbf{J}_{sys}^{T}$, after first obtaining the total inertia matrix from Eq. (48) and the total Coriolis, centrifugal and gravity torques from Eq. (49).
$\ddot{q}_e$: The generalized elastic accelerations of the platform, can be obtained using Eq. (44).

5.2. Direct dynamic problem

The direct dynamic model of the robot gives the platform accelerations and the accelerations of the generalized elastic coordinates as a function of the input torque of the motorized joints and the state of the robot, which consists of the positions and velocities of the rigid and elastic variables. The primary use of the direct dynamic model is the simulation of robotic systems.

Inputs:
Γ: The vector of motor torques.
$(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{q}_e, \dot{\mathbf{q}}_e)$: The state of the robot i.e. the position and velocity of the joint and the generalized position elastic variables respectively.
Outputs:
$\dot{\mathbf{V}}_p$: The rigid body Cartesian acceleration of the platform. It can be obtained by Eq. (46) after inverting the inertia matrix, $\mathbf{A}$, of the system. The inertia matrix, the vector containing the total Coriolis, centrifugal and gravity torques and the system Jacobian matrix are obtained in the same manner as in the inverse dynamic problem. Finally the rigid body velocity, $\mathbf{V}_p$, and the pose of the platform, ${}^{0}\mathbf{T}_p$, are obtained by integrating the rigid body Cartesian acceleration.
$\ddot{\mathbf{q}}_e$: The generalized elastic accelerations of the platform, can be obtained using Eq. (44).

6. Simulation

In order to validate the above model a simulation is carried out of the Gough–Stewart manipulator with flexible platform. The objective of the simulator is to follow a spatial trajectory. In the following, the steps taken to execute the simulation are described.

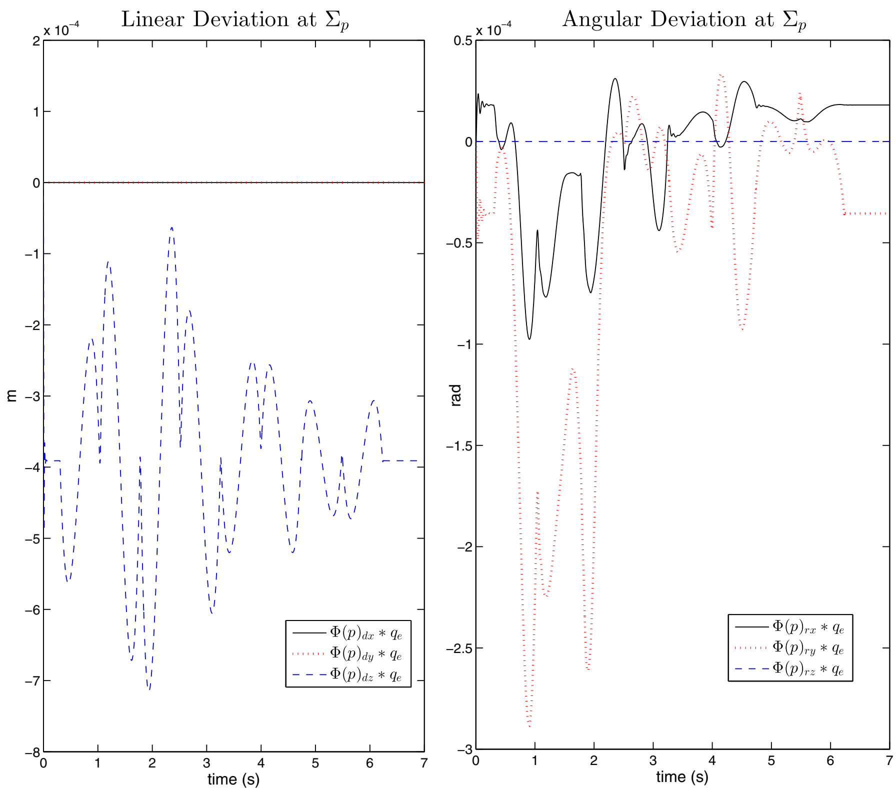

Fig. 6. Deformation at platform origin due to flexibility.

6.1. Simulation setup

The flexible platform is a regular hexagon that is inscribed in a circle of radius 0.5 m. The plate has a Young's Modulus of 2 $\times$ 10^{10} Nm, a uniform thickness 15 $\times$ 10^{-3} m, and a density of 7.5 $\times$ 10^3 kg/m^3. The total mass of the flexible platform is 73.069 kg. A modal analysis is carried out on the hexagonal plate using MSC Nastran©. The first ten non-rigid body modes are used to represent the flexibility of the platform. Table 2 contains the natural frequency, generalized stiffness and generalized damping of each mode. It should be noted that the modes are normalized such that the generalized mass, $\mathbf{m}_{ee}$, is the identity matrix.

The dynamic parameters of each leg are identical and are given in Table 3. The inertial parameters that are set to zero, either have no effect on the model or do not exist due to the symmetry of the links. The inertia tensor of link text j is given with respect to frame j as follows:

$$^{j}\mathbf{I}_j = \begin{bmatrix} XX_j & XY_j & XZ_j \\ XY_j & YY_j & YZ_j \\ XZ_j & YZ_j & ZZ_j \end{bmatrix}.$$

IA_j represents joint j's rotor inertia. The first moments of link j are calculated using the mass, denoted as Mj and the vector of center-of-mass coordinates denoted as Sj, as follows:

$$M\mathbf{S}_j = \begin{bmatrix} MX_j & MY_j & MZ_j \end{bmatrix}.$$

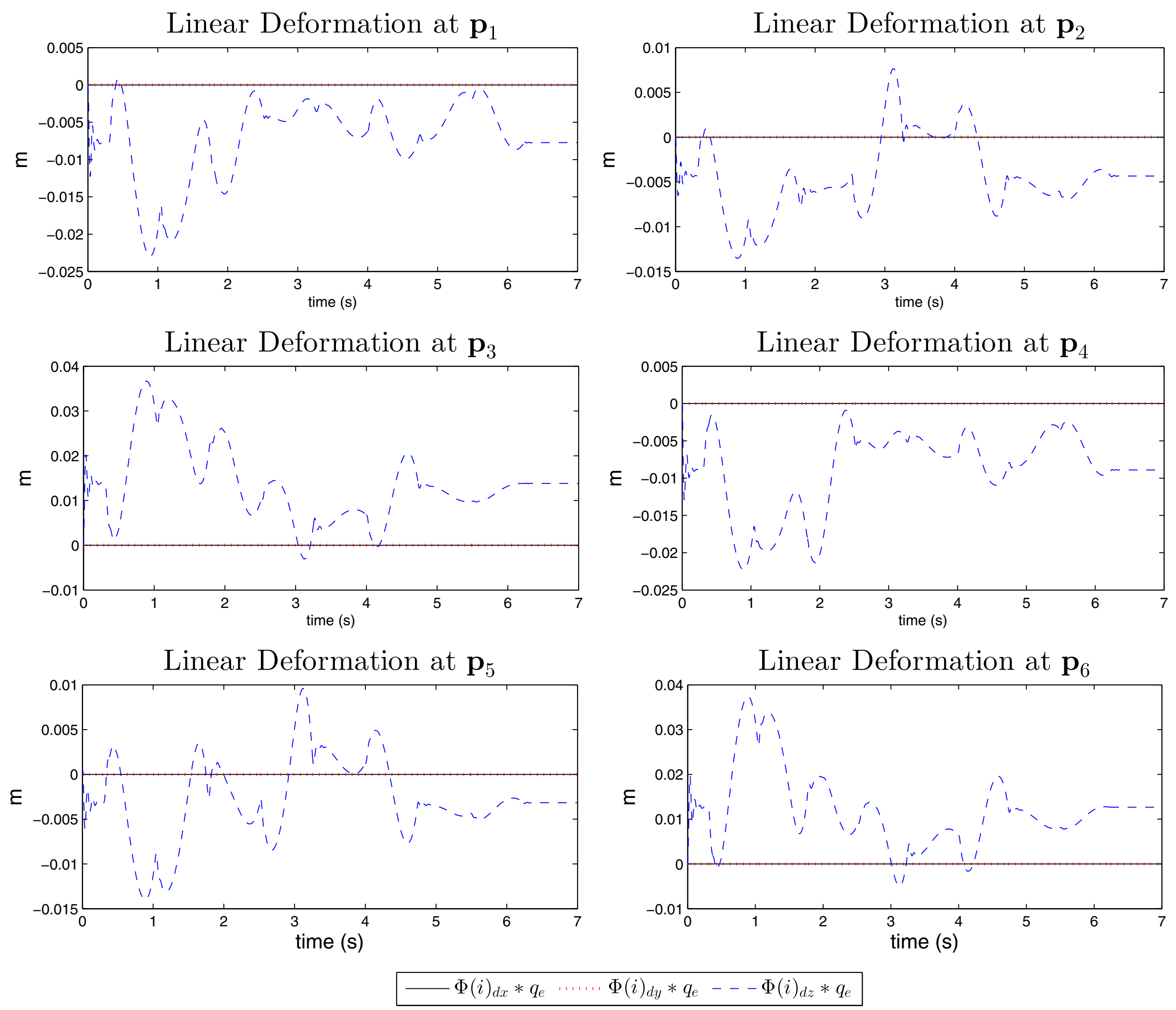

Fig. 7. Deformation at attachment points due to flexibility.

Table 4 gives the location of the base frame j, denoted as $\mathbf{b}_j$, for $j = 1 \ldots 6$ in the world frame. The components along z, $\mathbf{b}_j(z) = 0$, for all legs. Furthermore, in this table, the initial joint values of the robot are given.

6.2. Results

A spatial trajectory using 5*th* order polynomial interpolation, continuous in acceleration, is defined as the input to the system. Two rest periods are defined at the beginning and at the end of the trajectory, between which, the system must reach several poses that vary in both position and orientation.

The inverse dynamic model, denoted IDM, calculates the joint torques from the desired acceleration of the platform and the robot's state variables. The direct dynamic model, denoted DDM, is used to simulate the response of the platform by calculating the rigid body's acceleration and the acceleration of generalized elastic variables. From the acceleration, the state variables of the system are obtained via integration. An overview of the simulation is shown in Fig. 3.

From the inverse dynamic model the joint torques are calculated and are shown in Fig. 4.

The rigid linear and angular acceleration of the platform, as calculated by the direct dynamic model, are given in Fig. 5. In addition to this, Fig. 5 also shows the differences between the desired acceleration components obtained from the trajectory generator and the calculated accelerations, which are almost zero.

In Fig. 6, the linear and angular deformation of the platform at the platform origin ($\Phi_d(p)\mathbf{q}_e$ and $\Phi_r(p)\mathbf{q}_e$) are given. These variables are represented in the platform frame. The deformation is greatest in the z direction, which as shown in Fig. 1 corresponds to the normal of the platform plane. The angular deformation is greatest around the x and y axes.

Finally in Fig. 7, the linear deformation at each of the attachment points is given. This deformation is due to the free–free modeling strategy, and can be described as the deformation the platform undergoes to ensure the constraints at the attachment point are satisfied.

7. Conclusions

This work has presented a general strategy for modeling parallel manipulators with flexible platforms. The robot is decomposed into two subsystems, the first consisting of the rigid legs and base, the second of the flexible platform. The effects of the flexible subsystem on the rigid subsystem and vice versa are obtained by calculating the reaction forces at the connection points of the platform. A dynamic modeling equation is derived in terms of the Cartesian accelerations of the platform. The inverse dynamic model and the direct dynamic model can be calculated from this dynamic equation. A closed form solution, relating the torques to the Cartesian accelerations, is given in both cases in terms of the elastic and inertial parameters of the robot.

The model is validated using a numerical simulation of the Gough–Stewart manipulator with flexible platform, where the flexibility of the platform is represented by 10 modes. The simulation gives the output of the inverse and direct dynamic model while demonstrating the effects of flexibility on the platform.

In future work, using these models, control schemes can be constructed that would allow accurate positioning of a terminal frame while minimizing vibrations associated with the flexibility. Furthermore a complete description of the manipulator including the flexibility of the legs can be derived.

References

[1] K.-M. Lee, D.K. Shah, Dynamic analysis of a three-degrees-of-freedom in-parallel actuated manipulator, IEEE J. Robot. Autom. 4 (3) (1988) 361–367.

[2] Z. Geng, L.S. Haynes, J.D. Lee, R.L. Carroll, On the dynamic model and kinematic analysis of a class of Stewart platforms, Robot. Auton. Syst. 9 (4) (1992) 237–254.

[3] G. Lebret, K. Liu, F.L. Lewis, Dynamic analysis and control of a Stewart platform manipulator, J. Robot. Syst. 10 (5) (1993) 629–655.

[4] M. Ait-Ahmed, Contribution à la modélisation géométrique et dynamique des robots parallèles, Ph.D. thesis LAAS, Toulouse, 1993.

[5] S. Bhattacharya, H. Hatwal, A. Ghosh, An on-line parameter estimation scheme for generalized Stewart platform type parallel manipulators, Mech. Mach. Theory 32 (1) (1997) 79–89.

[6] S. Bhattacharya, D.N. Nenchev, M. Uchiyama, A recursive formula for the inverse of the inertia matrix of a parallel manipulator, Mech. Mach. Theory 33 (7) (1998) 957–964.

[7] M.-J. Liu, C.-X. Li, C.-N. Li, Dynamics analysis of the Gough–Stewart platform manipulator, IEEE Trans. Robot. Autom. 16 (1) (2000) 94–98.

[8] L.-W. Tsai, Solving the inverse dynamics of a Stewart–Gough manipulator by the principle of virtual work, J. Mech. Des. 122 (2000) 3.

[9] A. Codourey, E. Burdet, A body-oriented method for finding a linear form of the dynamic equation of fully parallel robots, Proceedings of the 1997 IEEE International Conference on Robotics and Automation, IEEE, vol. 2, 1997, pp. 1612–1618.

[10] S. Staicu, Dynamics of the 6–6 Stewart parallel manipulator, Robot. Comput. Integr. Manuf. 27 (1) (2011) 212–220.

[11] S. Staicu, Methodes matricielles en cinématique des mécanismes, UPB Scientific Bulletin, Mech. Eng. D, 62 (1), 2000, pp. 3–10.

[12] K. Sugimoto, Computational scheme for dynamic analysis of parallel manipulators, J. Mech. Des. Trans. ASME 111 (1989) 29–33.

[13] C. Reboulet, T. Berthomieu, Dynamic models of a six degree of freedom parallel manipulators, Fifth International Conference on Advanced Robotics, 'Robots in Unstructured Environments', ICAR 1991, IEEE, 1991, pp. 1153–1157.

[14] Z. Ji, Study of the effect of leg inertia in Stewart platforms, Proceedings of the 1993 IEEE International Conference on Robotics and Automation, IEEE, 1993, pp. 121–126.

[15] C.M. Gosselin, Parallel computational algorithms for the kinematics and dynamics of parallel manipulators, Proceedings of the 1993 IEEE International Conference on Robotics and Automation, IEEE, 1993, pp. 883–888.

[16] B. Dasgupta, T. Mruthyunjaya, A Newton–Euler formulation for the inverse dynamics of the Stewart platform manipulator, Mech. Mach. Theory 33 (8) (1998) 1135–1152.

[17] B. Dasgupta, P. Choudhury, A general strategy based on the Newton–Euler approach for the dynamic formulation of parallel manipulators, Mech. Mach. Theory 34 (6) (1999) 801–824.

[18] M. Carricato, C. Gosselin, On the modeling of leg constraints in the dynamic analysis of gough/stewart-type platforms, Journal of computational and nonlinear dynamics 4 (1).

[19] M. Afroun, A. Dequidt, L. Vermeiren, Revisiting the inverse dynamics of the Gough–Stewart platform manipulator with special emphasis on universal-prismatic-spherical leg and internal singularity, Proc. Inst. Mech. Eng. C J. Mech. Eng. Sci. 226 (10) (2012) 2422–2439.
[20] S. Fu, Y. Yao, Y. Wu, Comments on 'a Newton–Euler formulation for the inverse dynamics of the Stewart platform manipulator', Mech. Mach. Theory 42 (12) (2007) 1668–1671.
[21] M. Vakil, H. Pendar, H. Zohoor, Comments on 'closed-form dynamic equations of the general Stewart platform through the Newton–Euler approach' and 'a Newton–Euler formulation for the inverse dynamics of the Stewart platform manipulator', Mech. Mach. Theory 43 (10) (2008) 1349–1351.
[22] J. Gallardo, J. Rico, A. Frisoli, D. Checcacci, M. Bergamasco, Dynamics of parallel manipulators by means of screw theory, Mech. Mach. Theory 38 (11) (2003) 1113–1131.
[23] W. Khalil, S. Guegan, A novel solution for the dynamic modeling of Gough–Stewart manipulators, Proceedings of the 2002 IEEE International Conference on Robotics and Automation, IEEE, vol. 1, 2002, pp. 817–822.
[24] W. Khalil, O. Ibrahim, General solution for the dynamic modeling of parallel robots, J. Intell. Robot. Syst. 49 (1) (2007) 19–37.
[25] H. Abdellatif, B. Heimann, Computational efficient inverse dynamics of 6-dof fully parallel manipulators by using the lagrangian formalism, Mech. Mach. Theory 44 (1) (2009) 192–207.
[26] P. Mukherjee, B. Dasgupta, A. Mallik, Dynamic stability index and vibration analysis of a flexible Stewart platform, J. Sound Vib. 307 (3) (2007) 495–512.
[27] M. Mahboubkhah, M. Nategh, S.E. Khadem, A comprehensive study on the free vibration of machine tools' hexapod table, Int. J. Adv. Manuf. Technol. 40 (11–12) (2009) 1239–1251.
[28] O. Wallrapp, S. Wiedemann, Comparison of results in flexible multibody dynamics using various approaches, Nonlinear Dyn. 34 (1–2) (2003) 189–206.
[29] W. Khalil, M. Gautier, Modeling of mechanical systems with lumped elasticity, Proceedings of the 2000 IEEE International Conference on Robotics and Automation, IEEE, vol. 4, 2000, pp. 3964–3969.
[30] G. Rodriguez, Spatial operator approach to flexible manipulator inverse and forward dynamics, Proceedings of the 1990 IEEE International Conference on Robotics and Automation, IEEE, 1990, pp. 845–850.
[31] S. Briot, W. Khalil, Recursive and symbolic calculation of the elastodynamic model of flexible parallel robots, Int. J. Robot. Res. (2013). http://dx.doi.org/10.1177/0278364913507006.
[32] F. Boyer, W. Khalil, An efficient calculation of flexible manipulator inverse dynamics, Int. J. Robot. Res. 17 (3) (1998) 282–293.
[33] M. Rognant, E. Courteille, P. Maurine, A systematic procedure for the elastodynamic modeling and identification of robot manipulators, IEEE Trans. Robot. 26 (6) (2010) 1085–1093.
[34] W. Khalil, J. Kleinfinger, A new geometric notation for open and closed-loop robots, Proceedings of the 1986 IEEE International Conference on Robotics and Automation, IEEE, vol. 3, 1986, pp. 1174–1179.
[35] F. Boyer, W. Khalil, M. Benosman, G. Le Vey, Modeling and control of flexible robots, in: E. Dombre, W. Khalil (Eds.), Robot Manipulators: Modeling, Performance Analysis and Control, ISTE Ltd., Cambridge, Massachusetts, 2007, pp. 337–394, Ch. 7.