To generate identification data, we simulated this system using the feedback law

$$u(t) = r(t) - (-0.95q^{-2})y(t) = r(t) + 0.95y(t-2) \qquad (43)$$

which places the closed-loop poles in 0.8618 and 0.6382. In the simulation we used independent, zero-mean, Gaussian white noise reference and noise signals $\{r(t)\}$ and $\{e(t)\}$ with variances 1 and 0.01, respectively. $N = 200$ data samples were used.

In Table I, we have summarized the results of the identification, the numbers shown are the estimated parameter values together with their standard deviations. For comparison, we have, apart from the model structure (24), used a standard output error model model structure and a second-order ARMAX model structure. As can be seen, the standard output error model structure gives completely useless estimates, and the modified output error and the ARMAX model structures give similar and accurate results.

V. An Alternative Box–Jenkins Model Structure

The trick to include a modified noise model in the output error model structure is of course also applicable to the Box–Jenkins model structure. The alternative form will in this case be

$$y(t) = \frac{B(q)}{F(q)}\, u(t) + \frac{F_a^*(q)C(q)}{F_a(q)D(q)}\, e(t). \qquad (44)$$

An explicit expression for the gradient filters for this predictor can be derived similarly as in the output error case, albeit the formulas will be even messier. We leave the details to the reader.

VI. Conclusions

In this paper, we have proposed new versions of the well-known output error and Box–Jenkins model structures that can also be used for identification of unstable systems. The new model structures are equivalent to the standard ones, as far as number of parameters and asymptotical results are concerned, but guarantee stability of the predictors.

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Control of Nonlinear Chained Systems: From the Routh–Hurwitz Stability Criterion to Time-Varying Exponential Stabilizers

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***Abstract*—We show how any linear feedback that stabilizes the origin of a linear chain of integrators induces a simple, continuous time-varying feedback that exponentially stabilizes the origin of a nonlinear chained-form system. The design method is related to a method developed by M'Closkey and Murray to transform smooth feedback yielding slow polynomial convergence into continuous homogeneous ones that give exponential convergence.**

***Index Terms*—Asymptotic stability, nonholonomic system, time-varying feedback.**

I. Introduction

Control systems in the so-called *chained form* have been extensively studied in recent years. This research interest partly stems from the fact that the kinematic equations of many nonholonomic mechanical systems, such as these arising in mobile robotics (unicycle-type carts, car-like vehicles with trailers, etc.), can be converted into this form [12], [16], [18]. This paper addresses the problem of asymptotic stabilization of a given equilibrium point (which corresponds to a fixed configuration for a mechanical system).

Because chained systems do not satisfy Brockett's necessary condition [1], they cannot be asymptotically stabilized, with respect to any equilibrium point, by means of a continuous pure-state feedback $u(x)$. In [15], one of the authors proposed and derived smooth *time-varying* feedback laws $u(x,\, t)$ for the stabilization of a unicycle-type vehicle. This proposition showed how the topological obstruction raised by Brockett could be dodged and was the starting point of other studies on time-varying feedback. In [3] and [4], Coron established that most controllable systems can be asymptotically stabilized with this type of feedback. The literature on the subject has since then mostly focused on the explicit design of such stabilizing control laws. Smooth feedback laws yielding slow (polynomial) asymptotic convergence have first been designed (see, e.g., [13], [15]–[17], and [19]). More recently, properties associated with homogeneous systems have been used to obtain feedback laws only continuous but yielding an exponential convergence rate [7], [8], [10], [14].

Lately, M'Closkey and Murray have presented in [9] a method for transforming smooth time-varying stabilizers into homogeneous continuous ones. The method is best suited for driftless systems for which it applies systematically. The construction relies upon the initial knowledge of an adequate Lyapunov function coupled with a smooth stabilizing feedback law. More precisely, the exponential stabilizer is obtained by "scaling" the size of the smooth control inputs on a level set of the Lyapunov function. The feedbacks derived in the present paper have been obtained by adapting and combining the core of this method to the control design method earlier proposed by Samson in [16] for the smooth feedback stabilization of chained systems. Although our approach is specific to chained systems, it carries with it two important improvements with respect to [9]. The first one is that the knowledge

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of a (definite negative) Lyapunov function coupled with a smooth stabilizing feedback is not needed. This knowledge makes a significant difference because finding a "good" Lyapunov function for a chained system of order larger than three is not a simple task. The second improvement is related to the "scaling factor" used to transform the smooth feedback into a continuous exponentially stabilizing one. In [9], this factor is implicitly defined as the solution of an equation involving the considered Lyapunov function. Solving such an equation will usually have to be done numerically. The first feedback law proposed in the present study is of this type. We also show in a second result, however, that this scaling factor can be replaced by an adequate explicit function. The implementation of the resulting control law is consequently simplified.

This paper is organized as follows. In Section II, some results used further for the design of the control laws are recalled. The two main results and proposed control laws are presented in Section III in the form of two propositions. In the first one, the aforementioned scaling factor is still implicitly defined. The second proposition is an adaptation of the first one to get rid of the implicit definition of the scaling factor. Finally, a sketch of proof of these results is given in Section IV.

II. Preliminary Recalls

A. Stabilization of a Multi-Order Integrator and the Routh–Hurwitz Criterion

Consider the following linear $(n-1)$-order integrator $(d/dt)^{(n-1)}x_2 = u$ whose equivalent controllable state realization is:

$$\begin{cases} \dot{x}_i = x_{i+1}, & (i = 2, \cdots, n-1) \\ \dot{x}_n = u. \end{cases} \tag{1}$$

Any linear feedback control

$$u = -\sum_{i=2}^{n} a_i x_i \tag{2}$$

asymptotically stabilizes the origin of this system, provided all roots of the characteristic polynomial $p(s) = s^{n-1} + a_n s^{n-2} + \cdots + a_3 s + a_2$ associated with the closed-loop system have strictly negative real parts. The *Routh–Hurwitz table* associated with this polynomial is

$$\begin{array}{ccccc} 1 & a_{n-1} & a_{n-3} & \cdots & \cdots \\ a_n & a_{n-2} & a_{n-4} & \cdots & 0 \\ b_n & b_{n-2} & \cdots & \cdots & 0 \\ c_n & c_{n-2} & \cdots & \cdots & 0 \\ d_n & d_{n-2} & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 \\ \cdots & 0 & 0 & 0 & 0 \end{array}$$

with

$$\begin{aligned} a_k &= 0 \qquad \text{for } k < 2 \\ b_k &= -\frac{1}{a_n}(a_{k-2} - a_n a_{k-1}) \\ c_k &= -\frac{1}{a_n b_n}(a_n b_{k-2} - b_n a_{k-2}) \\ d_k &= -\frac{1}{b_n c_n}(b_n c_{k-2} - c_n b_{k-2}) \\ &\ \ddots \end{aligned}$$

Let $k \triangleq (k_2, \cdots, k_n)$ be defined from the first column of the Routh–Hurwitz table by

$$(k_n, k_{n-1}, k_{n-2}, \cdots) = (a_n, b_n, c_n, \cdots).$$

Then, we have the following two lemmas whose proofs can be found in several control textbooks (see [2], for example).

Lemma 1: Let $X_2 = (x_2, x_3, \cdots, x_n)^T$, and consider the linear change of coordinates ϕ: $X_2 \longmapsto Z_2 = (z_2, z_3, \cdots, z_n)^T = \Phi_k X_2$ defined by

$$\begin{aligned} z_2 &= x_2 \\ z_3 &= x_3 \\ z_{j+3} &= k_{j+1} z_{j+1} + L_f z_{j+2} \qquad (j = 1, \cdots, n-3) \end{aligned} \tag{3}$$

with $L_f \phi_i = (\partial \phi_i / \partial X_2) f$ the Lie derivative of ϕ_i along $f(X_2) = (x_3, x_4, \cdots, x_n, 0)^T$. Then, in the coordinates Z_2, the controlled system (1), (2) becomes

$$\begin{cases} \dot{z}_2 = z_3, \\ \dot{z}_{i+1} = -k_i z_i + z_{i+2} & (i = 2, \cdots, n-2) \\ \dot{z}_n = -k_{n-1} z_{n-1} - k_n z_n. \end{cases} \tag{4}$$

Using the fact that the time derivative of the quadratic function V_z defined by

$$\begin{aligned} V_z(Z_2) &\triangleq Z_2^T D_k Z_2 \\ D_k &\triangleq \mathrm{Diag}\left(\prod_{i=2}^{n-1} k_i, \prod_{i=3}^{n-1} k_i, \cdots, k_{n-1}, 1\right) \end{aligned} \tag{5}$$

along any solution of the system (4) is

$$\dot{V}_z(Z_2) = -2k_n z_n^2 \tag{6}$$

we easily establish the following Lemma.

Lemma 2: The origin $Z_2 = 0$ of the linear system (4) is asymptotically stable if and only if $k_i > 0$ for $i = 2, \cdots, n$.

A corollary of the above two lemmas is the well-known Routh–Hurwitz stability criterion.

Corollary 1 (Routh–Hurwitz): All roots of the polynomial $p(s) = s^{n-1} + a_n s^{n-2} + \cdots + a_3 s + a_2$ have strictly negative real parts if and only if $k_i > 0$ for $i = 2, \cdots, n$.

B. Nonexponential Time-Varying Feedback Stabilization of Chained Systems

The prime objective of the previous section was to point out the algebraic operations that transform the chain of integrators involved in the system (1), (2) into the *skew-symmetric* representation (4) to which the simple Lyapunov function (5) can be associated. In [16], the structural similitude between the linear n-order integrator system (1) and the following nonlinear chained system:

$$\begin{cases} \dot{x}_1 = u_1 \\ \dot{x}_i = u_1 x_{i+1}, & (i = 2, \cdots, n-1), \\ \dot{x}_n = u_2, \end{cases} \tag{7}$$

has been used, with the aforementioned transformations, to prove the following result.

Proposition 1 [16, Prop. 2.2]: Let $a_i\,(i = 2, \cdots, n)$ be a set of parameters for which the origin of the linear system (1), (2) is asymptotically stable. Then, the continuous time-varying feedback control

$$\begin{cases} u_1(x, t) = -k_1 x_1 + g(X_2) \sin t \\ u_2(x, t) = -u_1(x, t) \sum_{i=2}^{n} \text{sign}(u_1)^{n+1-i} a_i x_i \end{cases} \tag{8}$$

with $k_1 > 0$ and $g(X_2)$ a continuous function that vanishes at $X_2 = 0$ (i.e., $g(0) = 0$) and is strictly positive elsewhere, applied to the chained system (7):

1) makes the positive function

$$V_x(X_2) \equiv X_2^T \Phi_k^T D_k \Phi_k X_2 \qquad (= V_z(Z_2)) \tag{9}$$

nonincreasing along any solution of this system;

2) globally asymptotically stabilizes the origin $x = 0$ of this system.

This result clearly indicates how any linear stabilizing feedback for the linear $(n-1)$-order integrator system (1) induces a simple, continuous time-varying feedback law that asymptotically stabilizes the origin of the corresponding chained system (7). As pointed out in [16], however, a shortcoming of the feedback law (8) is that it yields slow (polynomial) asymptotic convergence to zero. The main contribution of this paper is to show how this time-varying control may itself be simply modified to make the controlled, chained system exponentially stable.

C. Homogeneity and Exponential Stabilization

The set of nonlinear systems homogeneous of degree zero with respect to some dilation constitutes a fairly natural extension of the set of linear systems. Some properties of these systems are briefly recalled hereafter. For more details, see, e.g., [5].

For any $\lambda > 0$ and any set of real parameters $r_i > 0\,(i = 1, \cdots, n)$, a "dilation" operator is a map $\delta(\lambda, .)$: $\mathbb{R}^n \longrightarrow \mathbb{R}^n$ defined by $\delta(\lambda, x_1, \cdots, x_n) = (\lambda^{r_1} x_1, \cdots, \lambda^{r_n} x_n)$.

A function $f \in C^0(\mathbb{R}^n \times \mathbb{R}; \mathbb{R})$ is homogeneous of degree $\tau \geq 0$ with respect to the (family of) dilation $\delta(\lambda, .$ if: $\forall\, \lambda > 0,\ f(\delta(\lambda, x), t) = \lambda^\tau f(x, t)$.

A *homogeneous norm* ρ associated with this dilation operator is a $\mathcal{C}^0$ function from $\mathbb{R}^n$ to $\mathbb{R}$, homogeneous of degree one with respect to the dilation, positive ($\rho(x) \geq 0, \forall\, x$), and proper. A consequence of this definition is that $\rho(x) = 0$ iff $x = 0$. An example of homogeneous norm is

$$\rho_p(x) = \left(\sum_{j=1}^{n} |x_j|^{p/r_j} \right)^{1/p}, \qquad \text{with } p > 0. \tag{10}$$

A differential system $\dot{x} = f(x, t)$ (or a vector field f), with $f \in C^0(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^n)$, is homogeneous of degree $\tau \geq 0$ with respect to the dilation $\delta(\lambda, .)$ if for any $i = 1, \cdots, n$, the ith component f_i of the vector field f is homogeneous of degree $\tau + r_i$.

Finally, let $f \in C^0(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^n)$, with $f(x, .)\ T$ periodic, define an homogeneous vector field of degree zero with respect to the dilation $\delta(\lambda, .)$. Then, the two following properties are equivalent:

1) the origin $x = 0$ of the system $\dot{x} = f(x, t)$ is asymptotically stable;

2) $x = 0$ is globally exponentially stable in the sense that there exist $\gamma > 0$ and, for any homogeneous norm ρ, a value K such that along any trajectory $x(t)\,(t \geq t_0)$ of the system $\dot{x} = f(x, t)$,

$$\rho(x(t)) \leq K \rho(x(t_0))\, e^{-\gamma(t-t_0)}.$$

III. MAIN RESULTS

Let us consider the chained system (7) and define a dilation $\delta_q(\lambda, X_2) = (\lambda^{r_2} x_2, \cdots, \lambda^{r_n} x_n)$ indexed by the integer $q \in \mathbb{N}$ via the *dilation weights* r_i chosen as follows:

$$r_i = n - i + q \qquad \text{for } i = 2, \cdots, n. \tag{11}$$

Let us also consider a set of parameters $a_i\ (i = 2, \cdots, n)$ chosen so the linear control (2) asymptotically stabilizes the origin of the linear system (1). The corresponding positive Routh–Hurwitz parameters are denoted as before as $k_i\ (i = 2, \cdots, n)$, and the matrix associated with the change of coordinates defined in Lemma 1 is again denoted as Φ_k. The first result involves a specific homogeneous norm ρ_q implicitly defined by

$$V_x(\delta_q(\rho_q(X_2)^{-1}, X_2)) = C, \qquad \forall\, X_2 \neq 0 \tag{12}$$

where C is a positive real number and V_x is the quadratic positive function introduced in Proposition 1. The next lemma asserts ρ_q is uniquely defined by the polynomial equation (12), provided q is chosen large enough.

Lemma 3: $q_0 > 1$ exists such that, for any $q \geq q_0$ ($q \in \mathbb{N}$),

1) $\forall\, X_2 \neq 0, \forall\, \lambda > 0,\ (\partial/\partial\lambda) V_x(\delta_q(\lambda, X_2)) > 0$, so the equation $V_x(\delta_q(\lambda, X_2)) = C$ admits a unique positive solution $\lambda(X_2)$;

2) the function ρ_q, from $\mathbb{R}^{n-1}$ to $\mathbb{R}^+$, defined by $\rho_q(X_2) = \lambda(X_2)^{-1}$ when $X_2 \neq 0$ and $\rho_q(0) = 0$, is smooth on $\mathbb{R}^{n-1} - \{0\}$ and homogeneous of degree one with respect to the dilation $\delta_q(\lambda, .)$.

We are now ready to state the first main result in the following proposition.

Proposition 2: The continuous time-varying feedback

$$\begin{cases} u_1(x, t) = -k_1 x_1 + \rho_q(X_2) \sin t, \qquad k_1 > 0,\ q \geq q_0 > 1 \\ u_2(x, t) = -u_1(x, t) \sum_{i=2}^{n} \left(\dfrac{\text{sign}(u_1)}{\rho_q(X_2)} \right)^{n+1-i} a_i x_i \end{cases} \tag{13}$$

applied to the chained system (7)

1) makes $\rho_q(X_2(t))$ nonincreasing along any solution of the controlled system;

2) globally exponentially stabilizes the origin $x = 0$ of this system.

Remark 1:

1) By imposing q to be larger than one, although the inverse of $\rho_q(X_2)$ is not defined for $X_2 = 0$, each term $x_i/\rho_q(X_2)^{n+1-i}$ involved in the control $u_2(x, t)$ is homogeneous of positive degree and tends to zero when $|X_2|$ tends to zero. Therefore, $u_2(x, t)$ is, by continuity, well defined on $\mathbb{R}^n \times \mathbb{R}$.

2) The condition imposed on the size of q_0 is directly related to the satisfaction of the *transversality condition* described in [9, Th. 4]. The connection appears explicitly in the proof of Proposition 2.

A practical difficulty with the control (13) is that the calculation of $\rho_q(X_2)$ requires solving the polynomial equation $V_x(\delta_q(\rho_q^{-1}, X_2)) = C$. In general, this will have to be done numerically. This difficulty, however, can be avoided by considering a homogeneous norm of the type (10), with the r_i's defined by (11), and using this function in the control expression, instead of $\rho_q(X_2)$. This statement is made precise in the following proposition, which is the second result of this paper.

Proposition 3: Let $\rho_{p,q}$ be the function on $\mathbb{R}^{n-1}$ defined by

$$\rho_{p,q}(X_2) = \left(\sum_{j=2}^{n} |x_j|^{p/r_j} \right)^{1/p} \qquad (r_j = n - j + q).$$

Then, $q_0 > 1$ exists such that, if $q \geq q_0$ and $p > n - 2 + q$, the continuous time-varying feedback control

$$\begin{cases} u_1(x, t) = -k_1 x_1 (\sin^2 t + \text{sign}(x_1) \sin t) \\ \qquad\qquad -k_{n+1} \rho_{p,q}(X_2) \sin t, \qquad k_1 > 0,\ k_{n+1} > 0 \\ u_2(x, t) = -u_1(x, t) \displaystyle\sum_{i=2}^{n} \left(\frac{\text{sign}(u_1)}{\rho_{p,q}(X_2)} \right)^{n+1-i} a_i x_i \end{cases} \tag{14}$$

applied to the chained system (7)

i) ensures along any solution of the controlled system,

$$V_x(Y_2((k+1)\pi)) \leq \alpha(q) V_x(Y_2(k\pi)) \qquad \forall k \in \mathbb{N} \tag{15}$$

with $\alpha(q) < 1$, and

$$Y_2 = \left(\frac{x_2}{\rho_{p,q}(X_2)^{n-2}}, \frac{x_3}{\rho_{p,q}(X_2)^{n-3}}, \cdots, \frac{x_{n-1}}{\rho_{p,q}(X_2)}, x_n \right)^T;$$

ii) globally exponentially stabilizes the origin $x = 0$ of this system.

Remark 2: The proof of this proposition uses the fact that $|u_1(x, t)| \geq k_{n+1}\rho_{p,q}(X_2)|\sin t|$, with the sign of $u_1(x, t)$ changing periodically as the sign of $\sin t$. Although the slightly more simple control $u_1(x, t) = -k_1 x_1 - k_{n+1}\rho_{p,q}(X_2) \sin t$ does not satisfy this inequality, we conjecture this control, combined with the control $u_2(x, t)$ of (14), also ensures the origin of the control system is *g.a.s.*

IV. Proofs of the Main Results

We report in this section the proofs of Propositions 2 and 3. For the sake of conciseness, the proofs of a few intermediary technical lemmas are omitted (they can be found in [11]).

A. Proof of Proposition 2

We first prove i). Let us assume $q > q_0$, so, according to Lemma 3, the equation $V_x(\delta_q(\lambda, X_2)) = C$ has a unique positive solution $\lambda(X_2)$ for any $X_2 \neq 0$. Differentiating with respect to time both members of the above equality, and denoting $W_2 = (w_2, \cdots, w_n) \triangleq \delta_q(\lambda(X_2), X_2)$, we obtain

$$\dot{\lambda} = -\left[\frac{\partial}{\partial \lambda} V_x(\delta_q(\lambda, X_2)) \right]^{-1} \frac{\partial V_x}{\partial x}(W_2) \delta_q(\lambda, \dot{X}_2). \tag{16}$$

In view of (7), (11), and (13),

$$\delta_q(\lambda, \dot{X}_2) = \lambda |u_1(x, t)| A_{\text{sign}(u_1)} W_2 \tag{17}$$

with $A_{\text{sign}(u_1)}$ one of the two matrices A_{+1} and A_{-1} defined by

$$A_{\text{sign}(u_1)} W_2 = \text{sign}(u_1) \left(w_3, \cdots, w_n, -\sum_{i=2}^{n} \text{sign}(u_1)^{n+1-i} a_i w_i \right)^T. \tag{18}$$

From (16) and (17), we deduce

$$\dot{\lambda} = -\lambda |u_1| \left[\frac{\partial}{\partial \lambda} V_x(\delta_q(\lambda, X_2)) \right]^{-1} \frac{\partial V_x}{\partial x}(W_2) A_{\text{sign}(u_1)} W_2. \tag{19}$$

Assume

$$\frac{\partial V_x}{\partial x}(W_2) A_{\text{sign}(u_1)} W_2 = -2k_n (\Phi_k W_2)_n^2, \tag{20}$$

for both values +1 and –1 of $\text{sign}(u_1)$. Then, (19) and (20) imply

$$\dot{\lambda} = 2\lambda k_n |u_1| \left[\frac{\partial}{\partial \lambda} V_x(\delta_q(\lambda, X_2)) \right]^{-1} (\Phi_k W_2)_n^2. \tag{21}$$

Because, from Lemma 3, $(\partial/\partial\lambda)V_x(\delta_q(\lambda, X_2)) > 0$ for any $X_2 \neq 0$, the above equality implies $\lambda(X_2(.))$ is nondecreasing—the inequality $(\partial/\partial\lambda)V_x(\delta_q(\lambda, X_2)) > 0$ corresponds to the transversality condition in [9]; this is the only place where this condition is used. There remains to prove (20). From Lemma 1, Φ_k is the matrix of the linear change of coordinates which transforms (1), (2) into (4). Therefore, we deduce from (6) and (9)

$$\begin{aligned} \frac{\partial V_x}{\partial x}(W_2) A_{+1} W_2 &= \frac{\partial V_z}{\partial z}(\Phi_k W_2) K \Phi_k W_2 \\ &= -2k_n (\Phi_k W_2)_n^2 \end{aligned} \tag{22}$$

with K the matrix associated with the right-hand side of (4). This proves (20) for $\text{sign}(u_1) = 1$. If $\text{sign}(u_1) = -1$, we consider the change of coordinates $\psi: \mathbb{R}^{n-1} \longrightarrow \mathbb{R}^{n-1}$ defined by $\psi_i(W_2) = (-1)^i w_i$ (remark that $\psi = \psi^{-1}$). It is simple to verify

$$\psi(A_{-1} W_2) = A_{+1} \psi(W_2). \tag{23}$$

Moreover, using the definition of the matrix Φ_k of change of coordinates introduced in Lemma 1, we can also verify

$$\Phi_k \psi(W_2) = \psi(\Phi_k W_2). \tag{24}$$

Using (22)–(24), and because, for any W_1 and W_2, $\psi^T(W_1) D_k \psi(W_2) = W_1^T D_k W_2$ because D_k is diagonal, we have

$$\begin{aligned} \frac{\partial V_x}{\partial x}(W_2) A_{-1} W_2 &= 2W_2^T \Phi_k^T D_k \Phi_k A_{-1} W_2 \\ &= 2\psi^T(\Phi_k W_2) D_k \psi(\Phi_k A_{-1} W_2) \\ &= 2\psi^T(W_2) \Phi_k^T D_k \Phi_k \psi(A_{-1} W_2) \\ &= 2\psi^T(W_2) \Phi_k^T D_k \Phi_k A_{+1} \psi(W_2) \\ &= \frac{\partial V_x}{\partial x}(\psi(W_2)) A_{+1} \psi(W_2) \\ &= -2k_n (\Phi_k \psi(W_2))_n^2 \\ &= -2k_n (\psi(\Phi_k W_2))_n^2 \\ &= -2k_n (\Phi_k W_2)_n^2. \end{aligned} \tag{25}$$

This proves (20) and completes the proof of i).

Part i) and (13) imply x_1 is bounded along the trajectories of the system, and because ρ_q is proper, all trajectories exist on $[0, +\infty)$ and are bounded. To show the asymptotic stability, we apply LaSalle's invariance principle [6] for time-periodic systems. First, remark, for $\nu > n - 2 + q$, ρ_q^ν is of class $\mathcal{C}^1$ on $\mathbb{R}^{n-1}$ because each partial derivative $(\partial \rho_q^\nu / \partial x_i)$ $(i = 2, \cdots, n)$ is homogeneous of degree $\nu - r_i > 0$, and therefore tends to zero as $|X_2|$ tends to zero. From what precedes, all solutions converge to the largest invariant set M contained in $E = \{x: (d/dt)\rho_q^\nu(X_2) = 0\}$. Let us show

$$M = \{(x_1, 0): x_1 \in \mathbb{R}\}. \tag{26}$$

Consider any solution $x(.)$ of the system contained in E, and assume $X_2(.)$ is not identically zero. Because $\rho_q(X_2(t))$ is constant and different from zero, this implies $X_2(t) \neq 0$ for all t. In view of (21) and Lemma 3, we deduce

$$|u_1(x(t), t)| (\Phi_k W_2)_n^2(t) \equiv 0. \tag{27}$$

From (13), $u_1(x(t), t)$ cannot be identically zero because ρ_q is constant and different from zero. Let (t_1, t_2) denote a nonempty time-interval on which $u_1(x(t), t) \neq 0$. Without loss of generality, we can assume $u_1(x(t), t)$ is positive on (t_1, t_2). Then, it comes that $(\Phi_k W_2)_n(t) = 0$ for $t \in (t_1, t_2)$. Because $W_2 = \delta_q(\lambda(X_2), X_2)$ and $\lambda(X_2(t))$ is constant, we deduce from (17) $\dot{W}_2 = \lambda u_1 A_{+1} W_2$, so

$$\begin{aligned}\Phi_k \dot{W}_2 &= \lambda u_1 \Phi_k A_{+1} W_2 \\ &= \lambda u_1 \Phi_k A_{+1} \Phi_k^{-1} \Phi_k W_2 = \lambda u_1 K \, \Phi_k W_2. \end{aligned} \quad (28)$$

Because both λ and u_1 do not vanish on (t_1, t_2), and in view of the structure of the matrix K [recall K is the matrix associated with the right-hand side of (4)], we easily show from (28) $(\Phi_k W_2)_n$ is identically zero on (t_1, t_2) only if all components of $\Phi_k W_2$ are identically zero, that is, if $X_2(.)$ is identically zero. This proves (26). From the expression of $u_1(x, t)$ and the system's equation $\dot{x}_1 = u_1$, $x_1(t)$ also converges to zero.

Finally, we easily show the closed-loop system (7)–(13) is homogeneous of degree zero with respect to the dilation $\overline{\delta}_q(\lambda, x) = (\lambda x_1, \delta_q(\lambda, X_2))$. In view of the results of Section II-C, this implies the exponential stability of this system. ■

B. Proof of Proposition 3

The proof uses the following Lemmas (see [11] for the proofs of these results).

Lemma 4: The map Y_2 defined by

$$X_2 \mapsto \left(\frac{x_2}{\rho_{p,q}(X_2)^{n-2}}, \frac{x_3}{\rho_{p,q}(X_2)^{n-3}}, \cdots, \frac{x_{n-1}}{\rho_{p,q}(X_2)}, x_n \right)^T$$

with $p > n-2+q$ and $q > 0$, is an homeomorphism on $\mathbb{R}^{n-1}$, and a C^1 diffeomorphism on $\mathbb{R}^{n-1} - \{0\}$ provided q is large enough. Moreover, $Y_2(X_2) = 0$ if and only if $X_2 = 0$, and $\lim_{|X_2| \to +\infty} |Y_2(X_2)| = +\infty$.

Lemma 5: Consider the function h defined by

$$h(u_1, X_2) = (u_1 x_3, u_1 x_4, \cdots, u_1 x_n, u_2(u_1, X_2))^T.$$

Two functions $\epsilon_{q,+1}$ and $\epsilon_{q,-1}$ in $\mathcal{C}^0(\mathbb{R}^{n-1} - \{0\}; \mathbb{R})$ exist such that

1) $\forall X_2 \neq 0$, $L_{h(u_1, X_2)} \rho_{p,q}(X_2) = |u_1| \epsilon_{q, \text{sign}(u_1)}(X_2)$;
2) $\lim_{q \to +\infty} \sup_{X_2 \neq 0} |\epsilon_{q,i}(X_2)| = 0$, $(i = +1, -1)$.

Lemma 6: Consider the system

$$\dot{y} = \gamma(t)(A + \epsilon(y)B)y \quad (29)$$

with $y \in \mathbb{R}^{n-1}$, $\gamma(.) \in \mathcal{C}^0(\mathbb{R}; \mathbb{R}^+)$, A a Hurwitz-stable matrix, $\epsilon \in \mathcal{C}^0(\mathbb{R}^{n-1} - \{0\}; \mathbb{R})$ bounded. Let P denote a symmetric positive-definite (s.p.d.) matrix such that $PA + A^T P \leq 0$ (such a matrix exists because A is stable), and $y(t)$ denote a maximal solution of (29).

Then, $\beta > 0$ exists such that the following property holds: for any function $\gamma_0(.) \in \mathcal{C}^0(\mathbb{R}; \mathbb{R}^+)$ strictly positive on some nonempty interval (t_1, t_2), and any $\delta \in (0, t_2 - t_1]$, $\alpha \in (0, 1)$ exists such that

$$\left. \begin{array}{c} \gamma(t) \geq \gamma_0(t), \forall t \in (t_1, t_2) \\ \|\epsilon\| \leq \beta \end{array} \right\} \Rightarrow \begin{cases} y(t)^T P y(t) \leq \\ (1-\alpha) y(t_1)^T P y(t_1) \\ \forall t \in [t_1 + \delta, t_2] \end{cases} \quad (30)$$

with $\|\epsilon\| \triangleq \text{Sup}\{|\epsilon(x)|: x \in \mathbb{R}^{n-1} - \{0\}\}$.

The proof of Proposition 3 involves two steps. First, we show, if q is large enough, any solution that crosses at some time the set $X_2 = 0$ (the same as the set $Y_2 = 0$, in view of Lemma 4) remains in this set ever after. For such a solution, the first state variable satisfies, after a finite time, the equation $\dot{x}_1 = -k_1 x_1 (\sin^2 t + \text{sign}(x_1) \sin t)$, and this implies $x_1(t)$ asymptotically converges to zero (see [10], for example). Therefore, the only solutions that may not converge to zero are those that never cross the set $X_2 = 0$. The second step of the proof thus consists in showing any of these solutions asymptotically converges to zero. Exponential stability then results because the controlled system is homogeneous of degree zero with respect to the dilation $\overline{\delta}_q(\lambda, x) = (\lambda x_1, \delta_q(\lambda, X_2))$.

Step 1: If $X_2(t) \neq 0$, the derivative of $Y_2(X_2)$ at time t is well defined and such that

$$\begin{aligned}\dot{Y}_2 = &\frac{u_1}{\rho_{p,q}} \left(y_3, \cdots, y_n, -\sum_{i=2}^{n} \text{sign}(u_1)^{n+1-i} a_i y_i \right) \\ &+ \frac{L_h \rho_{p,q}}{\rho_{p,q}} B Y_2 \end{aligned} \quad (31)$$

with $y_{i+1} = (x_{i+1}/\rho_{p,q}^{n-i-1})$ denoting the ith component of the vector Y_2, and $B = -\text{diag}(n-2, n-3, \cdots, 1, 0)$. In view of (18) and Lemma 5, we can rewrite (31) as

$$\dot{Y}_2 = \frac{|u_1|}{\rho_{p,q}(X_2)} (A_{\text{sign}(u_1)} + \epsilon_{q,\text{sign}(u_1)}(Y_2)B) Y_2. \quad (32)$$

Let us assume $X_2(t) \neq 0$ on some interval $[t_0, t_1)$. The function $u_1/\rho_{p,q}$ is well defined on this interval. Moreover, in view of the expression of the control $u_1(x, t)$

$$\gamma(t) \triangleq \frac{|u_1(x(t), t)|}{\rho_{p,q}(X_2(t))} \geq \gamma_0(t) \triangleq k_{n+1} |\sin t|, \quad (33)$$

with the sign of u_1 being the opposite of the sign of $\sin t$. The sign of u_1 thus changes periodically.

Because both A_{+1} and A_{-1} are Hurwitz-stable—from (23), the change of coordinates $Y_2 \longmapsto \psi Y_2$ transforms A_{-1} into A_{+1}—, s.p.d matrices $P_{\text{sign}(u_1)}$ and $Q_{\text{sign}(u_1)}$ exist such that, for each value of $\text{sign}(u_1)$, $P_{\text{sign}(u_1)} A_{\text{sign}(u_1)} + A_{\text{sign}(u_1)}^T P_{\text{sign}(u_1)} = -Q_{\text{sign}(u_1)}$. This readily implies the function $V_{+1}(Y_2) = Y_2^T P_{+1} Y_2$ (resp., $V_{-1}(Y_2) = Y_2^T P_{-1} Y_2$) is nonincreasing along the trajectories of system (32) for $\text{sign}(u_1) = 1$ and $\epsilon_{q,+1}$ bounded by a small enough value (resp. for $\text{sign}(u_1) = -1$ and $\epsilon_{q,-1}$ bounded by a small enough value). Let us now show, if $x(.)$ is a trajectory that crosses the set $X_2 = 0$ at some time t_0, $X_2(t) = 0$ for any $t \geq t_0$. Suppose on the contrary $|X_2(t_2)| > 0$ for some $t_2 > t_0$. For example, assume $X_2(t_2) > 0$. By continuity of $X_2(.)$, t_1 with $t_0 \leq t_1 < t_2$ exists such that

$$0 = X_2(t_1) < X_2(t), \qquad \text{for } t \in (t_1, t_2]. \quad (34)$$

By possibly decreasing the value of t_2, we can assume u_1 is of constant sign on $(t_1, t_2]$. We deduce from (34) $0 = V_{\text{sign}(u_1)}(Y_2(t_1)) < V_{\text{sign}(u_1)}(Y_2(t_2))$. This is clearly in contradiction with the fact that the function $V_{+1}(Y_2)$ (resp. $V_{-1}(Y_2)$) is nonincreasing on any set $Y_2 \neq 0$ and $\text{sign}(u_1) = 1$ (resp., $\text{sign}(u_1) = -1$).

Step 2: We now consider a solution such that $X_2(t) \neq 0, \forall t \geq 0$. From (20), $PA_{\text{sign}(u_1)} + A_{\text{sign}(u_1)}^T P \leq 0$, with $P = \Phi_k^T D_k \Phi_k$ the matrix associated with the function V_x. From Lemma 5, the upper bounds on the functions $\epsilon_{q,\text{sign}(u_1)}$ in (32) can be made arbitrary small by increasing q. By applying Lemma 5 to system (32) and because, from (33), $\gamma(t) > 0$ on any interval $(k\pi, (k+1)\pi)$, we deduce (using Lemma 5) the existence of $q_{\text{sign}(u_1)} > 0$ and $\alpha_{q_{\text{sign}(u_1)}} \in (0, 1)$ such that, for $q \geq q_{\text{sign}(u_1)}$,

$$\begin{aligned} &V_x(Y_2(k_{\text{sign}(u_1)}\pi)) \\ &\quad \leq (1 - \alpha_{q_{sign(u_1)}}) V_x(Y_2((k_{sign(u_1)} - 1)\pi)) \end{aligned} \quad (35)$$

with k_{+1} any even integer, and k_{-1} any odd integer. Property i) of Proposition 3 follows from (35) with $\alpha = \max(1 - \alpha_{q_{+1}}, 1 - \alpha_{q_{-1}})$ $(\in (0, 1))$, provided $q \geq \max(q_{+1}, q_{-1})$. This property, plus the proof of Step 1, clearly imply $Y_2(t)$ asymptotically converges to zero. The convergence of $x_1(t)$ to zero then easily follows from the first system's equation $\dot{x}_1 = u_1$ and the expression of the control $u_1(x, t)$ (see [10], for example). ■

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Fixed Zeros of Decentralized Control Systems

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***Abstract*—This paper considers the notion of *decentralized fixed zeros* for linear, time-invariant, finite-dimensional systems. For an N-channel plant that is free of unstable decentralized fixed modes, an unstable decentralized fixed zero of Channel i $(1 \le i \le N)$ is defined as an element of the closed right half-plane, which remains as a blocking zero of that channel under the application of every set of $N-1$ controllers around the other channels, which make the resulting single-channel system stabilizable and detectable. This paper gives a complete characterization of unstable decentralized fixed zeros in terms of system-invariant zeros.**

***Index Terms*—Decentralized control, fixed zeros, linear systems, stabilization.**

I. INTRODUCTION

The main objective of this paper is to give a definition and a characterization of unstable decentralized fixed zeros of a linear, time-invariant, finite-dimensional plant.

Consider the N-channel decentralized plant Z in Fig. 1, which is assumed to be free of unstable decentralized fixed modes [13]. Let $i \in \{1, \cdots, N\}$ be fixed. Assume, without loss of generality, $i = 1$. Let the closed-loop transfer matrix between u_1 and y_1 be denoted by $\hat{Z}_{11}$, where the dependence of $\hat{Z}_{11}$ on the controllers $Z_{c2}, \cdots, Z_{cN}$ is suppressed for simplicity.

An unstable decentralized fixed zero of Channel 1 is defined as an element of the closed right half-plane, which remains as a blocking zero [2], [3] of $\hat{Z}_{11}$ for the application of every collection of $N-1$ local controllers $Z_{c2}, \cdots, Z_{cN}$, which yield that the partially closed-loop system is stabilizable and detectable around Channel 1.

Decentralized fixed zeros deserve attention because of the performance limitations they impose on various sensitivity minimization problems, which can be explained by referring to Figs. 2 and 3, where $Z_{c1}, \cdots, Z_{cN}$ are local controllers to achieve two objectives: 1) closed-loop stability and 2) minimization of the H_∞ norm of the transfer matrix between w and z in Fig. 2.

In Fig. 2, the signal w is a noise affecting the first channel observation. In Fig. 3, the signal r is a reference signal to be tracked by the first channel output y_1. The transfer matrix between r and the error signal e is identical to the one between w and z in Fig. 2. It is easy to compute the transfer matrix between w and z (or the *sensitivity function around Channel 1*) equals $S := (I + \hat{Z}_{11}Z_{c1})^{-1}$. Let $Z_{c1}, Z_{c2}, \cdots, Z_{cN}$ be any collection of local controllers satisfying the closed-loop stability. From [8, Remark and Theorem 3.2] (see also Lemma 2 in the next section), the controllers $Z_{c2}, \cdots, Z_{cN}$ yield that the closed-loop system is stabilizable and detectable around Channel 1 in the partially closed-loop configuration of Fig. 1. Then, observe, at each unstable decentralized fixed zero s_0 of Channel 1, $\|S(s_0)\| = 1$, regardless of the controllers chosen. In other words, 1) the sensitivity of the closed-loop

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