Time-varying homogeneous feedback: design tools for the exponential stabilization of systems with drift

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Abstract

Over the past few years, important results have been obtained on time-varying feedback stabilization for strongly nonlinear systems. Research on this topic was motivated primarily by control problems for nonholonomic driftless mechanical systems —such as mobile robots. In this paper, we present recent results on *exponential* stabilization by time-varying homogeneous feedback, and investigate their applications to control systems with drift. Two general methods, originally developed for driftless systems, are presented. The first method relies on scaling asymptotically stabilizing feedback into exponential stabilizing feedback via a dilation. The second method applies a time-averaging technique to simplify the stability analysis of the closed-loop homogeneous system. These methods are demonstrated on two challenging stabilization problems: the planar rigid body with two forces and the spatial rigid body with one force and two torques.

1 Introduction

Renewed interest in using explicitly time-varying feedback for stabilizing certain nonlinear systems was prompted by the publication of a paper by Samson which showed how to construct time-periodic asymptotically stabilizing feedback laws for a nonholonomic cart [38]. The kinematic equations for this system failed to satisfy a necessary condition of Brockett for continuous stabilizability, i.e., stabilization of an equilibrium point with continuous feedback functions [5]. Since the class of driftless control systems —with fewer control inputs than state variables— suffered from the same obstruction to continuous stabilization, major research effort was directed toward deriving time-varying and continuous asymptotic stabilizers for these systems [9, 34, 39, 43, 44]. The analysis techniques used in these approaches, a combination of Lyapunov functions, center manifold theory and time averaging, produced differentiable control laws. A drawback of this important early research was the less-than-exponential convergence rate of the closed-loop system state to the desired equilibrium point. In a practical sense all that is required is convergence to a small neighborhood of the equilibrium point in a reasonable amount of time where the term "reasonable" is interpreted within the context of the application. However, simulations on many

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examples and experiments with actual devices revealed that the differentiable feedbacks often did not perform well.

Alternatives to this time-varying approach were also developed, using either discontinuous autonomous feedback, see e.g. [1, 6], or hybrid control laws, see e.g. [4, 40, 2]. The reader is referred to the comprehensive survey of Kolmanovsky and McClamroch in [17] for further references and results concerning these approaches.

In the system theoretic arena, Coron was proving results on what could be accomplished with continuous time-varying feedback. In particular, he gave general results on the global stabilization of controllable driftless systems [7], and, more recently, on the local stabilization of —a large class of— small time locally controllable systems [8]. Though the theory does not readily yield algorithms to apply to specific problems it encouraged the development of more design oriented methods.

Many results on homogeneous differential equations and control systems were pioneered by Hermes and Kawski. The paper [13] concisely lays out the theory of filtrations, dilations and homogeneous approximations of vector field sets and some specific examples are treated in [14, 15]. Kawski introduced a number of important definitions in [16], including the notion of exponential stability with respect to a homogeneous norm. Rosier proved a converse Lyapunov theorem for asymptotically stable homogeneous differential equations in [37]. Additional work was required to extend these results to the time-varying setting which is of interest to researchers in the driftless system community, and to develop design tools based on homogeneity properties.

M'Closkey and Murray investigated the application of homogeneous approximations and feedback to driftless systems in an effort to improve the rate of convergence of smooth feedbacks (references [21] to [24]). This motivated many other studies in this area [35, 30, 27, 12]. Homogeneous systems theory provides satisfying answers in this setting since an analytic and controllable driftless system in which the controls enter affinely, may always be approximated by a controllable nilpotent homogeneous system (this is a consequence of Proposition 2.3 in [13]). Many physical examples with nonholonomic constraints satisfy these conditions. Exponential stability (with respect to a homogeneous norm) of the approximation is systematically achieved with asymptotically stabilizing homogeneous feedbacks which render the closed-loop system homogeneous of degree zero. Even though it is generally difficult to specify *a priori* the rate of exponential convergence, the homogeneous feedbacks impart additional desirable properties to the closed-loop system including robustness to higher-order (with respect to a dilation) terms. Thus, stabilization of the approximation by homogeneous feedback ensures local stability of the original system. Another feature which has become apparent is the non-Lipschitz nature of the feedbacks required for exponential stabilization. The difficulties in analyzing the stability properties of these non-Lipschitz and time-varying systems has motivated the development of more analysis tools.

Researchers have recently extended ideas from driftless system control synthesis methods to systems with drift. For example, references [10, 29] address the exponential stabilization of the rigid body with two control torques with homogeneous feedback. In this paper we present two design methods for systems with drift that evolved from work originally developed for driftless systems. These methods do not capture all the work that has been done on time-varying stabilization by homogeneous feedback, however, they are general enough to apply to a large variety of systems —with or without drift. The first method is a Lyapunov approach that scales asymptotically stabilizing feedback into exponential stabilizers under certain conditions. The second method is based on time-averaging a time-periodic and homogeneous of degree zero system. Furthermore, we illustrate these methods on two problems: the stabilization of the planar rigid body with two control forces (a 6 dimensional system) and the stabilization of the spatial rigid body with one control force and two control torques (a 12 dimensional system). It should be emphasized that the tools presented here make the design task easier and less ad-hoc but designer intuition and expertise are still very important in producing successful solutions.

This paper is organized as follows. We recall in Section 2 some definitions and results about homogeneous systems. The two design methods are developed in Section 3, and illustrated on simple examples. The main applications are treated in Section 4. Finally, some concluding remarks are given.

2 Notation and Definitions

This section reviews dilations, homogeneous functions and vector fields, and introduces notation and several intermediate theorems required in the sequel.

The following average values are defined for any T-periodic functions α_1 and α_2,

$$\overline{\alpha_1} = \frac{1}{T}\int_0^T \alpha_1(t_1)\, dt_1\,, \qquad \overline{\alpha_2\overline{\alpha_1}} = \frac{1}{T}\int_0^T \int_0^{t_2} \alpha_2(t_2)\alpha_1(t_1)\, dt_1 dt_2.$$

This notation naturally extends to functions of several variables that are T-periodic with respect to one of their variables.

A dilation ${\Delta_\lambda}^r : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n$ is defined with respect to a fixed choice of coordinates $x = (x_1, x_2, \ldots, x_n)$ on $\mathbb{R}^n$ by assigning n positive rationals $r = (r_1, \ldots, r_n)$ and positive real parameter $\lambda > 0$ such that

$$\Delta_\lambda^r x = (\lambda^{r_1} x_1, \ldots, \lambda^{r_n} x_n), \quad \lambda > 0.$$

We often write Δ_λ in place of Δ_λ^r.

A continuous function $h : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is homogeneous of degree $l \geq 0$ with respect to Δ_λ if

$$h(\Delta_\lambda x, t) = \lambda^l h(x,t) \quad (\forall \lambda > 0\ , \forall (x,t)).$$

A homogeneous norm (with respect to the dilation Δ_λ) is any continuous, homogeneous of degree one, function that is strictly positive outside the origin. For example, one may take

$$\rho(x) = ||x_1|^{c/r_1} + |x_2|^{c/r_2} + \cdots + |x_n|^{c/r_n}|^{1/c}, \tag{1}$$

where c is some positive integer evenly divisible by r_i. Note that such an homogeneous norm satisfies the triangular inequality only when all weights are equal, i.e., generally it is not a norm in the classical sense. We are primarily interested in the convergence of time dependent functions using a homogeneous norm as a measure of their size.

A continuous vector field $f(x,t) = \sum a_i(x,t)\partial/\partial x_i + \partial/\partial t$ on $\mathbb{R}^n \times \mathbb{R}$ is homogeneous of degree d with respect to Δ_λ if a_i is degree $r_i + d$ for $i = 1, \ldots, m$. The variable t represents explicit time dependence and is not scaled. When a vector field is homogeneous it is most natural to use a corresponding homogeneous norm as the metric. The usual vector p-norms are homogeneous with respect to the standard dilation ($r_i = 1$). Lower case letters will be used to denote vector fields and functions and the interpretation that is desired should be apparent from the context.

The Euler vector field corresponding to a dilation Δ_λ is defined as

$$X_E(x) = \sum r_i x_i \frac{\partial}{\partial x_i}.$$

The images of trajectories of the system $\dot{x} = X_E(x)$ are the rays obtained by scaling the points on the sphere $\{x|\rho(x) = 1\}$ with the dilation.

Suppose the point $x = 0$ is an equilibrium point of the differential equation $\dot{x} = X(x,t)$. The concept of exponential stability of a vector field is now defined in the context of a homogeneous norm: the equilibrium point $x = 0$ is locally exponentially stable with respect to the homogeneous norm $\rho(\cdot)$ if there exist two constants $\gamma, K > 0$ and a neighborhood of the origin U such that any solution $x(t)$ with $x(0) \subset U$ satisfies

$$\rho(x(t)) \leq K\rho(x(0))e^{-\gamma t} \quad \forall t. \tag{2}$$

This stability type is denoted *ρ-exponential stability* to distinguish it from the usual definition of exponential stability. This definition was introduced by Kawski [16]. Note that this property does not depend on the particular choice of homogeneous norm (because two homogeneous norms w.r.t a given dilation are equivalent) but depends on the choice of dilation (i.e. on the weight vector r which defines this dilation). In the sequel the property of exponential stability for an homogeneous system will always mean ρ-exponential stability.

A result by Rosier establishes the existence of a homogeneous Lyapunov function for asymptotically stable homogeneous systems [37]. An extension of this result to time-periodic, homogeneous of degree zero systems is given in [35]. This converse Lyapunov theorem may be used to prove local asymptotic stability for an asymptotically stable degree zero system perturbed by higher order terms. In this case, however, the perturbing terms are bounded by the homogeneous norm. The local exponential stability of the detailed design examples in Section 4 rely on this result so we state it explicitly.

Theorem 1 *Let $x = 0$ be an asymptotically stable equilibrium point of the T-periodic continuous homogeneous of degree zero vector field $\dot{x} = f(x,t)$. Consider the perturbed system*

$$\dot{x} = f(x,t) + h(x,t). \tag{3}$$

Assume each component of $h(x,t)$ may be uniformly bounded by,

$$|h_i(x,t)| \leq K\rho^{r_i+\epsilon}(x) \quad i = 1,\ldots,n, \quad x \in U, \quad \epsilon > 0,$$

where U is an open neighborhood of the origin, ρ is any homogeneous norm, and K is some positive constant. Then $x = 0$ remains a locally exponentially stable equilibrium of the perturbed equation (3).

The proof follows the time-invariant case in [37].

Finally, the following theorem on stabilization of a system with additional integrators is useful for the design approaches presented in this paper. The result is due to Morin and Samson.

Theorem 2 ([29]) *Consider the system*

$$\dot{x} = f(x, v(x^{(m)},t),t), \tag{4}$$

where $f : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^n$ is of class C^0 and T-periodic w.r.t. t, $x^{(m)} := (x_1, \dots, x_m)$, $m \le n$, and $v : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ is of class C^0, T-periodic and differentiable w.r.t. t, of class C^1 on $(\mathbb{R}^m \setminus \{0\}) \times \mathbb{R}$, and homogeneous of degree q with respect to Δ_λ.

Assume further that (4) is homogeneous of degree zero with respect to the dilation Δ_λ and that the origin of this system is asymptotically stable.

Then, for sufficiently large $K > 0$, the origin $(x = 0, y = 0)$ of the system

$$\begin{aligned} \dot{x} &= f(x, y, t) \\ \dot{y} &= -K(y - v(x^{(m)}, t)) \end{aligned}$$

is asymptotically stable.

3 Synthesis tools for exponential stabilization of homogeneous systems

Consider the following control system

$$\dot{x} \;=\; b_0(x) + \sum_{i=1}^{m} u_i b_i(x)\,, \tag{5}$$

with b_i $(i = 0, \dots, m)$ smooth vector fields on $\mathbb{R}^n$. Throughout this section we make the following assumptions:

> Each b_i, $i = 1, \dots, m$ is homogeneous of degree $\deg(b_i) < 0$ with respect to a dilation Δ_λ and b_0 is homogeneous of degree zero with respect to the same dilation.

Equation (5) represents a very general class of control systems and our proposed techniques cannot address all stabilization problems for this class. Furthermore, there are instances where our approach is neither appropriate nor required. For example, if the linearization of (5) about an equilibrium point is controllable then there is no advantage to using the proposed techniques since linear state feedback can locally exponentially stabilize the nonlinear equations. However, if the linearization has uncontrollable modes at the origin then the results in this paper may be useful in deriving exponential stabilizers for the system. Within the set of systems with linearized uncontrollable modes at the origin we may also make a distinction between those systems which satisfy, and those which do not satisfy, Brockett's necessary condition for continuous asymptotic stabilization (see [5]). Some underactuated mechanical systems, rigid body stabilization, and mobile robot applications fall in the latter category. The techniques presented in this paper have found application to these systems.

The design tools we present constitute two different approaches. The first approach was originally developed by M'Closkey and Murray in [24] for the stabilization of driftless systems. It is relatively simple, however, to extend the basic idea to control systems with drift. In this approach, we suppose that a continuous, possibly time-varying, stabilizing feedback for (5) is known, together with an associated Lyapunov function —note that having a smooth (or even Lipschitz) controller in the situation when the linearization possesses an uncontrollable mode at zero excludes the possibility of exponential stability. Under a condition on the Lyapunov function (transversality condition) we show how to scale the original feedbacks so the closed loop system is homogeneous of degree zero and locally exponentially stable. This technique can be applied to systems with time-invariant or time-variant feedback. Other recent developments in this direction may be found in [36, 26].

The second approach was developed by the authors and colleagues in [22, 25] and [27], using important material from the works of Kurzweil and Jarnik [18], Sussmann and Liu [42] and Liu [20]. The basis for this technique is that of time-averaging and so it is immediately relevant to those systems which require explicit time dependence in continuous feedback functions to avoid violation of Brockett's condition. The time-averaging property states that if the frequency of the periodic terms increase for a homogeneous of degree zero system, then the system tends, in some sense, to an autonomous system. This property is exploited to simplify the task of designing asymptotically stabilizing feedbacks.

From a design point of view, these two approaches are quite different. In the first one, the difficulty consists of finding a smooth asymptotically stabilizing controller with a Lyapunov function, and especially in verifying that this function satisfies the transversality condition. In general, this condition gets more difficult to check as the system's dimension increases. Therefore, this approach might be difficult to apply for high dimensional systems (but this is not always the case, see e.g. [31]). A major advantage of this approach is that when this first step is achieved, the design of the exponential stabilizer is very simple and can be applied to both autonomous and non-autonomous feedback. In the second approach, we do not need an initial controller; the design relies primarily on the structure of the control Lie algebra. In this sense, this approach is much more intrinsic and self contained —for driftless systems, it applies systematically [27]. Also, having a high dimensional system does not *a priori* make the design significantly more difficult. However, this approach leads to extremely involved control laws when high-order Lie brackets of control vector fields are necessary to span the state space. Moreover, the fact that this approach is based on "high frequency averaging" often yields highly oscillatory trajectories. This may be viewed as undesirable for some applications.

3.1 Scaling control inputs to homogeneous functions

The synthesis result that relies on scaling feedback functions is now presented. The theorem is stated so as to be applicable to time-periodic feedback and time-periodic Lyapunov functions but applies equally to the case where there is no explicit time dependence in the control law or Lyapunov function.

Theorem 3 *Consider the system given in equation* (5) *with the dilation* Δ_λ *and its associated Euler vector field* X_E*. Assume there exist continuous, T-periodic feedback functions* $u_i(x,t)$: $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ *which asymptotically stabilize the origin of the system. If*

1. *there exists a* C^1*, positive definite, T-periodic function* $V(x,t)$ *such that* $\frac{d}{dt}V(x,t) \leq 0$ *along non-zero solutions of the closed-loop system, and*

2. *there exists* $C > 0$ *such that the level sets parameterized by* t,

$$G_t^C := \{x | V(x,t) = C\},$$

are bounded and satisfy the following transversality condition with the Euler vector field X_E,

$$L_{X_E} V(x,t) > 0 \quad x \in G_t^C,$$

then the original feedbacks may be modified to the following continuous ρ-exponentially stabilizing feedbacks:

$$\tilde{u}_i(x,t) = \tilde{\rho}^{-d_i}(x,t)u_i(\gamma_t(x),t) \qquad d_i := \mathit{deg}(b_i)\,, \quad i=1,\ldots,m,$$

where $\tilde{\rho}:\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}^+$ is a uniquely defined homogeneous of degree one function such that

$$\tilde{\rho}(x,t)|_{x\in G_t^C} = 1,$$

and the map $\gamma_t : (\mathbb{R}-\{0\}) \to G_t^C$ is defined as

$$\gamma_t(x) = \Delta_\lambda x = \overline{x} \in G_t^C \qquad \textit{for some } \lambda > 0.$$

(Proof in Appendix)

This theorem, originally developed for scaling asymptotic stabilizers into exponential stabilizers for driftless systems, can be immediately applied to asymptotically stabilized systems with drift provided the drift vector field is degree zero. The planar rigid body example in Section 4 is exponentially stabilized with this method. However, the following low dimensional example illustrates the essential features of the algorithm even though the system is "driftless."

Example 1. Consider the three-dimensional two input driftless system,

$$\dot{x}_1 = u_1, \quad \dot{x}_2 = u_2, \quad \dot{x}_3 = x_2u_1. \tag{6}$$

The control vector fields associated with this system are homogeneous of degree minus one with respect to the dilation $\Delta_\lambda(x) = (\lambda x_1, \lambda x_2, \lambda^2 x_3)$. This example does not have a drift vector field but nicely illustrates the process of checking the transversality condition and scaling the feedback. A smooth asymptotically stabilizing feedback for the system taken from [43] are the functions

$$u_1(x,t) = -x_1 + x_3\cos t, \tag{7}$$
$$u_2(x,t) = -x_2 + x_3^2\sin t. \tag{8}$$

Asymptotic stability of the closed-loop system can be shown using the Lyapunov function

$$V(x,t) = \Big(x_1 - \frac{x_3}{2}(\cos t + \sin t)\Big)^2 + \Big(x_2 - \frac{x_3^2}{2}(\sin t - \cos t)\Big)^2 + x_3^2.$$

Boundedness of the level sets V follows from the fact that V may be approximated by the quadratic form $\tilde{V} = \langle x, Bx\rangle$ for $\|x\|$ sufficiently small, where

$$B = \begin{bmatrix} 1 & 0 & -\frac{1}{2}\alpha \\ 0 & 1 & 0 \\ -\frac{1}{2}\alpha & 0 & 1+\frac{1}{4}\alpha^2 \end{bmatrix},$$

and $\alpha = \cos t + \sin t \in [-\sqrt{2}, \sqrt{2}]$. The transversality condition is satisfied for $\|x\|$ sufficiently small since the Lie derivative of $\tilde{V}$ with respect to the Euler vector field is $L_{X_E}\tilde{V} = \langle x, \tilde{B}x\rangle$, where $\tilde{B}$ is the symmetric matrix

$$\tilde{B} = \begin{bmatrix} 2 & 0 & -\frac{3}{2}\alpha \\ 0 & 2 & 0 \\ -\frac{3}{2}\alpha & 0 & 4+\alpha^2 \end{bmatrix}.$$

Since $\tilde{B}$ is positive definite for all $\alpha \in [-\sqrt{2}, \sqrt{2}]$ the Euler vector field is transverse to any level set of $\tilde{V}$ and hence any level set of V for $\|x\|$ sufficiently small. A bisection routine may be used to calculate the required λ since $V(\Delta_\lambda x, t)$ is a monotone increasing function of λ in a neighborhood of the origin. Once the value of λ has been computed which satisfies $V(\Delta_\lambda x, t) = 1$ then we set $\tilde{\rho}(x,t) = 1/\lambda$ and $\overline{x} = \gamma(x,t) = \Delta_\lambda x$. The modified feedbacks are

$$\begin{aligned}
\tilde{u}_1(x,t) &= \frac{1}{\lambda}\left(-\overline{x}_1 + \overline{x}_3 \cos t\right) \\
&= \frac{1}{\lambda}\left(-\lambda x_1 + \lambda^2 x_3 \cos t\right) \\
&= -x_1 + \lambda x_3 \cos t \\
\tilde{u}_2(x,t) &= \frac{1}{\lambda}\left(-\overline{x}_2 + \overline{x}_3^2 \sin t\right) \\
&= \frac{1}{\lambda}\left(-\lambda x_2 + \lambda^4 x_3^2 \sin t\right) \\
&= -x_2 + \lambda^3 x_3^2 \sin t.
\end{aligned}$$

The modified feedback is not computed explicitly in most circumstances but must be implemented in real-time. Experimental results demonstrating the feasibility of this approach are given in [24, 23]. Alternatively, a stable ODE can be used to track λ as pointed out by Praly [36].

3.2 Time-periodic averaging applied to stabilization

We present the design tool based on averaging of time-dependent ODE's. The following theorem combines results from [22, 25] and [27]. For the sake of simplicity, we only consider approximations of Lie-brackets of order two in this paper —see system (10) below. This is sufficient for the applications we want to consider in this paper. Extensions to higher-order brackets can be obtained using the algorithm in [27]. We also generalize these existing results by showing averaging properties for vector fields that are homogeneous of *non-zero degree*. This extension, in combination with Theorem 3, will prove useful in Section 4.

Theorem 4 *Consider the ODE*

$$\dot{x} = f_0(x, t/\epsilon) + \sum_{i=1}^{I} \epsilon^{-\frac{1}{2}} \alpha_i(t/\epsilon) f_i(x) \tag{9}$$

with $f_0(x,.)$ and α_i some T-periodic functions, and $\overline{\alpha_i} = 0$ for $i = 1, \ldots, I$. Assume also that

1. *f_0 is of class $\mathcal{C}^0$, $\mathcal{C}^1$ (resp. $\mathcal{C}^2$) w.r.t x on $(\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}$ if all α_i's are identically zero (resp. if some α_i is different from zero), and homogeneous of degree $d \geq 0$ w.r.t a dilation Δ_λ.*

2. *for any $i = 1, \ldots, I$, f_i is of class $\mathcal{C}^0$, $\mathcal{C}^2$ with respect to x on $(\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}$ and homogeneous of degree $d/2$ w.r.t. Δ_λ.*

3. *the origin of the "averaged system"*

$$\dot{y} = \overline{f}_0(y) + \sum_{i=1}^{I} \sum_{j>i} \overline{\alpha_j \overline{\alpha_i}}\, [f_i, f_j](y) \tag{10}$$

is asymptotically stable.

Then,

i) if $d = 0$, there exists $\epsilon_0 > 0$ such that the origin of (9) is (globally) exponentially stable for $0 < \epsilon < \epsilon_0$.

ii) if $d > 0$, the origin of (9) is locally asymptotically stable for any $\epsilon > 0$.

iii) let V_a be a homogeneous Lyapunov function for (10) and V be defined by

$$\begin{aligned} V(x,t) = \; & V_a(x) + \epsilon L_{\int_0^{t/\epsilon} \overline{f_0}(x) - f_0(x,\tau)\, d\tau} V_a(x) - \sum_{i=1}^{I} \epsilon^{\frac{1}{2}} \left(\int_0^{t/\epsilon} \alpha_i(\tau)\, d\tau \right) L_{f_i} V_a(x) \\ & - \sum_{i=1}^{I} \sum_{j=1}^{I} \epsilon \left(\int_0^{t/\epsilon} \overline{\alpha_j \overline{\alpha_i}} - \alpha_j(\tau_2) \int_0^{\tau_2} \alpha_i(\tau_1)\, d\tau_1 d\tau_2 \right) L_{f_j} L_{f_i} V_a(x)\,. \end{aligned} \tag{11}$$

Then, if all terms in the right-hand side are $\mathcal{C}^1$ on $(\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}$ —for instance if V_a is $\mathcal{C}^3$—, V is a global (resp. local) Lyapunov function for (9) for ϵ small enough if $d = 0$, (resp. for any $\epsilon > 0$ if $d > 0$).

(Proof in Appendix)

We now illustrate this theorem on two simple examples.

Example 1.(continued) Consider again system (6) and denote b_1 and b_2 the two vector fields: $b_1(x) = (1, 0, x_2)^T$, $b_2(x) = (0, 1, 0)^T$. Since $[b_1, b_2](x) = (0, 0, -1)^T$, the "controllability matrix" $(b_1(x), b_2(x), [b_1, b_2](x))$ is full rank everywhere. Let ρ be any homogeneous norm, smooth outside the origin (for instance $\rho(x) = (x_1^4 + x_2^4 + x_3^2)^{\frac{1}{4}}$). We consider two functions u_i^ϵ $(i = 1, 2)$ of the following form:

$$u_i^\epsilon(x,t) = u_{i,1}(x) + \epsilon^{-\frac{1}{2}} \alpha_i(t/\epsilon) u_{i,2}(x)\,, \quad (i = 1, 2)\,, \tag{12}$$

with α_1 and α_2 defined by

$$\alpha_1(t) = \sin t,\; \alpha_2(t) = -2\cos t\,, \tag{13}$$

so that $\overline{\alpha_2 \overline{\alpha_1}} = 1$. It is very easy to verify that the controlled system (6)-(12) is of the form (9) and that the corresponding averaged system (10) is given by

$$\begin{aligned} \dot{y} &= u_{1,1}(y) b_1(y) + u_{2,1}(y) b_2(y) + [b_1 u_{1,2}, b_2 u_{2,2}](y) \\ &= (u_{1,1} - L_{b_2 u_{2,2}} u_{1,2})(y) b_1(y) + (u_{2,1} + L_{b_1 u_{1,2}} u_{2,2})(y) b_2(y) + (u_{1,2} u_{2,2})(y) [b_1, b_2](y)\,. \end{aligned} \tag{14}$$

Let $u_{i,k}$ $(i, k = 1, 2)$ be defined by

$$\begin{aligned} & u_{1,2}(y) \; = \; \rho(y) \\ & ((u_{1,1} - L_{b_2 u_{2,2}} u_{1,2}), (u_{2,1} + L_{b_1 u_{1,2}} u_{2,2}), (u_{1,2} u_{2,2}))^T (y) = \\ & -(b_1(y), b_2(y), [b_1, b_2](y))^{-1}\, y \; = \; -(y_1, y_2, y_1 y_2 - y_3)^T\,. \end{aligned} \tag{15}$$

System (14) is then simply defined by $\dot{y} = -y$. Since this system is clearly stable, (15), (13), and (12) give the following feedback

$$\left\{ \begin{array}{lcl} u_1^\epsilon(x,t) & = & -x_1 + \dfrac{x_2^3(x_3 - x_1x_2)}{\rho^4(x)} + \epsilon^{-\frac{1}{2}} \sin(t/\epsilon)\, \rho(x) \\ u_2^\epsilon(x,t) & = & -x_2 + \dfrac{x_3 - x_1x_2}{\rho^4(x)}(x_1^3 + \dfrac{x_2x_3}{2}) - 2\epsilon^{-\frac{1}{2}} \cos(t/\epsilon)\, \dfrac{x_3 - x_1x_2}{\rho(x)}\,, \end{array} \right. \tag{16}$$

which, by application of Theorem 4, exponentially stabilizes the origin of (6) for ϵ small enough.

Example 2. The previous example shows how, from the knowledge of the controllability Lie algebra, some stabilizing feedbacks may be derived by application of Theorem 4. Another important application of Theorem 4, for which the role of the controllability Lie algebra is not completely obvious, is to non-affine control systems. Consider the following system in $\mathbb{R}^3$:

$$\begin{array}{l} \dot{x} \,=\, u_1b_1(x) + u_2b_2(x) + u_1u_2b_3(x) \\ b_1(x) = (1,0,0)^T,\ b_2(x) = (0,1,0)^T,\ b_2(x) = (0,0,1)^T\,, \end{array} \tag{17}$$

where b_1, b_2, and b_3 are homogeneous of degree -1, -1, and -2, respectively, with respect to the dilation with weight vector $r = (1,1,2)$. We define

$$\left\{ \begin{array}{lcl} u_1^\epsilon(x,t) & = & -x_1 + \sin(t/\epsilon)\, \rho(x) \\ u_2^\epsilon(x,t) & = & -x_2 - 2\sin(t/\epsilon)\, \dfrac{x_3}{\rho(x)}\,. \end{array} \right. \tag{18}$$

Let

$$u_3^\epsilon(x,t) := u_1^\epsilon(x,t)u_2^\epsilon(x,t) = x_1x_2 + \sin(t/\epsilon) \left(-x_2\rho(x) + 2\frac{x_1x_3}{\rho(x)} \right) - 2\sin^2(t/\epsilon)\, x_3\,. \tag{19}$$

Then, we can rewrite the controlled system (17)-(18) as the system $\dot{x} = u_1b_1(x) + u_2b_2(x) + u_3b_3(x)$ controlled by the feedback law $u^\epsilon(x,t) = (u_1^\epsilon(x,t), u_2^\epsilon(x,t), u_3^\epsilon(x,t)$. It is easy to verify that this system is of the form (9), and that the corresponding averaged system (10) is given by

$$\dot{y} = (-y_1, -y_2, -y_3 + y_1y_2)^T\,. \tag{20}$$

Since the origin of this system is clearly asymptotically stable, we deduce that the controlled system (17)-(18) is exponentially stable for ϵ sufficiently small.

4 Applications

In this section, we apply the results of Section 3 to the stabilization of two mechanical systems: the planar rigid body with two thrusters (two forces), and the spatial rigid body with three thrusters (one force and two torques). Until now, most of the literature on time-varying feedback stabilization has focused on driftless systems (in particular kinematic models of nonholonomic mobile robots). The methods of Section 3 can be applied to the stabilization of such systems, however, we consider some applications to systems with drift since the stabilization of these systems is still a very open problem. We hope to demonstrate that the techniques proposed in this paper have a broad field of application as well as motivate further studies in the development of these methods. The planar example is exponentially stabilized using the scaling procedure from Theorem 3 and the spatial rigid body is exponentially stabilized using the averaging approach for degree zero systems advocated in Theorem 4.

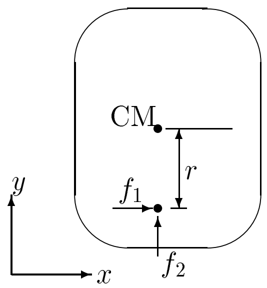

Figure 1: Planar rigid body model with body-fixed forces f_1 and f_2 and mass center at CM.

4.1 Exponential stabilization of the planar rigid body with two thrusters via input scaling

The planar rigid body shown in Figure 1 is a six state dynamic system. The two inputs are forces f_1 and f_2 whose line of action are fixed with respect to the body. The position of the mass center CM is given by (x, y) and the orientation of the body with respect to the horizontal is θ. The equations of motion are

$$\begin{aligned} M\ddot{x} &= f_1 \cos\theta - f_2 \sin\theta \\ M\ddot{y} &= f_1 \sin\theta + f_2 \cos\theta \\ J\ddot{\theta} &= r f_1, \end{aligned} \tag{21}$$

where M is the mass, J the moment of inertia, and r distance between the center of mass and the point where the forces act. Without loss of generality we choose these constants to be unity.

In order to apply the scaling theorem from Section 3.1 we require an asymptotically stabilizing control law for (21). As previously mentioned, this aspect of the design can be quite involved, however, once a stabilizer is found the transversality condition in Theorem 3 can be verified and, if true, the derivation of the exponential stabilizer is routine.

The steps in determining an exponential stabilizer are summarized for benefit of the reader:

1. First approximate (21) with a system of the form (5), i.e., define a dilation for which b_0 is degree zero and b_1 and b_2 are degree -1. This requires a coordinate change.
2. Stabilization of a four dimensional system may be considered since the high-gain feedback result of Theorem 2 will be used to eliminate two states from the six dimensional approximation. The stabilization of the four dimensional system requires several intermediate steps including derivation of a continuous asymptotic stabilizer and Lyapunov function.
3. The Lyapunov function for the stabilized four dimensional system is shown to satisfy the transversality condition. The feedbacks are scaled to exponential stabilizers.
4. The original equations in (21) are stabilized by combining the high gain feedback solution with the exponential stabilizers for the four dimensional system.

4.1.1 Homogeneous approximation of planar system

Define the following global coordinate change:

$$\begin{aligned} x_1 &:= \theta & x_4 &:= \dot{\theta} \\ x_2 &:= y & x_5 &:= \dot{y} \\ x_3 &:= \theta - x & x_6 &:= \dot{\theta} - \dot{x} - \theta\dot{y}. \end{aligned}$$

The system (21) is represented in these variables as

$$(\dot{x}_1, \dot{x}_2, \dot{x}_3, \dot{x}_4, \dot{x}_5, \dot{x}_6) = (x_4, x_5, x_6 + x_1 x_5, f_1, f_2 + x_1 f_1 + \mathcal{O}(3), -x_4 x_5 + \mathcal{O}(3)), \tag{22}$$

where, treating f_1 and f_2 as variables, $\mathcal{O}(3)$ represents order three and higher terms.

The next step is to identify a dilation for this system. Let $(r_1, r_2, r_3, r_4, r_5, r_6) := (1, 1, 2, 1, 1, 2)$ so that the dilation is

$$(x_1, x_2, x_3, x_4, x_5, x_6) \mapsto (\lambda x_1, \lambda x_2, \lambda^2 x_3, \lambda x_4, \lambda x_5, \lambda^2 x_6). \tag{23}$$

This dilation is suggested by the fact that the inputs directly force x_4 and x_5 and, since f_1 and f_2 will be degree one functions in the final closed-loop system, then x_4 and x_5 should have weight one. This choice induces weight one for x_1 and x_2 and weight two for x_6 and x_3 (the lowest order term in the $\dot{x}_6$ equation is $x_4 x_5$ with respect to this dilation). The $\mathcal{O}(2)$ terms in the $\dot{x}_5$ equation and $\mathcal{O}(3)$ terms in the $\dot{x}_6$ equation are higher order with respect to the dilation and the leading order homogeneous approximation is

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{pmatrix} = \underbrace{\begin{pmatrix} x_4 \\ x_5 \\ x_6 \\ 0 \\ 0 \\ -x_4 x_5 \end{pmatrix}}_{=:b_0(x)} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{=:b_1(x)} f_1 + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{=:b_2(x)} f_2. \tag{24}$$

Note that b_0 is degree 0 with respect to the dilation and b_1 and b_2 are both degree -1. If f_1 and f_2 are homogeneous degree one functions such that the closed-loop system in (24) is exponentially stable then these feedbacks will also locally exponentially stabilize (22) and hence the original system in (21). System (24) does not satisfy Brockett's necessary condition for continuous feedback stabilizability [5], however, we will construct a continuous, time-periodic, exponential stabilizer for this system.

4.1.2 Asymptotic (non-exponential) stabilization of a four dimensional system

Exponential stabilization of the six dimensional system in (24) can be simplified to the exponential stabilization of a four dimensional system by applying the results of Theorem 2. The idea behind the high-gain feedback approach is to replace x_4 and x_5, which are driven directly by f_1 and f_2, with *pseudo* inputs u_1 and u_2:

$$\begin{aligned}\dot{x}_1 &= u_1\\ \dot{x}_2 &= u_2\\ \dot{x}_3 &= x_6\\ \dot{x}_6 &= -u_1u_2.\end{aligned} \tag{25}$$

The objective is to derive an asymptotic stabilizer for (25) which can be scaled into an exponential stabilizer. Equation (24) can then be exponentially stabilized with high-gain feedback.

We will prove that the feedback

$$\begin{aligned}u_1 &= -x_1|x_1| + 2\sin(t)\left(\frac{x_3|x_3| + x_6|x_6|^{\frac{1}{3}}}{\rho_4^2(x)}\right) \quad \text{where } \rho_4(x) := \left(x_1^6 + x_2^6 + |x_3|^3 + x_4^2\right)^{\frac{1}{6}},\\ u_2 &= -x_1|x_1| + \sin(t)\rho_4^2(x)\end{aligned} \tag{26}$$

asymptotically stabilizes (25). Note that u_1 and u_2 are time periodic and homogeneous of degree 2 with respect to the dilation scaling powers $\tilde{r} = (1,1,2,3)$. The time periodicity avoids the topological obstruction for asymptotic stabilization with continuous, time-invariant feedback (system (25) does not satisfy Brockett's condition). Note that the system (25) cannot be *exponentially* stable with (26) because the closed-loop vector field is degree one with respect to $\tilde{r}$.

The proof that (26) asymptotically stabilizes (25) proceeds in several steps. First, consider the stability of the two dimensional *reduced* system

$$\begin{aligned}\dot{x}_3 &= x_6\\ \dot{x}_6 &= -2\sin^2(t)(x_3|x_3| + x_6|x_6|^{\frac{1}{3}}).\end{aligned} \tag{27}$$

This system is obtained by setting $x_1 \equiv 0$, $x_2 \equiv 0$ in (25) with the feedback (26). The time-averaged system associated with the reduced system is

$$\begin{aligned}\dot{x}_3 &= x_6\\ \dot{x}_6 &= -(x_3|x_3| + x_6|x_6|^{\frac{1}{3}}).\end{aligned} \tag{28}$$

This system is homogeneous degree one with respect to the dilation $r = (2,3)$. A continuously differentiable, homogeneous degree six Lyapunov function for the averaged system is

$$V_a = k_3\left(\frac{2}{3}|x_3|^3 + x_6^2\right) + x_6|x_3|^{1/2}x_3,$$

where k_3 is a parameter whose value is specified below. The time derivative of V_a along trajectories of the averaged system is homogeneous degree 7 (since (28) is degree 1 and V_a is degree 6),

$$\begin{aligned}\dot{V}_a\Big|_{(28)} &= -2k_3|x_6|^{\frac{7}{3}} - |x_3|^{\frac{7}{2}} - x_6|x_6|^{\frac{1}{3}}x_3|x_3|^{\frac{1}{2}} + \frac{3}{2}x_6^2|x_3|^{\frac{1}{2}}\\ &\le -2k_3|x_6|^{\frac{7}{3}} - |x_3|^{\frac{7}{2}} + |x_6|^{\frac{4}{3}}|x_3|^{\frac{3}{2}} + \frac{3}{2}|x_6|^2|x_3|^{\frac{1}{2}}\\ &\le -2k_3|x_6|^{\frac{7}{3}} - |x_3|^{\frac{7}{2}} + \frac{3}{7}\left(|x_3|^{\frac{3}{2}}\right)^{\frac{7}{3}} + \frac{4}{7}\left(|x_6|^{\frac{4}{3}}\right)^{\frac{7}{4}}\\ &\quad + \frac{3}{2}\left(\frac{1}{7}\left(|x_3|^{\frac{1}{2}}\right)^7 + \frac{6}{7}\left(x_6^2\right)^{\frac{7}{6}}\right)\\ &= \left(-2k_3 + \frac{13}{7}\right)|x_6|^{\frac{7}{3}} - \frac{5}{14}|x_3|^{\frac{7}{2}}.\end{aligned}$$

The second inequality in this expression follows from Young's inequality. The time derivative of V_a is negative definite when $k_3 > 13/14$.

A Lyapunov function for the time-periodic system (27) follows from Theorem 4. The idea behind this construction is to add a time-periodic degree 7 term to V_a. This term will not locally affect the positive definiteness property of V_a since V_a is degree 6. However, the partial derivative with respect to time of this term is chosen to cancel the time-periodic degree 7 term from $\partial V_a/\partial x|_{(27)}$. Thus, the sign of $\dot{V}$ is locally determined by the sign of $\dot{V}_a$. The Lyapunov function for (27) is

$$\begin{aligned} V :=& V_a - \left(\int_0^t \cos(2\tau)d\tau\right)\frac{\partial V_a}{\partial x_6}\left(x_3|x_3| + x_6|x_6|^{\frac{1}{3}}\right) \\ =& V_a - \frac{1}{2}\sin(2t)\underbrace{\left(2x_6 + |x_3|^{\frac{1}{2}}x_3\right)\left(x_3|x_3| + x_6|x_6|^{\frac{1}{3}}\right)}_{\text{degree 7}}. \end{aligned} \tag{29}$$

Note that $V > 0$ for $|x|$ sufficiently small. The time derivative of V along trajectories of (27) is

$$\begin{aligned} \dot{V}\Big|_{(27)} &= \dot{V}_a\Big|_{(28)} + \deg_8 \\ &< 0 \qquad \text{for sufficiently small } x \neq 0,\ k_3 > \frac{13}{14}. \end{aligned} \tag{30}$$

Now stability of the four dimensional system in (25) with the feedback defined in (26) is shown. The closed-loop four dimensional system is rewritten as

$$\begin{aligned} \dot{x}_1 &= -x_1|x_1| + \alpha_1(x,t) \\ \dot{x}_2 &= -x_1|x_1| + \alpha_2(x,t) \\ \dot{x}_3 &= x_6 \\ \dot{x}_6 &= -\left(-x_1|x_1| + \alpha_1(x,t)\right)\left(-x_1|x_1| + \alpha_2(x,t)\right)\,, \end{aligned} \tag{31}$$

where

$$\begin{aligned} \alpha_1(x,t) &= 2\sin(t)\left(\frac{x_3|x_3| + x_6|x_6|^{\frac{1}{3}}}{\rho_4^2(x)}\right) \\ \alpha_2(x,t) &= \sin(t)\,\rho_4^2(x)\,. \end{aligned}$$

This closed-loop system is degree one with respect to the scaling

$$(x_1, x_2, x_3, x_6) \mapsto (\lambda x_1, \lambda x_2, \lambda^2 x_3, \lambda^3 x_6). \tag{32}$$

A Lyapunov function for (31) may be derived by adding several terms to (29). Consider the function

$$W := V + k_1 x_1^6 + k_2 x_2^6 + 12k_1\cos(t)x_1^5\left(\frac{x_3|x_3| + x_6|x_6|^{\frac{1}{3}}}{\rho_4^2(x)}\right) + 6k_2\cos(t)x_2^5\rho_4^2(x),$$

where ρ_4 is defined in (26) and k_1 and k_2 are positive constants to be determined. Note that W is locally positive definite since the time periodic terms are degree 7.

The time derivative of W along trajectories of (31) is

$$
\begin{aligned}
\dot{W} &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x_3} x_6 - \frac{\partial V}{\partial x_6} \left(-x_1|x_1| + \alpha_1(x,t)\right)\left(-x_2|x_2| + \alpha_2(x,t)\right) \\
&\quad + 6k_1 x_1^5 \left(-x_1|x_1| + \alpha_1(x,t)\right) + 6k_2 x_2^5 \left(-x_2|x_2| + \alpha_2(x,t)\right) \\
&\quad - \underbrace{12k_1 \sin(t) x_1^5 \left(x_3|x_3| + x_6|x_6|^{\frac{1}{3}}\right) / \rho_4^2(x)}_{=6k_1 x_1^5 \alpha_1(x,t)} - \underbrace{6k_2 \sin(t) x_2^5 \rho_4^2(x)}_{=6k_2 x_5^2 \alpha_2(x,t)} + \mathrm{deg}_8 \\
&= \underbrace{\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x_3} x_6 - \frac{\partial V}{\partial x_6} \alpha_1(x,t)\alpha_2(x,t)}_{\dot{V}|_{(27)}} - \frac{\partial V}{\partial x_6} x_1|x_1|x_2|x_2| + \frac{\partial V}{\partial x_6} x_2|x_2|\alpha_1(x,t) \\
&\quad + \frac{\partial V}{\partial x_6} x_1|x_1|\alpha_2(x,t) - 6k_1 x_1^6 |x_1| - 6k_2 x_2^6 |x_2| + \mathrm{deg}_8.
\end{aligned}
$$

The homogeneous of degree 8 terms are denoted by deg_8. The degree 7 terms in $\dot{W}$, denoted $\mathrm{deg}_7(\dot{W})$, are negative definite for sufficiently large k_1 and k_2. This is argued as follows. The degree 7 terms in $\dot{W}$ are completely specified by the values they assume on the homogeneous sphere $S := \{x | \rho_4(x) = 1\}$. Now denote U the subset of S obtained by projecting onto S, along the homogeneous rays, the points in $\mathbb{R}^4$ with $x_1 = 0$ and $x_2 = 0$. The value of $\mathrm{deg}_7(\dot{W})$ is negative on S for sufficiently large k_1 and k_2 since

1. for $x \in U$, $\mathrm{deg}_7(\dot{W}) = \dot{V}|_{(28)} < 0$,
2. $\dot{W}$ is continuous on the sphere since it is the restriction of a continuous function on $\mathbb{R}^4$, so there is an open neighborhood of U when $k_1 = 0$ and $k_2 = 0$, denoted $n(U)$, such that $\mathrm{deg}_7(\dot{W}) < 0$, and,
3. on the set $S \setminus n(U)$ (so $x_1 \neq 0$ and/or $x_2 \neq 0$), the degree seven terms $6k_1 x_1^6 |x_1|$ and $6k_2 x_2^6 |x_2|$ can majorize all other degree seven terms by choosing k_1 and k_2 sufficiently large.

Thus, $\mathrm{deg}_7(\dot{W}) < 0$ on S for k_1 and k_2 sufficiently large. We conclude that $\dot{W} < 0$ in a neighborhood of the origin since the degree 8 terms do not locally affect the sign of $\dot{W}$ which is determined by $\mathrm{deg}_7(\dot{W})$. Thus, the zero solution of (31) is locally asymptotically stable.

We remark that it is possible to estimate lower bounds for k_1 and k_2 to ensure $\mathrm{deg}_7(\dot{W}) < 0$ in much the same way we estimated a lower bound for k_3. The analysis is quite involved and perhaps the most efficient means to obtain values is through simulation or numerical experimentation.

4.1.3 Exponential stabilization of the four dimensional system

Exponentially stabilizing feedbacks for (31) are derived by scaling the degree two feedback in (26) according to Theorem 3 if the transversality condition is satisfied. However, care must be exercised in choosing the appropriate dilation. The dilation in (32), which was instrumental in deriving the asymptotic stabilizer, cannot be the dilation we ultimately wish to use for obtaining exponential stability since the closed-loop system is not homogeneous of degree zero with respect to this dilation. The proper dilation to use is motivated by the dilation in (23) that was derived for the homogeneous approximation of the 6-dimensional system in (24). In our current situation, x_4 and x_5 are treated as pseudo-controls so if they are eliminated from (24) the dilation becomes:

$$(x_1, x_2, x_3, x_6) \mapsto (\lambda x_1, \lambda x_2, \lambda^2 x_3, \lambda^2 x_6). \tag{33}$$

If u_1 and u_2 are scaled to degree one functions according to (33), then x_4 and x_5 may be compensated in the full 6-dimensional equations using the high gain feedback from Theorem 2. In this case, the resulting 6-dimensional closed-loop system is homogeneous of degree zero with respect to (23). Although it may be possible to choose dilations other than (33) for scaling u_1 and u_2, this dilation is the only one which makes sense when compared to the dilation derived from the original homogeneous approximation of the 6-dimensional equations.

Note that the "b_0" vector field in (25)

$$(0, 0, x_6, 0)^T,$$

is degree zero with respect to (33).

The Euler vector field corresponding to (33) is

$$X_E = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + 2x_3 \frac{\partial}{\partial x_3} + 2x_6 \frac{\partial}{\partial x_6}. \tag{34}$$

In computing the derivative of W with respect to X_E we can exploit the fact that W is composed of homogeneous degree six and degree seven terms. To explore this idea further, note that X_E is degree zero with respect to *any* dilation so the derivative of the degree six terms in W with respect to X_E will remain degree six with respect to (32). Thus, the transversality condition will hold locally if the degree six terms in $L_{X_E}W$ are positive definite. Consider the degree 6 terms in W gathered together,

$$W = k_3 \left(\frac{2}{3}|x_3|^3 + |x_6|^2 \right) + x_6 |x_3|^{\frac{1}{2}} x_3 + k_1 x_1^6 + k_2 x_2^6 + \deg_7.$$

The derivative of W with respect to (34) is

$$L_{X_E}W = 4k_3|x_3|^3 + 4k_3|x_6|^2 + 2x_6|x_3|^{\frac{1}{2}}x_3 + 3x_6|x_3|^{\frac{1}{2}}x_3 + 6k_1x_1^6 + 6k_2x_2^6 + \deg_7.$$

The degree six (with respect to (32)) terms are positive definite for sufficiently large k_1, k_2 and k_3. Thus $L_{X_E}W > 0$, locally, with these choices.

The constraints placed on k_1, k_2 and k_3 in the previous steps do not conflict. In other words, in developing the stabilizing controls for the lower dimensional system in Section 4.1.2 it was shown that asymptotic stability is achieved for k_i sufficiently large. Presently, the tranversality condition also holds for k_i sufficiently large. Obtaining lower bounds for the k_i for which all constraints are satisfied is an arduous task and is left for the reader to complete. Furthermore, the scaling factor λ must be determined to map the current state to the level set. There are several methods to accomplish this task. For example, a bisection search routine may be used. This search is facilitated by the fact that W is an increasing function along trajectories of the Euler vector field in a neighborhood of the origin since the level sets of W are transversal to X_E. The bisection approach has been successfully implemented in real-time for a mobile robot stabilization problem [23].

The asymptotically stabilizing feedbacks in (26) are evaluated at the point the state maps to on the level set. These values are then divided by the scaling factor calculated with bisection and applied as the control. These controls will be denoted $\tilde{u}_1$ and $\tilde{u}_2$. A performance comparison between the asymptotically stabilizing feedback and scaled exponential stabilizer applied to (25) is shown in Figure 2 (only x_3 and x_6 are shown since these are the "hard" states to control). The closed-loop performance with the scaled feedbacks are shown with $k_1 = 2$, $k_2 = 3$, and $k_3 = 3$, a level set value of $W = 1 \times 10^{-5}$, and a series of initial conditions at the magnitude scales $\{1, 0.1, 0.01, 0.001\}$. The small value of the level set was chosen to ensure local stability of (25) with (26) in addition to satisfying the transversality requirement. Simulation was used for selecting k_1, k_2, k_3 and the level set of W.

These simulations reveal how the qualitative nature of the system response with the asymptotically stabilizing feedback changes as a function of initial condition size. In contrast to this behavior, the exponential stabilizer responses remain qualitatively similar regardless of the size of the initial condition. This is expected since the exponentially stable system is homogeneous degree zero and thus solutions of the closed-loop system scale to other solutions via the dilation, much like a linear system. On the other hand, the closed-loop system with asymptotically stabilizing feedback is *not* invariant with respect to a dilation so solutions do not scale to solutions. Another point worth noting is the similarity between the responses of both systems for the intermediate initial condition. This occurs because the exponential feedback is scaled from a level set near this initial condition and both closed-loop vector fields are close as a result (recall that the scaled exponentially stabilizing feedback is equal to the asymptotically stabilizing feedback on the level set chosen for the scaling). The scaled feedback is degree one and, although there is a higher order perturbing term in the $\dot{x}_3$ equation, its presence does not seem to adversely affect the closed-loop response even for the largest initial condition.

4.1.4 Exponential stabilization of the planar rigid body with two thrusters

The system in (24), and hence the planar rigid body equation in (21), may be stabilized by setting

$$\begin{aligned} f_1 &= -K(x_4 - \tilde{u}_1) \\ f_2 &= -K(x_5 - \tilde{u}_2), \end{aligned} \tag{35}$$

for K sufficiently large.

A simulation of (21) with (35) (after the appropriate coordinate transformations), with $K = 10$, is shown in Figure 3. The plot of $\log \|(x(t), y(t), \theta(t))\|$ is included to demonstrate exponential convergence of the system variables (only x, y and θ are graphed —their velocities converge exponentially too). The higher order terms neglected in the analysis do not affect the exponential stability of the closed-loop system in a neighborhood of the origin (refer to Theorem 1). This fact is evident from the time simulations of the full equations.

In conclusion, the algorithm of Theorem 3 is relatively easy to apply if an asymptotically stabilizing feedback and a Lyapunov function exist. The main difficulty with our example was first deriving a continuous, time-periodic, asymptotic stabilizer. Even in this case, the notions of dilations and homogeneous vector fields were indispensable tools for constructing the feedback and a Lyapunov function. Once the reduced four dimensional system was proven to be asymptotically stable, it was easy to verify the transversality condition and scale the feedback into exponential stabilizers. Checking the transversality condition was facilitated by considering the degree 6 terms

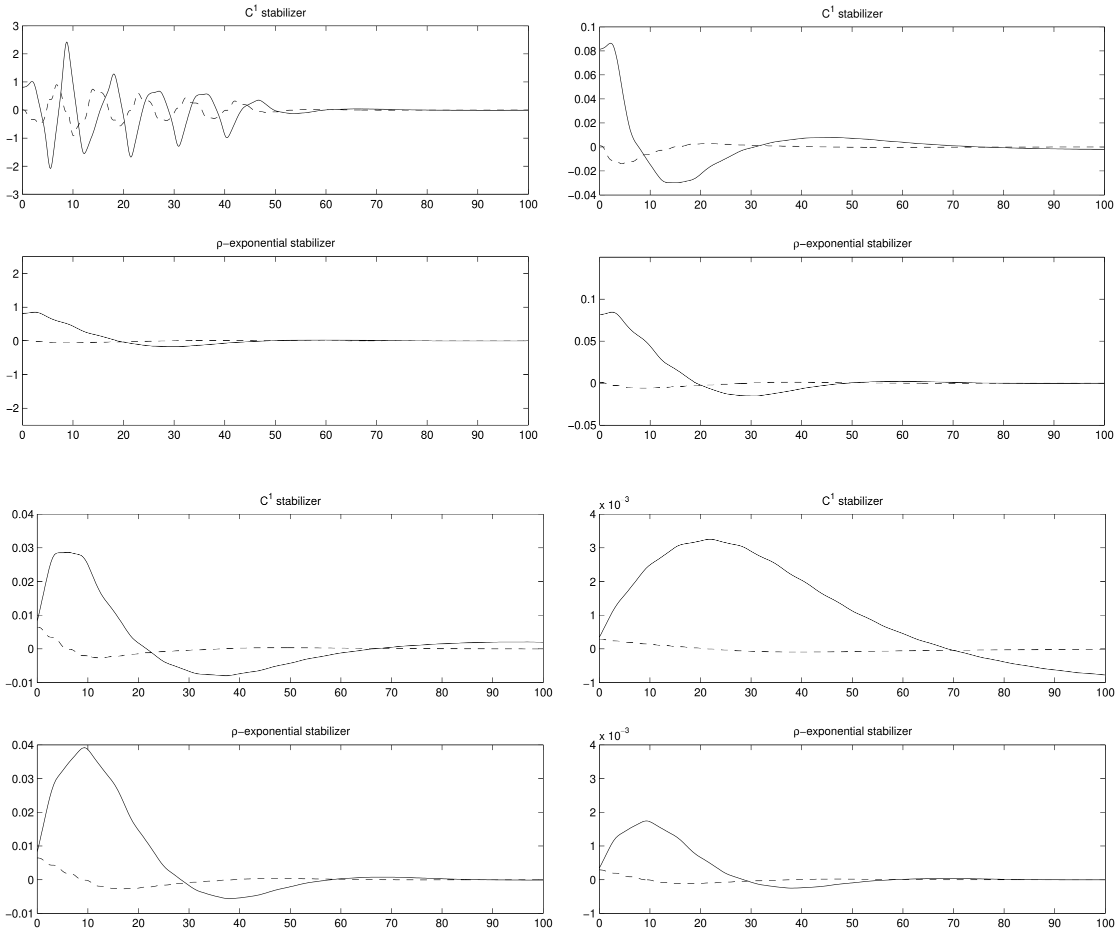

Figure 2: The response of (28) with the C^1 and scaled exponential stabilizers starting from four sets of initial conditions (x_3–solid; x_6–dashed)

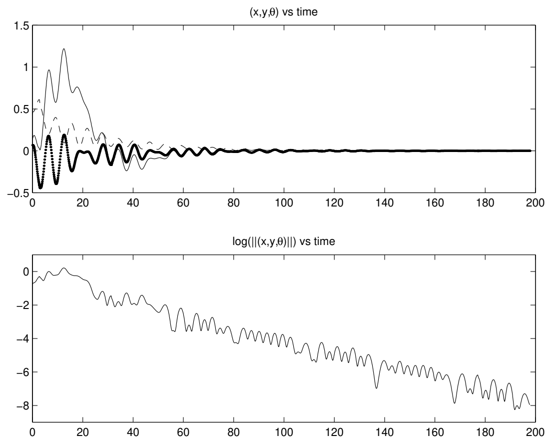

Figure 3: Response of planar system (21) with exponentially stabilizing feedback (x - solid line, y - dashed line, θ - dotted line)

of the Lyapunov function. The last step involved applying high-gain feedback to stabilize the full planar system with two thrusters.

4.2 Exponential stabilization of the spatial rigid body with three controls via time averaging

As a generalization of the previous example, we consider the spatial rigid body in Figure 4. Our objective is to stabilize this system with only three controls: one force and two torques. Our main tool in this section is Theorem 4. Other applications of this result (based on a weaker version) to systems with drift can be found for instance in [29] for the attitude stabilization only, or in [33] for the stabilization of both position and attitude but with four controls (one force and three torques). The spatial rigid body equations are

$$\left\{\begin{array}{rcl} \dot{R} & = & R\widehat{\omega} \\ \dot{x} & = & Rv \\ J\dot{\omega} & = & J\omega \times \omega + p\,Mv \times v + \tau \\ M\dot{v} & = & Mv \times \omega + f\,, \end{array}\right. \tag{36}$$

with

$\widehat{\omega}$: the matrix defined by $\widehat{\omega}x = \omega \times x$,

R: the rotation matrix from a body-fixed frame to an inertial frame. We assume that the body-fixed frame coincides with the frame of principal inertia axes,

x: the coordinates of the body in the inertial frame,

ω and v: the angular velocity, and velocity vectors expressed in the body-fixed frame,

$J = Diag(j_1, j_2, j_3)$ and $M = Diag(m_1, m_2, m_3)$: the inertia and mass matrices,

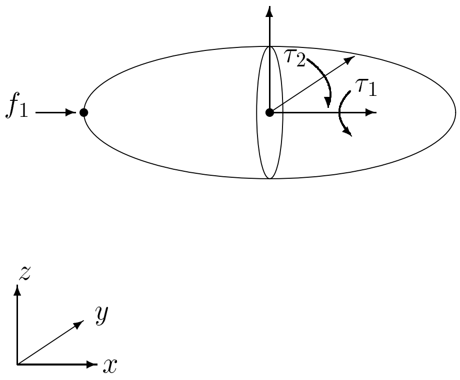

Figure 4: Spatial rigid body with body-fixed torques and force τ_1, τ_2, and f_1

τ and f: the torques and forces applied to the rigid body.

When p in the third equation of (36) is equal to zero, (36) represents for instance the equations of a rigid spacecraft. In this case, J is the inertia matrix of the rigid body, and $M = mI_d$ with m the mass of the spacecraft and I_d the identity matrix in $\mathbb{R}^3$. When p is equal to one, (36) can be used to represent the equations of a neutrally buoyant underwater vehicle in ideal fluid, with coincident center of mass and center of buoyancy (see e.g. [19]). In this case, J and M also include the fluid added inertia and mass. Our objective is to find a continuous feedback law which makes the equilibrium point $(R, x, \omega, v) = (I_d, 0, 0, 0)$ locally exponentially stable when only two torques and one force are available. For simplicity, we assume that

$$\tau = (\tau_1, \tau_2, 0)^T, \qquad f = (f_1, 0, 0)^T. \tag{37}$$

We also assume that

$$c := (j_1 - j_2)/j_3 \neq 0. \tag{38}$$

This assumption is a necessary controllability condition when $p = 0$ since otherwise, (36) and (37) imply that $\dot{\omega}_3 = c\,\omega_1\omega_2 = 0$.

We first choose a minimal parameterization of $S0(3)$: to any rotation R of angle $\phi \in]-\pi, \pi[$ and axis, the direction of which is defined by the vector $\vec{u}$, we associate the three-dimensional vector

$$\Theta = (\theta_1, \theta_2, \theta_3)^T = \tan\frac{\phi}{2}\,u, \tag{39}$$

with u the coordinate vector of $\vec{u}$ in the body-fixed frame. Using e.g. [11, eq. 2.25,2.35] , (36) can be rewritten, in the coordinates Θ, as:

$$\left\{\begin{array}{rcl} \dot{\Theta} & = & \frac{1}{2}(\omega + \Theta \times \omega + < \omega, \Theta > \Theta) \\ \dot{x} & = & R(\Theta)v \\ J\dot{\omega} & = & J\omega \times \omega + p\,Mv \times v + \tau \\ M\dot{v} & = & Mv \times \omega + f\,, \end{array}\right. \tag{40}$$

with $R(\Theta)$ the expression of R in the coordinates Θ, i.e.

$$R(\Theta) = (1+\theta_1^2+\theta_2^2+\theta_3^2)^{-1} \begin{pmatrix} 1+\theta_1^2-\theta_2^2-\theta_3^2 & 2(\theta_1\theta_2-\theta_3) & 2(\theta_1\theta_3+\theta_2) \\ 2(\theta_1\theta_2+\theta_3) & 1+\theta_2^2-\theta_1^2-\theta_3^2 & 2(\theta_2\theta_3-\theta_1) \\ 2(\theta_1\theta_3-\theta_2) & 2(\theta_2\theta_3+\theta_1) & 1+\theta_3^2-\theta_1^2-\theta_2^2 \end{pmatrix}. \quad (41)$$

Our feedback laws are defined by

$$\begin{cases} \tau_1^\epsilon(\Theta,x,\omega,v,t) &= -k_1(\omega_1-u_1^\epsilon(z,t)) \\ \tau_2^\epsilon(\Theta,x,\omega,v,t) &= -k_2(\omega_2-u_2^\epsilon(z,t)) \\ f_1^\epsilon(\Theta,x,\omega,v,t) &= -k_3(v_1-u_3^\epsilon(z,t)), \end{cases} \quad (42)$$

with $z=(\Theta,x,\omega_3,v_2,v_3)$, and $u_1^\epsilon, u_2^\epsilon$, and u_3^ϵ of the form

$$u_i^\epsilon(z,t) = u_{i,1}(z,t/\epsilon)+\epsilon^{-\frac{1}{2}}\alpha_i(t/\epsilon)u_{i,2}(z) \quad (i=1,2,3). \quad (43)$$

The different functions involved in (43) are defined as follows:

$$\begin{cases} \alpha_1(t) = -8\dfrac{\cos 2t}{2c+1},\ \alpha_2(t)=0,\ \alpha_3(t)=\cos t & (c\neq -1/2) \\ \alpha_1(t) = 8\dfrac{\cos t}{c+2},\ \alpha_2(t)=0,\ \alpha_3(t)=\cos 2t & (c=-1/2) \\ u_{1,1}(z,t/\epsilon) = \beta_1(z)+2\sin(t/\epsilon)\gamma_1(z) & u_{1,2}(z)=\delta_1(z) \\ u_{2,1}(z,t/\epsilon) = \beta_2(z)+\sin(t/\epsilon)\gamma_2(z) & u_{2,2}(z)=\delta_2(z)=0 \\ u_{3,1}(z,t/\epsilon) = \beta_3(z)+2\sin(t/\epsilon)\gamma_3(z) & u_{3,2}(z)=\delta_3(z), \end{cases} \quad (44)$$

with

$$\begin{cases} \rho(z) = (z_1^{12}+z_2^{12}+z_3^6+z_4^{12}+z_5^4+z_6^6+z_7^6+z_8^4+z_9^6)^{\frac{1}{12}} \\ \delta_1(z)=\rho(z),\ \delta_3(z) = -(\mu_5 z_5+\mu_8 z_8)/\rho^2(z) \\ \gamma_1(z) = \dfrac{1}{\rho(z)}(\overline{\alpha_3\overline{\alpha_4}}L_{\delta_3 b_3}\gamma_2\delta_1 - c(\mu_3 z_3+\mu_7 z_7)),\ \gamma_2(z)=\rho(z) \\ \gamma_3(z) = \dfrac{1}{\rho(z)}(\overline{\alpha_1\overline{\alpha_5}}L_{\delta_1 b_1}\gamma_2\delta_3 - (\mu_6 z_6+\mu_9 z_9)) \\ \beta_1(z) = -\overline{\alpha_1\overline{\alpha_5}}L_{b_5\gamma_2\delta_3}\delta_1-\mu_1 z_1,\ \beta_2(z)=-\mu_2 z_2,\ \beta_3(z) = -\overline{\alpha_3\overline{\alpha_4}}L_{b_4\gamma_2\delta_1}\delta_3-\mu_4 z_4, \end{cases} \quad (45)$$

$$\alpha_4(t) = \alpha_1(t)\sin t, \quad \alpha_5(t) = \alpha_3(t)\sin t, \quad (46)$$

μ_k ($k=1,\ldots,9$) any set of strictly positive constants, and b_1, b_3, b_4, and b_5 the vector fields defined by (50). With this definition we have

Proposition 1 *There exists $\epsilon_0>0$ such that for any $\epsilon\in(0,\epsilon_0)$, and for large enough values of k_1, k_2, and k_3, the continuous feedback law (42) makes the origin of the system (40) locally exponentially stable.*

Proof:
The proof is based on the homogeneity-based results reported in Sections 2 and 3. Accordingly,

we first exhibit an homogeneous approximation for system (40). Let $X = (\Theta, x, \omega, v)$, we rewrite system (40) as follows:

$$\begin{pmatrix} \dot\theta_1 \\ \dot\theta_2 \\ \dot\theta_3 \\ \dot x_1 \\ \dot x_2 \\ \dot x_3 \\ \dot\omega_1 \\ \dot\omega_2 \\ \dot\omega_3 \\ \dot v_1 \\ \dot v_2 \\ \dot v_3 \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{2}\omega_1 \\ \frac{1}{2}\omega_2 \\ \frac{1}{2}(\omega_3 + \omega_2\theta_1 - \omega_1\theta_2) \\ v_1 \\ v_2 + 2(\theta_1\theta_2 + \theta_3)v_1 - 2\theta_1 v_3) \\ v_3 - 2\theta_2 v_1 \\ 0 \\ 0 \\ c\,\omega_1\omega_2 \\ 0 \\ \frac{1}{m_2}(m_3 v_3\omega_1 - m_1 v_1\omega_3) \\ \frac{m_1}{m_3} v_1\omega_2 \end{pmatrix}}_{=:g_0(X)} + \underbrace{\begin{pmatrix} 0\\0\\0\\0\\0\\0\\1\\0\\0\\0\\0\\0 \end{pmatrix}}_{=:g_1(X)} \tau_1 + \underbrace{\begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\1\\0\\0\\0\\0 \end{pmatrix}}_{=:g_2(X)} \tau_2 + \underbrace{\begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\1\\0\\0 \end{pmatrix}}_{=:g_3(X)} f_1 + h(X, \tau, f) \quad (47)$$

Associated with this decomposition is the dilation Δ_λ with weight vector $r = (1, 1, 2, 1, 3, 2, 1, 1, 2, 1, 3, 2)$. Indeed, one easily checks that the vector fields g_0, g_1, g_2, and g_3 are homogeneous respectively of degree 0 -1, -1 and -1 with respect to Δ_λ. A tedious but simple examination also shows that h is a sum of vector fields homogeneous of degree one at least, when the weight 1 is assigned to each of the variables τ_1, τ_2, and f_1. Let us now assume that the following two properties are satisfied:

1. The feedback functions τ_1^ϵ, τ_2^ϵ, and f_1^ϵ defined by (42) are homogeneous of degree 1 w.r.t Δ_λ.
2. These feedback laws make the origin of the homogeneous approximation

$$\dot X = g_0(X) + g_1(X)\tau_1 + g_2(X)\tau_2 + g_3(X) f_1 \quad (48)$$

   asymptotically stable.

Then, Proposition 1 follows from Theorem 1 with $f = g_0 + g_1\tau_1^\epsilon + g_2\tau_2^\epsilon + g_3 f_1^\epsilon$ and $h = h(., \tau^\epsilon, f^\epsilon)$. We now concentrate on the proof of the two properties listed above. For the first property we have:

Lemma 1 *The feedback functions τ_1^ϵ, τ_2^ϵ, and f_1^ϵ defined by (42) are homogeneous of degree 1 w.r.t Δ_λ.*

(Proof in Appendix)

Before proceeding with the proof of the second property, it is worth noticing that the approximating system (48) retains local controllability properties of the original system (47). More precisely, one can show that Sussmann's sufficient condition for STLC [41, Th. 7.3] is satisfied for this system with $\theta = 1$ —here, θ denotes the one in [41, Th. 7.3].

To show the second property, we first apply Theorem 2. It is easy to check that the functions u_i^ϵ $(i = 1, \ldots, 3)$ defined by (43) are of class $\mathcal{C}^0$, and smooth outside $z = 0$. In fact, the continuity at $z = 0$ might appear not trivial since ρ vanishes at this point, however continuity directly follows from the fact that the u_i^ϵ's are homogeneous of degree 1. This property is shown in the proof of Lemma 1. Knowing that, and in view of the degrees of homogeneity of the vector fields $g_i\,(i = 0, \dots, 3)$, we deduce from Theorem 2 that the origin of the closed-loop system (48)–(42) is asymptotically stable, for large enough values of k_1, k_2, and k_3, provided that the functions $u_i^\epsilon\,(i = 1, \dots, 3)$ stabilize asymptotically the origin of the reduced order system

$$\dot{z} = b_0(z) + \sum_{i=1}^{3} u_i b_i(z) + u_1 u_2 b_4(z) + u_2 u_3 b_5(z)\,, \tag{49}$$

where the vector fields b_i are defined by:

$$\underbrace{\begin{pmatrix} 0 \\ 0 \\ \frac{1}{2}z_7 \\ 0 \\ z_8 - 2z_1 z_9 \\ z_9 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{b_0(z)}, \underbrace{\begin{pmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2}z_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{m_3}{m_2}z_9 \\ 0 \end{pmatrix}}_{b_1(z)}, \underbrace{\begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2}z_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{b_2(z)}, \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2(z_1 z_2 + z_3) \\ -2z_2 \\ 0 \\ -\frac{m_1}{m_2}z_7 \\ 0 \end{pmatrix}}_{b_3(z)}, \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ c \\ 0 \\ 0 \end{pmatrix}}_{b_4(z)}, \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{m_1}{m_3} \end{pmatrix}}_{b_5(z)}. \tag{50}$$

Note that system (49) is just the system obtained from (48) by taking $u_1 := \omega_1$, $u_2 := \omega_2$, and $u_3 := v_1$ as control variables. Note also that the vector fields b_i are homogeneous w.r.t. the dilation with weight vector $r' = (1, 1, 2, 1, 3, 2, 2, 3, 2)$. Setting

$$u_4 := u_1 u_2\,, \quad u_5 := u_2 u_3\,, \tag{51}$$

we rewrite (49) as

$$\dot{z} = b_0(z) + \sum_{i=1}^{5} u_i b_i(z)\,. \tag{52}$$

There remains to show that the origin of the closed-loop system (52)–(51)–(43) is asymptotically stable for ϵ small enough. The proof is based on Theorem 4. In particular, we use the fact that the Lie bracket $[b_1, b_5]$ can be used to yield movement in the "v_2-direction". From (51), (43), (44), and (46),

$$\begin{cases} u_4^\epsilon(z,t) := u_1^\epsilon(z,t) u_2^\epsilon(z,t) = u_{4,1}(z, t/\epsilon) + \epsilon^{-\frac{1}{2}}(\alpha_1(t/\epsilon)\,(\beta_2\delta_1)(z) + \alpha_4(t/\epsilon)\,(\gamma_2\delta_1)(z)) \\ u_5^\epsilon(z,t) := u_2^\epsilon(z,t) u_3^\epsilon(z,t) = u_{5,1}(z, t/\epsilon) + \epsilon^{-\frac{1}{2}}(\alpha_3(t/\epsilon)(\beta_2\delta_3)(z) + \alpha_5(t/\epsilon)(\gamma_2\delta_3)(z))\,, \end{cases} \tag{53}$$

with

$$\begin{cases} u_{4,1}(z, t/\epsilon) = -\cos(2t/\epsilon)\gamma_1\gamma_2 + \beta_1\beta_2 + \gamma_1\gamma_2 \\ u_{5,1}(z, t/\epsilon) = -\cos(2t/\epsilon)\gamma_2\gamma_3 + \beta_2\beta_3 + \gamma_2\gamma_3\,. \end{cases} \tag{54}$$

One can verify that the closed loop-system (52)–(43)–(53) is of the form (9) —with, in particular, $f_0 = b_0 + \sum_{i=1}^{5} b_i u_{i,1}^\epsilon$, and $d = 0$. In fact, by applying Theorem 4 to this system, one obtains:

Lemma 2 *The averaged system (10) associated with the closed-loop system (52)–(43)–(53) is given by*

$$\begin{aligned}\dot{y} = \ & b_0(y) + \sum_{i=1}^{3} b_i(y)\beta_i(y) + b_4(y)(\beta_1\beta_2 + \gamma_1\gamma_2)(y) + b_5(y)(\beta_2\beta_3 + \gamma_2\gamma_3)(y) \\ & + c_{1,5}(-[b_1\delta_1, b_5\gamma_2\delta_3] + [b_5\gamma_2\delta_3, b_4\beta_2\delta_1])(y) \\ & + c_{3,4}(-[b_3\delta_3, b_4\gamma_2\delta_1] + [b_4\gamma_2\delta_1, b_5\beta_2\delta_3])(y)\,,\end{aligned} \tag{55}$$

with $c_{1,5} := \overline{\alpha_1\overline{\alpha_5}}$ *and* $c_{3,4} := \overline{\alpha_3\overline{\alpha_4}}$ *such that*

$$c_{1,5} - c\, c_{3,4} = 1\,. \tag{56}$$

(Proof in Appendix)

To conclude, all we need to show is that the origin of system (55) is asymptotically stable. From (50),

$$[b_1, b_5](y) = (0,0,0,0,0,0,0,-\frac{m_1}{m_2},0)^T\,, \quad [b_3, b_4](y) = -c[b_1, b_5](y)\,, \quad [b_4, b_5](y) = 0\,. \tag{57}$$

Using (56) and (57), we can rewrite (55) —after some computations and using the relation $[u_1b_1, u_2b_2] = u_1u_2[b_1, b_2] + b_2L_{u_1b_1}u_2 - b_1L_{u_2b_2}u_1$ for arbitrary vector fields b_1, b_2, and arbitrary differentiable functions v_1, v_2— as:

$$\begin{aligned}\dot{y} = \ & b_0(y) + (\beta_1 + c_{1,5}L_{b_5\gamma_2\delta_3}\delta_1)b_1(y) + \beta_2 b_2(y) + (\beta_3 + c_{3,4}L_{b_4\gamma_2\delta_1}\delta_3)b_3(y) \\ & + (\beta_1\beta_2 + \gamma_1\gamma_2 - c_{3,4}L_{\beta_2\delta_3b_5+\delta_3b_3}\gamma_2\delta_1 + c_{1,5}L_{b_5\gamma_2\delta_3}\beta_2\delta_1)b_4(y) \\ & + (\beta_2\beta_3 + \gamma_2\gamma_3 - c_{1,5}L_{\delta_1b_1+\beta_2\delta_1b_4}\gamma_2\delta_3 + c_{3,4}L_{\gamma_2\delta_1b_4}\beta_2\delta_3)b_5(y) \\ & - \gamma_2\delta_1\delta_3[b_1, b_5](y)\,.\end{aligned} \tag{58}$$

In view of (45), (50), and (57), we first remark that

$$\dot{y}_2 = \tfrac{1}{2}\beta_2(y) = -\frac{\mu_2}{2}y_2\,. \tag{59}$$

This implies that the asymptotic stability of (58) is equivalent to the asymptotic stability of the system reduced to the invariant manifold $\{y : y_2 = 0\}$. Since on this manifold, $\beta_2 \equiv 0$, the dynamics of (58) can be simplified, using also (45) and the definition of $c_{1,5}$ and $c_{3,4}$ in Lemma 2, as

$$\begin{aligned}\dot{y} = \ & b_0(y) - \mu_1y_1b_1(y) - \mu_4y_4b_3(y) - c(\mu_3y_3 + \mu_7y_7)b_4(y) \\ & -(\mu_6y_6 + \mu_9y_9)b_5(y) + (\mu_5y_5 + \mu_8y_8)[b_1, b_5](y)\,.\end{aligned} \tag{60}$$

Using (50), this system can be expanded as

$$\left\{\begin{aligned}\dot{y}_1 &= -\tfrac{\mu_1}{2}y_1 \\ \dot{y}_3 &= \tfrac{1}{2}y_7 \\ \dot{y}_4 &= -\mu_4y_4 \\ \dot{y}_5 &= y_8 - 2y_1y_9 - 2\mu_4y_3y_4\end{aligned}\right. \qquad \left\{\begin{aligned}\dot{y}_6 &= y_9 \\ \dot{y}_7 &= -c^2(\mu_3y_3 + \mu_7y_7) \\ \dot{y}_8 &= -\tfrac{m_1}{m_2}(\mu_5y_5 + \mu_8y_8) - \tfrac{m_3}{m_2}\mu_1y_1y_9 + \tfrac{m_1}{m_2}\mu_4y_4y_7 \\ \dot{y}_9 &= -\tfrac{m_1}{m_3}(\mu_6y_6 + \mu_9y_9)\end{aligned}\right. \tag{61}$$

The origin of this system is clearly locally asymptotically stable since its linearization is — it is in fact globally asymptotically stable since this system is homogeneous of degree 0 with respect

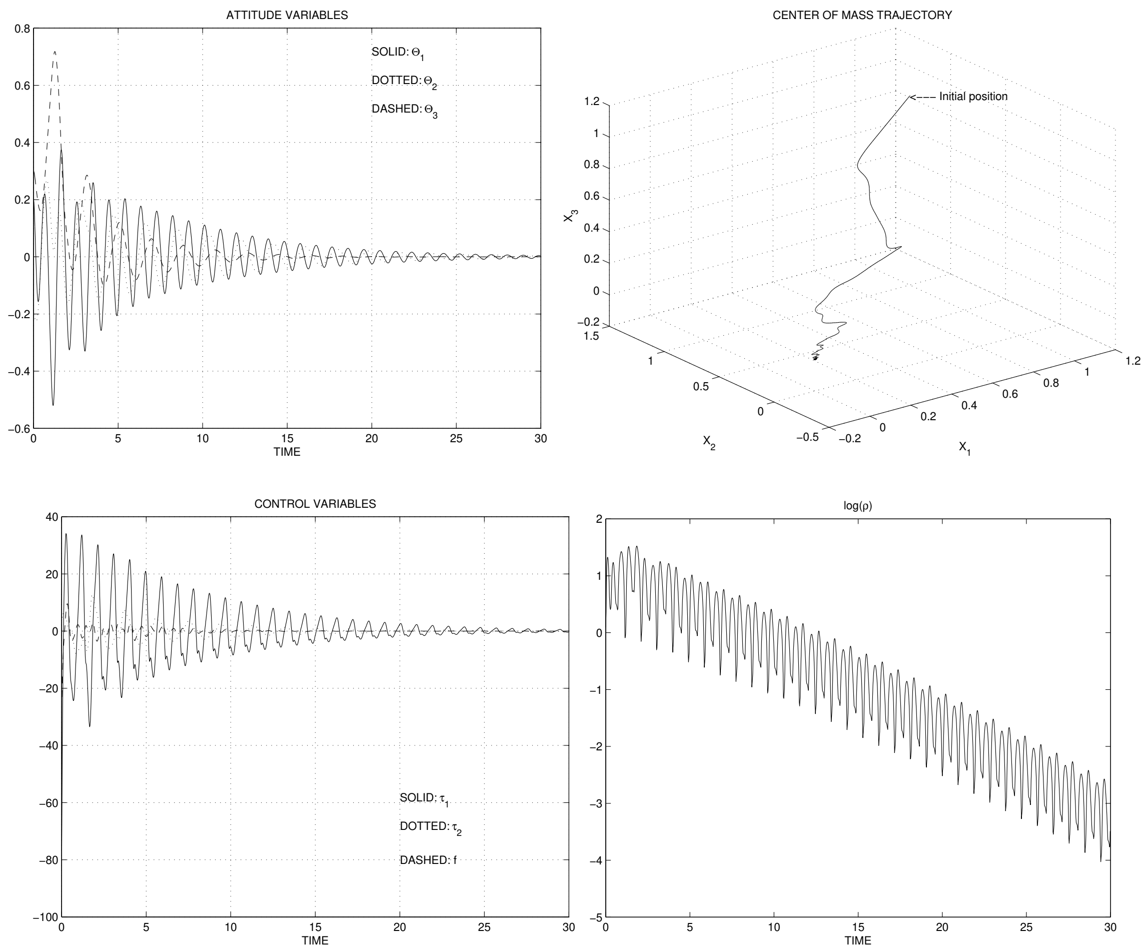

Figure 5: Simulation of the spatial rigid body

to the family of dilation with weight vector $r = (1, 2, 1, 3, 2, 2, 3, 2)^T$—. This concludes the proof of Proposition 1. ∎

Simulation results of system (40) controlled by (42) are given in Figure 5. This simulation was made with the following system parameters: $J = (4, 2, 2)$, $M = m\ I_d = 10\ I_d$, and $p = 0$. The control parameters used in this simulation are: $\epsilon = .3$, $k_1 = k_2 = k_3 = 15$, $\mu = (1, 1, 9/8, 1, 9/16, 9/16, 3/2, 3/2, 3/2)$. The initial conditions are $\Theta(0) = (.2, -.2, .3)$, $X(0) = (1, 1, 1)$, $\omega(0) = v(0) = (0, 0, 0)$.

5 Conclusion

In this paper we have brought together recent developments in exponential stabilization by homogeneous time-varying feedback. We have also shown that important design techniques, which had been used for the control of driftless systems, can be successfully applied to systems with drift. Still, many open questions need to be addressed concerning the development of these techniques.

First, for general controllable systems with drift, one needs to better understand the possibilities of approximation by homogeneous systems. Important results have already been obtained in this area (see e.g.[3]). However, it is not yet completely clear how these results can be used for exponential stabilization by homogeneous feedback.

For practical applications, a very important problem concerns the robustness of homogeneous feedback with respect to model errors. Homogeneous systems are robust with respect to perturbations by higher-order homogeneous vector fields. However, for actual systems —e.g. nonholonomic mechanical systems, under-actuated systems— uncertainties on the model often lead, in closed loop, to lower-order terms (see e.g. [28, Section 5]). Asymptotic stability is then no longer achieved.

Finally, concerning the application of the two design methods in Section 3, some issues need also to be considered. For the first approach (i.e., scaling), verification of the transversality condition is in general a hard problem. Alternative conditions could possibly simplify the control design. If the second approach allows control design for fairly general systems (for instance any driftless controllable system [27]), it becomes excessively involved when high-order Lie brackets are needed to ensure the controllability of the system. Simplifications can improve the appeal of this method for realistic applications.

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Appendix

Proof of Theorem 3

The idea in this proof is to rewrite equation (5) with its asymptotically stabilizing feedback into the form of a driftless system. The results of Theorem 4 in [24] can then be applied to obtain the result.

Assuming the conditions of the theorem are satisfied, define

$$\tilde{b}_0(x,t) := \frac{1}{\tilde{\rho}(x,t)} b_0(x),$$

where $\tilde{\rho}$ is defined in the theorem. The vector field $\tilde{b}_0$ is well-defined for $x \neq 0$ because $\tilde{\rho}$ is a degree 1 function such that $\tilde{\rho}(x,t) \neq 0$ for all $x \neq 0$ and for all t. Furthermore, $\tilde{b}_0$ is a degree -1 vector field, smooth on $\mathbb{R}^n \setminus \{0\}$. It is possible that the limit value at $x = 0$ of some components of $\tilde{b}_0$ are not defined. This occurrence, however, does not affect the stability proof. In the case when a component of $\tilde{b}_0$ has degree greater than zero then the limit value at $x = 0$ is always zero.

The closed-loop system with the asymptotically stabilizing feedback may be written as

$$\dot{x} \ = \ \tilde{b}_0(x)\tilde{\rho}(x,t) + \sum_{i=1}^{m} u_i(x,t) b_i(x), \tag{62}$$

which is a driftless system with the $m+1$ input vector fields $\{\tilde{b}_0, b_1, \ldots, b_m\}$ and the $m+1$ asymptotically stabilizing control functions $\{\tilde{\rho}, u_1, \ldots, u_m\}$. The remainder of the proof follows from applying Theorem 4 in [24] to system (62) (note —there is no accepted standard for defining the degree of a vector field and our definition is different than the one adopted in this reference).

The idea behind the proof of this theorem in this reference is to show that asymptotic stability still holds for the system with scaled inputs. Furthermore, since the closed-loop vector field is now homogeneous of degree zero, uniform asymptotic stability implies exponential stability. The reader is referred to [24] for details.

The smoothness condition on the asymptotically stabilizing feedback in Theorem 4 of [24] may be weakened to continuity w.r.t. x without changing the results. The system in (62) satisfies these conditions. The condition $\deg(b_i) = -1$ in this theorem may also be weakened to $\deg(b_i) = d_i < 0$. Note that the reformulated system "input" corresponding to $\tilde{\rho}(x,t)$ is invariant under the scaling process. That is, $\tilde{\rho}(x,t)$ is scaled to $\tilde{\rho}(x,t)$. Thus, the original "drift" vector field $b_0 = \tilde{b}_0\tilde{\rho}$ does not change with the scaling even though the remaining inputs u_i, $i = 1, \ldots, m$, are scaled to degree one functions (denoted $\tilde{u}_i$, $i = 1, \ldots, m$) so that the system

$$\begin{aligned} \dot{x} &= \tilde{b}_0(x)\tilde{\rho}(x,t) + \sum_{i=1}^{m} \tilde{u}_i(x,t)b_i(x) \\ &= b_0(x) + \sum_{i=1}^{m} \tilde{u}_i(x,t)b_i(x), \end{aligned}$$

is exponentially stable. This concludes the proof. ■

Proof of Theorem 4

Consider any homogeneous Lyapunov candidate function V_a, at least $\mathcal{C}^1$, and denote $\dot{V}_a$ its derivative along the solutions of (9). We have

$$\dot{V}_a = L_{f_0}V_a + \sum_{i=1}^{I} \epsilon^{-\frac{1}{2}}\alpha_i(t/\epsilon)\, L_{f_i}V_a\,. \tag{63}$$

Define

$$W_1(x,t,\epsilon) = V_a(x) + \epsilon L_{\int_0^{t/\epsilon} \overline{f}_0(x) - f_0(x,\tau)\, d\tau} V_a - \sum_{i=1}^{I} \epsilon^{\frac{1}{2}} \left(\int_0^{t/\epsilon} \alpha_i(\tau)\, d\tau \right) L_{f_i}V_a\,. \tag{64}$$

By the assumptions of Theorem 4 —recall that we assume all terms in (11) to be of class $\mathcal{C}^1$ for $x \neq 0$—, W_1 is $\mathcal{C}^1$ for $x \neq 0$. When $d > 0$, W_1 is strictly positive for $|x| \neq 0$ and small enough. Indeed, V_a in the right hand side of (64) is homogeneous, and the other terms are homogeneous of degree larger than $\deg(V_a)$ —recall that each $\partial V_a/\partial x_i$ is homogeneous of degree $deg(V_a) - r_i$ and that the product of two homogeneous functions k_1 and k_2 is homogeneous of degree $\deg(k_1) + \deg(k_2)$— and periodic with respect to t. These terms are therefore small w.r.t V_a for $|x|$ small enough. If $d = 0$, W_1 is strictly positive for any $|x| \neq 0$ provided that ϵ is small enough, because each terms in the right hand side of (64) is homogeneous of degree $deg(V_a)$ and periodic w.r.t. t. Differentiating W_1 along the solutions of (9), we obtain from (63) and (64)

$$\begin{aligned}\dot{W}_1 &= \dot{V}_a + L_{\overline{f}_0 - f_0} V_a - \sum_{i=1}^{I} \epsilon^{-\frac{1}{2}} \alpha_i(t/\epsilon) L_{f_i} V_a \\ &\quad - \sum_{i=1}^{I}\sum_{j=1}^{I} \alpha_j(t/\epsilon) \left(\int_0^{t/\epsilon} \alpha_i(\tau)\, d\tau \right) L_{f_j} L_{f_i} V_a + g_1(x, t/\epsilon, \epsilon) \\ &= L_{\overline{f}_0} V_a - \sum_{i=1}^{I}\sum_{j=1}^{I} \alpha_j(t/\epsilon) \left(\int_0^{t/\epsilon} \alpha_i(\tau)\, d\tau \right) L_{f_j} L_{f_i} V_a + g_1(x, t/\epsilon, \epsilon)\,,\end{aligned} \tag{65}$$

with g_1 continuous and periodic w.r.t. t. Moreover, if $d = 0$, one easily verifies that g_1 is homogeneous of degree $\deg(V_a)$ and tends to zero for any (x,t) as ϵ tends to zero. If $d > 0$, one can also verify that $\deg(g_1) > deg(V_a) + d$. Finally, we define

$$W_2(x,t,\epsilon) = W_1(x,t,\epsilon) - \sum_{i=1}^{I}\sum_{j=1}^{I} \epsilon \left(\int_0^{t/\epsilon} \overline{\alpha_j \overline{\alpha_i}} - \alpha_j(\tau_2) \int_0^{\tau_2} \alpha_i(\tau_1)\, d\tau_1 d\tau_2 \right) L_{f_j} L_{f_i} V_a\,. \tag{66}$$

Again, by assumption, W_2 is $\mathcal{C}^1$ for $x \neq 0$. By the same arguments as those used for W_1, one shows that W_2 is also strictly positive for $|x| \neq 0$ and small enough when $d > 0$, and strictly positive for any $|x| \neq 0$ provided that ϵ is small enough when $d = 0$. Differentiating W_2 along the solutions of (9), we obtain from (65) and (66):

$$\begin{aligned}\dot{W}_2 &= \dot{W}_1 - \sum_{i=1}^{I}\sum_{j=1}^{I} \left(\overline{\alpha_j \overline{\alpha_i}} - \alpha_j(t/\epsilon) \int_0^{t/\epsilon} \alpha_i(\tau_1)\, d\tau_1 \right) L_{f_j} L_{f_i} V_a + g_2(x, t/\epsilon, \epsilon) \\ &= L_{\overline{f}_0} V_a - \sum_{i=1}^{I}\sum_{j=1}^{I} \overline{\alpha_j \overline{\alpha_i}} L_{f_j} L_{f_i} V_a + \sum_{i=1}^{2} g_i(x, t/\epsilon, \epsilon)\,,\end{aligned} \tag{67}$$

with g_2 a function satisfying the same properties as g_1. First, we show that the proof will be complete if

$$\sum_{i=1}^{I}\sum_{j=1}^{I} \overline{\alpha_j \overline{\alpha_i}} L_{f_j} L_{f_i} = \sum_{i=1}^{I}\sum_{j>i} \overline{\alpha_j \overline{\alpha_i}} L_{[f_i, f_j]}\,. \tag{68}$$

Suppose indeed that (68) holds. Then, choosing V_a as a Lyapunov function for the system (10) —the existence of which is guaranteed by Rosier's theorem since (10) is homogeneous—, we obtain in view of (67) and (68):

$$\dot{W}_2 = -W(x) + \sum_{i=1}^{2} g_i(x, t/\epsilon, \epsilon)\,, \tag{69}$$

with W a proper function, homogeneous of degree $\deg(V_a) + d$. If $d = 0$, g_1 and g_2 are homogeneous of degree $\deg(V_a) = \deg(W)$ for any ϵ, and since W is proper, there exist some functions $\gamma_i(\epsilon)$ ($i = 1, 2$) such that

$$\sup_{x \neq 0} \frac{|g_i(x, t/\epsilon, \epsilon)|}{W(x)} \leq \gamma_i(\epsilon) \tag{70}$$

— the above inequality obviously holds, by compactness, on the unit sphere of any homogeneous norm, and therefore also holds on $\mathbb{R}^n \setminus \{0\}$ since the ratio g_i/W is homogeneous of degree zero . Using the fact that for any $x \neq 0$ and any t, $g_i(x, t/\epsilon, \epsilon)$ tends to zero as ϵ tends to zero, it follows that $\gamma_i(\epsilon)$ also tends to zero as ϵ does and therefore, we deduce from (69) and (70) that for ϵ small enough, the Lyapunov candidate function $V := W_2$ is strictly decreasing along the trajectories of (9), as long as $x \neq 0$. If $d > 0$, g_1 and g_2 are homogeneous of degree larger than $\deg(V_a) + d$ and therefore, can be neglected w.r.t. W for x small enough. Again this shows that V is strictly decreasing along the trajectories of (9), as long as $x \neq 0$.

All we need to do now is to prove (68). We rewrite the left-hand side of (68) as

$$\sum_{i=1}^{I}\sum_{j<i}\overline{\alpha_j\overline{\alpha_i}}L_{f_j}L_{f_i} + \sum_{i=1}^{I}\sum_{j=i}\overline{\alpha_j\overline{\alpha_i}}L_{f_j}L_{f_i} + \sum_{i=1}^{I}\sum_{j>i}\overline{\alpha_j\overline{\alpha_i}}L_{f_j}L_{f_i} \,. \tag{71}$$

Since each α_i is of zero average, $\overline{\alpha_i\overline{\alpha_j}} = -\overline{\alpha_j\overline{\alpha_i}}$, and in particular $\overline{\alpha_i\overline{\alpha_i}} = 0$. This allows to rewrite (71) (after a change of indices) as

$$\sum_{i=1}^{I}\sum_{j>i}\overline{\alpha_j\overline{\alpha_i}}(L_{f_j}L_{f_i} - L_{f_i}L_{f_j}) = \sum_{i=1}^{I}\sum_{j>i}\overline{\alpha_j\overline{\alpha_i}}L_{[f_j,f_i]} \,. \tag{72}$$

This shows (68) and concludes the proof. ∎

Proof of Lemma 1

The proof only consists in a careful examination. Let us first recall the weight vector $r = (1,1,2,1,3,2,1,1,2,1,3,2)$. Since ω_1, ω_2, and v_1 have weight one, it is sufficient to show, in view of (42), that the u_i^ϵ $(i = 1, \ldots, 3)$ are homogeneous of degree one. From (43) and (44), we only need to show that the β_i's, γ_i's, and δ_i's are also homogeneous of degree one. From the definition of the weight vector r, the weights associated with the components z_i of z are $(1,1,2,1,3,2,2,3,2)$. This readily implies that ρ in (45) is homogeneous of degree one, and therefore that the δ_i's are homogeneous of degree one too. Then, from (50), it is easy to verify that —in the reduced coordinates z—,

$$\deg(b_1) = \deg(b_2) = \deg(b_3) = -1\,, \quad \deg(b_4) = \deg(b_5) = -2\,. \tag{73}$$

Using the fact that

$$\deg(L_f g) = \deg(g) + \deg(f)\,,$$

for any homogeneous function g and vector field f, the rest of the proof follows from (73) and from the homogeneity degree of ρ and of the δ_i's. ∎

Proof of Lemma 2

The closed-loop system (52)–(43)–(53) can be rewritten as:

$$\begin{aligned} \dot z = \;& b_0 + \textstyle\sum_{i=1}^5 b_i u_{i,1} \\ & + \epsilon^{-\frac{1}{2}}\left(\textstyle\sum_{i=1}^3 \alpha_i b_i u_{i,2} + \alpha_1 b_4 \beta_2 \delta_1 + \alpha_4 b_4 \gamma_2 \delta_1 + \alpha_3 b_5 \beta_2 \delta_3 + \alpha_5 b_5 \gamma_2 \delta_3\right)\,. \end{aligned} \tag{74}$$

We identify this system with system (9), with $I = 7$. Because the α_i's in (74) do not correspond to those in (9), we add a $'$ sign to the latter, i.e.

$$\begin{array}{l} f_0 = b_0 + \sum_{i=1}^5 b_i u_{i,1}, \\ f_i = b_i u_{i,2}\ (i = 1,2,3),\ \ f_4 = b_4\beta_2\delta_1, f_5 = b_4\gamma_2\delta_1, f_6 = b_5\beta_2\delta_3, f_7 = b_5\gamma_2\delta_3, \\ \alpha_i' = \alpha_i\ (i = 1,2,3), \quad \alpha_4' = \alpha_1, \alpha_5' = \alpha_4, \alpha_6' = \alpha_3, \alpha_7' = \alpha_5\,. \end{array} \tag{75}$$

We now compute the averaged system (10). In view of (75), this system is of the form

$$\dot{y} = \overline{f}_0(y) + \sum_{i=1}^{7}\sum_{j>i} \overline{\alpha_j'\overline{\alpha_i'}}[f_i, f_j](y)\,. \tag{76}$$

We first deduce from (44) and (54) that:

$$\begin{array}{ll} \overline{f}_0 & = b_0 + \sum_{i=1}^5 b_i \overline{u}_{i,1} \\ & = b_0 + \sum_{i=1}^3 b_i\beta_i + b_4(\beta_1\beta_2 + \gamma_1\gamma_2) + b_5(\beta_2\beta_3 + \gamma_2\gamma_3)\,. \end{array} \tag{77}$$

We now consider the terms in (76) involving Lie brackets of the f_k's. Using (44) and (46), it is easy to verify the following equalities:

$$\overline{\alpha_j\overline{\alpha_i}} = 0 \quad \text{for } j \geq i \text{ and } (i,j) \notin \{(1,5),(3,4)\}\,, \tag{78}$$

and

$$\begin{array}{l} c_{1,5} := \overline{\alpha_1\overline{\alpha_5}} = \begin{cases} \frac{1}{2c+1} & (c \neq -\frac{1}{2}) \\ \frac{2}{c+2} & (c = -\frac{1}{2}) \end{cases} \\ c_{3,4} := \overline{\alpha_3\overline{\alpha_4}} = \begin{cases} \frac{-2}{2c+1} & (c \neq -\frac{1}{2}) \\ \frac{-1}{c+2} & (c = -\frac{1}{2})\,. \end{cases} \end{array} \tag{79}$$

We deduce from (79) that, for any value of c,

$$c_{1,5} - c\, c_{3,4} = 1\,. \tag{80}$$

Let us recall at this point that for two functions α' and α'' of zero average, $\overline{\alpha'\overline{\alpha''}} = -\overline{\alpha''\overline{\alpha'}}$. As a consequence, using (75), (78) and (79), we obtain:

$$\begin{array}{rcl} \sum_{i=1}^{7}\sum_{j>i} \overline{\alpha_j'\overline{\alpha_i'}}\,[f_i, f_j](y) & = & \overline{\alpha_7'\overline{\alpha_1'}}\,[f_1, f_7](y) + \overline{\alpha_5'\overline{\alpha_3'}}\,[f_3, f_5](y) \\ & & +\overline{\alpha_7'\overline{\alpha_4'}}\,[f_4, f_7](y) + \overline{\alpha_6'\overline{\alpha_5'}}\,[f_5, f_6](y) \\ & = & c_{1,5}(-[b_1\delta_1, b_5\gamma_2\delta_3] + [b_5\gamma_2\delta_3, b_4\beta_2\delta_1])(y) \\ & & +c_{3,4}(-[b_3\delta_3, b_4\gamma_2\delta_1] + [b_4\gamma_2\delta_1, b_5\beta_2\delta_3])(y)\,. \end{array} \tag{81}$$

The proof of the lemma directly follows from (76), (77), (81), and (80). ∎

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