Trajectory tracking for non-holonomic vehicles

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1 Introduction

For many years, the control of non-holonomic vehicles has been a very active research field. At least two reasons account for this fact. On one hand, wheeled-vehicles constitute a major and ever more ubiquitous transportation system. Previously restricted to research laboratories and factories, automated wheeled-vehicles are now envisioned in everyday life (e.g. through car-platooning applications or urban transportation services), not to mention the military domain. These novel applications, which require coordination between multiple agents, give rise to new control problems. On the other hand, the kinematic equations of non-holonomic systems are highly nonlinear, and thus of particular interest for the development of nonlinear control theory and practice. Furthermore, some of the control methods initially developed for non-holonomic systems have proven to be applicable to other physical systems (e.g. underactuated mechanical systems), as well as to more general classes of nonlinear systems.

The present paper addresses feedback motion control of non-holonomic vehicles, and more specifically *trajectory tracking*, by which we mean the problem of stabilizing the state, or an output function of the state, to a desired reference value, possibly time-varying. So defined the trajectory tracking problem incorporates most of the problems addressed in the control literature: output feedback regulation, asymptotic stabilization of a fixed-point and, more generally, of admissible non-stationary trajectories, practical stabilization of general trajectories. A notable exception is the *path following* problem, which will not be considered here because it is slightly different in nature. This problem is nonetheless important for applications, and we refer the reader to e.g. [29, 7] for related control design results.

The methods reviewed in the paper cover a large range of applications: position control of vehicles (e.g. car-platooning, "cruising mode" control), position and orientation control (e.g. parking, stabilization of pre-planned reference trajectories, tracking of moving targets). For controllable linear systems, linear state feedbacks provide simple, efficient, and robust control solutions. By contrast, for non-holonomic systems, different types of feedback laws have been proposed, each one carrying its specific advantages and limitations. This diversity is partly justified by several theoretical results, recalled further in the paper, which account for the difficulty/impossibility of deriving feedback laws endowed with all the good properties of linear feedbacks for linear control systems. As a consequence, the choice of a control approach for a given application is a matter of compromise, depending on the system characteristics and the performance requirements. At this point, simulations can provide useful complementary guidelines for the choice of the control law. Due to space limitations, we were not able to include simulation results in this paper, but we refer the interested reader to [23] where a detailed simulation study for a car like-vehicle, based on the control laws here proposed, is given.

While the present paper reviews most of the classical trajectory tracking problems for non-holonomic vehicles, it is by no means a complete survey of existing control methods. Besides paper size considerations, those here discussed are primarily based on our own experience, and reflect our preferences. For survey-like expositions, we refer the reader to e.g. [12, 7]. The paper's scope is also limited to "classical" non-holonomic vehicles, for which the "hard" non-linearities arise exclusively from the kinematics. More general non-holonomic mechanical systems, in the sense of e.g. [4], are not considered here.

The paper is organized as follows. Several models are introduced in Section 2, with some of their properties being recalled. Section 3 is the core of the paper: the main trajectory tracking problems are reviewed from both the application and control design viewpoints. In particular, advantages and limitations inherent to specific types of feedback controllers are discussed. Finally, some concluding remarks are provided.

2 Modeling of vehicles' kinematics

In this section, some aspects of the modeling of non-holonomic vehicles are recalled and illustrated in the case of unicycle and car-like vehicles. The properties reviewed in this section apply (or extend) to most wheeled vehicles used in real-life applications.

2.1 Kinematics w.r.t. an inertial frame

Wheeled mechanical systems are characterized by non-completely integrable velocity constraints $\langle \alpha_j(q), \dot{q} \rangle = 0\,, \quad q \in Q$, with Q the mechanical configuration space (manifold) and the α_j's denoting smooth mappings. These constraints are derived from the usual wheel's rolling-without-slipping assumption. Under very mild conditions (satisfied for most systems of practical interest), these constraints are equivalent to

$$\dot{q} = \sum_{i=1}^{m} u_i X_i(q) \tag{1}$$

where the u_i's denote "free" variables, the X_i's are smooth vector fields (v.f.) orthogonal to the α_j's, and $m < \dim(Q)$. A state space reduction to a submanifold M of the mechanical configuration space Q (see e.g. [6] for more details) yields a control model in the same form, with the following properties which will be assumed to hold from now on.

Properties:
P.1 $m < n := \dim(M)$,
P.2 the X_i's are linearly independent at any $q \in M$,
P.3 the X_i's satisfy the Lie Algebra Rank Condition on M, i.e. for any $q \in M$,

$$\mathrm{span}\{X_i(q), [X_i, X_j](q), [X_i, [X_j, X_k]](q), \ldots\} = \mathbb{R}^n$$

Recall that Property **P.3** ensures that System (1) is locally controllable at any point, and globally controllable if M is connected (see e.g. [25, Prop. 3.15]).

A basic example is the unicycle-like robot of Fig. 1, whose kinematic model with respect to the inertial frame $\mathcal{F}_0 = (0, i_0, j_0)$ is given by:

$$\dot{q} = u_1 \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} + u_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{2}$$

with $q = (x, y, \theta)'$, u_1 the signed longitudinal velocity of the vehicle's body, and u_2 its angular velocity.

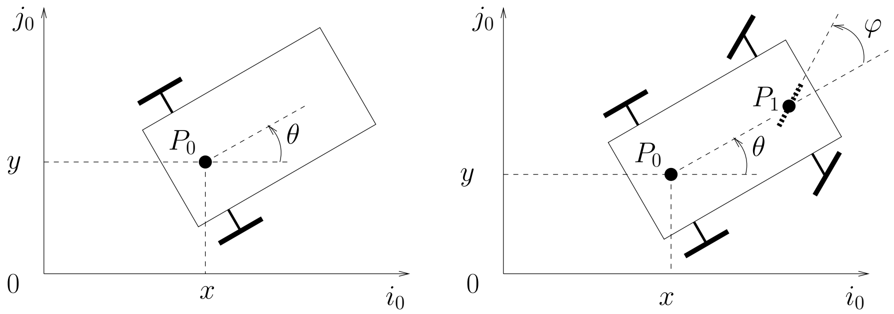

Fig. 1. The unicycle (l) and car (r)-like vehicles

A second example is the car-like vehicle of Fig. 1. A kinematic model for this system is

$$\dot{q} = u_1 \begin{pmatrix} \cos\theta \\ \sin\theta \\ \dfrac{\tan\varphi}{\ell} \\ 0 \end{pmatrix} + u_2 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{3}$$

with $q = (x, y, \theta, \varphi)'$, φ the steering wheel angle, and ℓ the distance between P_0 and P_1. An equivalent, but slightly simpler, model is given by

$$\dot{q} = u_1 \begin{pmatrix} \cos\theta \\ \sin\theta \\ \zeta \\ 0 \end{pmatrix} + u_2 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{4}$$

with $q = (x, y, \theta, \zeta)'$ and $\zeta := (\tan\varphi)/\ell$. Note that the control variable u_2 in (4) differs from the one in (3) by a factor $(1 + \tan^2\varphi)/\ell$.

2.2 Kinematics w.r.t. a moving frame

A generic property of vehicles is the invariance (or symmetry) with respect to the Lie group of rigid motions in the plane. More precisely, following [4], the state space M can usually be decomposed as a product $M = G \times S$, where $G = \mathbb{R}^2 \times \mathbb{S}^1 \approx SE(2)$ is associated with the vehicle's body configuration (position and orientation) in the plane, and S is associated with "internal" state variables of the vehicle. With this decomposition, one has

$$q = \begin{pmatrix} g \\ s \end{pmatrix}, \qquad g = (x, y, \theta)' \in G, s \in S \tag{5}$$

and System (1) can be written as

$$\begin{aligned} \dot{g} &= \sum_{i=1}^{m} u_i X_i^g(g, s) \\ \dot{s} &= \sum_{i=1}^{m} u_i X_i^s(s) \end{aligned} \tag{6}$$

For example, $S = \varnothing$ for the unicycle-like vehicle whereas, in the case of the car-like vehicle, $S = \mathbb{S}^1$ with $s = \varphi$ for System (3), and $S = \mathbb{R}$ with $s = \zeta$ for System (4).

G is endowed with the group product

$$g_1 g_2 := \begin{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + R(\theta_1) \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \\ \theta_1 + \theta_2 \end{pmatrix} \tag{7}$$

with $g_i = (x_i, y_i, \theta_i)' \in G$ $(i = 1, 2)$, and $R(\theta)$ the rotation matrix of angle θ. For any fixed s the "vector fields" $X_i^g(g, s)$ are left-invariant w.r.t. this group product, i.e. for any fixed $g_0 \in G$ and any solution $t \longmapsto (g(t), s(t))$ to (6), $t \longmapsto (g_0 g(t), s(t))$ is also a solution to (6), associated with the same control input. By using this invariance property, it is simple to show that the kinematics w.r.t. a moving frame $\mathcal{F}_r = (0_r, i_r, j_r)$ (see Fig. 2) is given by

$$\begin{aligned} \dot{g}_e &= \sum_{i=1}^{m} u_i X_i^g(g_e, s) - \bar{R}(g_e, g_r)\dot{g}_r \\ \dot{s} &= \sum_{i=1}^{m} u_i X_i^s(s) \end{aligned} \tag{8}$$

with

$$g_e := g_r^{-1} g = \begin{pmatrix} R(-\theta_r) \begin{pmatrix} x - x_r \\ y - y_r \end{pmatrix} \\ \theta - \theta_r \end{pmatrix} \text{ and } \bar{R}(g_e, g_r) := \begin{pmatrix} R(-\theta_r) & \begin{pmatrix} -y_e \\ x_e \end{pmatrix} \\ 0 & 1 \end{pmatrix} \tag{9}$$

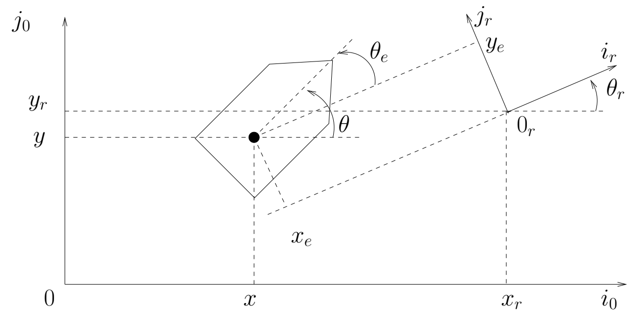

Fig. 2. Kinematics w.r.t. a moving frame

Note that System (8) can also be written as

$$\dot{q}_e = \sum_{i=1}^{m} u_i X_i(q_e) + P(q_e, t) \tag{10}$$

with $q_e = (g_e, s)$ and $P(q_e, t) = (-\bar{R}(g_e, g_r)\dot{g}_r, 0)$.

System (8) is a generalization of System (6), and corresponds to the kinematic model w.r.t. the moving frame $\mathcal{F}_r$. Let us mention a few important properties of this system. First of all, it is defined for any configuration of the vehicle. Then, as a consequence of the invariance property evoked above, the control v.f. of this system are the same as those of System (6).

2.3 Tracking error models

A trajectory tracking problem typically involves a reference trajectory $q_r : t \longmapsto q_r(t) = (g_r, s_r)(t)$, with $t \in \mathbb{R}_+$. Then, one has to define a suitable representation for the tracking error. This step is all the more important that an adequate choice significantly facilitates the control design. As pointed out earlier, g_e given by (9) is a natural choice for the tracking error associated with g. Usually the set S is a product such as $\mathbb{S}^1 \times \mathbb{S}^1 \times \cdots$ or $\mathbb{R} \times \mathbb{R} \times \cdots$, so that it is also endowed with a "natural" (abelian) group structure. A simple choice for the tracking error associated with s is $s_e := s - s_r$. Note, however, that depending on the control v.f.'s structure, this is not always the best choice. Now, with the tracking error defined by

$$q_e := \begin{pmatrix} g_e \\ s_e \end{pmatrix} := \begin{pmatrix} g_r^{-1} g \\ s - s_r \end{pmatrix}$$

we obtain the following *tracking error model*, deduced from (8),

$$\begin{aligned} \dot{g}_e &= \sum_{i=1}^{m} u_i X_i^g(g_e, s_e + s_r) - \bar{R}(g_e, g_r)\dot{g}_r \\ \dot{s}_e &= \sum_{i=1}^{m} u_i X_i^s(s_e + s_r) - \dot{s}_r \end{aligned} \tag{11}$$

This model can be further particularized in the case when the reference trajectory is "feasible" (or "admissible"), i.e. when there exist smooth time functions u_i^r such that

$$\forall t, \quad \dot{q}_r(t) = \sum_{i=1}^{m} u_i^r(t) X_i(q_r(t))$$

Then, (11) becomes

$$\begin{aligned} \dot{g}_e &= \sum_{i=1}^{m} u_i^e X_i^g(g_e, s_e + s_r) + \sum_{i=1}^{m} u_i^r \left(X_i^g(g_e, s_e + s_r) - \tilde{A}(g_e) X_i^g(0, s_r) \right) \\ \dot{s}_e &= \sum_{i=1}^{m} u_i^e X_i^s(s_e + s_r) + \sum_{i=1}^{m} u_i^r \left(X_i^s(s_e + s_r) - X_i^s(s_r) \right) \end{aligned} \tag{12}$$

with $u_i^e := u_i - u_i^r$ and

$$\tilde{A}(g_e) := \begin{pmatrix} I_2 & \begin{pmatrix} -y_e \\ x_e \end{pmatrix} \\ 0 & 1 \end{pmatrix}$$

The choice $s_e = s - s_r$ is natural when the v.f. X_i are affine in s. This property is satisfied for several vehicles' models, under an appropriate choice of coordinates, and in particular by the car's model (4). In this latter case, Eq. (12) is given by

$$\begin{aligned} \dot{g}_e &= u_1^e \begin{pmatrix} \cos\theta_e \\ \sin\theta_e \\ \zeta_e + \zeta_r \end{pmatrix} + u_1^r \begin{pmatrix} \cos\theta_e - 1 + y_e\zeta_r \\ \sin\theta_e - x_e\zeta_r \\ \zeta_e \end{pmatrix} \\ \dot{\zeta}_e &= u_2^e \end{aligned} \tag{13}$$

2.4 Linearized systems

A classical way to address the control of a nonlinear error model like (12) is to consider its linearization at the equilibrium $(g_e, s_e, u^e) = 0$. It is given by

$$\dot{q}_e = \sum_{i=1}^{m} u_i^r(t) A_i(s_r(t)) q_e + B(s_r(t)) u^e \tag{14}$$

with $B(s_r) = (X_1(0, s_r) \cdots X_m(0, s_r))$ and the A_i's some matrices easily determined from (12). A first observation is that System (14) is independent of g_r. This is again a consequence of the invariance property recalled in Section 2.2. Another well known property is that this linearized system is neither controllable nor stabilizable at fixed points (i.e. $u^r = 0$), since in this case the system reduces to $\dot{q}_e = Bu^e$ with B a $n \times m$ constant matrix, and from Property **P.1**, $m < n$. However, along non-stationary reference trajectories, the linearized system can be controllable. Consider for instance the car's error model (13). In this case, the matrices A_1, A_2, and B in (14) are given by

$$A_1(s_r) = \begin{pmatrix} 0 & \zeta_r & 0 & 0 \\ -\zeta_r & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B(s_r) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ \zeta_r & 0 \\ 0 & 1 \end{pmatrix}$$

and $A_2(s_r) = 0$. By applying classical results from linear control theory (see e.g. [8, Sec. 5.3]), one can show for example that if, on a time-interval $[t_0, t_1]$, ζ_r and u_1^r are smooth functions of time and u_1^r is not identically zero, then the linearized system is controllable on $[t_0, t_1]$. In other words, this system is controllable for "almost all" reference inputs u^r. As shown in [31], this is a generic property for analytic Systems (1) satisfying Property **P.3**.

2.5 Transformations into chained systems

Let us close this section with a few remarks on local models. It is well known, from [24, 32], that the kinematic models (1) of several non-holonomic vehicles, like unicycle and car-like vehicles, can be locally transformed into *chained systems* defined by

$$\begin{cases} \dot{x}_1 = v_1 \\ \dot{x}_2 = v_2 \\ \dot{x}_k = v_1 x_{k-1} \quad (k = 3, \ldots, n) \end{cases} \tag{15}$$

For example, the car's model (4) can be transformed into System (15) with $n = 4$, with the coordinates x_i $(i = 1, \ldots, 4)$ and inputs v_1, v_2 defined by

$$\begin{aligned} (x_1, x_2, x_3, x_4) &= (x, \zeta/(\cos^3\theta), \tan\theta, y) \\ (v_1, v_2) &= (u_1 \cos\theta, (u_2 + 3u_1\zeta^2 \tan\theta)/(\cos^3\theta)) \end{aligned} \tag{16}$$

Beside its (apparent) simplicity, an important property of System (15) is that its v.f. are left-invariant, in the sense of Section 2.2, w.r.t to the group operation on $\mathbb{R}^n$ defined by

$$xy = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_k + y_k + \sum_{j=2}^{k-1} \frac{y_1^{k-j} x_j}{(k-j)!} \quad (k = 3, \dots, n) \end{pmatrix}$$

with $x, y \in \mathbb{R}^n$. In these new coordinates, the group invariance suggests to define the tracking error vector as $x_e := x_r^{-1} x$ (compare with (9)). A difficulty, however, comes from that the change of coordinates is only locally defined, on a domain related to the inertial frame $\mathcal{F}_0$. While this is not a strong limitation for fixed-point stabilization, it becomes a major issue when the reference trajectory is not compelled to stay within the domain of definition of the change of coordinates. A way to handle this difficulty consists in considering a local transformation associated with the tracking error model (8). More precisely, whenever System (1) can be transformed into a chained system, then it follows from the formulation (10) of System (8) that the latter can also be transformed into a chained system with an added perturbation term $P(x_e, t)$. This new error model is then well defined whenever the vehicle's configuration g is in a (semi-global) neighborhood of the configuration g_r associated with the reference frame.

3 An overview of trajectory tracking problems

Consider a linear control system

$$\dot{x} = Ax + Bu \tag{17}$$

with the pair (A, B) controllable, and a matrix K such that $A+BK$ is Hurwitz-stable. Consider also an admissible reference trajectory $t \mapsto x_r(t)$, with $\dot{x}_r = Ax_r + Bu^r$. Then, the feedback law

$$u(x, x_r, u^r) := u^r + K(x - x^r) \tag{18}$$

applied to System (17) yields $\dot{x}_e = (A + BK)x_e$, with $x_e := x - x_r$. Since $A + BK$ is Hurwitz-stable, the feedback law (18) asymptotically stabilizes any admissible reference trajectory. Does there exist similar "universal" continuous feedback controls $u(x, x_r, u^r)$ for System (1)? It has been known for a long time (Brockett [5]) that the answer to this question is negative, because fixed-points (for which $u^r = 0$) cannot be asymptotically stabilized by continuous pure-state feedbacks. However, they can always be asymptotically stabilized by (periodic) *time-varying* continuous state feedbacks [9]. Then, we may ask whether there exist universal time-varying continuous feedback laws $u(x, x_r, u^r, t)$ that make *any* admissible reference trajectory asymptotically stable. A recent result shows that the answer to this question is again negative.

Theorem 1. *[13] Consider System* (1) *with $m = 2$, and assume Properties* **P.1-P.3**. *Then, given a continuous feedback law $u(x, x_r, u^r, t)$, with $\partial u/\partial t$ and $\partial^2 u/(\partial u^r \partial t)$ well defined everywhere and bounded on $\{(x, x_r, u^r, t) : x = x_r, u^r = 0\}$, there exist admissible reference trajectories which are not asymptotically stabilized by this control*[1].

Since universal asymptotic stabilization of admissible reference trajectories is not possible, what else can be done? Three major possibilities have been explored in the dedicated control literature.

1. Output feedback control. This consists in stabilizing only a part of the system's state. A typical application example is the car-platooning problem in the cruising mode for which controlling the vehicle's orientation directly is not compulsory (more details in the next section).
2. Asymptotic stabilization of specific trajectories. By restricting the set of admissible trajectories, via the imposition of adequate extra conditions, the problem of asymptotic stabilization considered earlier becomes amenable. Two types of trajectories have been more specifically addressed: trajectories reduced to fixed points ($u^r = 0$), corresponding to parking-like applications, and trajectories for which u^r does not converge to zero.
3. Practical stabilization. The idea is to relax the asymptotic stabilization objective. For many applications, practical stabilization yielding ultimately bounded and small tracking errors is sufficient. Not only the theoretical obstruction revealed by Theorem 1 no longer holds in this case, but also any trajectory, not necessarily admissible, can be stabilized in this way.

We now review in more details these different trajectory tracking problems and strategies.

3.1 Output feedback control

As pointed out in Section 2.4, the linearization of a control system (1) satisfying Property **P.1**, at any equilibrium $(q, u) = (q_0, 0)$, is neither controllable nor asymptotically stabilizable. This accounts for the difficulty of controlling such a system. In some applications, however, it is not necessary to control the full state q, but only a vector function $h := (h_1, \ldots, h_p)'$ of q and, possibly, t. In particular, if $p \leq m$, then the mapping

$$u \longmapsto \frac{\partial h}{\partial q}\dot{q} = \frac{\partial h}{\partial q}\sum_{i=1}^{m} u_i X_i(q) = \frac{\partial h}{\partial q}X(q)u \tag{19}$$

with $X(q) = (X_1(q) \cdots X_m(q))$, may be onto, as a result of the full rankedness of the matrix $\frac{\partial h}{\partial q}X(q)$. If this is the case, h can be easily controlled via its time-derivative. To illustrate this fact, consider the car-platooning problem, for

[1] A few (weak) uniformity assumptions w.r.t. x_r are also required in the definition of asymptotic stability (see [13] for more details).

which a vehicle is controlled so as to follow another vehicle which moves with a positive longitudinal velocity. This problem can be solved via the asymptotic stabilization of a point P attached to the controlled vehicle to a point P_r attached behind the leading reference vehicle (see Fig. 3). With the notation of this figure, the function h above can then be defined as $h(q,t) = (x_P(q) - x_r(t), y_P(q) - y_r(t))'$ or, if only relative measurements are available, by

$$h(q,t) = R(-\theta_r) \begin{pmatrix} x_P(q) - x_r(t) \\ y_P(q) - y_t(t) \end{pmatrix} \tag{20}$$

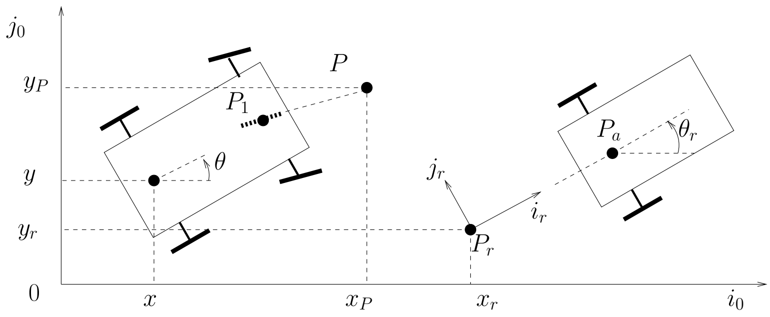

Fig. 3. Car-platooning

The control objective is to asymptotically stabilize h to zero. With the output function h defined by (20) and the control model (3), a direct calculation shows that the determinant of the square matrix $\frac{\partial h}{\partial q}X(q)$ in (19) is equal to $d/(\cos\varphi)$ with d the distance between P_1 and P. Since

$$\dot{h} = \frac{\partial h}{\partial q}X(q)u + \frac{\partial h}{\partial t}$$

the feedback law

$$u = \left(\frac{\partial h}{\partial q}X(q)\right)^{-1}\left(Kh - \frac{\partial h}{\partial t}\right)$$

with K any Hurwitz stable matrix, is well defined when $d > 0$ and $|\varphi| < \pi/2$, and yields $\dot{h} = Kh$, from which the asymptotic and exponential stability of $h = 0$ follows. Such a control strategy, which basically consists in virtually "hooking" the controlled vehicle to the leader by tying the points P and P_r together, is simple and works well as long as the longitudinal velocity of the reference vehicle remains positive. However, it is also intuitively clear that it does not work when the reference vehicle moves backward (with a negative longitudinal velocity), because the orientation angle difference $\theta - \theta_r$ and the steering wheel angle φ then tend to grow away from zero, so as to enter new stability regions near π. This is similar to the jack-knife effect for a tractor trailer. To avoid this effect, an active orientation control is needed.

3.2 Stabilization of specific trajectories

Stabilization of both the position and the orientation implies controlling the full state q. Asymptotic stabilization of trajectories has been successfully addressed in two major cases:

1. The reference trajectory is reduced to a fixed point. This corresponds to a parking-like problem.
2. The reference trajectory is admissible (feasible) and the longitudinal velocity along this trajectory does not tend to zero.

Fixed-point stabilization

There is a rich control literature on the fixed point stabilization problem for non-holonomic vehicles. Brockett's theorem [5], (and its extensions [34, 27]...) has first revealed the non-existence of smooth pure-state feedback asymptotic stabilizers.

Theorem 2. *[5, 27] Consider a system* (1) *satisfying the properties* **P.1**–**P.2***, and an equilibrium point q_0 of this system. Then, there exists no continuous feedback $u(q)$ that makes q_0 asymptotically stable.*

Following [28], in which time-varying continuous feedback was used to asymptotically stabilize a unicycle-like robot, the genericity of this type of control law has been established.

Theorem 3. *[9] Consider a system* (1) *satisfying the property* **P.3***, and an equilibrium point q_0 of this system. Then, there exist smooth time-varying feedbacks $u(x, t)$, periodic w.r.t. t, that make q_0 asymptotically stable.*

Since then, many studies have been devoted to the design of feedback laws endowed with similar stabilizing properties. Despite a decade of research effort in this direction, it seems that the following dilemma cannot be avoided:

- Smooth (i.e. differentiable or at least Lipschitz-continuous) asymptotic stabilizers can be endowed with good robustness[2] properties and low noise sensitivity, but they yield slow (not exponential) convergence to the considered equilibrium point (see e.g. [16]).
- Stability and uniform exponential convergence can be obtained with feedback laws which are only continuous, but such controllers suffer from their lack of robustness, and their high sensitivity to noise.

[2] The type of robustness we are more specifically considering here is the property of preserving the closed-loop stability of the desired equilibrium against small structured modeling errors, control delays, sampling of the control law, fluctuations of the sampling period, etc...

To illustrate this dilemma, we provide and discuss below different types of asymptotic stabilizers which have been proposed in the literature. Explicit control laws are given for the car-like vehicle. For simplicity, the desired equilibrium is chosen as the origin of the frame $\mathcal{F}_0$, and the coordinates (16) associated with the chained system (15) are used. Let us start with smooth time-varying feedbacks, as proposed e.g. in [28, 26, 33]...

Proposition 1. *[29] Consider some constants* $k_i > 0$ $(i = 1, \ldots, 4)$ *such that the polynomial* $p(s) := s^3 + k_2 s^2 + k_3 s + k_4$ *is Hurwitz. Then, for any Lipschitz-continuous mapping* $g : \mathbb{R}^3 \longrightarrow \mathbb{R}$ *such that* $g(0) = 0$ *and* $g(y) > 0$ *for* $y \neq 0$*, the Lipschitz-continuous feedback law*

$$\begin{cases} v_1(x,t) = -k_1 x_1 + g(x_2, x_3, x_4) \sin t \\ v_2(x,t) = -|v_1| k_2 x_2 - v_1 k_3 x_3 - |v_1| k_4 x_4 \end{cases} \tag{21}$$

makes $x = 0$ *asymptotically stable for the chained system* (15) *with* $n = 4$.

While Lipschitz-continuous feedbacks are reasonably insensitive to measurement noise, and may be robust w.r.t. modeling errors, their main limitation is their slow convergence rate. Indeed, most trajectories converge to zero only like $t^{-1/\alpha}$ $(\alpha \geq 1)$. This has been the main motivation for investigating feedbacks which are only continuous. In this context, homogeneous (polynomial) v.f. have played an important role, especially those of degree zero which yield exponential convergence once asymptotic stability is ensured. It is beyond the scope of this paper to provide an introduction to homogeneous systems, and we refer the reader to [11] for more details. Let us only mention that the control theory for homogeneous systems is an important and useful extension of the classical theory for linear systems. The association of homogeneity properties with a time-varying feedback, for the asymptotic stabilization of non-holonomic vehicles, was first proposed in [15]. This approach is quite general since it applies to any system (1) with the property **P.3** [17]. An example of such a controller is proposed next (see e.g. [16] for complementary results).

Proposition 2. *[19] Consider some constants* $k_i > 0$ $(i = 1, \ldots, 5)$ *such that the polynomial* $p(s) := s^3 + k_2 s^2 + k_3 s + k_4$ *is Hurwitz. For any* $p, d \in \mathbb{N}^*$*, denote by* $\rho_{p,d}$ *the function defined on* $\mathbb{R}^3$ *by*

$$\rho_{p,q}(\bar{x}_2) := \left(|x_2|^{p/r_2(d)} + |x_3|^{p/r_3(d)} + |x_4|^{p/r_4(d)} \right)^{1/p}$$

with $\bar{x}_2 := (x_2, x_3, x_4)$*,* $r(d) := (1, d, d+1, d+2)'$*. Then, there exists* $d_0 > 0$ *such that, for any* $d \geq d_0$ *and* $p > d + 2$*, the continuous feedback law*

$$\begin{cases} v_1(x,t) = -\left(k_1(x_1 \sin t + |x_1|) + k_5 \rho_{p,d}(\bar{x}_2)\right) \sin t \\ v_2(x,t) = -\dfrac{|v_1| k_2 x_2}{\rho_{p,d}(\bar{x}_2)} - \dfrac{v_1 k_3 x_3}{\rho_{p,d}^2(\bar{x}_2)} - \dfrac{|v_1| k_4 x_4}{\rho_{p,d}^3(\bar{x}_2)} \end{cases} \tag{22}$$

makes $x = 0$ $\mathcal{K}$*-exponentially stable for the chained system* (15) *with* $n = 4$.

The property of $\mathcal{K}$-exponential stability evoked in the above proposition means that along any solution to the controlled system, $|x(t)| \leq k(|x_0|)e^{-\gamma t}$, with $\gamma > 0$ and k some continuous, positive, strictly increasing function from $\mathbb{R}_+$ to $\mathbb{R}_+$, with $k(0) = 0$. While this property ensures uniform exponential convergence of the trajectories to the origin, it is not equivalent to the classical definition of uniform exponential stability because k is not necessarily smooth. It is also interesting to compare the feedbacks (21) and (22). In particular, the second component v_2 of (22) is very similar to the component v_2 of (21), except that the constant gains k_i are replaced by the state dependent control gains $k_i/\rho_{p,d}^{i-1}(\bar{x}_2)$. The fact that these "gains" tend to infinity when x tends to zero (even though the overall control is well defined by continuity at $x = 0$), accounts for the lack of robustness and high noise sensitivity of this type of feedback. A more specific statement concerning the robustness issue is as follows.

Proposition 3. *[14] For any $\varepsilon > 0$, there exist v.f. $Y_1^\varepsilon, Y_2^\varepsilon$ on $\mathbb{R}^4$, with $\|Y_i^\varepsilon(x)\| \leq \varepsilon$ $(i = 1, 2; x \in \mathbb{R}^4)$, such that the origin of the system*

$$\dot{x} = v_1(x,t)(X_1 + Y_1^\varepsilon) + v_2(x,t)(X_2 + Y_2^\varepsilon) \tag{23}$$

with X_1 and X_2 the v.f. of the chained system of dimension 4, and v_1, v_2 given by (22)*, is not stable.*

In other words, whereas the origin of System (23) is asymptotically stable when $Y_1^\varepsilon = Y_2^\varepsilon = 0$, the slightest perturbation on the control v.f. may invalidate this result. Note that a small inaccuracy about the robot's geometry may very well account for such a perturbation in the modeling of the system. The practical consequence is that, instead of converging to the origin, the state will typically converge to a limit cycle contained in a neighborhood of the origin. It is possible to give an order of magnitude for the size of such a limit cycle, as a function of ε (assumed small). In terms of the original coordinates (x, y, θ, φ) of the car's model (3), we obtain for the control law (22)

$$\text{size}(x) \approx \varepsilon^{1/(d+1)}, \text{size}(y) \approx \varepsilon^{(d+2)/(d+1)}, \text{size}(\theta) \approx \varepsilon, \text{size}(\varphi) \approx \varepsilon^{d/(d+1)} \tag{24}$$

with size(z) denoting the order of magnitude of the limit cycle in the z-direction. Relation (24) shows how differently the state components are affected, and also how the modeling errors effects are amplified for the components x and φ. Note, however, that (24) corresponds to the worst case so that there also exist structured uncertainties which preserve the asymptotic stability of the origin. An analysis of the sensitivity to state measurement noise would yield similar results, with ε interpreted as the maximum amplitude of the noise.

In [3], a control approach based on the use of hybrid feedbacks has been proposed to ensure exponential convergence to the origin, with the stability of the origin being preserved against small perturbations of the control v.f., like those considered in (23). The hybrid continuous/discrete feedbacks there considered are related to time-varying continuous feedback $v(x,t)$, except that the dependence on the state x is updated only periodically. The result given in [3], devoted to the class of chained systems, has been extended in [18] to any analytic system (1) satisfying Property **P.3**. For example, the following result is shown in this latter reference.

Proposition 4. *[18] Let $T, k_1, \dots, k_4$ be some constants such that $T > 0$ and $|k_i| < 1\ \forall i$. Then the hybrid-feedback $v(x(kT),t),\ k \in \mathbb{N} \cap (t/T - 1, t/T]$ with v defined, in the coordinates x_i of the chained system of dimension 4, by*

$$\begin{cases} v_1(x,t) = \left((k_1-1)x_1 + 2\pi\rho(x)\sin(2\pi t/T)\right)/T \\ v_2(x,t) = \Big((k_2-1)x_2 + 2(k_3-1)\frac{x_3}{\rho(x)}\cos(2\pi t/T) \\ \qquad\qquad +8(k_4-1)\frac{x_4}{\rho^2(x)}\cos(4\pi t/T)\Big)/T \end{cases} \tag{25}$$

and $\rho(x) = a_3|x_3|^{1/3} + a_4|x_4|^{1/4}$ $(a_3, a_4 > 0)$ is a $\mathcal{K}(T)$-exponential stabilizer for the car, robust w.r.t. unmodeled dynamics.

The property of $\mathcal{K}(T)$-exponential stability means that, for some constants $K, \eta > 0$ and $\gamma < 1$, each solution $x(t,0,x_0)$ of the controlled system with initial condition x_0 at $t=0$ satisfies, for any $k \in \mathbb{N}$ and $s \in [0,T)$, $|x((k+1)T,0,x_0)| \le \gamma|x(kT,0,x_0)|$ and $|x(kT+s,0,x_0)| \le K|x(kT,0,x_0)|^\eta$. While it implies the exponential convergence of the solutions to the origin, it is neither equivalent to the classical uniform exponential stability, nor to the $\mathcal{K}$-exponential stability.

Unfortunately, the robustness to unmodeled dynamics evoked in Proposition 4 relies on the perfect timing of the control implementation. The slightest control delay, or fluctuation of the sampling period, destroys this property with the same effects as those discussed previously for continuous homogeneous feedbacks like (22).

To summarize, asymptotic fixed-point stabilization is theoretically possible but, in practice, no control solution designed to this goal is entirely satisfactory due to the difficulty, inherent to this type of system, of ensuring robust stability and fast convergence simultaneously.

Non-stationary admissible trajectories

The difficulty of stabilizing fixed-points comes from the fact that the linearized system at such equilibria is not controllable (and not stabilizable either). However, as shown in Section 2.4, the system obtained by linearizing the tracking error equations around other reference trajectories may be controllable. For a car-like vehicle, this is the case for example when the longitudinal velocity u_1^r associated with the reference trajectory is different from zero. Under some extra conditions on u_1^r (e.g. if $|u_1^r|$ remains larger than a positive constant), such trajectories can then be *locally* asymptotically stabilized by using feedbacks calculated for the linearized error system. The following proposition provides a nonlinear version of such a feedback for the car-like vehicle, with a proven large domain of stability. This result can be found in [21] (modulo a slight extension of the stability domain), following a technique already used in [30].

Proposition 5. *[21] Consider the tracking error model* (13) *associated with the car model* (4)*, and assume that along the reference trajectory ζ_r is bounded. Then, the feedback law*

$$\begin{cases} u_1^e = -k_1|u_1^r|\left(x_e\cos\theta_e + y_e\sin\theta_e + \frac{\zeta}{k_2}\frac{\sin}{\cos^3}(\frac{\theta_e}{2})\right) \\ u_2^e = -k_3u_1^r\frac{\sin}{\cos^3}(\frac{\theta_e}{2}) - k_4|u_1^r|z + k_2F_x\cos^2\frac{\theta_e}{2}\sin\theta_e - 2k_2F_y\cos^4\frac{\theta_e}{2} \\ -k_2F_\theta\left(x_e(\frac{3}{4}\sin^2\theta_e - \cos^4\frac{\theta_e}{2}) - 2y_e\sin\theta_e\cos^2\frac{\theta_e}{2}\right) \end{cases} \tag{26}$$

with $k_1,\ldots,k_4>0$, $z:=\zeta_e+2k_2(-x_e\sin\frac{\theta_e}{2}+y_e\cos\frac{\theta_e}{2})\cos^3\frac{\theta_e}{2}$, and

$$F_x = u_1^e\cos\theta_e + u_1^r(\cos\theta_e - 1 + y_e\zeta_r),\ F_y = u_1^e\sin\theta_e + u_1^r(\sin\theta_e - x_e\zeta_r)$$
$$F_\theta = u_1^e\zeta + u_1^r\zeta_e$$

makes the origin of System (13) *stable. Furthermore, if u^r is differentiable with u^r and $\dot{u}^r$ bounded, and u_1^r does not tend to zero as t tend to infinity, then the origin is also globally asymptotically stable on the set $\mathbb{R}^2\times(-\pi,\pi)\times\mathbb{R}$.*

Note that the sign of u_1^r is not required to be constant, so that reference trajectories involving both forward and backward motions can be stabilized (compare with Section 3.1).

A simpler controller can be obtained either by working on a linearized error system, or by linearizing the expression (26) w.r.t. the state variables. This yields the feedback control

$$\begin{cases} u_1^e = -k_1|u_1^r|(x_e + \zeta_r\theta_e/(2k_2)) \\ u_2^e = 2k_2u_1^r\zeta_rx_e - 2k_2k_4|u_1^r|y_e - u_1^r(2k_2+k_3/2)\theta_e - k_4|u_1^r|\zeta_e \end{cases} \tag{27}$$

and one can show that it is a *local* asymptotic stabilizer for System (13) if, for example, $|u_1^r(t)|>\delta>0\ \forall t$.

Finally, let us mention that, for both controllers (26) and (27), if u_1^r is a constant different from zero, then the convergence of the tracking error to zero is exponential with a rate proportional to $|u_1^r|$.

3.3 Practical stabilization

Asymptotic stability is certainly a desirable property for a controlled system, but several results recalled in the previous sections point out that it cannot always be obtained, or robustly ensured, in the case of non-holonomic systems.

i) Theorem 1 basically tells us that no continuous feedback can asymptotically stabilize all admissible trajectories. This leads to the difficult question of choosing a controller when the reference trajectory is not known *a priori*.

ii) The difficult compromise between stability robustness and fast convergence, arising when trying to asymptotically stabilize a desired fixed configuration, was pointed out in Section 3.2.

iii) Asymptotic stabilization of non-admissible trajectories is not possible, by definition, although it may be useful, for some applications, to achieve some type of tracking of such trajectories. Consider for example a two-car platooning situation with the leading car engaged in a sequence of maneuvers. A way of addressing this problem consists in trying to stabilize a virtual frame attached to the leading car, at a certain distance behind it. This corresponds to the situation of Fig. 3 with (P_r, i_r, j_r) representing the virtual frame. The reference trajectory is then $g_r = (x_r, y_r, \theta_r)$ and $s_r = 0$, with (x_r, y_r) the coordinates of the point P_r. It is simple to verify that, except for pure longitudinal displacements of the leading car, the resulting trajectory of the virtual frame is not admissible.

We present in this section a control approach that we have been developing for a few years, and which allows to address the trajectory tracking problem in a novel way. This method is potentially applicable to all non-holonomic vehicles, and it has already been tested experimentally on a unicycle-like robot [1]. We illustrate it below for the problem of tracking another vehicle with a car, and refer to [22] for the general setting. As indicated in Point *iii)* above, this problem can be addressed by defining a reference trajectory (g_r, s_r), with $s_r = 0$ and g_r the configuration of a frame located behind the reference vehicle (Fig. 3). No assumption is made on g_r so that the associated reference trajectory may, or may not, be admissible. It may also be reduced to a fixed-point.

The control approach is based on the concept of transverse function [20].

Definition 1. *Let $p \in \mathbb{N}$ and $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$. A smooth function $f : \mathbb{T}^p \longrightarrow M$ is a* transverse function *for System* (1) *if,*

$$\forall \alpha \in \mathbb{T}^p, \quad \text{rank} H(\alpha) = n \quad (= \dim(M)) \tag{28}$$

with

$$H(\alpha) := \left(X_1(f(\alpha)) \cdots X_m(f(\alpha)) \ \frac{\partial f}{\partial \alpha_1}(\alpha) \cdots \frac{\partial f}{\partial \alpha_p}(\alpha) \right) \tag{29}$$

Transverse functions allow to use $\dot{\alpha}_1, \ldots, \dot{\alpha}_p$ as additional (virtual) control inputs. This is related to the idea of controlled oscillator in [10]. For the car model (4), the usefulness of these complementary inputs is explicited in the following lemma.

Lemma 1. *Consider the tracking error model* (10) *associated with the car model* (4) *and the reference trajectory $(g_r, s_r = 0)$. Let $f : \mathbb{T}^p \longrightarrow \mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{R}$ denote a smooth function. Define the "neighbor" state z by*

$$z := \begin{pmatrix} \begin{pmatrix} x_e \\ y_e \end{pmatrix} - R(\theta_e - f_3) \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \\ \theta_e - f_3 \\ \zeta - f_4 \end{pmatrix} \tag{30}$$

and the augmented control vector $\bar{u} := (u_1, u_2, -\dot{\alpha}_1, \ldots \ldots, -\dot{\alpha}_p)'$*. Then,*

$$\dot{z} = A(z,f)\left(H(\alpha)\bar{u} + B(z)P(q_e,t) + u_1 C(z)\right) \tag{31}$$

with $H(\alpha)$ *the matrix specified in relation* (29)*,*

$$A(z,f) := \begin{pmatrix} R(z_3) & \left(R(z_3)\begin{pmatrix} f_2 \\ -f_1 \end{pmatrix} \; 0\right) \\ 0 & I_2 \end{pmatrix}, \quad B(z) := \begin{pmatrix} R(-z_3) & 0 \\ 0 & I_2 \end{pmatrix}$$

and $C(z) := (0, 0, z_4, 0)'$*.*

If f in (30) is a transverse function, then $A(z,f)H(\alpha)$ is a full-rank matrix and it is simple to use (31) in order to derive a control which asymptotically stabilizes $z = 0$. Such a control law is pointed out by the following proposition.

Proposition 6. *With the notations of Lemma 1, assume that* f *is a transverse function for the car model* (4)*, and consider the dynamic feedback law*

$$\bar{u} := H^{\dagger}(\alpha)\left(-B(z)P(q_e,t) - (A(z,f))^{-1}Z(z)\right) \tag{32}$$

with $H^{\dagger}(\alpha)$ *a right-inverse of* $H(\alpha)$ *and*

$$Z(z) := (k_1 z_1, k_2 z_2, 2k_3 \tan(z_3/2), k_4 x_4)' \quad (k_i > 0)$$

Then, for any reference trajectory g_r *such that* $\dot{g}_r$ *is bounded,*

1. $z = 0$ *is exponentially stable for the controlled system* (31)*,*
2. *any trajectory* q_e *of the controlled tracking error model* (10) *converges to* $f(\mathbb{T}^p)$ *and, with an adequate choice of* $\alpha(0)$*, this set is exponentially stable.*

This result shows that it is possible to stabilize the tracking error q_e to the set $f(\mathbb{T}^p)$ for *any* reference trajectory g_r. Since $\mathbb{T}^p$ is compact and f is smooth, $f(\mathbb{T}^p)$ is bounded. In particular, if $f(\mathbb{T}^p)$ is contained in a small neighborhood of the origin, then the tracking error q_e is ultimately small, *whatever* the trajectory g_r. This leaves us with the problem of determining transverse functions. In [22], a general formula is proposed for driftless systems. A family of transverse functions, computed in part from this formula, is pointed out below. These functions are defined on $\mathbb{T}^2$, i.e. they depend on two variables α_1, α_2. It is clear that for the car model, two is the smallest number of variables for which (28) can be satisfied. However, using more variables can also be of interest in practice (see [2] for complementary results in this direction), and some research effort could be devoted to explore this issue further.

Lemma 2. *For any* $\varepsilon > 0$*, and* $\eta_1, \eta_2, \eta_3 > 0$ *such that* $6\eta_2\eta_3 > 8\eta_3 + \eta_1\eta_2$*, the function* f *defined by*

$$f(\alpha) = \left(\bar{f}_1(\alpha), \bar{f}_4(\alpha), \arctan(\bar{f}_3(\alpha)), \bar{f}_2(\alpha)\cos^3 f_3(\alpha)\right)' \tag{33}$$

with $\bar{f} : \mathbb{T}^2 \longrightarrow \mathbb{R}^4$ *defined by*

$$\begin{aligned}
\bar{f}_1(\alpha) &= \varepsilon(\sin\alpha_1 + \eta_2 \sin\alpha_2)\\
\bar{f}_2(\alpha) &= \varepsilon\eta_1 \cos\alpha_1\\
\bar{f}_3(\alpha) &= \varepsilon^2\left(\tfrac{\eta_1 \sin 2\alpha_1}{4} - \eta_3 \cos\alpha_2\right)\\
\bar{f}_4(\alpha) &= \varepsilon^3\left(\eta_1 \tfrac{\sin^2\alpha_1 \cos\alpha_1}{6} - \tfrac{\eta_2\eta_3 \sin 2\alpha_2}{4} - \eta_3 \sin\alpha_1 \cos\alpha_2\right)
\end{aligned}$$

is a transverse function for the car model (4).

4 Conclusion

Trajectory stabilization for nonholonomic systems is a multi-faceted problem. In the first place, the classical objective of *asymptotic stabilization* (combining stability and convergence to the desired trajectory) can be considered only when the reference trajectory is known to be *admissible*. The difficulties, in this favorable case, are nonetheless numerous and epitomized by the non-existence of universal stabilizers. For some trajectories, endowed with the right properties (related to motion persistency, and fortunately often met in practice) the problem can be solved by applying classical control techniques. This typically yields linear controllers derived from a linear approximation of the trajectory error system, or slightly more involved nonlinear versions in order to expand the domain of operation for which asymptotic stabilization can be proven analytically. For other trajectories, such as fixed configurations, less classical control schemes (time-varying ones, for instance) have to be used, with mitigated practical success though, due to the impossibility of complying with the performance/robustness compromise as well as in the linear case. Basically, one has to choose between fast convergence, accompanied with high sensitivity to modeling errors and measurement noise, and slow convergence, with possibly more robustness. In the end, when the reference trajectory and its properties are not known in advance (except for its admissibility), so that it may not belong to the two categories evoked above, the practitioner has no other choice than trying to guess which control strategy will apply best to its application, or work out some empirical switching strategy, with no absolute guarantee of success in all situations. This is not very satisfactory, all the more so because there are applications for which the control objective is more naturally expressed in terms of tracking a *non-admissible* trajectory (we gave the example of tracking a maneuvering vehicle). In this latter case, none of the control techniques extensively studied during these last fifteen years towards the goal of asymptotic stabilization is suitable. These considerations led us to propose another point of view according to which the relaxation of this stringent, and sometimes unworkable, goal into a more pragmatic one of *practical stabilization*, with ultimate boundedness of the tracking errors instead of convergence to zero, can be beneficial to enlarge the set of control possibilities and address the trajectory tracking problem in a re-unified way. The *transverse function* approach described in Section 3.3 has been developed with this point of view. It allows to derive smooth feedback controllers which uniformly ensure ultimate boundedness of the tracking errors, with arbitrary pre-specified tracking precision, *whatever the reference trajectory* (admissibility is no longer a prerequisite).

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