SCALAR CONSERVATION LAWS WITH MOVING DENSITY CONSTRAINTS ARISING IN TRAFFIC FLOW MODELING

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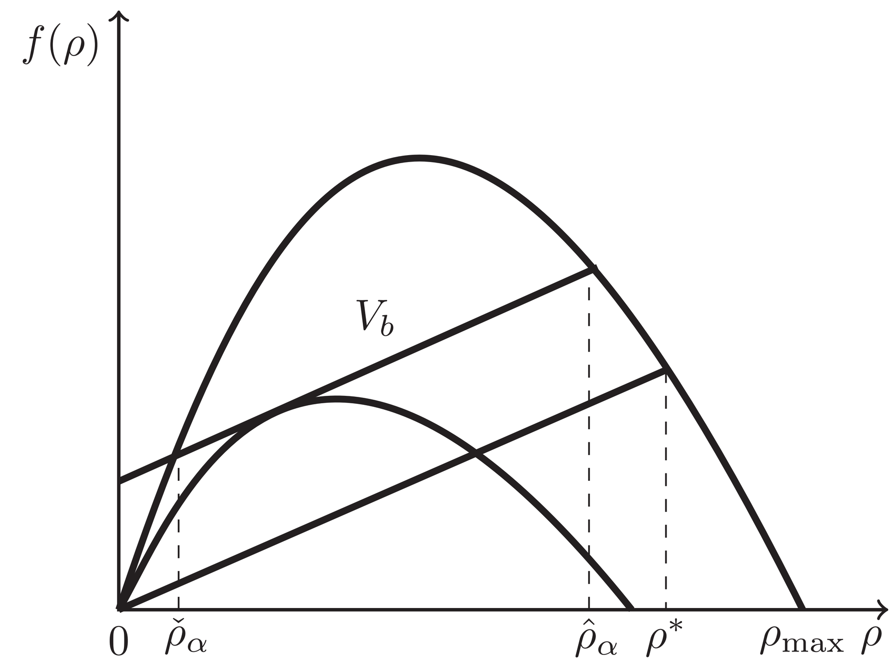
Model

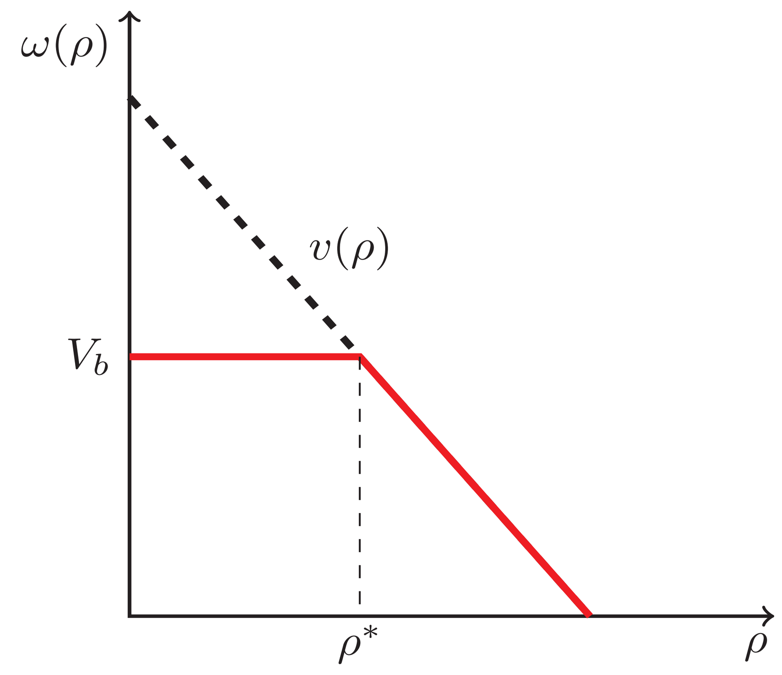
A slow moving large vehicle, e.g., a bus or truck, reduces the road capacity and thus generates a moving bottleneck for the surrounding traffic flow. From the macroscopic point of view this can be modeled by a PDE-ODE coupled system introduced in [3]

$$\begin{cases} \partial_t \rho + \partial_x f(\rho) = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}, \\ \rho(t, y(t)) \leq \alpha R, & t \in \mathbb{R}^+, \\ \dot{y}(t) = \omega(\rho(t, y(t)+)), & t \in \mathbb{R}^+, \\ y(0) = y_0. \end{cases}$$

Fundamental Diagram



Velocity



Main Definitions

Definition 1 (Riemann Solver) *The constrained Riemann solver $\mathcal{R}^\alpha$ for the Cauchy problem is defined as follows:*

1. *If $f(\mathcal{R}(\rho_L, \rho_R)(V_b)) > F_\alpha + V_b \mathcal{R}(\rho_L, \rho_R)(V_b)$, then*

$$\mathcal{R}^\alpha(\rho_L, \rho_R)(x) = \begin{cases} \mathcal{R}(\rho_L, \hat{\rho}_\alpha) & \text{if } x < V_b t, \\ \mathcal{R}(\check{\rho}_\alpha, \rho_R) & \text{if } x \geq V_b t, \end{cases} \quad \text{and} \quad y(t) = V_b t.$$

2. *If $V_b \mathcal{R}(\rho_L, \rho_R)(V_b) \leq f(\mathcal{R}(\rho_L, \rho_R)(V_b)) \leq F_\alpha + V_b \mathcal{R}(\rho_L, \rho_R)(V_b)$, then*

$$\mathcal{R}^\alpha(\rho_L, \rho_R) = \mathcal{R}(\rho_L, \rho_R) \quad \text{and} \quad y(t) = V_b t.$$

3. *If $f(\mathcal{R}(\rho_L, \rho_R)(V_b)) < V_b \mathcal{R}(\rho_L, \rho_R)(V_b)$, then*

$$\mathcal{R}^\alpha(\rho_L, \rho_R) = \mathcal{R}(\rho_L, \rho_R) \quad \text{and} \quad y(t) = v(\rho_R) t.$$

Note: When the constraint is enforced, a nonclassical shock arises, which satisfies the Rankine-Hugoniot condition but violates the Lax entropy condition.

Definition 2 (Weak solution) *A couple $(\rho, y) \in \mathcal{C}^0\left(\mathbb{R}^+; \mathbf{L}^1 \cap BV(\mathbb{R})\right) \times \mathbf{W}^{1,1}(\mathbb{R}^+)$ is a solution to the Cauchy Problem if*

1. *ρ is a weak solution of the conservation law, i.e. for all $\varphi \in \mathcal{C}^1_c(\mathbb{R}^2)$*

$$\int_{\mathbb{R}^+} \int_{\mathbb{R}} (\rho \partial_t \varphi + f(\rho) \partial_x \varphi) \, dx \, dt + \int_{\mathbb{R}} \rho_0(x) \varphi(0,x) \, dx = 0 \, ;$$

2. *y is a Carathéodory solution of the ODE, i.e. for a.e. $t \in \mathbb{R}^+$*

$$y(t) = y_0 + \int_0^t \omega(\rho(s, y(s)+)) \, ds \, ;$$

3. *the constraint is satisfied, in the sense that for a.e. $t \in \mathbb{R}^+$*

$$\lim_{x \to y(t)\pm} \left(f(\rho) - \omega(\rho)\rho\right)(t,x) \leq F_\alpha .$$

Existence Theorem

Theorem 1 (Existence of solutions) *For every initial data $\rho_0 \in BV(\mathbb{R})$ such that $TV(\rho_0) \leq C$ is bounded, the Cauchy problem admits a weak solution in the sense of Definition 2.*

Sketch of the proof:

Lemma 2 (Bound on the total variation) *Define the Glimm type functional*

$$\Upsilon(t) = \Upsilon(\rho^n(t,\cdot)) = TV(\rho^n) + \gamma = \sum_j \left|\rho^n_{j+1} - \rho^n_j\right| + \gamma,$$

with $\gamma = \gamma(t) = \begin{cases} 0 & \text{if } \rho^n(t, y_n(t)-) = \hat{\rho}_\alpha, \ \rho^n(t, y_n(t)+) = \check{\rho}_\alpha \\ 2|\hat{\rho}_\alpha - \check{\rho}_\alpha| & \text{otherwise.} \end{cases}$

Then, for any $n \in \mathbb{N}$, $t \mapsto \Upsilon(t) = \Upsilon(\rho^n(t,\cdot))$ at any interaction either decreases by at least 2^{-n}, or remains constant and the number of waves does not increase.

Lemma 3 (Convergence of approximate solutions) *Let ρ^n and y_n, $n \in \mathbb{N}$, be the wave front tracking approximations of the Cauchy Problem, and assume $TV(\rho_0) \leq C$ be bounded, $0 \leq \rho_0 \leq 1$. Then, up to a subsequence, we have the following convergences*

$$\begin{aligned} \rho^n &\to \rho && \text{in } \mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}^+ \times \mathbb{R}); \\ y_n(\cdot) &\to y(\cdot) && \text{in } \mathbf{L}^\infty([0,T]), \text{ for all } T > 0; \\ \dot{y}_n(\cdot) &\to \dot{y}(\cdot) && \text{in } \mathbf{L}^1([0,T]), \text{ for all } T > 0; \end{aligned}$$

for some $\rho \in \mathcal{C}^0\left(\mathbb{R}^+; \mathbf{L}^1 \cap BV(\mathbb{R})\right)$ and $y \in \mathbf{W}^{1,1}(\mathbb{R}^+)$.

Note: For the full proof see [2].

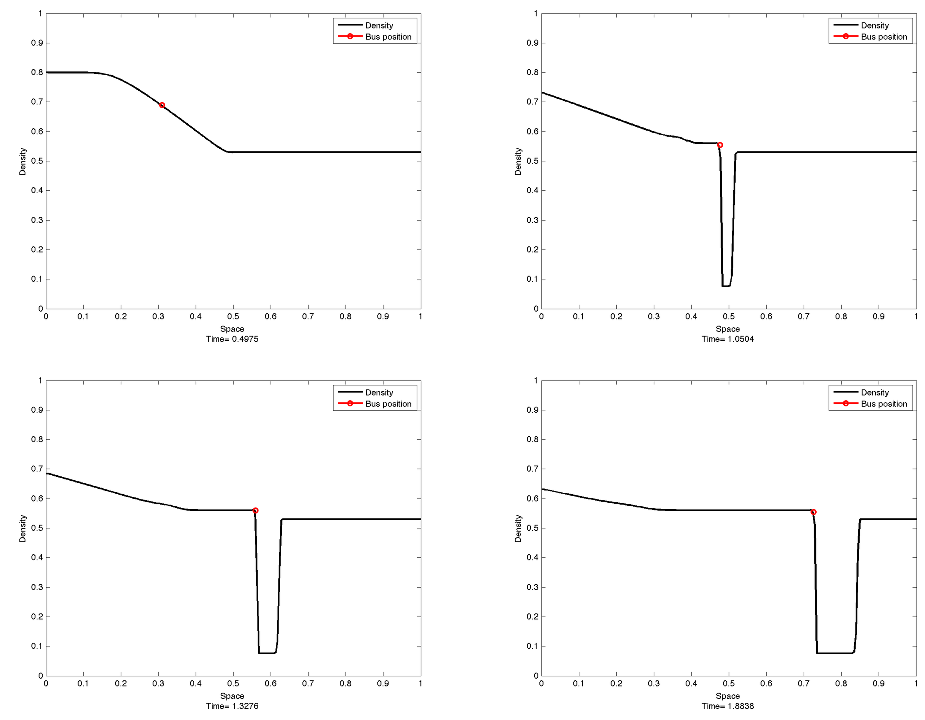
Preliminary numerical results

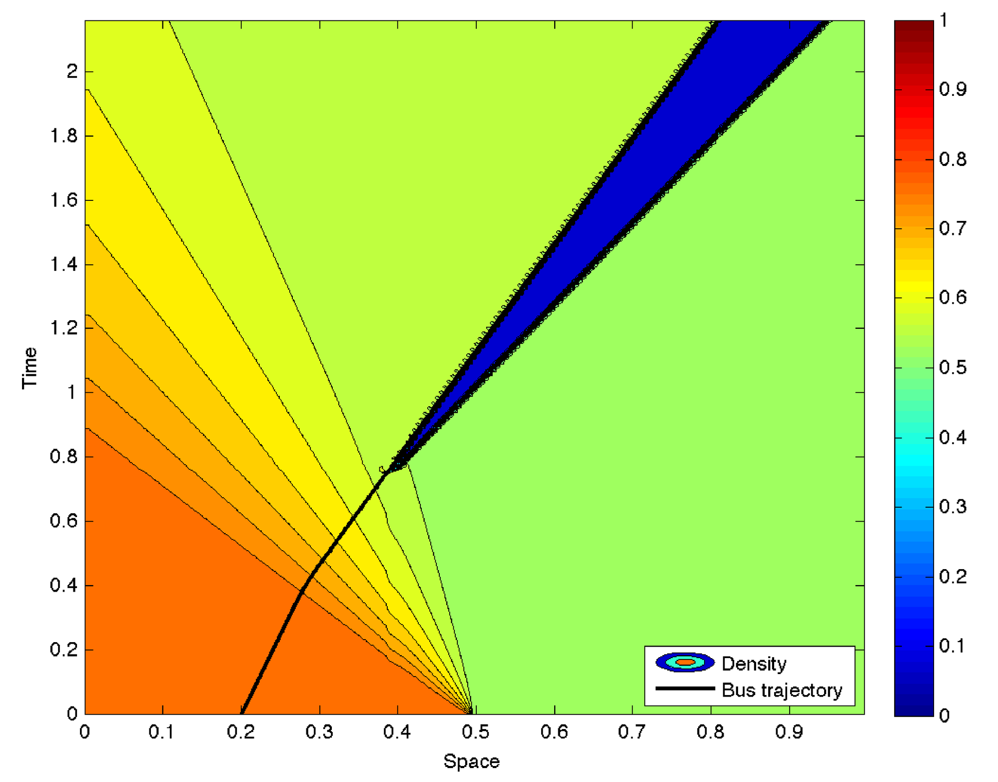
Numerical simulations were carried out using a front capturing scheme [1] with moving space grid as in [4]. In particular, the space discretization follows the bus trajectory.

For the numerical simulations we used the following flux function $f(\rho) = \rho(1-\rho)$ and the following parameters $V_b = 0.3$, $\alpha = 0.6$. The (reference) space step is $\Delta x = 0.005$, CFL $= \frac{1}{2}$ and the time step is computed accordingly. In the following figure, we show the evolution in time of the density, corresponding to the following Riemann type initial data

$$\rho(0,x) = \begin{cases} 0.8 & \text{if } x < 0.5, \\ 0.53 & \text{if } x > 0.5, \end{cases}$$

and $y_0 = 0.2$.





References

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[2] M. L. Delle Monache and P. Goatin. Scalar conservation laws with moving density constraints arising in traffic flow modeling. Research Report 8119, Inria, http://hal.inria.fr/docs/00/74/56/81/PDF/RR-8119.pdf, October 2012.

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[4] X. Zhong, T. Y. Hou, and P. G. LeFloch. Computational methods for propagating phase boundaries. *Journal of Computational Physics*, 124(1):192–216, 1996.