TRB 2014 - Workshop "Crowd Flow Dynamics, Modeling and Management"

Mathematical modeling of crowds

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Outline of the talk

1. Mathematical models
2. Numerical tests (M. Twarogowska, P.Goatin and R. Duvigneau, submitted)
3. Conclusion and perspectives

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Mathematical modeling of pedestrian motion: frameworks

Microscopic

- individual agents
- ODEs system
- many parameters
- low and high densities
- comp. cost $\sim$ ped. number.

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Macroscopic

- continuous fluid
- PDEs
- few parameters
- very high densities
- analytical theory
- comp. cost $\sim$ domain size

Macroscopic models

- Pedestrians as "thinking fluid"[1]
- Averaged quantities:
  - $\rho(t, \mathbf{x})$ pedestrians density
  - $\vec{v}(t, \mathbf{x})$ mean velocity

Mass conservation

$$\begin{cases} \partial_t \rho + \mathrm{div}_{\mathbf{x}}(\rho \vec{v}) = 0 \\ \rho(0, \mathbf{x}) = \rho_0(\mathbf{x}) \end{cases}$$

for $\mathbf{x} \in \Omega \subset \mathbb{R}^2$, $t > 0$

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Two classes

- **1st order models:** velocity given by a phenomenological *speed-density relation* $\vec{v} = V(\rho)\vec{\nu}$
- **2nd order models:** velocity given by a *momentum balance equation*

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- Density must stay non-negative and bounded: $0 \leq \rho(t,\mathbf{x}) \leq \rho_{\max}$
- Different from fluid dynamics:
  - preferred direction
  - no conservation of momentum / energy
  - $n \ll 6 \cdot 10^{23}$

[1] R.L. Hughes, Transp. Res. B, 2002

Desired direction of motion $\vec{\nu}$[1]

Pedestrians:

- seek the shortest route to destination
- try to avoid high density regions

$$\vec{\nu} = -\frac{\nabla_{\mathbf{x}}\phi}{|\nabla_{\mathbf{x}}\phi|}$$

[1] Hughes (macro 1st), Jiang et al. (macro 2nd), Colombo (non-local), Hartmann (micro)...

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The potential $\phi : \Omega \to \mathbb{R}$ is given by the Eikonal equation

$$\begin{cases} |\nabla_{\mathbf{x}}\phi| = C(t, \mathbf{x}, \rho) & \text{in } \Omega \\ \phi(t, \mathbf{x}) = 0 & \text{for } \mathbf{x} \in \Gamma_{outflow} \end{cases}$$

where $C = C(t, \mathbf{x}, \rho) \geq 0$ is the *running cost*

$\Longrightarrow$ the solution $\phi(t, \mathbf{x})$ represents the weighted distance of the position $\mathbf{x}$ from the target $\Gamma_{outflow}$

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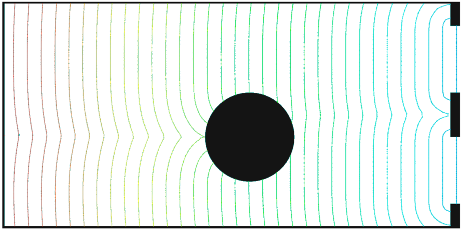
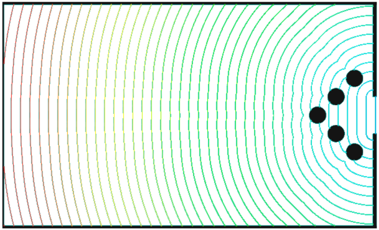
Eikonal equation: level set curves for $|\nabla_{\mathbf{x}}\phi| = 1$

In an empty space: potential is proportional to distance to destination





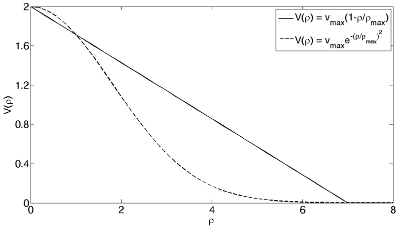
Speed-density relation

Speed function $V(\rho)$:

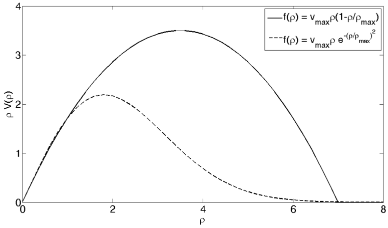
- decreasing function wrt density
- $V(0) = v_{\max}$ **free flow**

  $V(\rho_{\max}) = 0$ **congestion**

Examples:



speed $V(\rho)$



flux $\rho V(\rho)$

First order models

- Hughes' model[2]

$$V(\rho) = v_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right) \qquad |\nabla_{\mathbf{x}}\phi| = \frac{1}{V(\rho)}$$

  - pedestrians tend to minimize their estimated travel time to the exit
  - pedestrians temper their estimated travel time avoiding high densities
  - CRITICISM: instantaneous global information on entire domain

[2] R.L. Hughes, Transp. Res. B, 2002
[3] Y. Xia, S.C. Wong and C.-W. Shu, Physical Review E, 2009

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- Dynamic model with memory effect[3]

$$V(\rho) = v_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right) \qquad \vec{\nu} = -\frac{\nabla_{\mathbf{x}}(\phi + \omega D)}{|\nabla_{\mathbf{x}}(\phi + \omega D)|}$$

where

$$|\nabla_{\mathbf{x}}\phi| = \frac{1}{v_{\max}}, \quad D(\rho) = \frac{1}{v(\rho)} + \beta\rho^2 \quad \textit{discomfort}$$

  - pedestrians seek to minimize their estimated travel time based on their knowledge of the walking domain
  - pedestrians temper their behavior locally to avoid high densities

[2] R.L. Hughes, Transp. Res. B, 2002

[3] Y. Xia, S.C. Wong and C.-W. Shu, Physical Review E, 2009

Other first order models

- Non-local flow:[4]

$$\vec{v} = V(\rho * \eta)\ \vec{\nu}(\mathbf{x}) \qquad \text{OR} \qquad \vec{v} = V(\rho)\left(\vec{\nu}(\mathbf{x}) - \varepsilon \frac{\nabla(\rho * \eta)}{\sqrt{1 + |\nabla(\rho * \eta)|^2}}\right)$$

[4] R.M. Colombo, Garavello and M. Lécureux-Mercier, M3AS, 2012

[5] E. Cristiani, B. Piccoli and A. Tosin, Multiscale Model. Simul., 2011

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- Multi-scale time-evolving measures:[5]
  Probability distribution of pedestrians

$$\mu_t = \theta m_t + (1-\theta) M_t \quad \text{where} \begin{cases} m_t = \sum_{j=1}^{N} \delta_{P_j(t)} & \text{microscopic mass} \\ dM_t(\mathbf{x}) = \rho(t,\mathbf{x}) d\mathbf{x} & \text{macroscopic mass} \end{cases}$$

  Governing equation: probability conservation deduced from individual-based modeling

$$\partial_t \mu_t + \mathrm{div}_{\mathbf{x}}(\mu_t \vec{v}_t) = 0$$

$$\vec{v}_t(\mathbf{x}) = v_{\max} \vec{\nu}(\mathbf{x}) + N \int_{\mathcal{B}_R(\mathbf{x})} K(\mathbf{x},\mathbf{y}) d\mu_t(\mathbf{y})$$

[4] R.M. Colombo, Garavello and M. Lécureux-Mercier, M3AS, 2012
[5] E. Cristiani, B. Piccoli and A. Tosin, Multiscale Model. Simul., 2011

Second order model

Momentum balance equation[6][7]

$$\partial_t(\rho\vec{v}) + \mathrm{div}_{\mathbf{x}}(\rho\vec{v}\otimes\vec{v}) + \nabla_{\mathbf{x}}P(\rho) = \frac{1}{\tau}(\rho V(\rho)\vec{\nu} - \rho\vec{v})$$

where

- $V(\rho) = v_{\max} e^{-\alpha\left(\frac{\rho}{\rho_{\max}}\right)^2}$
- $|\nabla_{\mathbf{x}}\phi| = 1/V(\rho)$
- $P(\rho) = p_0\rho^\gamma$, $p_0 > 0$, $\gamma > 1$ internal pressure
- τ response time

[6] Payne-Whitham, 1971

[7] Y.Q. Jiang, P. Zhang, S.C. Wong and R.X. Liu, Physica A, 2010

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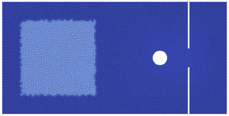
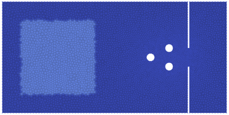
Hughes' VS second order model: obstacles effect

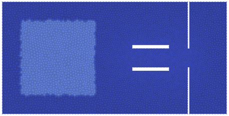
Obstacles:



1. one column $r = 0.3m$



2. three columns $r = 0.2m$



3. two walls $1.2m \times 0.2m$

What is the effect of the obstacles on the outflow?

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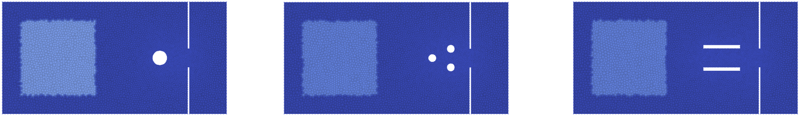

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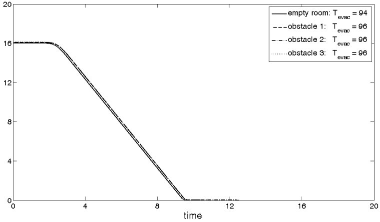
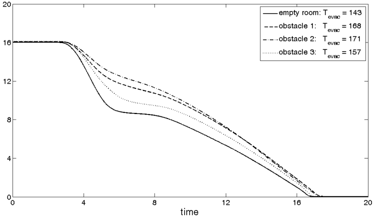


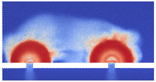
Hughes' model



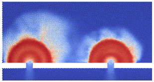
Second order model

Fig. Time evolution of the total mass of pedestrians inside the room.

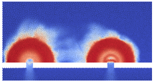
Second order model: stop-and-go waves



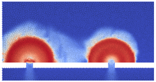
t=90



t=100



t=110



t=120

$$P(\rho) = 0.005\rho^2, \qquad v_{\max} = 2, \qquad \rho_{\max} = 7$$

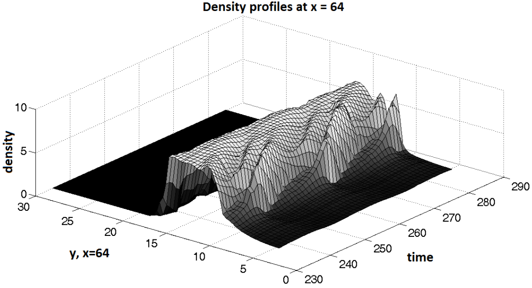
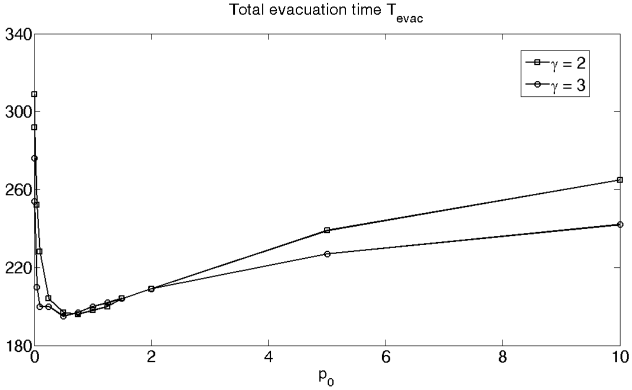


Fig. Time evolution of density profile at $x = 64$ (left exit)

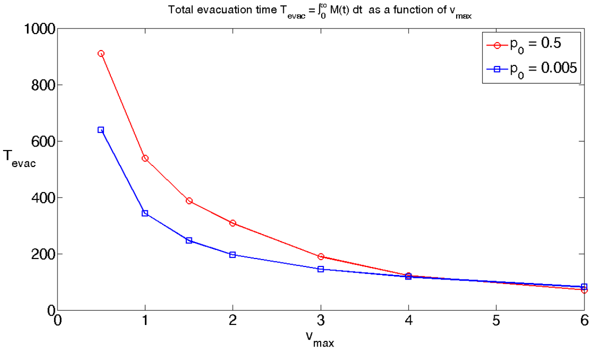
Second order model: dependence on p_0

$P(\rho) = p_0\rho^\gamma$: total evacuation time optimal for $p_0 \simeq 0.5$



Second order model: dependence on $v_{\max}$

Total evacuation time

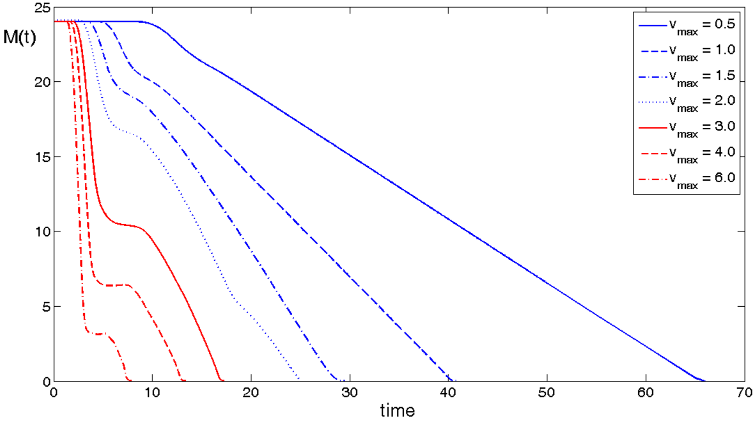


Social force models[8] show a minimum for $v_{\max} \simeq 1.4\ m/s$
$\Longrightarrow$ **faster-is-slower effect**[9]
Accounting for inter-pedestrian friction?

[8] D. Helbing, I. Farkas and T. Vicsek, Nature, 2000
[9] D.R. Parisi and C.O. Dorso, Physica A, 2007

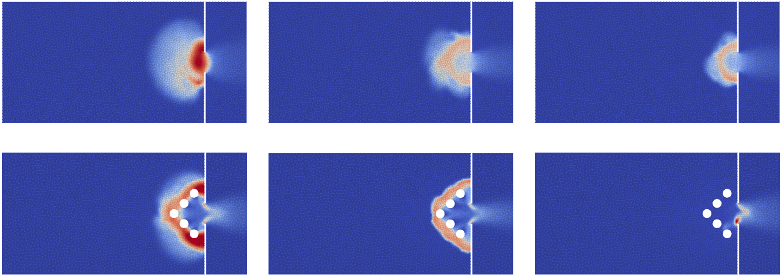
Second order model: dependence on $v_{\max}$

Total mass evolution



Evacuation optimization: Braess' paradox[10] ?

Problem: **clogging** at exit

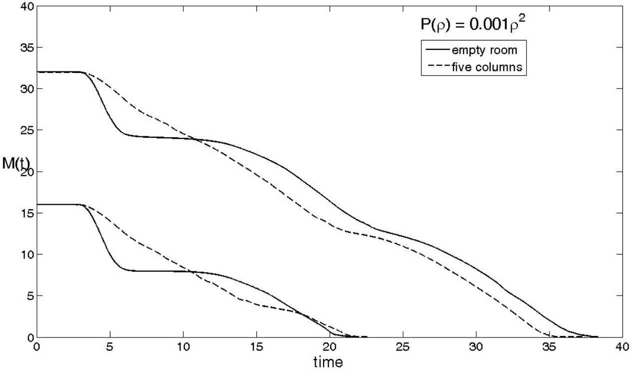


Can obstacles reduce the evacuation time?

[10] Braess, D. *Über ein Paradoxon aus der Verkehrsplanung*, Unternehmensforschung, 1968

Evacuation optimization: Braess' paradox?

Time evolution of the total mass of pedestrians inside the room



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Strengths:

- lower computational cost for large crowds
- global description of spatio-temporal evolution
- mathematical tools for well-posedness and numerical approximation
- suitable for posing control and optimization problems

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Aspects to be addressed:

- reproduce emerging phenomena observed in real situations
- account for individual choices that may affect the whole system
- validation on empirical data

Thank you for your attention!