Congrès SMAI 2013

Modèles macroscopiques de trafic routier et piétonnier

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Outline of the talk

1. Traffic flow models
2. Phase transition models
3. Flux constraints
4. Crowd dynamics
5. Conclusion

Traffic flow models

Three possible scales:

- Microscopic:
  - ODEs system
  - numerical simulations
  - many parameters
- Kinetic:
  - distribution function of the microscopic states
  - Boltzmann-like equations
- Macroscopic:
  - PDEs from fluid dynamics
  - analytical theory
  - few parameters
  - suitable to formulate control and optimization problems

Macroscopic models

$$\Big[\text{number of vehicles in } [a,b] \text{ at time } t\Big] = \int_a^b \rho(t,x)\ dx$$

must be conserved!

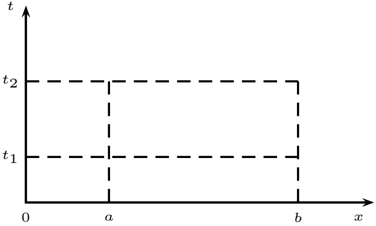
$$\begin{aligned} \int_a^b \rho(t_2-,x)dx &= \int_a^b \rho(t_1+,x)dx \\ &+ \\ \int_{t_1}^{t_2} f(t,a+)dt &- \int_{t_1}^{t_2} f(t,b-)dt \end{aligned}$$

$\Downarrow$

divergence theorem for (ρ, f)

$\Downarrow$

$$\int_{t_1}^{t_2} \int_a^b \partial_t \rho + \partial_x f\ dx\ dt = 0$$



Conservation laws

(System of) PDEs of the form

$$\partial_t u(t, \mathbf{x}) + \mathrm{div}_{\mathbf{x}} f(u(t, \mathbf{x})) = 0 \qquad t > 0,\ \mathbf{x} \in \mathbb{R}^D$$

where $u \in \mathbb{R}^n$ conserved quantities
$f : \mathbb{R}^{n \times D} \to \mathbb{R}$ smooth **strictly hyperbolic** flux

Basic facts:

- No classical smooth solutions $\Longrightarrow$ **weak solutions**
- No uniqueness $\Longrightarrow$ **entropy conditions**
- Well posedness known only if $\min\{n, D\} = 1$

Requirements

- No information propagates faster than vehicles (anisotropy)
- Flux-density relation: $f(t,x) = \rho(t,x)v(t,x)$.
- Density and mean velocity must be non-negative and bounded: $0 \leq \rho(t,x),\ v(t,x) < +\infty,\ \forall x, t > 0$.
- Different from fluid dynamics:
  - preferred direction
  - no conservation of momentum / energy
  - no viscosity
  - Avogadro number for vehicles: 106 vh/lane$\times$km $\ll 6 \cdot 10^{23}$

Macroscopic models

$n \ll 6 \cdot 10^{23}$ but ...

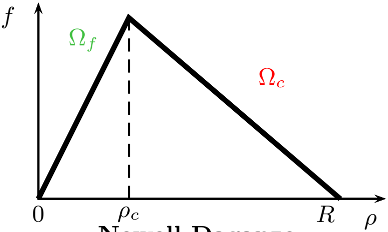
First order models

Lighthill-Whitham '55, Richards '56, Greenshields '35:

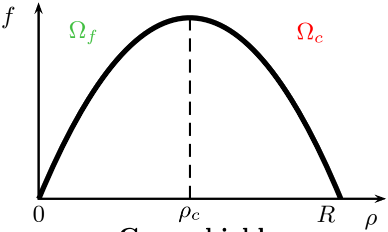
- Non-linear transport equation: scalar conservation law

$$\partial_t \rho + \partial_x f(\rho) = 0, \quad f(\rho) = \rho v(\rho)$$

- Empirical flux function: fundamental diagram



Newell-Daganzo



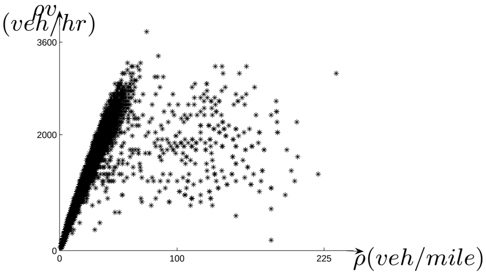
Greenshields

with R the maximal or *jam* density and ρ_c the critical density:

- flux is increasing for $\rho \leq \rho_c$: free-flow phase
- flux is decreasing for $\rho \geq \rho_c$: congestion phase

Motivation for higher order models

- Traffic satisfies “mass" conservation. What about other fundamental conservation principles from fluid dynamics: conservation of momentum, conservation of energy?
- Experimental observations of fundamental diagrams are more complex than postulated by first order traffic models



Viale Muro Torto, Roma

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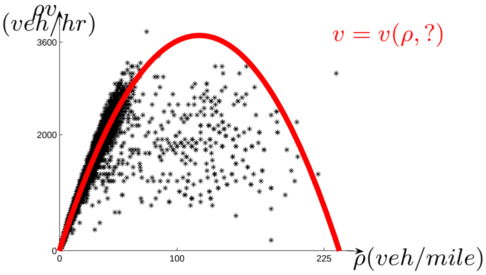

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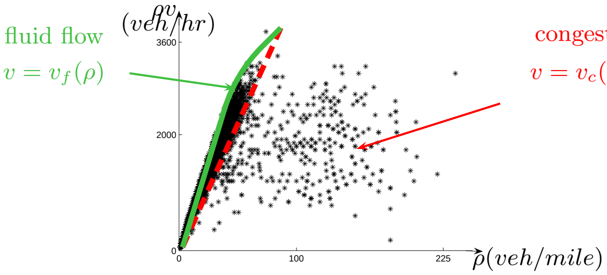

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Second order models

- **Payne '71:**

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t v + v \partial_x v = -\dfrac{c_0^2}{\rho} \partial_x \rho + \dfrac{v_*(\rho) - v}{\tau} \end{cases}$$

Critics (Del Castillo et al. '94, Daganzo '95):

- drivers should have only positive speeds;
- anisotropy: drivers should react only to stimuli from the front.

Second order models

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$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t v + v\partial_x v = -\dfrac{c_0^2}{\rho}\partial_x \rho + \dfrac{v_*(\rho) - v}{\tau} \end{cases}$$

- **Aw-Rascle '00:**

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t(\rho w) + \partial_x(\rho v w) = 0 \end{cases} \qquad v = v(\rho, w)$$

$w = v + p(\rho)$ Lagrangian marker, $p = p(\rho)$ “pressure"

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$w = v + p(\rho)$ Lagrangian marker, $p = p(\rho)$ "pressure"

- **Colombo '02:**

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t q + \partial_x((q - Q)v) = 0 \end{cases} \qquad v = v(\rho, q)$$

q "momentum", Q road parameter

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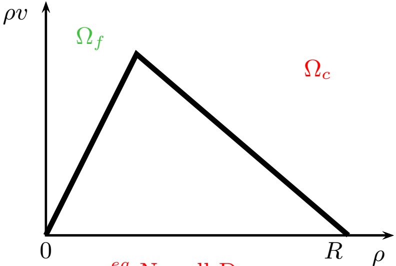
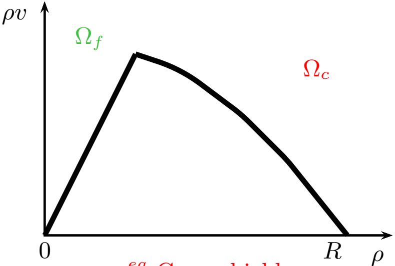
A general traffic flow model with phase transition

congestion = perturbation from equilibrium



v_c^{eq} Newell-Daganzo



v_c^{eq} Greenshields

Fluid flow: in Ω_f

$$\partial_t \rho + \partial_x(\rho v_f) = 0$$

$$v_f(\rho) = V$$

Congestion: in Ω_c

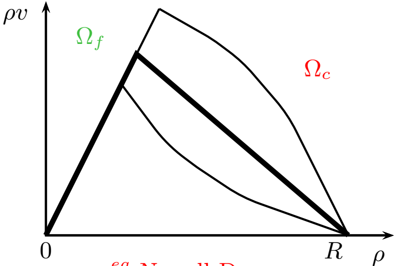
$$\begin{cases} \partial_t \rho + \partial_x(\rho v_c) = 0 \\ \end{cases}$$

$$v_c(\rho, q) = v_c^{eq}(\rho)$$

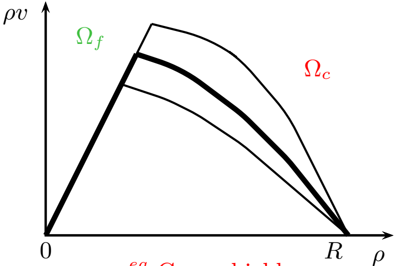
(Blandin-Work-Goatin-Piccoli-Bayen, 2011)

A general traffic flow model with phase transition

congestion = perturbation from equilibrium



v_c^{eq} Newell-Daganzo



v_c^{eq} Greenshields

Fluid flow: in Ω_f

$$\partial_t \rho + \partial_x(\rho v_f) = 0$$

$$v_f(\rho) = V$$

Congestion: in Ω_c

$$\begin{cases} \partial_t \rho + \partial_x(\rho v_c) = 0 \\ \partial_t q + \partial_x(q v_c) = 0 \end{cases}$$

$$v_c(\rho, q) = v_c^{eq}(\rho)(1+q)$$

(Blandin-Work-Goatin-Piccoli-Bayen, 2011)

Analytical study

Analysis of congestion phase:

Eigenvalues	$\lambda_1(\rho,q) = v_c^{eq}(\rho)\,(1+q) + q\,v_c^{eq}(\rho) + \rho\,(1+q)\partial_\rho\, v_c^{eq}(\rho)$	$\lambda_2(\rho,q) = v_c^{eq}(\rho)(1+q)$
Eigenvectors	$r_1 = \begin{pmatrix} \rho \\ q \end{pmatrix}$	$r_2 = \begin{pmatrix} v_c^{eq}(\rho) \\ -(1+q)\,\partial_\rho\, v_c^{eq}(\rho) \end{pmatrix}$
Nature of the Lax-curves	$\nabla\lambda_1.r_1 = \rho^2\,(1+q)\partial^2_{\rho\rho}\, v_c^{eq}(\rho) + 2\,\rho\,(1+2\,q)\,\partial_\rho\, v_c^{eq}(\rho) + 2\,q\,v_c^{eq}(\rho)$	$\nabla\lambda_2.r_2 = 0$
Riemann invariants	$v_c^{eq}(\rho)\,(1+q)$	q/ρ

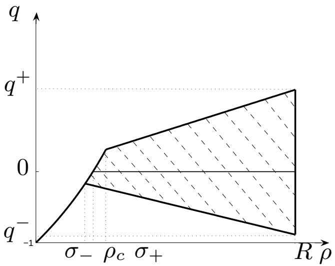
- Across first family waves (shocks or rarefactions) ρ/q is conserved, which models the average driver aggressiveness
- Across second family waves (contact discontinuities) v is conserved (like in free flow)

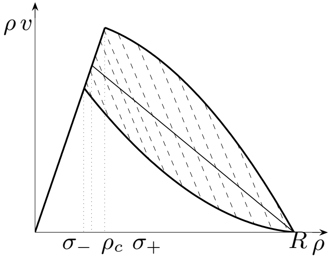
Domain definition

Definition of free-flow and congestion phases as invariant domains of dynamics

$$\Omega_f = \{(\rho, q) \mid (\rho, q) \in [0, R] \times [0, +\infty[\ , v_c(\rho, q) = V \ , \ 0 \le \rho \le \sigma_+\}$$

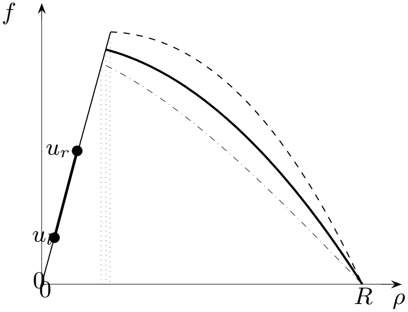
$$\Omega_c = \left\{(\rho, q) \mid (\rho, q) \in [0, R] \times [0, +\infty[\ , v_c(\rho, q) < V \ , \ \frac{q^-}{R} \le \frac{q}{\rho} \le \frac{q^+}{R}\right\}$$





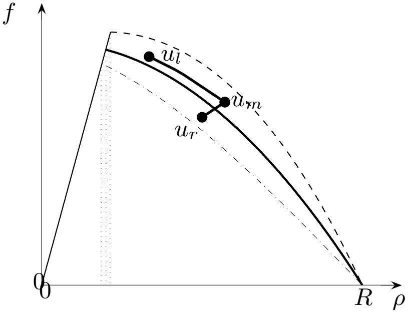
Model parameters: free-flow speed V, jam density R, critical density ρ_c, upper and lower bound for perturbation q^- and q^+

Riemann solver



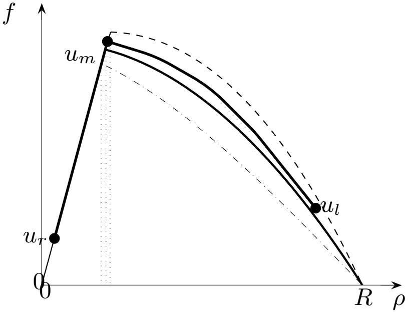
- free-flow to free-flow: contact discontinuity

Riemann solver



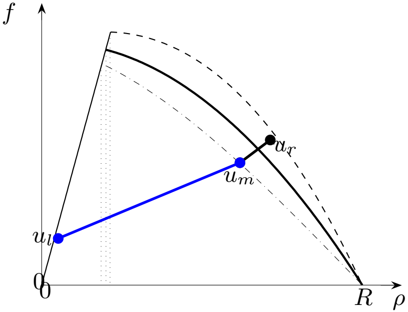
- free-flow to free-flow: contact discontinuity
- congestion to congestion: shock or rarefaction + contact discontinuity

Riemann solver



- free-flow to free-flow: contact discontinuity
- congestion to congestion: shock or rarefaction + contact discontinuity
- congestion to free-flow: rarefaction + contact discontinuity

Riemann solver



- free-flow to free-flow: contact discontinuity
- congestion to congestion: shock or rarefaction + contact discontinuity
- congestion to free-flow: rarefaction + contact discontinuity
- free-flow to congestion: phase transition + contact discontinuity

Mass conservation across phase transitions

Phase transition speed Λ must satisfy Rankine-Hugoniot conditions

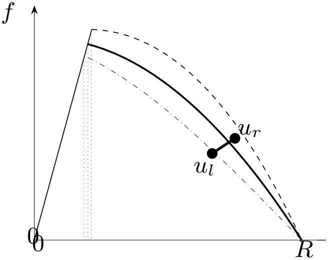
$$\Lambda(\rho_+ - \rho_-) = F_+ - F_-$$

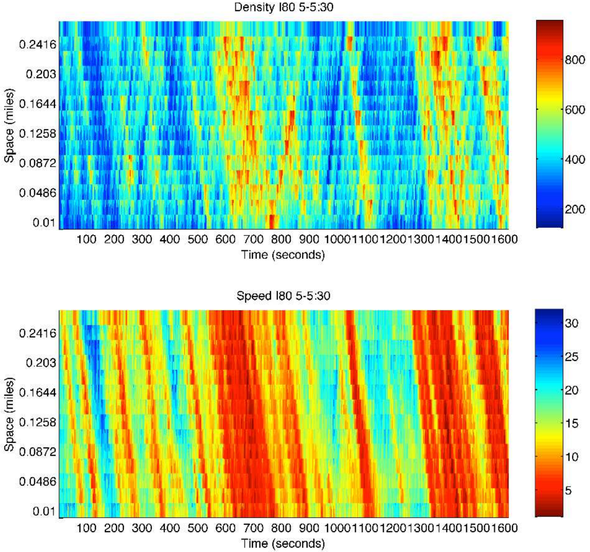
with

$$F_- = \begin{cases} \rho_- \, v_f(\rho_-) & \text{if } \rho_- \in \Omega_f \\ \rho_- \, v_c(\rho_-, q_-) & \text{if } \rho_- \in \Omega_c \end{cases}$$

$$F_+ = \begin{cases} \rho_+ \, v_f(\rho_+) & \text{if } \rho_+ \in \Omega_f \\ \rho_+ \, v_c(\rho_+, q_+) & \text{if } \rho_+ \in \Omega_c \end{cases}$$

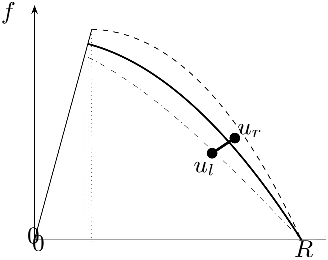
Modeling forward moving discontinuities in congestion

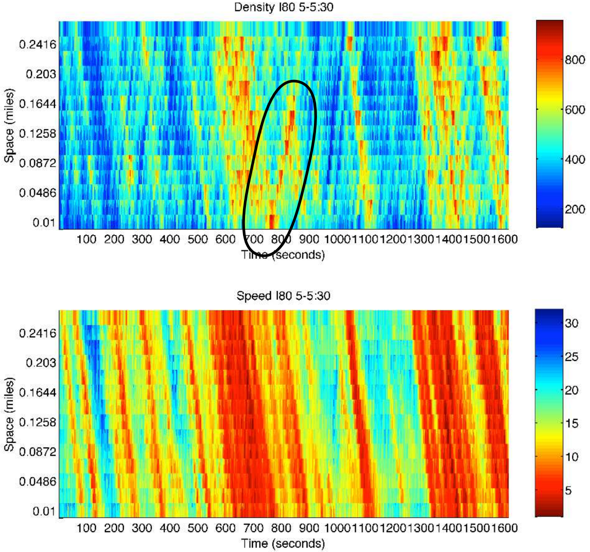




NGSIM data of I-80, CA (Blandin-Argote-Bayen-Work, TR-B, 2011)

Modeling forward moving discontinuities in congestion





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Cauchy problem - well posedness

Theorem (Colombo-Goatin-Priuli, 2006)

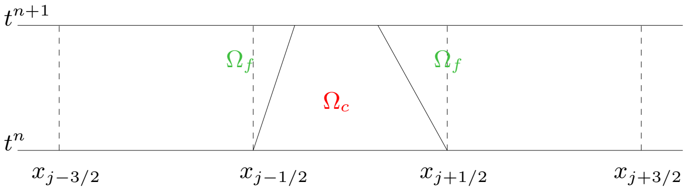
$\forall M > 0$,there exists a semigroup $S : \mathbb{R}^+ \times \mathcal{D} \mapsto \mathcal{D}$ s.t.

- $\mathcal{D} \supseteq \{\mathbf{u} \in \mathbf{L^1} : \mathrm{TV}(\mathbf{u}) \leq M\}$;
- $\|S_{t_1}\mathbf{u}_1 - S_{t_2}\mathbf{u}_2\|_{\mathbf{L^1}} \leq L(M) \cdot \left(\|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathbf{L^1}} + |t_1 - t_2|\right) \quad \forall \mathbf{u}_1, \mathbf{u}_2 \in \mathcal{D}$.

Sketch of proof

- Existence:
  - Construction of sequence of approximate solutions by wave-front tracking method (piecewise constant approximations: Dafermos '72, DiPerna '76, Bressan '92, Risebro '93)
  - Proof of convergence of the sequence of approximate solutions using BV compactness result (Helly's theorem)
  - Show that limit is a weak solution to the Cauchy problem
- Uniqueness: shift differentials

Finite volume numerical schemes

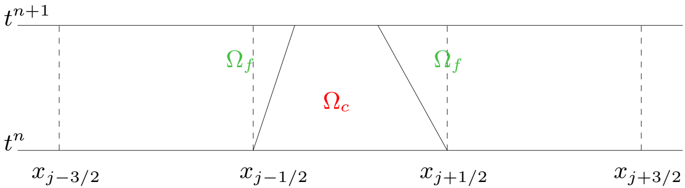


Problem:

$\Omega_f \cup \Omega_c$ is not convex $\longrightarrow$ Godunov method doesn't work in general

$$\mathbf{u}_j^n \in \Omega_c, \mathbf{u}_{j+1}^n \in \Omega_f \quad \not\Rightarrow \quad \mathbf{u}_j^{n+1} \in \Omega_f \cup \Omega_c$$

Finite volume numerical schemes



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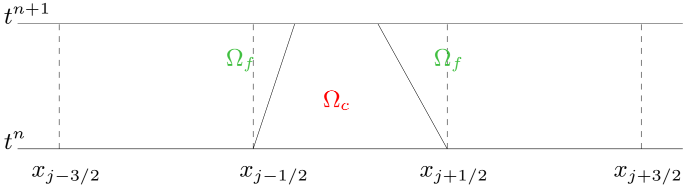
Solutions

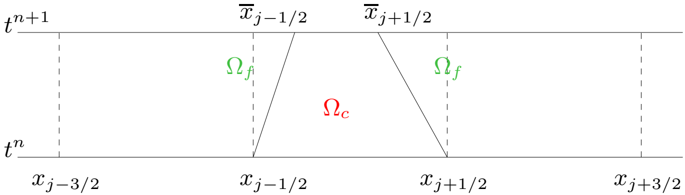
- **moving meshes for phase transitions:** Zhong - Hou - LeFloch '96;
- **transport-equilibrium method:** Chalons '07.

Godunov method



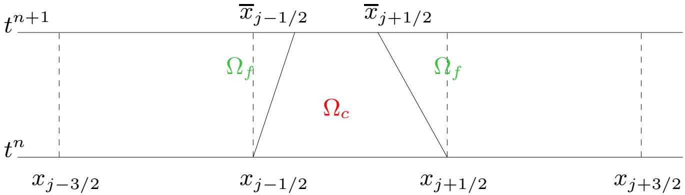
$$\mathbf{u}_j^{n+1} = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} \mathbf{v}(\Delta t, x) dx$$

Modified Godunov method (Chalons-Goatin, 2008)



$$\overline{\mathbf{u}}_j^{n+1} = \frac{1}{\overline{\Delta x}_j} \int_{\overline{x}_{j-1/2}}^{\overline{x}_{j+1/2}} \mathbf{v}(\Delta t, x) dx$$

Modified Godunov method (Chalons-Goatin, 2008)



$$\overline{\mathbf{u}}_j^{n+1} = \frac{1}{\overline{\Delta x}_j} \int_{\overline{x}_{j-1/2}}^{\overline{x}_{j+1/2}} \mathbf{v}(\Delta t, x) dx$$

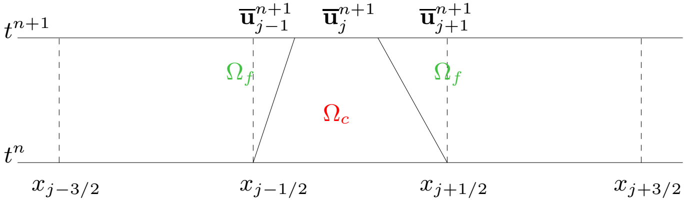
Green's formula:

$$\overline{\mathbf{u}}_j^{n+1} = \frac{\Delta x}{\overline{\Delta x}_j} \mathbf{u}_j^n - \frac{\Delta t}{\overline{\Delta x}_j} (\overline{\mathbf{f}}_{j+1/2}^{n,-} - \overline{\mathbf{f}}_{j-1/2}^{n,+})$$

with numerical flux

$$\overline{\mathbf{f}}_{j+1/2}^{n,\pm} = \mathbf{f}(\mathbf{v_r}(\sigma_{j+1/2}^{\pm}; \mathbf{u}_j^n, \mathbf{u}_{j+1}^n)) - \sigma_{j+1/2} \mathbf{v_r}(\sigma_{j+1/2}^{\pm}; \mathbf{v}_j^n, \mathbf{v}_{j+1}^n)$$

Random sampling



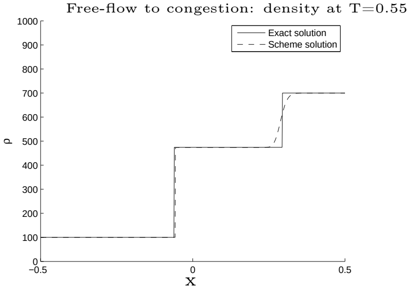
(a_n) **equi-distributed random sequence** in $]0,1[$ (ex. Van der Corput)

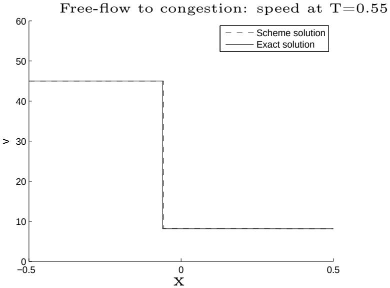
$$\mathbf{u}_j^{n+1} = \begin{cases} \overline{\mathbf{u}}_{j-1}^{n+1} & \text{si} \quad a_{n+1} \in \left]0, \frac{\Delta t}{\Delta x}\sigma_{j-1/2}^{+}\right[\\ \overline{\mathbf{u}}_{j}^{n+1} & \text{si} \quad a_{n+1} \in \left[\frac{\Delta t}{\Delta x}\sigma_{j-1/2}^{+}, 1 + \frac{\Delta t}{\Delta x}\sigma_{j+1/2}^{-}\right[\\ \overline{\mathbf{u}}_{j+1}^{n+1} & \text{si} \quad a_{n+1} \in \left[1 + \frac{\Delta t}{\Delta x}\sigma_{j+1/2}^{-}, 1\right[\end{cases}$$

$\sigma_{j+1/2}$ = phase transition speed at $x_{j+1/2}$
$\sigma_{j+1/2}^{+} = \max\{\sigma_{j+1/2}, 0\}$, $\sigma_{j+1/2}^{-} = \min\{\sigma_{j+1/2}, 0\}$

Benchmark test

Newell-Daganzo with $V = 45$, $R = 1000$, $\rho_c = 220$, $\sigma_- = 190$, $\sigma_+ = 270$:





Initial data: $\mathbf{u}_l = (100, 0) \in \Omega_f$, $\mathbf{u}_r = (700, 0.5) \in \Omega_c$ above equilibrium.
Gives: phase transition + 2-contact discontinuity linked by $\mathbf{u}_m = (474, -0.42) \in \Omega_c$.

(Blandin-Work-Goatin-Piccoli-Bayen, 2011)

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4 Crowd dynamics

5 Conclusion

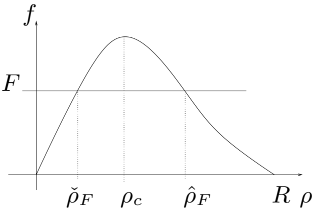
A toll gate

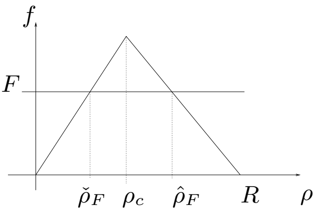
May be written as a conservation law with unilateral constraint:

$$\partial_t \rho + \partial_x f(\rho) = 0 \quad x \in \mathbb{R},\ t > 0$$

$$\rho(0, x) = \rho_0(x) \qquad x \in \mathbb{R}$$

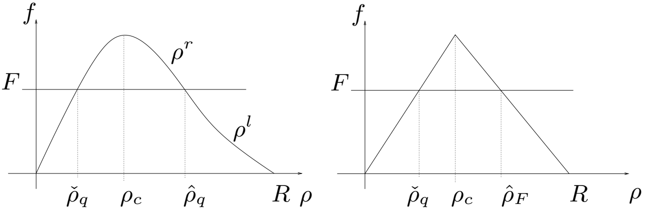
$$f(\rho(t, 0)) \leq F(t) \quad t > 0$$





The Constrained Riemann Solver $\mathcal{R}^F$

$$(\text{CRP}) \quad \begin{cases} \partial_t \rho + \partial_x f(\rho) = 0 \\ \rho(0,x) = \rho_0(x) \\ f\left(\rho(t,0)\right) \leq F \end{cases} \qquad \rho_0(x) = \begin{cases} \rho^l & \text{if } x < 0 \\ \rho^r & \text{if } x > 0 \end{cases}$$



Definition (Colombo-Goatin, 2007)

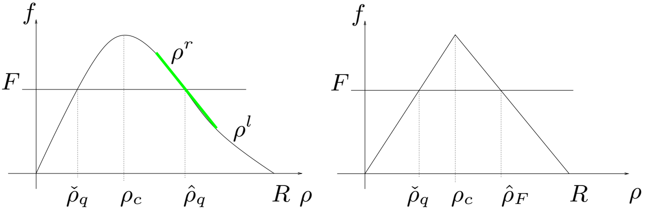
If $f\left(\mathcal{R}(\rho^l,\rho^r))(0)\right) \leq F$, then $\mathcal{R}^F(\rho^l,\rho^r) = \mathcal{R}(\rho^l,\rho^r)$.

Otherwise, $\mathcal{R}^F(\rho^l,\rho^r)(x) = \begin{cases} \mathcal{R}(\rho^l,\hat\rho_F)(x) & \text{if } x < 0\,, \\ \mathcal{R}(\check\rho_F,\rho^r)(x) & \text{if } x > 0\,. \end{cases}$

$\Longrightarrow$ non-classical shock at $x = 0$

The Constrained Riemann Solver $\mathcal{R}^F$

$$(\mathrm{CRP}) \quad \begin{cases} \partial_t \rho + \partial_x f(\rho) = 0 \\ \rho(0,x) = \rho_0(x) \\ f(\rho(t,0)) \leq F \end{cases} \qquad \rho_0(x) = \begin{cases} \rho^l & \text{if } x < 0 \\ \rho^r & \text{if } x > 0 \end{cases}$$



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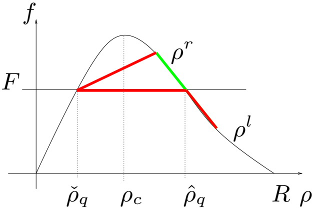

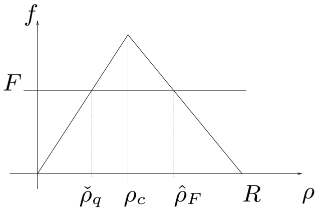

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$\Longrightarrow$ non-classical shock at $x = 0$

Entropy conditions

Definition (Colombo-Goatin, 2007)

$\rho \in \mathbf{L}^\infty$ is **weak entropy solution** if

- $\forall \phi \in \mathcal{C}_c^1$, $\phi \geq 0$, and $\forall k \in [0, R]$

$$\int_0^{+\infty} \int_{\mathbb{R}} \left(|\rho - \kappa| \partial_t + \Phi(\rho, \kappa) \partial_x \right) \phi \, dx \, dt + \int_{\mathbb{R}} |\rho_0 - \kappa| \, \phi \, dx$$
$$+ 2 \int_0^{+\infty} \left(1 - \frac{F(t)}{f(\rho_c)} \right) f(\kappa) \, \phi(t, 0) \, dt \geq 0$$

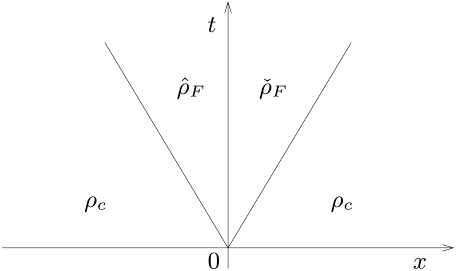
- $f(\rho(t, 0-)) = f(\rho(t, 0+)) \leq F(t)$ a.e. $t > 0$

where $\Phi(a, b) = \mathrm{sgn}(a - b)(f(a) - f(b))$

(Cfr. conservation laws with discontinuous flux function:
Karlsen-Risebro-Towers '03, Karlsen-Towers '04, Coclite-Risebro '05...)

Well-posedness in BV

constraint $\longrightarrow$ TV(ρ) explosion



We consider the function

$$\Psi(\rho) = \mathrm{sgn}(\rho - \rho_c)(f(\rho_c) - f(\rho))$$

(cfr. Temple '82, Coclite-Risebro '05 ...)

Well-posedness in BV

Theorem (Colombo-Goatin, 2007)

$F \in \mathrm{BV}$. There exists a semigroup $S^F : \mathbb{R}^+ \times \mathcal{D} \mapsto \mathcal{D}$ s.t.

- $\mathcal{D} \supseteq \{\rho \in \mathbf{L^1} : \Psi(\rho) \in \mathrm{BV}\}$;
- $\left\| S_t^F \rho_1 - S_t^F \rho_2 \right\|_{\mathbf{L^1}} \leq \|\rho_1 - \rho_2\|_{\mathbf{L^1}} \quad \forall \rho_1, \rho_2 \in \mathcal{D}$.

Proof

1. Wave-front tracking.
2. Glimm functional *ad hoc*

$$\Upsilon(\rho^n, F^n) = \sum_{\alpha} |\Psi(\rho^n_{\alpha+1}) - \Psi(\rho^n_\alpha)| + 5 \sum_{t_\beta \geq 0} \left| F^n_{\beta+1} - F^n_\beta \right| + \gamma(\rho^n)$$

3. Doubling of variables method with constraint.

Well-posedness in $\mathbf{L}^\infty$

If $F^1, F^2 \in \mathbf{L}^\infty$, $\rho_1, \rho_2 \in \mathbf{L}^\infty$ and $\rho_1 - \rho_2 \in \mathbf{L}^\mathbf{1}$:

$$\int_{\mathbb{R}} |\rho^1 - \rho^2|(T,x)\, dx \;\leq\; 2 \int_0^T |F^1 - F^2|(t)\, dt + \int_{\mathbb{R}} |\rho_0^1 - \rho_0^2|(x)\, dx$$

Theorem (Andreianov-Goatin-Seguin, 2010)

$\forall \rho_0 \in \mathbf{L}^\infty$ and $\forall F \in \mathbf{L}^\infty$ $\exists!$ weak entropy solution.

Proof

Truncation + regularization + finite propagation speed.

Finite volume schemes

Constraint at $i = 0$:

$$u_i^{n+1} = u_i^n - \frac{k}{h_i}\left(g(u_i^n, u_{i+1}^n, F_{i+1/2}^n) - g(u_{i-1}^n, u_i^n, F_{i-1/2}^n)\right)$$

with numerical flux

$$g(u,v,F) = \begin{cases} \min(h(u,v), F) & \text{if interface } i = 0 \\ h(u,v) & \text{otherwise} \end{cases}$$

h classical numerical flux:

- **regular:** Lipschitz L;
- **consistent:** $h(s,s) = f(s)$;
- **monotone:** $u \nearrow$, $v \searrow$.

(Andreianov-Goatin-Seguin, 2010)

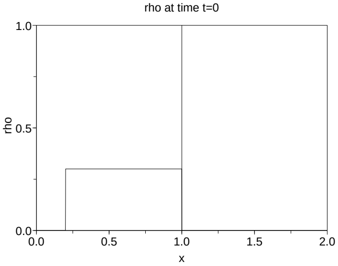
Example: toll gate

We consider

$$\partial_t \rho + \partial_x \left(\rho(1-\rho)\right) = 0$$

$$\rho(0,x) = 0.3\chi_{[0.2,1]}(x)$$

$$f(\rho(t,1)) \leq 0.1$$



Extensions

- Second order models (Aw-Rascle)
  (Garavello-Goatin, 2011)
- Rigorous study of general fluxes and non-classical problems
  (Chalons-Goatin-Seguin, 2013)
- Improved numerical techniques for non-classical problems
  (Chalons-Goatin-Seguin, 2013)
- Moving bottlenecks
  (DelleMonache-Goatin, 2012)

Outline of the talk

1. Traffic flow models
2. Phase transition models
3. Flux constraints
4. Crowd dynamics
5. Conclusion

Crowd dynamics

2D system modeling a crowd in a confined space:

$$\begin{cases} \partial_t \rho(t,\mathbf{x}) + \mathrm{div}_{\mathbf{x}} \mathbf{f}(t,\mathbf{x}) = 0 \quad t > 0,\ \mathbf{x} \in \Omega \subset \mathbb{R}^2 \\ + \text{ boundary conditions} \\ + \text{ closure equation for the flux } \mathbf{f} \end{cases}$$

to reproduce known pedestrian behavior:

- seeking the *fastest* route
- avoiding high densities and borders
- lines formation in opposite fluxes
- collective auto-organization at intersections
- behavior changes in **panic** situations and becomes irrational
- etc ...

Hughes' model (2002)

Mass conservation

$$\partial_t \rho + \mathrm{div}_{\mathbf{x}} \left(\rho \vec{V}(\rho) \right) = 0 \quad \text{in } \mathbb{R}^+ \times \Omega$$

where

$$\vec{V}(\rho) = v(\rho)\vec{N} \quad \text{and} \quad v(\rho) = v_{\max} \left(1 - \frac{\rho}{\rho_{\max}} \right)$$

Direction of the motion: $\vec{N} = -\dfrac{\nabla \phi}{|\nabla \phi|}$ is given by

$$|\nabla \phi| = \frac{1}{v(\rho)} \quad \text{in } \Omega$$

$$\phi(t, \mathbf{x}) = 0 \qquad \text{for } \mathbf{x} \in \partial\Omega_{exit}$$

- pedestrians tend to minimize their estimated travel time to the exit
- pedestrians temper their estimated travel time avoiding high densities
- CRITICS: instantaneous global information on entire domain

Dynamic model with memory effect

Mass conservation

$$\partial_t \rho + \mathrm{div}_{\mathbf{x}} \left(\rho \vec{V}(\rho) \right) = 0 \quad \text{in } \mathbb{R}^+ \times \Omega$$

where

$$\vec{V}(\rho) = v(\rho) \vec{N} \quad \text{and} \quad v(\rho) = v_{\max} \left(1 - \frac{\rho}{\rho_{\max}} \right)$$

Direction of the motion: $\vec{N} = -\dfrac{\nabla_{\mathbf{x}}(\phi + \omega D)}{|\nabla_{\mathbf{x}}(\phi + \omega D)|}$ where

$$|\nabla_{\mathbf{x}} \phi| = \frac{1}{v_{\max}} \quad \text{in } \Omega, \quad \phi(\mathbf{x}) = 0 \text{ for } \mathbf{x} \in \partial\Omega_{exit},$$

$$D = D(\rho) = \frac{1}{v(\rho)} + \beta \rho^2 \quad \text{discomfort}$$

- pedestrians seek to minimize their estimated travel time based on their knowledge of the walking domain
- pedestrians temper their behavior locally to avoid high densities

(Xia-Wong-Shu, 2009)

Second order model

Euler equations with relaxation

$$\partial_t \rho + \nabla \cdot (\rho \vec{V}) = 0$$

$$\partial_t (\rho \vec{V}) + \nabla \cdot (\rho \vec{V} \otimes \vec{V}) = \underbrace{\frac{1}{\tau}(\rho v_e(\rho)\vec{N} - \rho \vec{V})}_{relaxation\ term} + \underbrace{\nabla P(\rho)}_{anticipation\ factor}$$

where

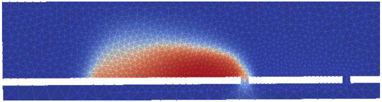
$$v_e(\rho) = v_{\max} \exp\left(-\alpha \left(\frac{\rho}{\rho_{\max}}\right)^2\right), \qquad P(\rho) = p_0 \rho^\gamma$$

and boundary conditions: $\nabla_{\mathbf{x}} \rho \cdot \vec{n} = 0$ and $\vec{V} \cdot \vec{n} = 0$

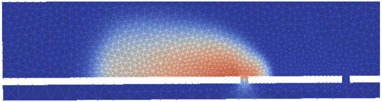
(Jiang-Zhang-Wong-Liu, 2010)

The fastest route ...

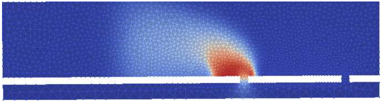
... depends on the model!



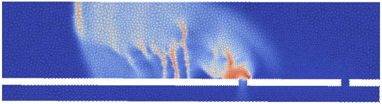
$|\nabla_{\mathbf{x}}\phi| = 1$



$\nabla_{\mathbf{x}}(\phi + \omega D)$



$|\nabla_{\mathbf{x}}\phi| = 1/v(\rho)$



second order

(Twarogowska-Duvigneau-Goatin, 2013)

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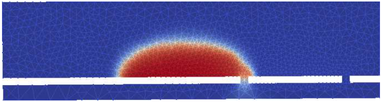

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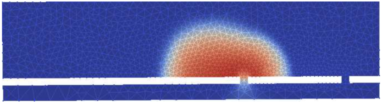

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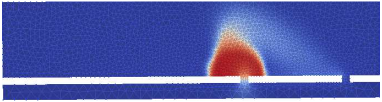

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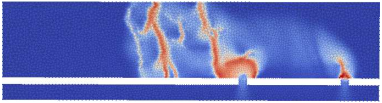

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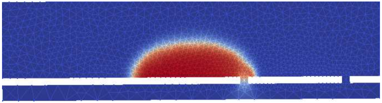

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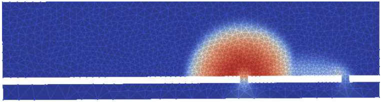

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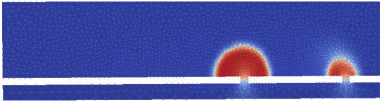

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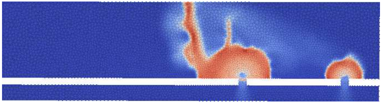

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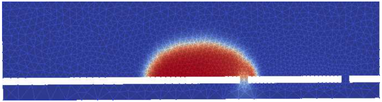

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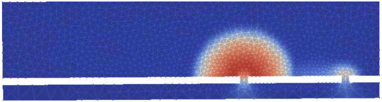

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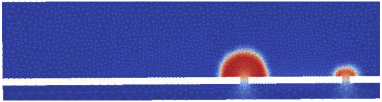

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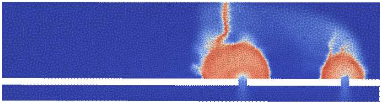
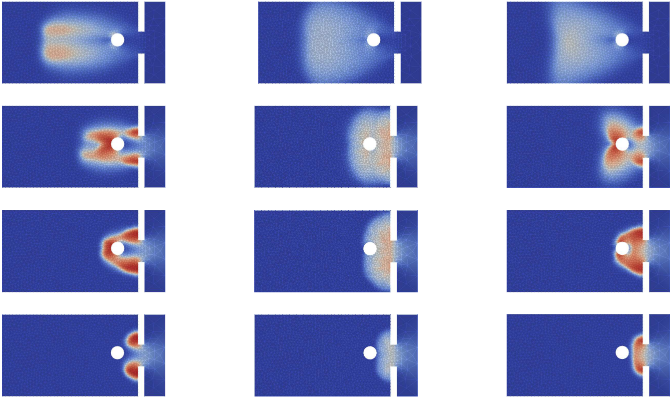


second order

(Twarogowska-Duvigneau-Goatin, 2013)

Braess' paradox?

A column in front of the exit can reduce inter-pedestrians pressure and evacuation time?



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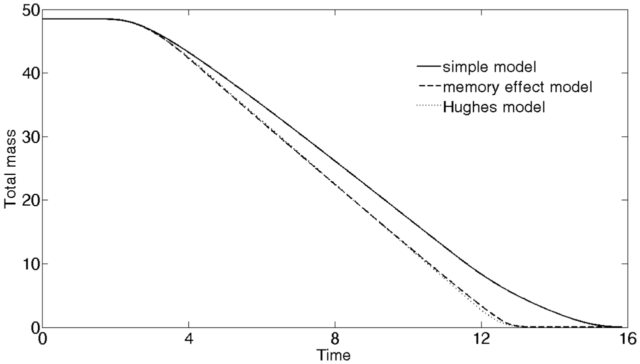
(Twarogowska-Duvigneau-Goatin, 2013)

Braess' paradox?

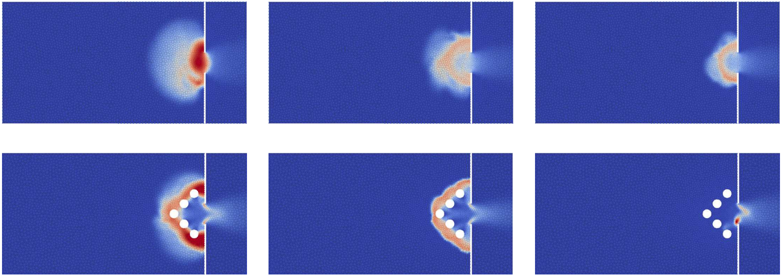
Evacuation time:



(Twarogowska-Duvigneau-Goatin, 2013)

Braess' paradox?

The second order model displays a better behavior:



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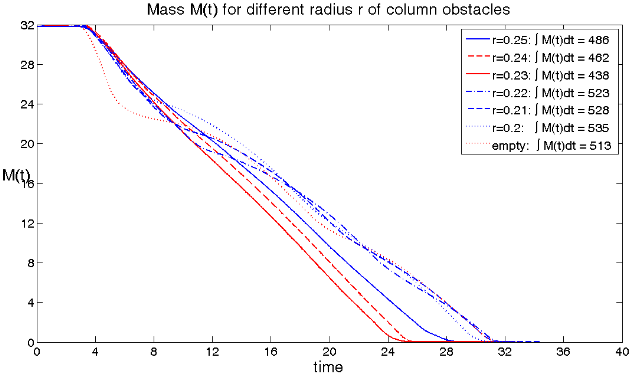

(Twarogowska-Duvigneau-Goatin, 2013)

The 1D case: statement of the problem

Rigorous (preliminary) results:
We consider the initial-boundary value problem

$$\begin{aligned} &\rho_t - \left(\rho(1-\rho)\frac{\phi_x}{|\phi_x|}\right)_x = 0 \\ &|\phi_x| = c(\rho) \end{aligned} \qquad x \in \Omega = \,]-1,1[,\ t > 0$$

with initial density $\rho(0,\cdot) = \rho_0 \in \mathrm{BV}(]0,1[)$
and *absorbing* boundary conditions

$$\begin{aligned} &\rho(t,-1) = \rho(t,1) = 0 \qquad \text{(weak sense)} \\ &\phi(t,-1) = \phi(t,1) = 0 \end{aligned}$$

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General cost function $c\colon [0,1[\,\to [1,+\infty[$ smooth s.t. $c(0) = 1$ and $c'(\rho) \geq 0$
(e.g. $c(\rho) = 1/v(\rho)$)

The 1D case: statement of the problem

The problem can be rewritten as

$$\rho_t - \Big(\operatorname{sgn}(x - \xi(t))\ f(\rho)\Big)_x = 0$$

where the *turning point* is given by

$$\int_{-1}^{\xi(t)} c\left(\rho(t,y)\right)\ dy = \int_{\xi(t)}^{1} c\left(\rho(t,y)\right)\ dy$$

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$\longrightarrow$ the discontinuity point $\xi = \xi(t)$ is not fixed *a priori*, but depends *non-locally* on ρ

The 1D case: preliminary results

- **existence and uniqueness** of Kruzkov's solutions for an elliptic regularization of the eikonal equation and $c = 1/v$ (DiFrancesco-Markowich-Pietschmann-Wolfram, 2011)

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- **wave-front tracking algorithm** and convergence of finite volume schemes
(Goatin-Mimault, 2013)

The 1D case: entropy condition

Definition: entropy weak solution (ElKhatib-Goatin-Rosini, 2012)

$\rho \in \mathbf{C^0}\left(\mathbb{R}^+;\mathbf{L^1}(\Omega)\right) \cap \mathrm{BV}\left(\mathbb{R}^+ \times \Omega;[0,1]\right)$ s.t. for all $k \in [0,1]$ and $\psi \in \mathbf{C_c^\infty}(\mathbb{R} \times \Omega;\mathbb{R}^+)$:

$$\begin{aligned} 0 \leq & \int_0^{+\infty}\int_{-1}^{1} \left(|\rho - k|\psi_t + \Phi(t,x,\rho,k)\psi_x\right)\, dx\, dt + \int_{-1}^{1} |\rho_0(x) - k|\psi(0,x)\, dx \\ & + \mathrm{sgn}(k)\int_0^{+\infty} \left(f\left(\rho(t,1-)\right) - f(k)\right)\psi(t,1)\, dt \\ & + \mathrm{sgn}(k)\int_0^{+\infty} \left(f\left(\rho(t,-1+)\right) - f(k)\right)\psi(t,-1)\, dt \\ & + 2\int_0^{+\infty} f(k)\psi\left(t,\xi(t)\right)\, dt. \end{aligned}$$

where $\Phi(t,x,\rho,k) = \mathrm{sgn}(\rho - k)\left(F(t,x,\rho) - F(t,x,k)\right)$

The 1D case: maximum principle

Proposition (ElKhatib-Goatin-Rosini, 2012)

Let $\rho \in \mathbf{C^0}\left(\mathbb{R}^+; \mathrm{BV}(\Omega) \cap \mathbf{L^1}(\Omega)\right)$ be an entropy weak solution. Then

$$0 \leq \rho(t,x) \leq \|\rho_0\|_{\mathbf{L}^\infty(\Omega)}.$$

Characteristic speeds satisfy

$$\begin{aligned} f'\left(\rho^+(t)\right) &\leq \dot{\xi}(t), \text{ if } \rho^-(t) < \rho^+(t), \\ -f'\left(\rho^-(t)\right) &\geq \dot{\xi}(t), \text{ if } \rho^-(t) > \rho^+(t). \end{aligned}$$

Outline of the talk

1. Traffic flow models
2. Phase transition models
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Perspectives

Sound analytical basis for practical implementation:

ROAD TRAFFIC :

- finite acceleration for pollution models
- ramp metering and rerouting models
- optimal control techniques for traffic management

(ORESTE Associated Team with UC Berkeley)

PEDESTRIANS :

- well-posedness
- validation against empirical data
- shape optimization for architecture and urban planning

Thank you for your attention!