Scalar conservation laws with moving density constraints arising in traffic flow modeling

Maria Laura Delle Monache

Email: maria-laura.delle_monache@inria.fr.

Joint work with Paola Goatin

14th International Conference on Hyperbolic Problems:Theory, Numerics, Applications
Università di Padova

28, June 2012

Outline

Outline

Mathematical Model I

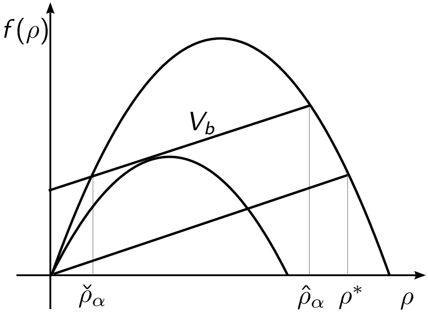
A slow moving large vehicle along a road reduces its capacity and generates a moving bottleneck for the cars flow.
From a macroscopic point of view this can be modeled by a PDE-ODE coupled model consisting in a scalar conservation law with moving density constraint and an ODE describing the slower vehicle.[1]

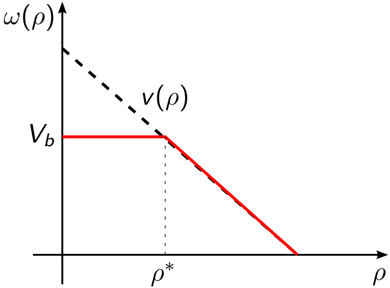
$$\begin{cases} \partial_t \rho + \partial_x f(\rho) = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}, \\ \rho(t, y(t)) \leq \alpha R, & t \in \mathbb{R}^+, \\ \dot{y}(t) = \omega(\rho(t, y(t)+)), & t \in \mathbb{R}^+, \\ y(0) = y_0. & \end{cases} \tag{1}$$

- $\rho = \rho(t,x) \in [0, R]$ mean traffic density.
- $y = y(t) \in \mathbb{R}$ bus position.

[1] F. Giorgi. *Prise en compte des transports en commun de surface dans la modélisation macroscopique de l' écoulement du trafic*. 2002. Thesis (Ph.D.) - Institut National des Sciences Appliquées de Lyon

Mathematical Model II





- $v(\rho) = V(1 - \frac{\rho}{R})$ mean traffic velocity, smooth decreasing.
- $f : [0, R] \to \mathbb{R}^+$ flux function, strictly concave $f(\rho) = \rho v(\rho)$.
- $\omega(\rho) = \begin{cases} V_b & \text{if } \rho \leq \rho^* \doteq R(1 - V_b/V), \\ v(\rho) & \text{otherwise,} \end{cases}$

  with $V_b \leq V$ lower vehicle speed.

Mathematical Model III

Fixing the value of the parameters:

- $\alpha \in]0,1[$ reduction rate of the road capacity due to the presence of the bus.
- $R = V = 1$ respectively the maximal density and the maximal velocity allowed on the road.

We obtain

$$\begin{cases} \partial_t \rho + \partial_x(\rho(1-\rho)) = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}, \\ \rho(t,y(t)) \leq \alpha, & t \in \mathbb{R}^+, \\ \dot{y}(t) = \omega(\rho(t,y(t)+)), & t \in \mathbb{R}^+, \\ y(0) = y_0. \end{cases} \tag{2}$$

Lattanzio, Maurizi and Piccoli Model

$$\begin{cases} \partial_t \rho + \partial_x f(x, y(t), \rho) = 0, \\ \rho(0, x) = \rho_0(x), \\ \dot{y}(t) = \omega(\rho(t, y(t))), \\ y(0) = y_0. \end{cases} \tag{3}$$

The model [2] gives a similar approach to the traffic flow problem even though with specific differences:

- Use of a cut-off function for the capacity dropping of car flows against constrained conservation laws with non-classical shocks.
  $\rightarrow$ $f(x, y, \rho) = \rho \cdot v(\rho) \cdot \varphi(x - y(t))$.
- Assumption that the slower vehicle has a velocity $\omega(\rho)$ such that $\omega(0) = V$ and $\omega(R) = 0$.
- ODE considered in the Filippov sense against Carathéodory approach.

[2] C.Lattanzio, A. Maurizi and B. Piccoli. Moving bottlenecks in car traffic flow: A PDE-ODE coupled model. *SIAM J. Math. Anal.,43(1):50-67, 2011.*

Colombo and Marson Model

$$\begin{cases} \partial_t \rho + \partial_x[\rho \cdot v(\rho)] = 0, \\ \rho(0,x) = \bar{\rho}(x), \\ \dot{p}(t) = \omega(\rho(t,p)), \\ p(0) = \bar{p}. \end{cases} \tag{4}$$

The model [3] is a coupled ODE-PDE problem with:

- Assumption that $\omega(\rho) \geq v(\rho)$.
- Weak coupling between the ODE and the PDE.
- Dependence of Filippov solutions to the ODE from the initial datum both of the ODE and of the conservation law.
- Hölder dependence on $\bar{p}$.

[3] R. M. Colombo and A. Marson. A Hölder continuous ODE related to traffic flow. *Proc. Roy. Soc. Edinburgh Sect. A, 133(4):759-772, 2003.*

Outline

Riemann Problem I

Consider (2) with the particular choice[4]

$$y_0 = 0 \quad \text{and} \quad \rho_0(x) = \begin{cases} \rho_L & \text{if } x < 0, \\ \rho_R & \text{if } x > 0. \end{cases} \tag{5}$$

Rewriting equations in the bus reference frame i.e., setting $X = x - V_b t$. We get

$$\begin{cases} \partial_t \rho + \partial_X \left(f(\rho) - V_b \rho \right) = 0, \\ \rho(0,x) = \begin{cases} \rho_L & \text{if } X < 0, \\ \rho_R & \text{if } X > 0, \end{cases} \end{cases} \tag{6}$$

under the constraint

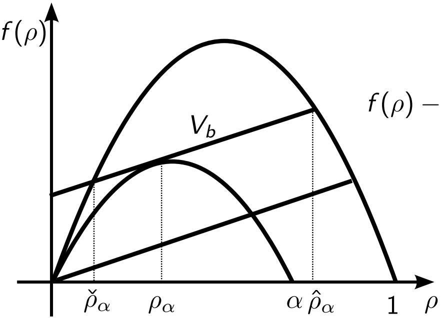
$$\rho(t,0) \leq \alpha. \tag{7}$$

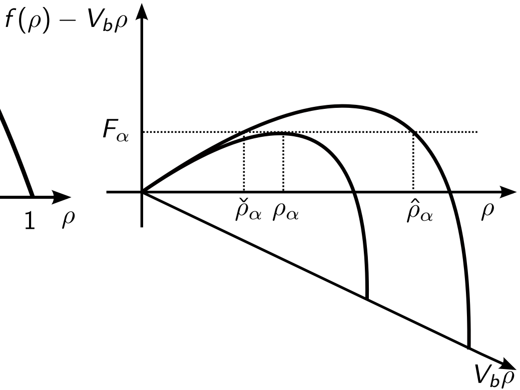
Solving problem (6), (7) is equivalent to solving (6) under the corresponding constraint on the flux

$$\boxed{f(\rho(t,0)) - V_b \rho(t,0) \leq f_\alpha(\rho_\alpha) - V_b \rho_\alpha \doteq F_\alpha}.$$

[4] R. M. Colombo and P. Goatin, A well posed conservation law with a variable unilateral constraint. *J. Differential Equations*, 234(2):654-675, 2007.

Riemann Problem II





Riemann Solver

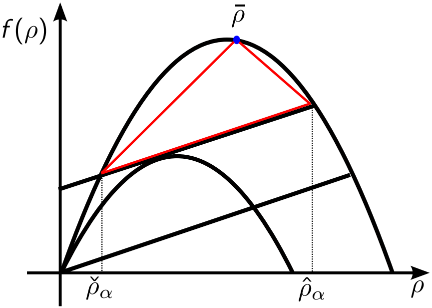
Definition (Riemann Solver)

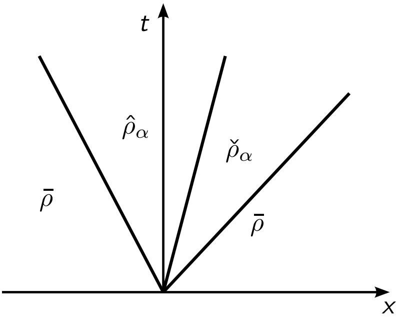
The constrained Riemann solver $\mathcal{R}^\alpha$ for (2), (5) is defined as follows.

1. If $f(\mathcal{R}(\rho_L,\rho_R)(V_b)) > F_\alpha + V_b\mathcal{R}(\rho_L,\rho_R)(V_b)$, then
$$\mathcal{R}^\alpha(\rho_L,\rho_R)(x) = \begin{cases} \mathcal{R}(\rho_L,\hat{\rho}_\alpha) & \text{if } x < V_b t, \\ \mathcal{R}(\check{\rho}_\alpha,\rho_R) & \text{if } x \geq V_b t, \end{cases} \quad \text{and} \quad y(t) = V_b t.$$
2. If $V_b\mathcal{R}(\rho_L,\rho_R)(V_b) \leq f(\mathcal{R}(\rho_L,\rho_R)(V_b)) \leq F_\alpha + V_b\mathcal{R}(\rho_L,\rho_R)(V_b)$, then
$$\mathcal{R}^\alpha(\rho_L,\rho_R) = \mathcal{R}(\rho_L,\rho_R) \quad \text{and} \quad y(t) = V_b t.$$
3. If $f(\mathcal{R}(\rho_L,\rho_R)(V_b)) < V_b\mathcal{R}(\rho_L,\rho_R)(V_b)$, then
$$\mathcal{R}^\alpha(\rho_L,\rho_R) = \mathcal{R}(\rho_L,\rho_R) \quad \text{and} \quad y(t) = v(\rho_R)t.$$

Note: when the constraint is enforced, a nonclassical shock arises, which satisfies the Rankine-Hugoniot condition but violates the Lax entropy condition ... .

Riemann Solver: Example





Outline

Cauchy Problem

A bus travels along a road modelled by

$$\begin{cases} \partial_t \rho + \partial_x(\rho(1-\rho)) = 0, \\ \rho(0,x) = \rho_0(x), \\ \rho(t, y(t)) \leq \alpha. \end{cases} \tag{8}$$

The bus influences the traffic along the road but it is also influenced by it. The bus position $y = y(t)$ then solves

$$\begin{cases} \dot{y}(t) = \omega(\rho(t, y(t)+)), \\ y(0) = y_0. \end{cases} \tag{9}$$

Solutions to (9) are intended in Carathéodory sense, i.e., as absolutely continuous functions which satisfy (9) for a.e. $t \geq 0$.

Cauchy Problem: Solution

Definition (Weak solution)

A couple $(\rho, y) \in \mathcal{C}^0\left(\mathbb{R}^+; \mathbf{L^1} \cap \mathrm{BV}(\mathbb{R})\right) \times \mathbf{W^{1,1}}(\mathbb{R}^+)$ is a solution to (2) if

1. ρ is a weak solution of the conservation law, i.e. for all $\varphi \in \mathcal{C}_c^1(\mathbb{R}^2)$

$$\int_{\mathbb{R}^+} \int_{\mathbb{R}} (\rho \partial_t \varphi + f(\rho) \partial_x \varphi) \, dx \, dt + \int_{\mathbb{R}} \rho_0(x) \varphi(0, x) \, dx = 0 \; ; \tag{10a}$$

2. y is a Carathéodory solution of the ODE, i.e. for a.e. $t \in \mathbb{R}^+$

$$y(t) = y_0 + \int_0^t \omega(\rho(s, y(s)+)) \, ds \; ; \tag{10b}$$

3. the constraint is satisfied, in the sense that for a.e. $t \in \mathbb{R}^+$

$$\lim_{x \to y(t)\pm} \left(f(\rho) - \omega(\rho)\rho\right)(t, x) \leq F_\alpha . \tag{10c}$$

Note: The above traces exist because $\rho(t, \cdot) \in \mathrm{BV}(\mathbb{R})$ for all $t \in \mathbb{R}^+$.

Wave-Front Tracking Method I

- Fix $n \in \mathbb{N}$, $n > 0$ and introduce in $[0,1]$ the mesh $\mathcal{M}_n$ by

$$\mathcal{M}_n = \left(2^{-n}\mathbb{N} \cap [0,1]\right) \cup \{\check{\rho}_\alpha, \hat{\rho}_\alpha\}.$$

- Let f_n be the piecewise linear function which coincides with f on $\mathcal{M}_n$.
- Let

$$\rho_0^n = \sum_{j \in \mathbb{Z}} \rho_{0,j}^n \, \chi_{]x_{j-1}, x_j]} \quad \text{with } \rho_{0,j}^n \in \mathcal{M}_n,$$

  such that

$$\lim_{n \to \infty} \|\rho_0^n - \rho_0\|_{\mathbf{L}^1(\mathbb{R})} = 0,$$

  and $\mathrm{TV}(\rho_0^n) \leq \mathrm{TV}(\rho_0)$.
- y_0 is given.

Wave-Front Tracking Method II

For small times $t > 0$, a piecewise approximate solution (ρ^n, y_n) to (2) is constructed piecing together the solutions to the Riemann problems

$$\begin{cases} \partial_t \rho + \partial_x (f^n(\rho)) = 0, \\ \rho(0,x) = \begin{cases} \rho_0 & \text{if } x < y_0, \\ \rho_1 & \text{if } x > y_0, \end{cases} \\ \rho(t, y_n(t)) \leq \alpha, \end{cases} \qquad \begin{cases} \partial_t \rho + \partial_x (f^n(\rho)) = 0, \\ \rho(0,x) = \begin{cases} \rho_j & \text{if } x < x_j, \\ \rho_{j+1} & \text{if } x > x_j, \end{cases} \\ j \neq 0, \end{cases} \tag{11}$$

where y_n satisfies

$$\begin{cases} \dot{y}_n(t) = \omega(\rho^n(t, y_n(t)+)), \\ y_n(0) = y_0. \end{cases} \tag{12}$$

Bounds on the total variation

Define the Glimm type functional

$$\Upsilon(t) = \Upsilon(\rho^n(t,\cdot)) = \mathrm{TV}(\rho^n) + \gamma = \sum_j \left|\rho^n_{j+1} - \rho^n_j\right| + \gamma, \tag{13}$$

with

$$\gamma = \gamma(t) = \begin{cases} 0 & \text{if } \rho^n(t, y_n(t)-) = \hat{\rho}_\alpha,\ \rho^n(t, y_n(t)+) = \check{\rho}_\alpha \\ 2|\hat{\rho}_\alpha - \check{\rho}_\alpha| & \text{otherwise.} \end{cases} \tag{14}$$

Lemma (Decreasing functional)

For any $n \in \mathbb{N}$, the map $t \mapsto \Upsilon(t) = \Upsilon(\rho^n(t,\cdot))$ at any interaction either decreases by at least 2^{-n}, or remains constant and the number of waves does not increase.

Proof

Convergence of approximate solutions

Lemma (Convergence of approximate solutions)

Let ρ^n and y_n, $n \in \mathbb{N}$, be the wave front tracking approximations to (1) constructed as detailed in Section 2, and assume $TV(\rho_0) \leq C$ be bounded, $0 \leq \rho_0 \leq 1$. Then, up to a subsequence, we have the following convergences

$$\rho^n \to \rho \qquad \text{in } \mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}^+ \times \mathbb{R}); \tag{15a}$$

$$y_n(\cdot) \to y(\cdot) \qquad \text{in } \mathbf{L}^\infty([0, T]), \text{ for all } T > 0; \tag{15b}$$

$$\dot{y}_n(\cdot) \to \dot{y}(\cdot) \qquad \text{in } \mathbf{L}^1([0, T]), \text{ for all } T > 0; \tag{15c}$$

for some $\rho \in \mathcal{C}^0\left(\mathbb{R}^+; \mathbf{L}^1 \cap BV(\mathbb{R})\right)$ and $y \in \mathbf{W}^{1,1}(\mathbb{R}^+)$.

Sketch of the proof:

- (15a): $\mathrm{TV}(\rho^n(t, \cdot)) \leq \Upsilon(t) \leq \Upsilon(0)$ + Helly's theorem.
- (15b): $|\dot{y}_n(t)| \leq V_b$ + Ascoli-Arzelà theorem.
- (15c): $\mathrm{TV}\left(\dot{y}_n; [0, T]\right) \leq 2\,\mathrm{NV}\left(\dot{y}_n; [0, T]\right) + \|\dot{y}_n\|_{\mathbf{L}^\infty([0,T])} \leq 2\,\mathrm{TV}(\rho_0) + V_b$

Complete Proof

Convergence of weak solution

Theorem (Existence of solutions)

For every initial data $\rho_0 \in BV(\mathbb{R})$ such that $TV(\rho_0) \leq C$ is bounded, problem (1) *admits a weak solution in the sense of Definition (Weak Solution).*

Sketch of the proof:

- $\int_{\mathbb{R}^+} \int_{\mathbb{R}} (\rho \partial_t \varphi + f(\rho) \partial_x \varphi) \, dx \, dt + \int_{\mathbb{R}} \rho_0(x) \varphi(0, x) \, dx = 0$:

  $\rho^n \to \rho$ in $\mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}^+ \times \mathbb{R}) \Rightarrow$ limit in the weak formulation of the conservation law.
- $\dot{y}(t) = \omega(\rho(t, y(t)+))$ for a.e. $t > 0$:

  $\lim_{n\to\infty} \rho^n(t, y_n(t)+) = \rho^+(t) = \rho(t, y(t)+)$ for a.e. $t \in \mathbb{R}^+$ [5] + (15c).
- $\lim_{x \to y(t)\pm} (f(\rho) - \omega(\rho)\rho)(t, x) \leq F_\alpha$:

  direct use of the convergence result already proved.

Complete Proof

[5] A. Bressan and P.G. LeFloch. Structural stability and regularity of entropy solutions to hyperbolic systems of conservation laws. *Indiana Univ. Math. J.*, 48(1):43-84, 1999, Section 4

Outline

Conclusions and Future work

- The coupled PDE-ODE model represents a good approach to the problem of moving bottleneck, as shown by different numerical approaches (works by F.Giorgi, J. Laval, L. Leclerq, C.F. Daganzo).
- We were able to develop a strong coupling between the PDE and ODE.
- We proved the existence of solutions for this model.
- Stability of solution is currently a work in progress.

Thank you for your attention.

- **Remark 1** Definition 1 is well posed even if the classical solution $\mathcal{R}(\rho_L, \rho_R)(x/t)$ displays a shock at $x = V_b t$. In fact, due to Rankine-Hugoniot equation, we have

$$f(\rho_L) = f(\rho_R) + V_b(\rho_L - \rho_R)$$

and hence

$$f(\rho_L) > f_\alpha(\rho_\alpha) + V_b(\rho_L - \rho_\alpha) \quad \Longleftrightarrow \quad f(\rho_R) > f_\alpha(\rho_\alpha) + V_b(\rho_R - \rho_\alpha).$$

- **Remark 2** The density constraint $\rho(t, y(t)) \leq \alpha$ is handled by the corresponding condition on the flux

$$f(\rho(t, y(t))) - \omega(\rho(t, y(t)))\rho(t, y(t)) \leq F_\alpha. \tag{16}$$

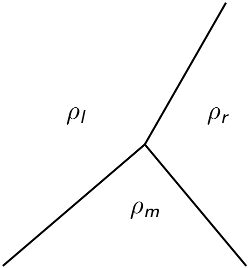
The corresponding density on the reduced roadway at $x = y(t)$ is found taking the solution to the equation

$$f(\rho_y) + \omega(\rho_y)(\rho - \rho_y) = \rho \left(1 - \frac{\rho}{\alpha}\right)$$

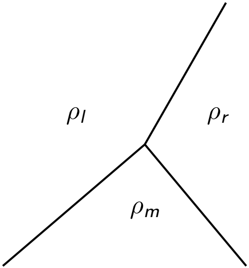
closer to $\rho_y \doteq \rho(t, y(t))$.

Back

Lemma 3: Proof I

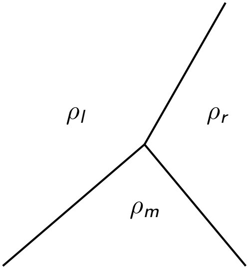


Lemma 3: Proof I

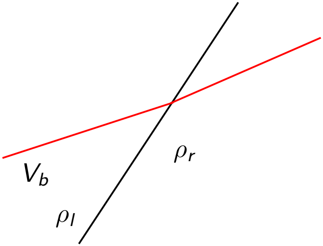


Either two shocks collide (which means that the number of waves diminishes) or a shock and a rarefaction cancel.

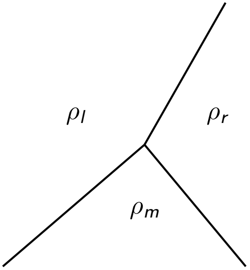
Lemma 3: Proof I



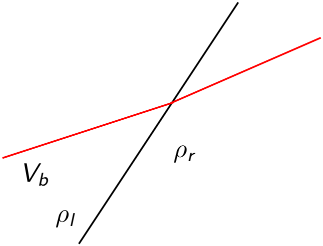
Either two shocks collide (which means that the number of waves diminishes) or a shock and a rarefaction cancel.



Lemma 3: Proof I



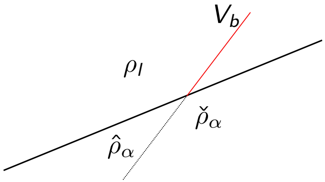
Either two shocks collide (which means that the number of waves diminishes) or a shock and a rarefaction cancel.



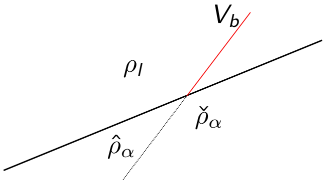
$\mathrm{TV}(\rho^n)$, Υ and the number of waves remain constant.

Back

Lemma 3: Proof II



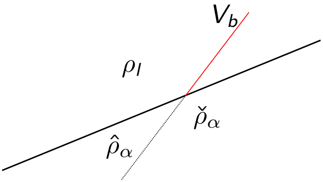
Lemma 3: Proof II

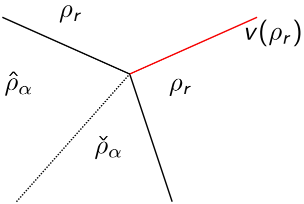


After the collision, the number of discontinuities in ρ^n diminishes and the functional Υ remains constant:

$$\begin{aligned}\Delta\Upsilon(\bar{t}) &= \Upsilon(\bar{t}+) - \Upsilon(\bar{t}-) \\ &= |\rho_l - \check{\rho}_\alpha| + 2|\hat{\rho}_\alpha - \check{\rho}_\alpha| - (|\rho_l - \hat{\rho}_\alpha| + |\hat{\rho}_\alpha - \check{\rho}_\alpha|) \\ &= 0.\end{aligned}$$

Lemma 3: Proof II

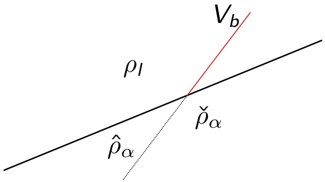




After the collision, the number of discontinuities in ρ^n diminishes and the functional Υ remains constant:

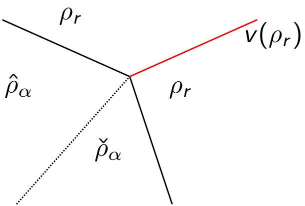
$$
\begin{aligned}
\Delta\Upsilon(\bar{t}) &= \Upsilon(\bar{t}+) - \Upsilon(\bar{t}-) \\
&= |\rho_l - \check{\rho}_\alpha| + 2|\hat{\rho}_\alpha - \check{\rho}_\alpha| - (|\rho_l - \hat{\rho}_\alpha| + |\hat{\rho}_\alpha - \check{\rho}_\alpha|) \\
&= 0.
\end{aligned}
$$

Lemma 3: Proof II



After the collision, the number of discontinuities in ρ^n diminishes and the functional Υ remains constant:

$$\begin{aligned}\Delta\Upsilon(\bar{t}) &= \Upsilon(\bar{t}+) - \Upsilon(\bar{t}-)\\ &= |\rho_l - \check{\rho}_\alpha| + 2|\hat{\rho}_\alpha - \check{\rho}_\alpha| - (|\rho_l - \hat{\rho}_\alpha| + |\hat{\rho}_\alpha - \check{\rho}_\alpha|)\\ &= 0.\end{aligned}$$

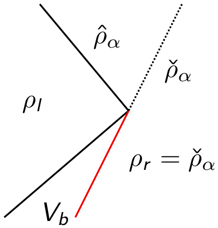


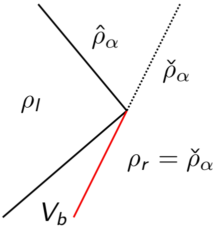
The number of discontinuities in ρ^n diminishes and the functional Υ remains constant:

$$\begin{aligned}\Delta\Upsilon(\bar{t}) &= \Upsilon(\bar{t}+) - \Upsilon(\bar{t}-)\\ &= |\hat{\rho}_\alpha - \rho_r| + 2|\hat{\rho}_\alpha - \check{\rho}_\alpha| - (|\check{\rho}_\alpha - \rho_r| + |\hat{\rho}_\alpha - \check{\rho}_\alpha|)\\ &= 0.\end{aligned}$$

Back

Lemma 3: Proof III

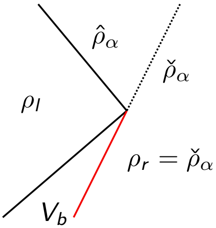




New waves are created at $\bar{t}$ and the total variation is given by:

- $\mathrm{TV}(\bar{t}-) = |\check{\rho}_\alpha - \rho_l| \leq 2^{-n}$;
- $\mathrm{TV}(\bar{t}+) = |\hat{\rho}_\alpha - \check{\rho}_\alpha| + |\hat{\rho}_\alpha - \rho_l| \leq 2|\hat{\rho}_\alpha - \check{\rho}_\alpha|$,

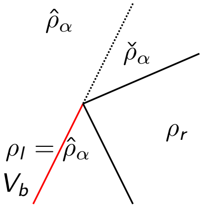
$$\Delta\Upsilon(\bar{t}) = \Upsilon(\bar{t}+) - \Upsilon(\bar{t}-) \leq -2^{-n}$$

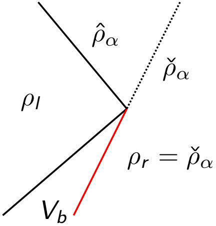


New waves are created at $\bar{t}$ and the total variation is given by:

- $\mathrm{TV}(\bar{t}-) = |\check{\rho}_\alpha - \rho_l| \leq 2^{-n}$;
- $\mathrm{TV}(\bar{t}+) = |\hat{\rho}_\alpha - \check{\rho}_\alpha| + |\hat{\rho}_\alpha - \rho_l| \leq 2|\hat{\rho}_\alpha - \check{\rho}_\alpha|$,

$$\Delta\Upsilon(\bar{t}) = \Upsilon(\bar{t}+) - \Upsilon(\bar{t}-) \leq -2^{-n}$$

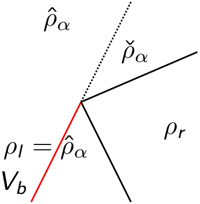




New waves are created at $\bar{t}$ and the total variation is given by:

- $\mathrm{TV}(\bar{t}-) = |\check{\rho}_\alpha - \rho_l| \leq 2^{-n}$;
- $\mathrm{TV}(\bar{t}+) = |\hat{\rho}_\alpha - \check{\rho}_\alpha| + |\hat{\rho}_\alpha - \rho_l| \leq 2|\hat{\rho}_\alpha - \check{\rho}_\alpha|$,

$$\Delta\Upsilon(\bar{t}) = \Upsilon(\bar{t}+) - \Upsilon(\bar{t}-) \leq -2^{-n}$$



New waves are created at $\bar{t}$ and the total variation is given by:

- $\mathrm{TV}(\bar{t}-) = |\hat{\rho}_\alpha - \rho_r| \leq 2^{-n}$;
- $\mathrm{TV}(\bar{t}+) = |\hat{\rho}_\alpha - \check{\rho}_\alpha| + |\check{\rho}_\alpha - \rho_r| \leq 2|\hat{\rho}_\alpha - \check{\rho}_\alpha|$,

$$\Delta\Upsilon(\bar{t}) = \Upsilon(\bar{t}+) - \Upsilon(\bar{t}-) \leq -2^{-n}$$

Back

Convergence of approximate solutions II

Proof:
From 3 we have $\mathrm{TV}(\rho^n(t,\cdot)) \leq \Upsilon(t) \leq \Upsilon(0)$. Using Helly's Theorem we ensure the existence of a subsequence converging to some function $\rho \in \mathcal{C}^0\left(\mathbb{R}^+; \mathbf{L}^1 \cap \mathrm{BV}(\mathbb{R})\right)$, proving (15a).

Since $|\dot{y}_n(t)| \leq V_b$, the sequence $\{y_n\}$ is uniformly bounded and equicontinuous on any compact interval $[0, T]$. By Ascoli-Arzelà Theorem, there exists a subsequence converging uniformly, giving (15b).

We can estimate the speed variation at interactions times $\bar{t}$ by the size of the interacting front:

$$|\dot{y}_n(\bar{t}+) - \dot{y}_n(\bar{t}-)| = |\omega(\rho_l) - \omega(\rho_r)| \leq |\rho_l - \rho_r|.$$

$\dot{y}_n$ increases only at interactions with rarefaction fronts, which must be originated at $t = 0$. Therefore,

$$\mathrm{TV}\left(\dot{y}_n; [0, T]\right) \leq 2\,\mathrm{NV}\left(\dot{y}_n; [0, T]\right) + \|\dot{y}_n\|_{\mathbf{L}^\infty([0,T])} \leq 2\,\mathrm{TV}(\rho_0) + V_b$$

is uniformly bounded, proving (15c). Back

Convergence of weak solution I

$$\boxed{\int_{\mathbb{R}^+}\int_{\mathbb{R}}(\rho\partial_t\varphi + f(\rho)\partial_x\varphi)\,dx\ dt + \int_{\mathbb{R}}\rho_0(x)\varphi(0,x)\ dx = 0\ ;} \qquad (17)$$

Proof:

Since ρ^n converge strongly to ρ in $\mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}^+ \times \mathbb{R})$, it is straightforward to pass to the limit in the weak formulation of the conservation law, proving that the limit function ρ satisfies (17).

$$\boxed{\int_{\mathbb{R}^+}\int_{\mathbb{R}} (\rho\partial_t\varphi + f(\rho)\partial_x\varphi)\, dx\ dt + \int_{\mathbb{R}} \rho_0(x)\varphi(0,x)\ dx = 0\ ;} \tag{17}$$

Proof:
Since ρ^n converge strongly to ρ in $\mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}^+ \times \mathbb{R})$, it is straightforward to pass to the limit in the weak formulation of the conservation law, proving that the limit function ρ satisfies (17).

$$\boxed{\dot{y}(t) = \omega(\rho(t, y(t)+)) \quad \text{for a. e.} t > 0.} \tag{18}$$

Proof:
We prove that

$$\lim_{n\to\infty} \rho^n(t, y_n(t)+) = \rho^+(t) = \rho(t, y(t)+) \quad \text{for a. e. } t \in \mathbb{R}^+. \tag{19}$$

By pointwise convergence a. e. of ρ^n to ρ, $\exists$ a sequence $z_n \geq y_n(t)$ s.t. $z_n \to y(t)$ and $\rho^n(t, z_n) \to \rho^+(t)$.

Back

Convergence of weak solution II

Cont.

For a. e. $t > 0$, the point $(t, y(t))$ is for $\rho(t, \cdot)$ either a continuity point, or it belongs to a discontinuity curve (either a classical or a non-classical shock).

Fix $\epsilon^* > 0$ and assume $\mathrm{TV}\left(\rho(t,\cdot);]y(t)-\delta, y(t)+\delta[\right) \leq \epsilon^*$, for some $\delta > 0$.

Then by weak convergence of measure[6] $\mathrm{TV}\left(\rho^n(t,\cdot);]y(t)-\delta, y(t)+\delta[\right) \leq 2\epsilon^*$ for n large enough, and

$$\left|\rho^n(t, y_n(t)+) - \rho^+(t)\right| \leq \left|\rho^n(t, y_n(t)+) - \rho^n(t, z_n)\right| + \left|\rho^n(t, z_n) - \rho^+(t)\right| \leq 3\epsilon^*$$

for n large enough.

If $\rho(t, \cdot)$ has a discontinuity of strength greater than ϵ^* at $y(t)$, then $\left|\rho^n(t, y_n(t)+) - \rho^n(t, y_n(t)-)\right| \geq \epsilon^*/2$ for n sufficiently large.

Back

[6] Lemma 15, A. Bressan and P.G. LeFloch. Structural stability and regularity of entropy solutions to hyperbolic systems of conservation laws. *Indiana Univ. Math. J.*,48(1):43-84, 1999

Cont.

We set $\rho^{n,+} = \rho^n(t, y_n(t)+)$ and we show that for each $\varepsilon > 0$ there exists $\delta > 0$ such that for all n large enough there holds

$$\left|\rho^n(s,x) - \rho^{n,+}\right| < \varepsilon \quad \text{for } |s-t| \leq \delta,\ |x - y(t)| \leq \delta,\ x > y_n(s). \tag{20}$$

In fact, if (20) does not hold, we could find $\varepsilon > 0$ and sequences $t_n \to t$, $\delta_n \to 0$ s. t. $\mathrm{TV}\left(\rho^n(t_n,\cdot);]y_n(t_n), y_n(t_n) + \delta_n[\right) \geq \varepsilon$.
By strict concavity of the flux function f, there should be a uniformly positive amount of interactions in an arbitrarily small neighborhood of $(t, y(t))$, giving a contradiction. Therefore (20) holds and we get

$$\left|\rho^n(t, y_n(t)+) - \rho^+(t)\right| \leq |\rho^n(t, y_n(t)+) - \rho^n(t, z_n)| + \left|\rho^n(t, z_n) - \rho^+(t)\right| \leq 2\varepsilon$$

for n large enough, thus proving (19).
Combining (15c) and (19) we get (18).

Back

$$\boxed{\lim_{x \to y(t)\pm} \left(f(\rho) - \omega(\rho)\rho\right)(t,x) \leq F_\alpha.} \tag{21}$$

Proof:

Introduce the sets

$$\Omega^{\pm} = \{(t,x) \in \mathbb{R}^+ \times \mathbb{R} \colon x \lessgtr y(t)\}$$

and

$$\Omega_n^{\pm} = \{(t,x) \in \mathbb{R}^+ \times \mathbb{R} \colon x \lessgtr y_n(t)\},$$

Consider a test function $\varphi \in \mathcal{C}_c^1(\mathbb{R}^+ \times \mathbb{R})$, $\varphi \geq 0$, s. t. $\text{supp}(\varphi) \cap \{(t, y(t)) \colon t > 0\} \neq 0$ and $\text{supp}(\varphi) \cap \{(t, y_n(t)) \colon t > 0\} \neq 0$.

Back

Convergence of weak solution V

Cont.

Then by conservation on Ω_n^+ we have

$$\begin{aligned}
&\iint \chi_{\Omega_n^+}\left(\rho^n\partial_t\varphi + f^n(\rho^n)\partial_x\varphi\right)\,dx\,dt \\
&= \int_0^{+\infty}\int_{y_n(t)}^{+\infty}\left(\rho^n\partial_t\varphi + f^n(\rho^n)\partial_x\varphi\right)\,dx\,dt \\
&= \int_0^{+\infty}\left(f^n(\rho^n(t,y_n(t)+)) - \dot{y}_n(t)\rho^n(t,y_n(t)+)\right)\varphi(t,y_n(t))\,dt \\
&= \int_0^{+\infty}\left(f^n(\rho^n(t,y_n(t)+)) - \omega(\rho^n(t,y_n(t)+))\rho^n(t,y_n(t)+)\right)\varphi(t,y_n(t))\,dt \\
&\leq \int_0^{T} F_\alpha\varphi(t,y_n(t))\,dt, \qquad (22)
\end{aligned}$$

Back

Convergence of weak solution VI

Cont.

The same can be done for the limit solutions ρ and $y(t)$ that, by conservation on Ω, satisfy

$$\iint \chi_{\Omega^+} \left(\rho \partial_t \varphi + f(\rho) \partial_x \varphi\right) \, dx \, dt = \int_0^{+\infty} \int_{y(t)}^{+\infty} \left(\rho \partial_t \varphi + f(\rho) \partial_x \varphi\right) \, dx \, dt$$

$$= \int_0^{+\infty} \left(f(\rho(t, y(t)+)) - \dot{y}(t) \rho(t, y(t)+)\right) \varphi(t, y(t)) \, dt$$

$$= \int_0^{+\infty} \left(f(\rho(t, y(t)+)) - \omega(\rho(t, y(t)+)) \rho(t, y(t)+)\right) \varphi(t, y(t)) \, dt \tag{23}$$

By (15a) and (15b) we can pass to the limit in (22) and (23), which gives

$$\int_0^{+\infty} \left(f(\rho(t, y(t)+)) - \omega(\rho(t, y(t)+)) \rho(t, y(t)+) - F_\alpha\right) \varphi(t, y(t)) \, dt \leq 0.$$

Since the above inequality holds for every test function $\varphi \geq 0$,
we have proved (21).

Back