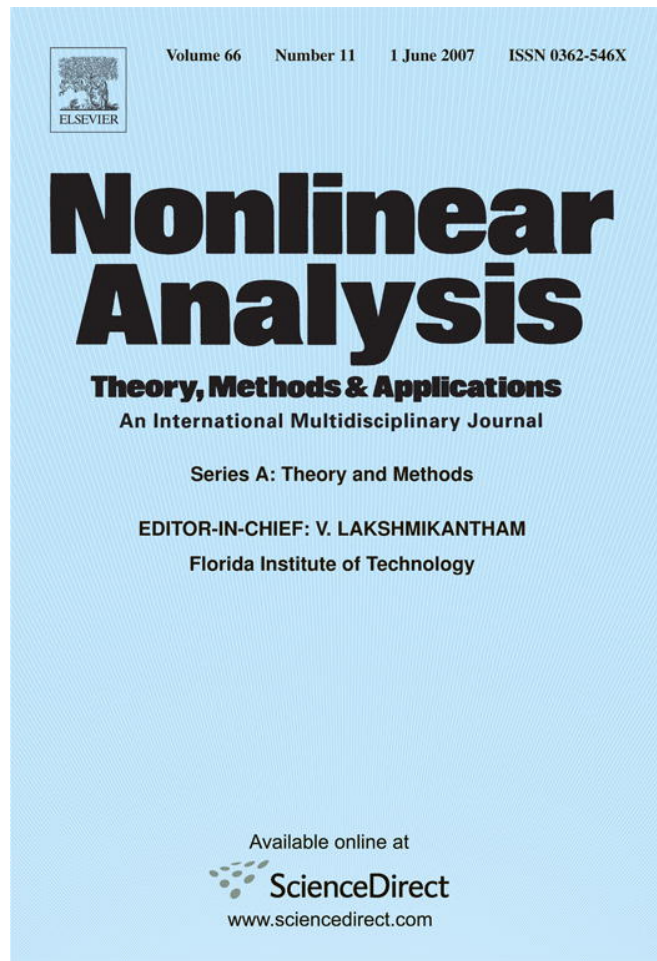


Provided for non-commercial research and educational use only.  
Not for reproduction or distribution or commercial use.



This article was originally published in a journal published by Elsevier, and the attached copy is provided by Elsevier for the author's benefit and for the benefit of the author's institution, for non-commercial research and educational use including without limitation use in instruction at your institution, sending it to specific colleagues that you know, and providing a copy to your institution's administrator.

All other uses, reproduction and distribution, including without limitation commercial reprints, selling or licensing copies or access, or posting on open internet sites, your personal or institution's website or repository, are prohibited. For exceptions, permission may be sought for such use through Elsevier's permissions site at:

<http://www.elsevier.com/locate/permissionusematerial>

# Global well posedness of traffic flow models with phase transitions

Rinaldo M. Colombo<sup>a,\*</sup>, Paola Goatin<sup>b</sup>, Fabio S. Priuli<sup>c</sup>

<sup>a</sup> *Department of Mathematics, Via Branze 38, 25123 Brescia, Italy*

<sup>b</sup> *I.S.I.T.V., Université du Sud - Toulon - Var, B.P. 56, 83162 La Valette du Var Cedex, France*

<sup>c</sup> *S.I.S.S.A., Via Beirut 2–4, 34100 Trieste, Italy*

Received 16 January 2006; accepted 17 March 2006

---

## Abstract

This note is devoted to the study of traffic flow models that develop phase transitions. From the analytical point of view, this is a first example of a well posedness result for conservation laws developing phase transitions, which is independent from the number of phase boundaries in the initial data or in the solutions. We consider below the Cauchy problem as well as the problem with boundaries.

© 2006 Elsevier Ltd. All rights reserved.

MSC: 35L65; 90B20

Keywords: Hyperbolic conservation laws; Phase transitions; Continuum traffic models

---

## 1. Introduction

The aim of this note is to prove the well posedness of traffic flow models developing phase transitions and their stability with respect to the parameters. First, we briefly deal with the scalar model by Drake et al. [16] whose analytical properties have not yet been apparently considered in the literature. Then, we consider the model introduced in [7] and obtain a first example of a system of conservation laws developing phase transitions whose well posedness is proved *globally*, i.e. for all initial data attaining values in a given set and with bounded total variation. In the literature, several results deal with the solution to Riemann problems in the presence of phase

---

\* Corresponding author. Tel.: +39 30 371 5730; fax: +39 30 371 5745.

E-mail addresses: [rinaldo@ing.unibs.it](mailto:rinaldo@ing.unibs.it) (R.M. Colombo), [goatin@univ-tln.fr](mailto:goatin@univ-tln.fr) (P. Goatin), [priuli@sissa.it](mailto:priuli@sissa.it) (F.S. Priuli).

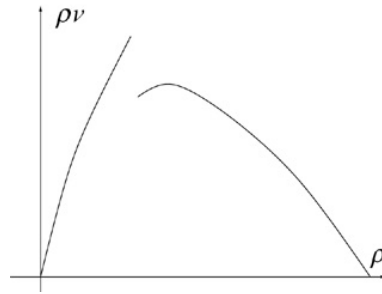


Fig. 1. The fundamental diagram in the Edie hypothesis, see [16,17].

transitions, see for instance [13,19,20]. Other works prove the global in time well posedness of the Cauchy problem, but with initial data that are perturbations of a given phase boundary, see for instance [8,9]. On the contrary, here the number of phase boundaries that are present in the data and in the solution is not a priori fixed.

From the traffic point of view, the well posedness proved below allows us to consider various control and optimization problems, see [11].

### 2. The scalar model

In [16] the authors select a variation of the “Edie formulation” [17] as the best among several traffic models, see also [22] or [24, Model B]. Essentially, it consists of the Lighthill–Whitham [21] and Richards [23] (LWR) model with a fundamental diagram as in Fig. 1. Then, the conservation of the total number of vehicles along any road segment reads

$$\partial_t \rho + \partial_x (\rho v(\rho)) = 0, \tag{2.1}$$

where  $\rho$  is the traffic density and  $v$  the traffic speed. In this model, the speed  $v$  and the flow  $\rho v$  are defined on a *disconnected* set, its two connected components being two disjoint intervals representing the *free* and the *congested* phase.

**Proposition 2.1.** *Let  $\Omega_f = [0, \check{R}]$ ,  $\Omega_c = [\hat{R}, R]$ , with  $0 < \check{R} < \hat{R} < R$ , and  $v: \Omega_f \cup \Omega_c \mapsto \mathbb{R}$  be smooth, decreasing and such that  $v(R) = 0$ . Then, for all  $\rho_o \in \mathbf{L}^1(\mathbb{R}; \Omega_f \cup \Omega_c)$ , (2.1) admits a unique weak entropy solution  $\rho \in \mathbf{C}^0(\mathbb{R}^+; \mathbf{L}^1(\mathbb{R}; \Omega_f \cup \Omega_c))$  attaining  $\rho_o$  as initial data and which is non-expansive with respect to the  $\mathbf{L}^1$  norm.*

The proof of Proposition 2.1 follows from the slightly more general Proposition 4.1 proved below. Note that the invariance of  $\Omega_f \cup \Omega_c$  implies that density and speed remain positive and bounded.

### 3. The 2 x 2 model

We consider now the model introduced in [7]. It consists of a scalar LWR model coupled with the  $2 \times 2$  system presented in [6]. The former applies to the states of *free* flow, while the latter to the *congested* states. A *phase transition* is a discontinuity separating a state of free traffic from one in the congested phase. More precisely, the model in [7] reads

$$\begin{array}{ll} \text{Free flow: } (\rho, q) \in \Omega_f & \text{Congested flow: } (\rho, q) \in \Omega_c \\ \partial_t \rho + \partial_x [\rho \cdot v_f(\rho)] = 0 & \begin{cases} \partial_t \rho + \partial_x [\rho \cdot v_c(\rho, q)] = 0 \\ \partial_t q + \partial_x [(q - Q) \cdot v_c(\rho, q)] = 0 \end{cases} \\ v_f(\rho) = \left(1 - \frac{\rho}{R}\right) \cdot V & v_c(\rho, q) = \left(1 - \frac{\rho}{R}\right) \cdot \frac{q}{\rho}. \end{array} \tag{3.1}$$

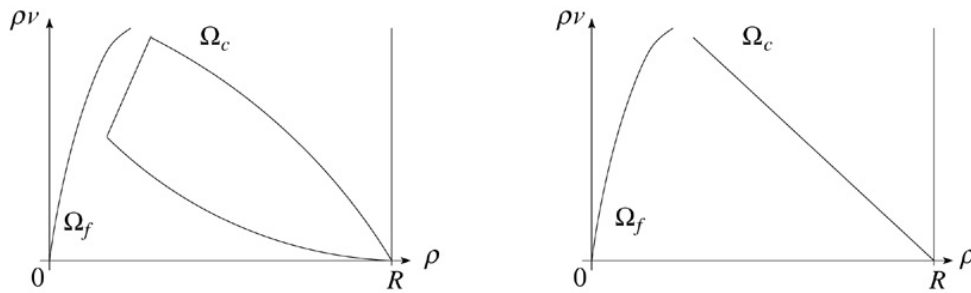


Fig. 2. Left, a generic fundamental diagram for (3.1) and, right, the limiting case  $Q^- = Q = Q^+$ .

$\rho$  and  $v$  are as above,  $q$  is the weighted linear momentum,  $R$  is the maximal traffic density,  $V$  is the maximal traffic speed and  $Q$  is characterized by the phenomenon of *wide jams*, see [7]. System (3.1) is studied in  $\Omega_f \cup \Omega_c$ , where

$$\begin{aligned} \Omega_f &= \{(\rho, q) \in [0, R] \times \mathbb{R}^+ : v_f(\rho) \geq V_f, q = \rho \cdot V\} \\ \Omega_c &= \left\{(\rho, q) \in [0, R] \times \mathbb{R}^+ : v_c(\rho, q) \leq V_c, \frac{q - Q}{\rho} \in \left[ \frac{Q^- - Q}{R}, \frac{Q^+ - Q}{R} \right] \right\}, \end{aligned}$$

where we denote  $\mathbb{R}^+ = [0, +\infty[$ . Here,  $V_f$  and  $V_c$  are the threshold speeds, i.e. above  $V_f$  the flow is free, while below  $V_c$  the flow is congested. Following [7], throughout the present note we assume that the various parameters are strictly positive and satisfy

$$\begin{aligned} V > V_f > V_c, \quad \frac{Q^+ - Q}{RV} < 1, \quad V_f = \frac{V - Q^+/R}{1 - (Q^+ - Q)/(RV)}. \end{aligned} \tag{3.2}$$

In the limiting case  $Q^- = Q = Q^+$ , (3.1) essentially reduces to the scalar model in Fig. 2, right, see also [22] or [24, Model B], which falls within the scope of Proposition 2.1.

### 3.1. The Cauchy problem

Introduce the notations:

$$\begin{aligned} X &= \mathbf{L}^1(\mathbb{R}; \Omega_f \cup \Omega_c), \quad \|u\|_{\mathbf{L}^1} = \|\rho\|_{\mathbf{L}^1(\mathbb{R})} + \|q\|_{\mathbf{L}^1(\mathbb{R})}, \\ u &= (\rho, q), \quad \text{TV}(u) = \text{TV}(\rho) + \text{TV}(q). \end{aligned} \tag{3.3}$$

**Definition 3.1.** Fix  $M > 0$  and  $X$  as above. A map  $S: \mathbb{R}^+ \times \mathcal{D} \mapsto \mathcal{D}$  is an  $M$ -Riemann semigroup ( $M$ -RS) if the following holds:

- (RS1)  $\mathcal{D} \supseteq \{u \in X : \text{TV}(u) \leq M\}$ ;
- (RS2)  $S_0 = \text{Id}$  and  $S_{t_1} \circ S_{t_2} = S_{t_1+t_2}$ ;
- (RS3) there exists an  $L = L(M)$  such that for  $t_1, t_2$  in  $\mathbb{R}^+$  and  $u_1, u_2$  in  $\mathcal{D}$ ,

$$\|S_{t_1}u_1 - S_{t_2}u_2\|_{\mathbf{L}^1} \leq L \cdot (\|u_1 - u_2\|_{\mathbf{L}^1} + |t_1 - t_2|);$$

- (RS4) if  $u \in \mathcal{D}$  is piecewise constant, then for  $t$  small,  $S_t u$  coincides with the gluing of solutions to Riemann problems.

By “solutions to Riemann problems” we refer here to those defined in [7, Section 3], recalled here in the proof of Proposition 4.2. Properties (RS1)–(RS4) provide the natural extension of [4, Definition 9.1] to the present case.

We are now ready to state the main result of this note, namely the existence of an  $M$ -RS generated by the Cauchy problem for (3.1).

**Theorem 3.2.** *For any positive  $M$ , the system (3.1) generates an  $M$ -RS  $S: \mathbb{R}^+ \times \mathcal{D} \mapsto \mathcal{D}$ . Moreover*

- (CP1) *for all  $(\rho_o, q_o) \in \mathcal{D}$ , the orbit  $t \mapsto S_t(\rho_o, q_o)$  is a weak entropic solution to (3.1) with initial data  $(\rho_o, q_o)$ ;*  
 (CP2) *any two  $M$ -RS coincide up to the domain;*  
 (CP3) *the solutions yielded by  $S$  can be characterized as viscosity solutions, in the sense of [4, Theorem 9.2].*  
 (CP4)  $\mathcal{D} \subseteq \{u \in X: \text{TV}(u) \leq \widehat{M}\}$  *for a positive  $\widehat{M}$  dependent only on  $M$ .*

The proof is deferred to Section 4. Here we observe that the description of several realistic situations requires suitable source terms in the right hand sides of (3.1). The techniques in [10] can then be applied.

### 3.2. The initial-boundary value problem

From the point of view of traffic flow, it is natural to consider the case of a road starting at  $x = 0$  where the inflow at time  $t$  is the prescribed quantity  $\tilde{f}(t)$ . This leads to the Initial-Boundary Value Problem (IBVP) consisting of (3.1) with initial and boundary data

$$\begin{cases} (\rho, q)(0, x) = (\bar{\rho}, \bar{q})(x) & x \geq 0, \\ (\rho v)(t, 0) = \tilde{f}(t) & t \geq 0. \end{cases} \quad (3.4)$$

If, besides (3.2), also

$$\left(1 - \frac{Q^+}{RV}\right) \cdot \left(\frac{Q^+}{Q} - 1\right) < 1 \quad (3.5)$$

holds, then the definition of the solution introduced in [18], see also [1, Definition NC], applies: the boundary data  $\tilde{f}$  is attained in the sense that

$$\lim_{x \rightarrow 0^+} \rho(t, x) \cdot v(\rho(t, x), q(t, x)) = \tilde{f}(t) \quad \text{for a.e. } t \geq 0.$$

Denote the maximum possible traffic flow by  $F = R_f V_f$ .

**Proposition 3.3.** *If (3.2) and (3.5) hold, then for all  $(\bar{\rho}, \bar{q}) \in \Omega_f \cup \Omega_c$ , there exists a threshold  $f^{\max} = f^{\max}(\bar{\rho}, \bar{q})$  such that for all  $\tilde{f} \in [0, f^{\max}]$  the Riemann problem made by (3.1) with data*

$$\begin{cases} (\rho, q)(0, x) = (\bar{\rho}, \bar{q}) & x \geq 0, \\ (\rho v)(t, 0) = \tilde{f} & t \geq 0 \end{cases} \quad (3.6)$$

*admits a solution in the sense of [1, Definition NC]. More precisely, there exists a unique state  $(\tilde{\rho}, \tilde{q}) \in \Omega_f \cup \Omega_c$  such that the flow at  $(\tilde{\rho}, \tilde{q})$  is  $\tilde{f}$  and the solution to the standard Riemann problem (3.1) with data  $(\tilde{\rho}, \tilde{q})$  and  $(\bar{\rho}, \bar{q})$  consists of waves having only positive speed. Furthermore,*

- (1) *If  $(\bar{\rho}, \bar{q}) \in \Omega_f$ , then  $f^{\max} = F$  and  $(\tilde{\rho}, \tilde{q})$  is in  $\Omega_f$ . The solution consists of a 2-wave in the free phase.*

- (2) If  $(\bar{\rho}, \bar{q}) \in \Omega_c$ , then there exists a  $f^{\min} = f^{\min}(\bar{\rho}, \bar{q})$  such that:
- (a) If  $f^{\min} \leq \tilde{f} \leq f^{\max}$ ,  $(\tilde{\rho}, \tilde{q})$  is the unique intersection between the curve  $\rho v_c(\rho, q) = \tilde{f}$  and the 2-wave through  $(\bar{\rho}, \bar{q})$ . The solution consists of a simple 2-wave.
  - (b) If  $\tilde{f} < f^{\min}$ , then  $(\tilde{\rho}, \tilde{q})$  is the unique state in  $\Omega_f$  such that  $\tilde{\rho} v_f(\tilde{\rho}) = \tilde{f}$ . The solution consists of a phase boundary and a 2-wave.

The thresholds  $f^{\min}$  and  $f^{\max}$  are given explicitly in Section 4, in the proof of Proposition 3.3. Here we stress that, differently from what happens in the LWR model, the incoming flow  $\tilde{f}$  can be greater than the flow  $\bar{\rho}v(\bar{\rho})$  present on the road.

We remark that, due to the presence of phase boundaries, the number of waves entering the domain  $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^+$  cannot be a priori established.

Once the Riemann problem (3.1)–(3.6) is solved, the full IBVP (3.1)–(3.4) can be considered. Now, in Definition 3.1, (3.3) is substituted by

$$\begin{aligned}
 u &= (\rho, q, f) \\
 X &= \mathbf{L}^1(\mathbb{R}^+; (\Omega_f \cup \Omega_c) \times [0, F]) \\
 \|u\|_{\mathbf{L}^1} &= \|\rho\|_{\mathbf{L}^1(\mathbb{R}^+)} + \|q\|_{\mathbf{L}^1(\mathbb{R}^+)} + \|f\|_{\mathbf{L}^1(\mathbb{R}^+)} \\
 \text{TV}(u) &= \text{TV}(\rho) + \text{TV}(q) + \text{TV}(f) + |(\rho v)(0) - f(0)|.
 \end{aligned}
 \tag{3.7}$$

**Theorem 3.4.** Let (3.2) and (3.5) hold. For every positive  $M$ , the IBVP (3.1)–(3.4) generates a  $M$ -RS

$$\begin{aligned}
 S : \mathbb{R}^+ \times \mathcal{D} &\mapsto \mathcal{D} \\
 t, (\bar{\rho}, \bar{q}, \tilde{f}) &\mapsto (\rho(t), q(t), \mathcal{T}_t \tilde{f})
 \end{aligned}$$

in the sense of Definition 3.1 and (3.7). Moreover

- (IBVP1) for all  $(\bar{\rho}, \bar{q}, \tilde{f}) \in \mathcal{D}$ , the map  $t \mapsto (\rho(t), q(t))$  is a solution to (3.1)–(3.4) with initial data  $(\bar{\rho}, \bar{q})$  and boundary data  $\tilde{f}$ ;
- (IBVP2) any two  $M$ -RS coincide up to the domain;
- (IBVP3) the solutions yielded by  $S$  can be characterized as viscosity solutions, in the sense of [2, Section 5];
- (IBVP4)  $\mathcal{D} \subseteq \{u \in X: \text{TV}(u) \leq \widehat{M}\}$  for an  $\widehat{M} > 0$  depended only on  $M$ .

Above,  $\mathcal{T}$  is the translation operator, i.e.  $(\mathcal{T}_t f)(s) = f(t + s)$ . In the present case, (RS3) of Definition 3.1 implies that

$$\begin{aligned}
 &\|(\rho_1, q_1)(t_1) - (\rho_2, q_2)(t_2)\|_{\mathbf{L}^1} \\
 &\leq L \cdot \left( \|(\bar{\rho}_1, \bar{q}_1) - (\bar{\rho}_2, \bar{q}_2)\|_{\mathbf{L}^1} + \|\tilde{f}_1 - \tilde{f}_2\|_{\mathbf{L}^1} + |t_1 - t_2| \right).
 \end{aligned}$$

Following [1], the techniques in [3,12,14] can be extended to deal with the present case. Hence the proof of Theorem 3.4 is omitted.

## 4. Proofs

### 4.1. The scalar case

Let  $f: \Omega \mapsto \mathbb{R}$  be smooth with  $\Omega$  being the union of two separated closed real intervals  $\Omega_f$  and  $\Omega_c$ , the phases. The case of more than 2 phases is entirely similar. The standard Kruřkov Theorem, see for instance [4, Paragraphs 6.2 and 6.3], is directly extended to the present situation.

**Proposition 4.1.** *Let  $f: \Omega \mapsto \mathbb{R}$  be locally Lipschitz. Then,*

(1) *for all  $u_o \in \mathbf{L}^1(\mathbb{R}; \Omega) \cap \mathbf{BV}(\mathbb{R}; \Omega)$ , the Cauchy problem (2.1) with initial datum  $u_o$  admits a weak entropy solution  $u: \mathbb{R}^+ \times \mathbb{R} \mapsto \mathbb{R}$  with*

$$\text{TV}(u(t, \cdot)) \leq \text{TV}(u_o) \quad \text{and} \quad \|u(t, \cdot)\|_{\mathbf{L}^\infty} \leq \|u_o\|_{\mathbf{L}^\infty} \quad \forall t \geq 0;$$

(2) *if  $u_o$  and  $w_o$  are in  $\mathbf{L}^1(\mathbb{R}; \Omega) \cap \mathbf{BV}(\mathbb{R}; \Omega)$ , then for all  $t \geq 0$*

$$\|u(t, \cdot) - w(t, \cdot)\|_{\mathbf{L}^1} \leq \|u_o - w_o\|_{\mathbf{L}^1}.$$

It is straightforward to extend the proof in [4] to the present situation. Indeed, assume for simplicity that  $\Omega_f = [a, b]$  and  $\Omega_c = [c, d]$ , with  $-\infty < a \leq b < c \leq d < +\infty$ , the other cases being entirely analogous. Consider the following extension  $\bar{f}$  of  $f$  to the whole of  $\mathbb{R}$ :

$$\bar{f}(u) = \begin{cases} f(a) & \text{if } u \in ]-\infty, a[ \\ f(u) & \text{if } u \in [a, b] \\ \frac{c-u}{c-b} f(b) + \frac{u-b}{c-b} f(c) & \text{if } u \in ]b, c[ \\ f(u) & \text{if } u \in [c, d] \\ f(d) & \text{if } u \in ]d, +\infty[. \end{cases}$$

Then, it is immediate to prove that if  $u_o \in \mathbf{L}^1(\mathbb{R}, \Omega) \cap \mathbf{BV}(\mathbb{R}, \Omega)$ , then the weak entropy solution  $u = u(t, x)$  of

$$\begin{cases} \partial_t u + \partial_x [\bar{f}(u)] = 0 \\ u(0, x) = u_o(x) \end{cases}$$

attains values in  $\Omega$ . Indeed, if  $u_o(\mathbb{R}) \subseteq [a, d]$ , then also  $u(t, \mathbb{R}) \subseteq [a, d]$  for all  $t \geq 0$  by the “maximum principle” [4, (iv), Theorem 6.3, Chapter 6]. Moreover,  $u(t, x)$  is the limit of piecewise constant solutions  $u^\nu(t, x)$  to conservation laws

$$\begin{cases} \partial_t u + \partial_x [\bar{f}^\nu(u)] = 0 \\ u(0, x) = u_o^\nu(x) \end{cases}$$

with  $f^\nu$  being a piecewise linear and continuous approximation of  $\bar{f}$  and  $u_o^\nu$  a piecewise constant approximation of  $u_o$ , see [4, Sections 6.1 and 6.2]. For all  $\nu$ ,  $\bar{f}^\nu$  can be chosen so that  $\bar{f}^\nu(u) = \bar{f}(u)$  for  $u \in [b, c]$ . Hence, if  $u_o^\nu(\mathbb{R})$  does not intersect  $[b, c]$ , also  $u^\nu(t)$  does not attain values in  $[b, c]$ .

For the sake of completeness, we recall that in the present section we used Liu’s entropy condition, see [15, Section 8.4].

#### 4.2. The $2 \times 2$ model

The proof of Theorem 3.2 is achieved through the construction of exact weak solutions to (3.1) that are only approximately entropic, built by means of wave front tracking.

Recall first the following basic information on the  $2 \times 2$  system on the right hand side of (3.1):

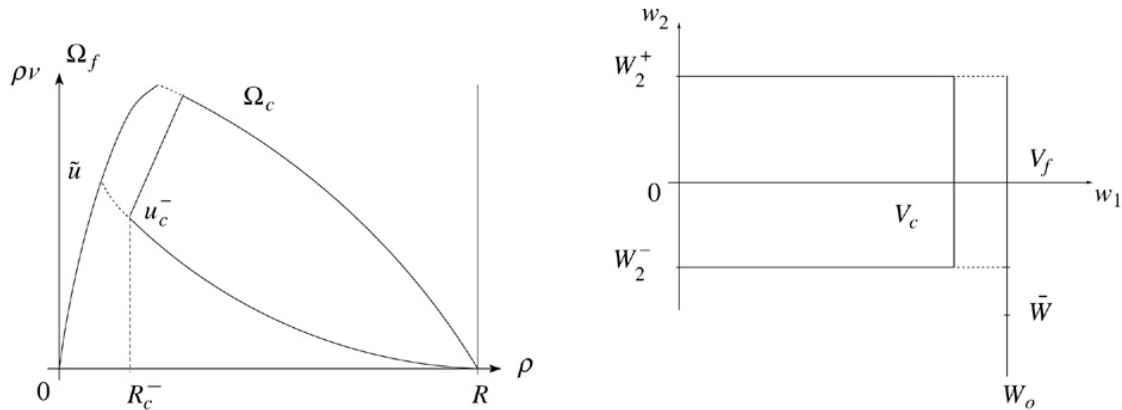


Fig. 3. Notation used in the proof of Proposition 4.2.

$$\begin{aligned}
 r_1(\rho, q) &= \begin{bmatrix} \rho \\ q - Q \end{bmatrix}, & r_2(\rho, q) &= \begin{bmatrix} R - \rho \\ \frac{R}{\rho}q \end{bmatrix}, \\
 \lambda_1(\rho, q) &= \left( \frac{2}{R} - \frac{1}{\rho} \right) \cdot (Q - q) - \frac{Q}{R}, & \lambda_2(\rho, q) &= v_c(\rho, q), \\
 \nabla \lambda_1 \cdot r_1 &= 2 \frac{Q - q}{R}, & \nabla \lambda_2 \cdot r_2 &= 0, \\
 \mathcal{L}_1(\rho; \rho_o, q_o) &= Q + \frac{q_o - Q}{\rho_o} \rho, & \mathcal{L}_2(\rho; \rho_o, q_o) &= \frac{\rho}{\rho_o} \frac{R - \rho_o}{R - \rho} q_o, \\
 w_1 &= v_c(\rho, q), & w_2 &= \frac{q - Q}{\rho},
 \end{aligned} \tag{4.1}$$

where  $r_i$  is the  $i$ -th right eigenvector,  $\lambda_i$  the corresponding eigenvalue and  $\mathcal{L}_i$  is the  $i$ -Lax curve. In the Riemann coordinates  $(w_1, w_2)$ ,  $\Omega_c = [0, V_c] \times [W_2^-, W_2^+]$ . For  $(\rho, q) \in \Omega_f$ , we extend the corresponding Riemann coordinates  $(w_1, w_2)$  as follows. Let  $\tilde{u} = (\tilde{\rho}, \tilde{\rho}V)$  be the point in  $\Omega_f$  defined by  $\tilde{\rho} = Q/(V - W_2^-)$ . Define

$$w_1 = V_f \quad \text{and} \quad w_2 = \begin{cases} V - Q/\rho & \text{if } \rho \geq \tilde{\rho}, \\ v_f(\tilde{\rho}) - v_f(\rho) + V - Q/\tilde{\rho} & \text{if } \rho < \tilde{\rho}, \end{cases} \tag{4.2}$$

so that, in the Riemann coordinates,  $\Omega_f = \{V_f\} \times [W_o, W_2^+]$ , see Fig. 3. Introduce for later use the functions

$$\begin{aligned}
 \bar{\Lambda}(\rho^l, \rho) &= \frac{\rho^l v_f(\rho^l) - \rho v_c(\rho, \mathcal{L}_1(\rho; R, Q^-))}{\rho^l - \rho}, \\
 \bar{\lambda}_1(\rho) &= \lambda_1(\rho, \mathcal{L}_1(\rho; R, Q^-)), \\
 \varphi(\rho; \rho_o, q_o) &= \int_{\rho_o}^{\rho} \lambda_1(r, \mathcal{L}_1(r; \rho_o, q_o)) \, dr \\
 &= \left( \frac{q_o - Q}{\rho_o} \left( 1 - \frac{\rho + \rho_o}{R} \right) - \frac{Q}{R} \right) (\rho - \rho_o).
 \end{aligned}$$

The former is the speed of the phase boundary joining  $(\rho^l, \rho^l V) \in \Omega_f$  to  $(\rho, \mathcal{L}_1(\rho; R, Q^-)) \in \Omega_c$ .  $\bar{\lambda}_1$  is the 1-characteristic speed of  $(\rho, \mathcal{L}_1(\rho; R, Q^-)) \in \Omega_c$ . The latter map  $\varphi$  is defined for  $(\rho_o, q_o) \in \Omega_c$  and  $\rho \in [0, R]$ .



**Proposition 4.2.** Consider the Riemann problem made by (3.1) with data

$$(\rho, q)(0, x) = \begin{cases} (\rho^l, q^l) & \text{if } x < 0 \\ (\rho^r, q^r) & \text{if } x > 0. \end{cases} \quad (4.3)$$

For every  $(\rho^l, q^l), (\rho^r, q^r)$  in  $\Omega_f \cup \Omega_c$ , there exist smooth scalar functions defining Riemann problems for scalar conservation laws whose solutions, juxtaposed, yield a solution to (3.1)–(4.3).

**Proof.** Following the definition of a solution to (3.1)–(4.3) given in [7], we consider several different cases.

(A): The data in (4.3) are in the same phase.

The standard procedure in [3, Section 1] works also in the present case, since no phase boundary is present.

(B):  $(w_1^l, w_2^l) \in \Omega_c$  and  $(w_1^r, w_2^r) \in \Omega_f$ .

By [7, Section 3], the intermediate state  $(\rho^m, q^m)$  in the Riemann coordinates reads  $(w_1^r, w_2^l)$ , so that  $\rho^m = Q / \left( V - \frac{q^l - Q}{\rho^l} \right)$ , and  $q^m = \rho^m V$ . Then, the  $\rho$  component in the solution to (3.1)–(4.3) is obtained as the juxtaposition of the solutions to the scalar Riemann problems

$$\begin{cases} \partial_t \rho + \partial_x (\varphi(\rho; \rho^l, q^l)) = 0 \\ \rho(0, x) = \begin{cases} \rho^l & \text{if } x < 0 \\ \rho^m & \text{if } x > 0 \end{cases} \end{cases} \quad \text{and} \quad \begin{cases} \partial_t \rho + \partial_x (\rho \cdot v_f(\rho)) = 0 \\ \rho(0, x) = \begin{cases} \rho^m & \text{if } x < 0 \\ \rho^r & \text{if } x > 0, \end{cases} \end{cases}$$

while  $q$  is computed through

$$\begin{aligned} q(t, x) &= \mathcal{L}_1(\rho(t, x); \rho^l, q^l) & \text{in } \Omega_c \\ q(t, x) &= V\rho(t, x) & \text{in } \Omega_f. \end{aligned}$$

The former Riemann problem displays a scalar phase transition, see Proposition 4.1.

(C):  $(w_1^l, w_2^l) \in \Omega_f$  and  $(w_1^r, w_2^r) \in \Omega_c$  with  $w_2^l \in [W_2^-, W_2^+]$ .

Similarly to the previous case, the solution to (3.1)–(4.3) has the following structure:

If  $w_2^l < 0$ , phase boundary between  $(w_1^l, w_2^l)$  and  $(V_c, w_2^l)$ , a rarefaction up to  $(w_1^r, w_2^l)$  and a 2-Lax wave.

If  $w_2^l = 0$ , a phase transition acting as a contact discontinuity between  $(w_1^l, w_2^l)$  and  $(w_1^r, w_2^l)$  followed by a 2-Lax wave.

If  $w_2^l > 0$ , a shock-like phase transition between  $(w_1^l, w_2^l)$  and  $(w_1^r, w_2^l)$  followed by a 2-Lax wave.

In each of these cases, the intermediate state is  $(w_1^r, w_2^l) \in \Omega_c$ . Call  $(\rho^m, q^m)$  its coordinates in the  $(\rho, q)$  plane. As before, the first component in the solution to (3.1)–(4.3) is obtained as the juxtaposition of the solutions to the scalar Riemann problems

$$\begin{cases} \partial_t \rho + \partial_x (\varphi(\rho; \rho^l, q^l)) = 0 \\ \rho(0, x) = \begin{cases} \rho^l & \text{if } x < 0 \\ \rho^m & \text{if } x > 0 \end{cases} \end{cases} \quad \begin{cases} \partial_t \rho + \partial_x (\rho \cdot v_c(\rho^r, q^r)) = 0 \\ \rho(0, x) = \begin{cases} \rho^m & \text{if } x < 0 \\ \rho^r & \text{if } x > 0, \end{cases} \end{cases}$$

the other conserved variable being

$$\begin{aligned} q(t, x) &= \mathcal{L}_1(\rho(t, x), \rho^l, q^l) & \text{for } \rho^l \leq \rho \leq \rho^m, \\ q(t, x) &= \mathcal{L}_2(\rho(t, x), \rho^m, q^m) & \text{for } \rho^m \leq \rho \leq \rho^r. \end{aligned}$$

The former Riemann problem displays a phase transition.

(D):  $(w_1^l, w_2^l) \in \Omega_f$  with  $w_2^l < W_2^-$  and  $(w_1^r, w_2^r) \in \Omega_c$ .

The solution to (3.1)–(4.3) consists of a phase boundary followed by a 1-Lax wave in  $\Omega_c$ , and a 2-Lax wave in  $\Omega_c$  between the states  $(w_1^r, W_2^-)$  and  $(w_1^r, w_2^r)$ . Call  $(\rho', q')$  the  $(\rho, q)$ -coordinates of  $(w_1^r, W_2^-) \in \Omega_c$ . Define

$$\psi(\rho) = \begin{cases} \varphi(\rho; R, Q^-) & \text{if } \rho \in [R_c^-, R] \\ (\rho^l - R_c^-)\bar{\lambda}(\rho^l, R_c^-) + \varphi(R_c^-, R, Q^-) & \text{if } \rho = \rho^l. \end{cases}$$

The solution to (3.1)–(4.3) is obtained as the juxtaposition of the solutions to the scalar Riemann problems

$$\begin{cases} \partial_t \rho + \partial_x (\psi(\rho)) = 0 \\ \rho(0, x) = \begin{cases} \rho^l & \text{if } x < 0 \\ \rho^r & \text{if } x > 0 \end{cases} \end{cases} \quad \begin{cases} \partial_t \rho + \partial_x (\rho \cdot v_c(\rho^r, q^r)) = 0 \\ \rho(0, x) = \begin{cases} \rho^l & \text{if } x < 0 \\ \rho^r & \text{if } x > 0. \end{cases} \end{cases} \tag{4.4}$$

The former Riemann problem presents a phase transition. To check that the solution provided by the Riemann problems (4.4) coincide with that of (3.1)–(4.3), we consider in the three cases below only the phase boundary and the 1-wave, the equivalence for the 2-wave being immediate.

- If  $\bar{\lambda}(R_c^-, \rho^l) \leq \bar{\lambda}_1(R_c^-)$ , the solution is a phase transition from  $u^l$  to  $u_c^-$ , followed by a 1-Lax rarefaction from  $u_c^-$  to  $u^r$ , see Fig. 3, left. The Rankine–Hugoniot condition between  $u^l$  to  $u_c^-$  given by the scalar problem on the left in (4.4) gives the speed of the phase boundary

$$\bar{\lambda}(R_c^-, \rho^l) = \frac{\psi(\rho^l) - \psi(R_c^-)}{\rho^l - R_c^-}.$$

- If  $\bar{\lambda}(\rho, \rho^l) > \bar{\lambda}_1(\rho)$ , for all  $\rho \in [R_c^-, \rho^l]$ , the solution is a shock-like phase transition. Consider the Riemann problems (4.4). Again the speed given by the Rankine–Hugoniot condition between  $u^l$  and  $u^r$  is the speed of the phase boundary:

$$\bar{\lambda}(\rho^r, \rho^l) = \frac{\psi(\rho^r) - \psi(\rho^l)}{\rho^r - \rho^l}.$$

To check this, it is sufficient to verify that

$$\begin{aligned} & \rho^r v_c(\rho^r, \mathcal{L}_1(\rho^r, R, Q)) - R_c^- v_c(R_c^-, \mathcal{L}_1(\rho^l, R, Q)) \\ &= \int_{R_c^-}^{\rho^r} \lambda_1(r, \mathcal{L}_1(r; R, Q^-)) dr. \end{aligned}$$

- Otherwise, let  $\rho_m \in [R_c^-, \rho^l]$  be the smallest density such that  $\bar{\lambda}(\rho^l, \rho_m) = \bar{\lambda}_1(\rho_m)$ . Then, the solution is a right-sonic phase transition attached to a 1-Lax rarefaction along the lower boundary of  $\Omega_c$ . Indeed,  $\bar{\lambda}_1(\rho_m) = (\psi(\rho_m) - \psi(\rho^l)) / (\rho_m - \rho^l)$ , since  $\bar{\lambda}(\rho_m, \rho^l) = \bar{\lambda}_1(\rho_m)$ .  $\square$

Following [3], we now construct piecewise constant weak solutions to (3.1). First, for  $\nu \in \mathbb{N}$ , we introduce a mesh  $\Omega^\nu$  in  $\Omega$ . In Riemann coordinates, let

$$\Omega_c^\nu = \{(i2^{-\nu}V_c, W_2^- + j2^{-\nu}(W_2^+ - W_2^-)) \in \Omega_c : i, j = 0, \dots, 2^\nu\}$$

where  $W_2^-$  and  $W_2^+$  are as in Fig. 3. Now, let

$$I_1^\nu = \{W_2^- + j2^{-\nu}(W_2^+ - W_2^-) : j = 0, \dots, 2^\nu\}$$

$$\begin{aligned}
 I_2^\nu &= \left\{ w \in [W_o, W_2^-]: w = V - Q/\rho \text{ where } \bar{\lambda}(\rho', \rho) = \bar{\lambda}_1(\rho') \right. \\
 &\quad \left. \text{for some } (\rho', \mathcal{L}_1(\rho'; R, Q^-)) \in \Omega_c^\nu \right\} \\
 \bar{W} &= \min I_2^\nu \\
 I_3^\nu &= \begin{cases} \{W_o\} & \text{if } \bar{W} - W_o < 2^{-\nu} \\ \{W_o + j2^{-\nu}(\bar{W} - W_o): j = 0, \dots, 2^\nu\} & \text{if } \bar{W} - W_o > 2^{-\nu} \end{cases} \\
 \Omega_f^\nu &= \{V_f\} \times (I_1^\nu \cup I_2^\nu \cup I_3^\nu) \\
 \Omega^\nu &= \Omega_f^\nu \cup \Omega_c^\nu.
 \end{aligned}$$

The construction above leads to a mesh  $\Omega^\nu$  with several properties. First, for any point  $w \in \Omega$  there is a point  $w^\nu \in \Omega^\nu$  such that  $\|w - w^\nu\| = O(2^{-\nu})$ . Secondly, there is a positive  $\delta^\nu$  such that any two points in the mesh are distant more than  $\delta^\nu$ . Moreover,  $\Omega^{\nu+1} \subseteq \Omega^\nu$ . Finally, the following key proposition holds.

**Proposition 4.3.** *The Riemann problem (3.1)–(4.3) with data in  $\Omega^\nu$  admits a piecewise constant weak solution attaining values in  $\Omega^\nu$ .*

**Proof.** Following [3], simply substitute in the construction of Proposition 4.2 the functions  $\varphi$  and  $\psi$  with functions  $\varphi^\nu$  and  $\psi^\nu$  defined by these two properties:  $\varphi^\nu(w) = \varphi(w)$  and  $\psi^\nu(w) = \psi(w)$  for every  $w \in \Omega^\nu$ ;  $\varphi^\nu$  and  $\psi^\nu$  are piecewise linear and continuous.  $\square$

Note that the exact weak solutions yielded by Proposition 4.3 may well be non-entropic.

An approximate solution  $u^\nu = u^\nu(t, x)$  to the Cauchy problem for (3.1) is now constructed by means of the standard wave front tracking technique, see [3,4]. The initial data  $u_o$  in  $\mathbf{L}^1$ , with  $\text{TV}(u_o) \leq M$ , is substituted by a piecewise constant  $u_o^\nu$  such that  $u_o^\nu(\mathbb{R}) \subseteq \Omega^\nu$ ,  $\|u_o^\nu - u_o\|_{\mathbf{L}^1} \leq 1/\nu$  and  $\text{TV}(u_o^\nu) \leq \text{TV}(u_o)$ . The Riemann problems centered at each point of jump in  $u_o^\nu$  are solved through the approximate procedure described above. The corresponding solutions are glued together and a piecewise constant approximate solution  $u^\nu$  on all  $\mathbb{R}$  is obtained up to a first time  $t_1$  when two waves interact. At time  $t_1$  a new Riemann problem arises and it is again solved through the same procedure.  $u^\nu$  can be defined up to any positive time provided the number of interaction points is finite on any compact subset of  $\mathbb{R}^+ \times \mathbb{R}$  and the range of  $u^\nu$  remains in  $\Omega^\nu$ .

The latter requirement is met. Indeed, according the approximation procedure defined above, the approximate solution to Riemann problems with data in the mesh  $\Omega^\nu \subset \Omega$  attains values in  $\Omega^\nu$ . The former requirement is obtained through suitable interaction estimates, that also ensure the existence of a bound on  $\text{TV}(u^\nu(t))$  uniform in  $\nu$  and  $t$ .

To this aim, we assign a *strength* to each simple wave. Let  $u^l, u^r$  be the states on the sides of the wave and call  $(w_1^l, w_2^l), (w_1^r, w_2^r)$  the corresponding Riemann coordinates, see (4.2). Then, the strength of the wave is

$$\tau = \left| w_1^r - w_1^l \right| + \left| w_2^r - w_2^l \right|. \tag{4.5}$$

Note that only in case (D) above these summands are both non-zero. Let  $\tau_{i,\alpha}$  be the strength of the wave of the  $i$ -th family exiting from the  $\alpha$ -th point of jump  $x_\alpha$  in  $u^\nu(t, \cdot)$ . With this choice, define the usual Glimm functionals

$$V^\nu(t) = \sum_{i,\alpha} |\tau_{i,\alpha}|, \quad Q^\nu(t) = \sum_{\alpha,\beta: x_\alpha < x_\beta} |\tau_{2,\alpha} \tau_{1,\beta}|. \tag{4.6}$$

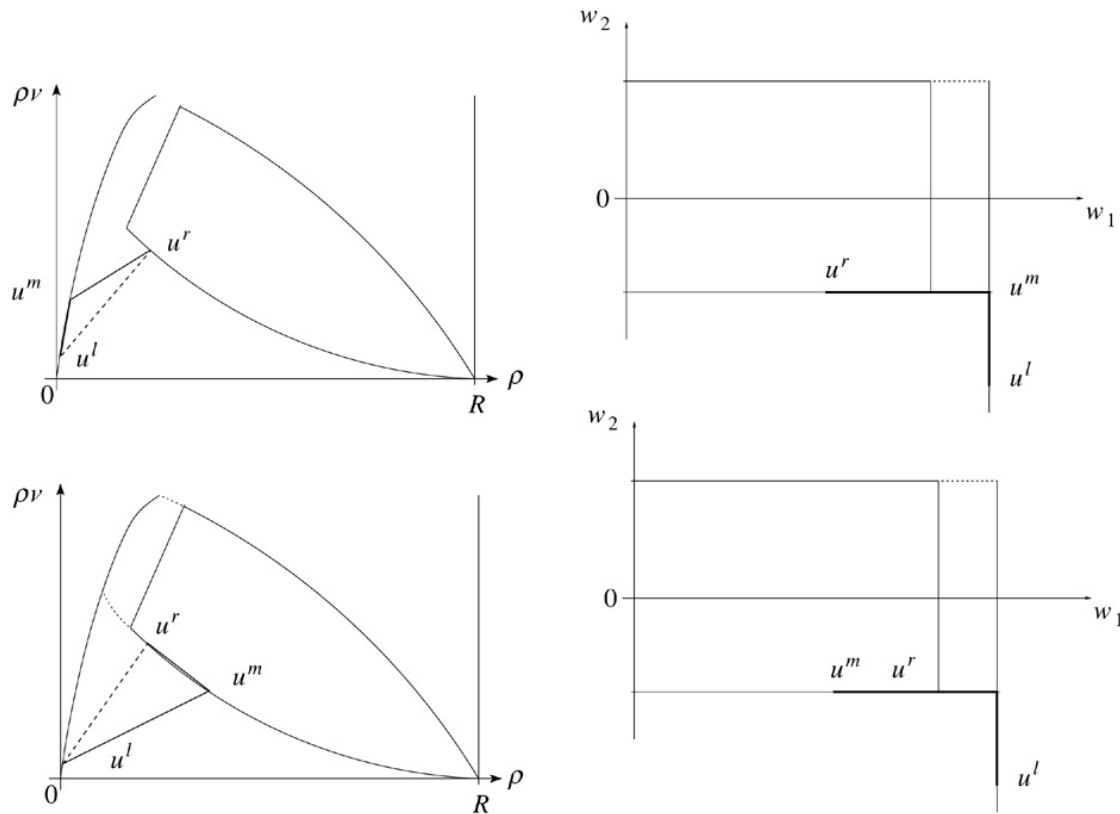


Fig. 4. Interactions against a phase boundary of type **(D)**.

It is immediate to prove that along any approximate solution, the maps  $t \mapsto V^\nu(t)$  and  $t \mapsto Q^\nu(t)$  are both non-increasing and, at each interaction, at least one of them decreases by at least  $2^{-\nu}$ . Hence there is a finite number of interactions on all  $\mathbb{R}^+ \times \mathbb{R}$ . When waves of types **(A)**, **(B)** and **(C)** only take part to the interaction, this is proved as in [3, a), b) and c) in Section 2]. In the case of interactions with phase boundaries of type **(D)**, the choice (4.5) is exactly the one that is additive/subtractive with respect to wave sizes, see Fig. 4.

We prove the  $L^1$  Lipschitz dependence using pseudo-polygons, as in [5], see also [1,3,8, 12,14]. We introduce a class of curves (pseudo-polygons) that connect any two initial data in  $\mathcal{D}_M^\nu = \{u: \mathbb{R} \rightarrow \Omega^\nu : V^\nu(u) \leq M\}$ .

Let  $]a, b[$  be an open interval and **PC** denote the set of piecewise constant functions with finitely many jumps. An *elementary path* is a map  $\gamma: ]a, b[ \mapsto \mathbf{PC}$  of the form

$$\gamma(\theta) = \sum_{\alpha=1}^N u^\alpha \cdot \chi_{[x_{\alpha-1}^\theta, x_\alpha^\theta[}, \quad x_\alpha^\theta = x_\alpha + \xi_\alpha \theta,$$

with  $x_{\alpha-1}^\theta \leq x_\alpha^\theta$  for all  $\theta \in ]a, b[$  and  $\alpha = 1, \dots, N$ .

A continuous map  $\gamma: ]a, b[ \mapsto \mathcal{D}_M^\nu$  is a pseudo-polygon if there exist countably many disjoint open intervals  $J_h \subseteq ]a, b[$  such that  $]a, b[ \setminus \bigcup_h J_h$  is countable and the restriction of  $\gamma$  to each  $J_h$  is an elementary path. Moreover, any two elements of  $\mathcal{D}_M^\nu$  can be joined by a pseudo-polygon  $\gamma$  entirely contained in  $\mathcal{D}_M^\nu$ .

As shown in [3–5], the semigroup  $S^\nu$  preserves pseudo-polygons in the sense that if  $\gamma$  is a pseudo-polygon then  $S_t^\nu \circ \gamma$  is also a pseudo-polygon, for all  $t \geq 0$ .

**Proposition 4.4.** Consider a point  $P_* = (t_*, x_*)$  of interaction. Let  $u^\nu(t, x)$  be the approximate solution to (3.1) defined for  $t < t_*$  by extending backward the shocks and for  $t \geq t_*$  by solving the approximate Riemann problem. Then

$$\sum_{\alpha>0} |\sigma_{i,\alpha}^+ \eta_{i,\alpha}^+| \leq \left( 1 + K \sum_{k \neq i} \left| \sum_{\alpha>0} \tau_{k,\alpha}^- \right| \right)^2 \sum_{\alpha>0} |\sigma_{i,\alpha}^- \eta_{i,\alpha}^-| + K \left| \sum_{\alpha} \tau_{i,\alpha}^- \left| \sum_{k \neq i} \sum_{\alpha>0} \sigma_{k,\alpha}^- \eta_{k,\alpha}^- \right| \right|. \tag{4.7}$$

Once Proposition 4.2 allows us to reduce the solution of the general Riemann problem (3.1)–(4.3) to that of scalar Riemann problems, the proof of Proposition 4.4 follows exactly the same lines of [3, Proposition 5.1].

Define the length of (the evolution of) a curve  $\gamma \subseteq \mathcal{D}_M^\nu$  of approximate solutions as  $\|\gamma\|_\nu = \int_a^b \Upsilon^\nu[\gamma(\theta)] d\theta$ , where  $\Upsilon^\nu = \sum_{i,\alpha} |\sigma_{i,\alpha} \xi_{i,\alpha} W_{i,\alpha}|$ ,  $W_{i,\alpha}$  being a suitable weight and  $\sigma_{i,\alpha}$  being the strength of a jump measured in the conserved coordinates. Proposition 4.4 ensures that using [3, (6.1)] one can define weights  $W_{i,\alpha}$  such that  $W_{i,\alpha} \in [1, W]$  and the map  $t \mapsto \Upsilon^\nu(u(t))$  is non-increasing. The first requirement implies that the metric

$$d_\eta^\nu(u, w) = \inf \{ \|\gamma\|_\nu : \gamma \text{ pseudo-polygonal joining } u \text{ to } w \}$$

is equivalent to the  $L^1$ -distance uniformly in  $\nu$ , see also [1,3,4]; the latter ensures that the  $\nu$ -approximate semigroup  $S^\nu$  is non-expansive with respect to  $d_\eta^\nu$ . This finally ensures that the approximate semigroup is  $L^1$ -Lipschitz, uniformly with respect to  $\nu$ , completing the proof of Theorem 3.2.

The following consequence of the particular form of (3.1) is useful toward the proof of Proposition 3.3.

**Lemma 4.5.** With the notation in (4.1),

$$\frac{d(\rho v_c(\mathcal{L}_1(\rho; \rho_o, q_o)))}{d\rho}(\rho_o; \rho_o, q_o) = \lambda_1(\rho_o, q_o).$$

The proof is straightforward and, hence, omitted.

**Proof of Proposition 3.3.** Note that condition (3.5) ensures that  $\sup_{\Omega_f \cup \Omega_c} \lambda_1 < 0$ . Indeed, referring to the notation in Fig. 5, if  $\lambda_1(\rho_f^+, q_f^+) < 0$ , then  $\lambda_1 < 0$  on all  $\Omega_f \cup \Omega_c$ , by the convexity of the regions  $\{(\rho, q) : \lambda_1 > 0\}$  and  $\{(\rho, q) : q < \rho V\}$ . A simple computation leads to

$$\lambda_1(\rho_f^+, q_f^+) = \frac{(RV - Q^+)(Q^+ - Q) - RVQ}{R(RV - Q^+ + Q)}$$

whose denominator is positive by (3.2). We thus obtain  $\lambda_1(\rho_f^+, q_f^+) < 0$  if and only if (3.5) holds.

1. If  $(\bar{\rho}, \bar{q})$  is in  $\Omega_f$ , then for any  $\tilde{f} \in [0, F]$ , the line  $\rho v = \tilde{f}$  intersects  $\Omega_f$  at a unique point  $(\tilde{\rho}, \tilde{q})$ . The standard Riemann problem with data  $(\tilde{\rho}, \tilde{q})$ ,  $(\bar{\rho}, \bar{q})$  admits a solution consisting of a simple wave with positive speed. The restriction of this solution to  $x \geq 0, t \geq 0$  is a solution to the Riemann problem for (3.1)–(3.6).

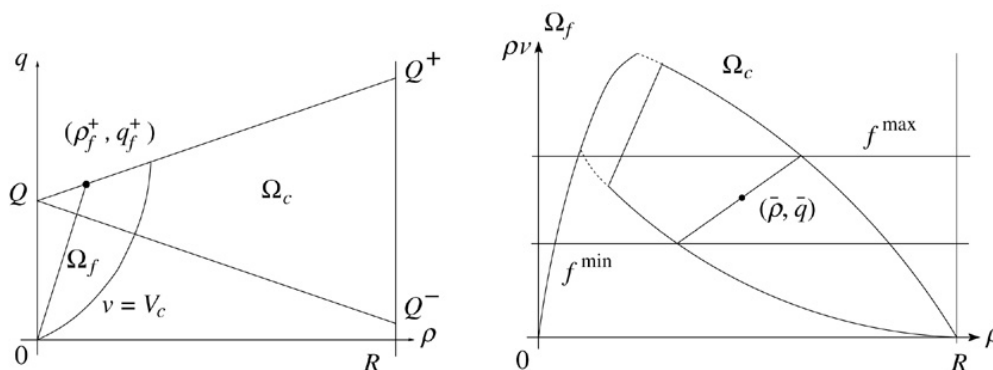


Fig. 5. Notation for the proof of Proposition 3.3.

2. (a) If  $(\bar{\rho}, \bar{q})$  is in  $\Omega_c$ , then the 2-Lax curve through  $(\bar{\rho}, \bar{q})$  has a unique intersection with the line  $\rho v = \tilde{f}$  at a point  $(\tilde{\rho}, \tilde{q})$  if and only if  $\tilde{f} \in [f^{\min}, f^{\max}]$ , see Fig. 5.
- (b) If  $(\bar{\rho}, \bar{q})$  is in  $\Omega_c$  and  $\tilde{f} \in [0, f^{\min}[$ , then the line  $\rho v = \tilde{f}$  intersects  $\Omega_f$  at a single point, say  $(\tilde{\rho}, \tilde{q})$ . Lemma 4.5 ensures that the standard Riemann problem with data  $(\tilde{\rho}, \tilde{q}), (\bar{\rho}, \bar{q})$  has a solution consisting of a phase boundary having positive speed, no 1 wave and a 2 contact discontinuity. The restriction of this solution to  $x \geq 0, t \geq 0$  is a solution to the Riemann problem for (3.1)–(3.6).  $\square$

## Acknowledgment

During this work, the first author was supported by the Université du Sud Toulon - Var.

## References

- [1] D. Amadori, R.M. Colombo, Continuous dependence for  $2 \times 2$  conservation laws with boundary, *J. Differential Equations* 138 (2) (1997) 229–266.
- [2] D. Amadori, R.M. Colombo, Viscosity solutions and standard Riemann semigroup for conservation laws with boundary, *Rend. Sem. Mat. Univ. Padova* 99 (1998) 219–245.
- [3] S. Bianchini, The semigroup generated by a Temple class system with non-convex flux function, *Differential Integral Equations* 13 (10–12) (2000) 1529–1550.
- [4] A. Bressan, *Hyperbolic Systems of Conservation Laws*, Oxford University Press, 2000.
- [5] A. Bressan, R.M. Colombo, The semigroup generated by  $2 \times 2$  conservation laws, *Arch. Ration. Mech. Anal.* 133 (1) (1995) 1–75.
- [6] R.M. Colombo, A  $2 \times 2$  hyperbolic traffic flow model, *Math. Comput. Modeling* 35 (5–6) (2002) 683–688. Traffic flow—modeling and simulation.
- [7] R.M. Colombo, Hyperbolic phase transitions in traffic flow, *SIAM J. Appl. Math.* 63 (2) (2002) 708–721.
- [8] R.M. Colombo, A. Corli, Continuous dependence in conservation laws with phase transitions, *SIAM J. Math. Anal.* 31 (1) (1999) 34–62.
- [9] R.M. Colombo, A. Corli, Sonic hyperbolic phase transitions and Chapman–Jouguet detonations, *J. Differential Equations* 184 (2) (2002) 321–347.
- [10] R.M. Colombo, A. Corli, On a class of hyperbolic balance laws, *J. Hyperbolic Differ. Equ.* 1 (4) (2004) 725–745.
- [11] R.M. Colombo, A. Groli, Minimising stop and go waves to optimise traffic flow, *Appl. Math. Lett.* 17 (6) (2004) 697–701.
- [12] R.M. Colombo, A. Groli, On the initial boundary value problem for Temple systems, *Nonlinear Anal.* 56 (4) (2004) 569–589.
- [13] R.M. Colombo, F.S. Priuli, Characterization of Riemann solvers for the two phase  $p$ -system, *Comm. Partial Differential Equations* 28 (7–8) (2003) 1371–1390.
- [14] R.M. Colombo, M.D. Rosini, Well posedness of balance laws with boundary, *J. Math. Anal. Appl.* (ISSN: 0022-247X) 311 (2) (2005) 683–702.

- [15] C.M. Dafermos, *Hyperbolic Conservation Laws in Continuum Physics*, 1st edition, Springer-Verlag, New York, 2000.
- [16] J.S. Drake, J.L. Schofer, A.D. May, A statistical Analysis of Speed Density Hypothesis, in: *Highway Research Record*, vol. 154, 1967, pp. 53–87.
- [17] L.C. Edie, Car following and steady-state theory for non-congested traffic, *Oper. Res.* 9 (1961) 66–76.
- [18] J. Goodman, Initial boundary value problems for hyperbolic systems of conservation laws, Ph.D. Thesis, California University, 1982.
- [19] P.G. Lefloch, M.D. Thanh, Nonclassical Riemann solvers and kinetic relations. I. A nonconvex hyperbolic model of phase transitions, *Z. Angew. Math. Phys.* 52 (4) (2001) 597–619.
- [20] P.G. Lefloch, M.D. Thanh, Non-classical Riemann solvers and kinetic relations. II. An hyperbolic–elliptic model of phase-transition dynamics, *Proc. Roy. Soc. Edinburgh Sect. A* 132 (1) (2002) 181–219.
- [21] M.J. Lighthill, G.B. Whitham, On kinematic waves. II. A theory of traffic flow on long crowded roads, *Proc. R. Soc. Lond. Ser. A.* 229 (1955) 317–345.
- [22] G.F. Newell, A simplified car-following theory: a lower order model, *Transport. Res. B* 36 (3) (2002) 195–205.
- [23] P.I. Richards, Shock waves on the highway, *Oper. Res.* 4 (1956) 42–51.
- [24] H.M. Zhang, T. Kim, A car-following theory for multiphase vehicular traffic flow, *Transport. Res. B* 39 (5) (2005) 385–399.