Coping with correlations in the analysis of the thermodynamic limit of neuronal networks

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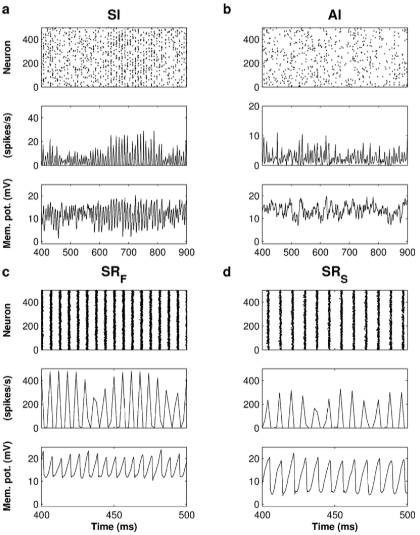
The question

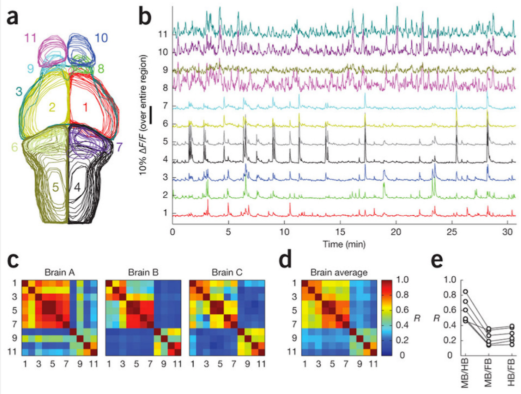
- Find concise mathematical descriptions of large networks of neurons



From A. Kumar et al., Neural Computation, 2008.

The question

- Find concise mathematical descriptions of large networks of neurons



From M. B. Ahrens et al., Nature Methods, 2013.

Technique: Light-sheet microscopy

This talk

- Randomly connected networks of rate neurons
- Random and **correlated** synaptic weights
- Networks can be multipopulation and balanced
- Annealed and quenched results

The mathematical model

- Intrinsic dynamics:

$$\mathcal{S} := \begin{cases} dV_t & = \quad -\alpha V_t dt + \sigma dW_t, \ 0 \leq t \leq T \\ \text{Law of } V_0 & = \quad \mu_0, \end{cases}$$

- There is a unique solution to $\mathcal{S}$ (Ornstein-Uhlenbeck process):

$$V_t = \exp(-\alpha t) V_0 + \sigma \int_0^t \exp(\alpha(s-t)) dW_s$$

- Note P its law on the set $\mathcal{T} := \mathcal{C}([0, T]; \mathbb{R})$ of trajectories

The mathematical model

- N neurons, $N = 2n + 1$ arranged in a circle
- Coupled dynamics

$$\mathcal{S}(J^N) :=$$
$$\begin{cases} dV_t^i & = \left(-\alpha V_t^i + \sum_{j=1}^N J_{ij}^N f(V_t^j)\right) dt + \sigma dW_t^i \\ \text{Law of} & \\ V_N(0) & = (V_0^1, \cdots, V_0^N) = \mu_0^{\otimes N} \end{cases}$$

$i \in I_n := [-n, \cdots, n]$.

- f is bounded, Lipschitz continuous (usually a sigmoid), defining the firing rate: activity function
- W^i: independent Brownians: intrinsic noise on the membrane potentials

The mathematical model

- There is a unique solution to $\mathcal{S}(J^N)$
- Note $P(J^N)$ its law on the set $\mathcal{T}^N$ of N-tuples of trajectories.

Modeling the synaptic weights

- J_{ij}^N: stationary Gaussian field: random synaptic weights

$$\mathbb{E}[J_{ij}^N] = \frac{\overline{J}}{N}$$

$$cov(J_{ij}^N J_{kl}^N) = \frac{\Lambda(k-i, l-j)}{N}$$

- $\Lambda(k, l)$ is a covariance function.
- Analogy with random media

Consequences

- $P(J^N)$ is a random law on $\mathcal{T}^N$
- Consider the law $P^{\otimes N}$ of N independent uncoupled neurons
- Girsanov theorem allows us to compare the law of the solution to the coupled system, $P(J^N)$, with the law of the uncoupled system, $P^{\otimes N}$:

$$\frac{dP(J^N)}{dP^{\otimes N}} = \exp\left\{ \sum_{i \in I_n} \frac{1}{\sigma} \int_0^T \left(\sum_{j \in I_n} J_{ij}^N f(V_t^j) \right) dW_t^i - \frac{1}{2\sigma^2} \int_0^T \left(\sum_{j \in I_n} J_{ij}^N f(V_t^j) \right)^2 dt \right\}$$

- Consider the "empirical" measure
$$\hat{\mu}_N(V_N) = \frac{1}{N} \sum_{i \in I_n} \delta_{S^i(V_{N,p})},$$
a probability measure on $\mathcal{T}^{\mathbb{Z}}$.
- $V_{N,p}$ is the periodic extension of the finite sequence of trajectories $V_N = (V^{-n}, \cdots, V^n)$.
- $V_{N,p} = (\cdots, V_N, V_N, \cdots)$
- S is the shift operator acting on elements of $\mathcal{T}^{\mathbb{Z}}$.
- Case $N = 3$
$$V_{3,p} = (\cdots, V^{-1}, V^0, V^1, V^{-1}, V^0, V^1, \cdots)$$
$$S^1(V_{3,p}) = (\cdots, V^1, V^{-1}, V^0, V^1, V^{-1}, V^0, \cdots)$$
$$S^2(V_{3,p}) = (\cdots, V^0, V^1, V^{-1}, V^0, V^1, V^{-1}, \cdots)$$
- It defines the mapping
$$\hat{\mu}_N : \mathcal{T}^N \to \mathcal{P}_S(\mathcal{T}^{\mathbb{Z}})$$

An example:
Let $g \in C_b(\mathcal{T}^{\mathbb{Z}})$:

$$\int_{\mathcal{T}^{\mathbb{Z}}} g(v)\, d\hat{\mu}_N(V_N)(v) = \frac{1}{N}\sum_{1}^{N} g(S^i V_{N,p})$$

Let us specialize to $g(v) = v_t^1 v_s^2$

$$\int_{\mathcal{T}^{\mathbb{Z}}} g(v)\, d\hat{\mu}_N(V_N)(v) =$$
$$\frac{1}{N}\left(V_t^{-n} V_s^{-n+1} + V_t^{-n+2} V_s^{-n+3} + \cdots V_t^{n-1} V_s^{n} + V_t^{n} V_s^{-n}\right)$$

$\hat{\mu}_N$ captures all correlations between the neurons' activities.

- We are interested in the law of $\hat{\mu}_N$ under $P(J^N)$
- Define
$$Q^N = \int_\Omega P(J^N(\omega))\, d\omega,$$
the average of $P(J^N)$ w.r.t. to the "random medium", i.e. the synaptic weights.
- We study the law of $\hat{\mu}_N$ under Q^N: annealed results.
- We then mention some results about the law of $\hat{\mu}_N$ under $P(J^N)$: quenched results.

The strategy

- Consider the law Π^N of $\hat{\mu}_N$ under Q^N: it is a probability measure on $\mathcal{P}_S(\mathcal{T}^{\mathbb{Z}})$:

$$\Pi^N(B) = Q^N(\hat{\mu}_N \in B),$$

B measurable set of $\mathcal{P}_S(\mathcal{T}^{\mathbb{Z}})$
This is also noted $Q^N \circ (\hat{\mu}_N)^{-1}$.

The strategy

- Establish a Large Deviation Principle for the sequences of probability measures $(\Pi^N)_{N\geq 1}$, i.e.
- Design a rate function (non-negative lower semi-continuous) H on $\mathcal{P}_S(\mathcal{T}^{\mathbb{Z}})$
- The intuitive meaning of H is the following

$$Q^N(\hat{\mu}_N \simeq Q) \simeq e^{-NH(Q)}$$

- The measures $\hat{\mu}_N$ concentrate on the measures Q such that $H(Q) = 0$.
- If H reaches 0 at a single measure Q then Π^N converges in law toward the Dirac mass δ_Q

Exponential approximation

1. Note that the sequence $\Pi_0^N = P^{\otimes N} \circ (\hat{\mu}_N)^{-1}$ satisfies the LDP with good rate function [Ell85]

$$I^{(3)}(\mu; P^{\mathbb{Z}}) = \lim_{N \to \infty} \frac{1}{N} I^{(2)}(\mu^N; P^{\otimes N})$$

$I^{(2)}(\mu^N; P^{\otimes N})$ is the relative entropy.

2. Show that there exists a sequence Ψ_m of continuous functions $\mathcal{P}_S(\mathcal{T}^{\mathbb{Z}}) \to \mathcal{P}_S(\mathcal{T}^{\mathbb{Z}})$ and a measurable map $\Psi : \mathcal{P}_S(\mathcal{T}^{\mathbb{Z}}) \to \mathcal{P}_S(\mathcal{T}^{\mathbb{Z}})$ such that for every $\alpha < \infty$

$$\limsup_{m \to \infty} \sup_{\mu : I^{(3)}(\mu) \leq \alpha} D_T(\Psi_m(\mu), \Psi(\mu)) = 0$$

3. D_T is a distance on the set of measures (Wasserstein).

Exponential approximation

1. Show that the family $\Pi_0^N \circ \Psi_m^{-1}$ is an exponentially good approximation of the family Π^N,
2. and conclude that Π^N satisfies the LDP with good rate function

$$H(\mu) = \inf \left\{ I^{(3)}(\nu) : \mu = \Psi(\nu) \right\}$$

Convergence results

Theorem (O.F., J. Maclaurin, E. Tanré)

H achieves its minimum at a unique point μ_* of $\mathcal{P}_S(\mathcal{T}^{\mathbb{Z}})$.

Annealed results

Corollary

The law of the empirical measure $\hat{\mu}_N$ under Q^N converges weakly to δ_{μ_*}

This means that

$$\forall F \in C_b(\mathcal{P}(\mathcal{T}^{\mathbb{Z}}))$$

$$\lim_{N\to\infty} \mathbb{E}\left[\int_{\mathcal{T}^Z} F\left(\frac{1}{N}\sum_{i\in I_n} \delta_{S^i v_{N,p}}\right) P(J^N(\omega))(dv_N)\right] = F(\mu_*)$$

Annealed results

Specializing F to

$$F(\mu) = \int_{\mathcal{T}^{\mathbb{Z}}} g(v)\, d\mu(v)$$

for $g \in C_b(\mathcal{T}^{\mathbb{Z}})$:

$$\lim_{N\to\infty} \mathbb{E}\left[\int_{\mathcal{T}^Z} \frac{1}{N}\left(\sum_{i\in I_n} g(S^i v_{N,p})\right) P(J^N(\omega))(dv_N)\right] = \int_{\mathcal{T}^{\mathbb{Z}}} g(v)\, d\mu_*(v)$$

Annealed results

Specializing even further with $g \in C_b(\mathcal{T}^M)$, $M \geq 1$

$$\lim_{N\to\infty} \mathbb{E}\left[\int_{\mathcal{T}^Z} \frac{1}{N}\left(\sum_{i \in I_n} g(S^i v_{N,p})\right) P(J^N(\omega))(dv_N)\right] = \int_{\mathcal{T}^M} g(v)\, d\mu_*^M(v),$$

where μ_*^M is the Mth-order marginal of μ_*.

Quenched results

- The existence of an LDP for the annealed law of the empirical measure $\hat{\mu}_N$ implies that "half" the same principle applies to the quenched law.
- This implies that the quenched law of the empirical measure converges exponentially fast to δ_{μ_*} and therefore

for all $g \in C_b(\mathcal{T}^M)$, $M \geq 1$, for almost all ω:

$$\lim_{N \to \infty} \int_{\mathcal{T}^N} \frac{1}{N} \left(\sum_{i \in I_n} g(S^i v_{N,p}) \right) P(J^N(\omega))(dv_N) = \int_{\mathcal{T}^M} g(v)\, d\mu_*^M(v),$$

where μ_*^M is the Mth-order marginal of μ_*.

The thermodynamic limit equations

Theorem (O.F., J. Maclaurin, E. Tanré)

μ_* is the law of the solution to an infinite dimensional linear non-Markovian stochastic system. It is a Gaussian measure (in $\mathcal{P}_S(\mathcal{T}^{\mathbb{Z}})$) if the initial condition is Gaussian.

The thermodynamic limit equations

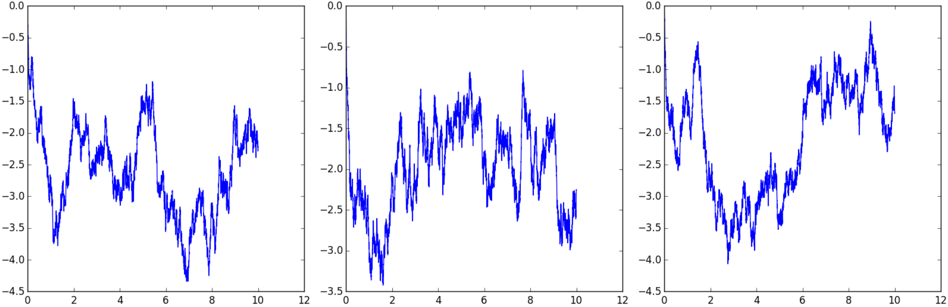
$$a_t^j = \sigma^{-2} \sum_{k \in \mathbb{Z}} \int_0^t L_{\mu_*}^{k-j}(t,s)\big(c_{\mu_*}(s) + a_s^k\big)\,ds + \sigma^{-2} \sum_{k \in \mathbb{Z}} \int_0^t L_{\mu_*}^{k-j}(t,s)\,dB_s^k$$

$$Z_t^j = Z_0^j + B_t^j + \int_0^t \big(a_s^j + c_{\mu_*}(s)\big)\,ds$$

- μ_* is the law of Z.
- $L_{\mu_*}^k(t,s)$ is a correlation function defined from μ_*.
- $c_{\mu_*}(t)$ is also defined by μ_*.
- The B_t^ks are i.i.d. as the uncoupled neuron.

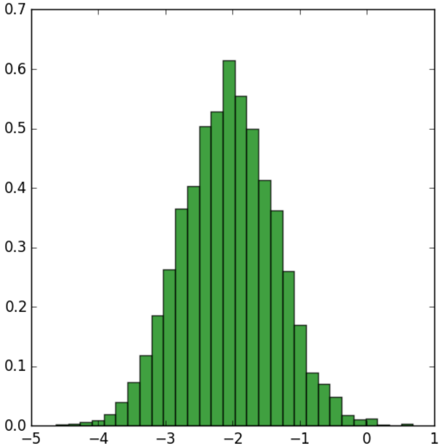
Numerical results (nearest neighbours correlations)

Examples of time variations of membrane potentials:



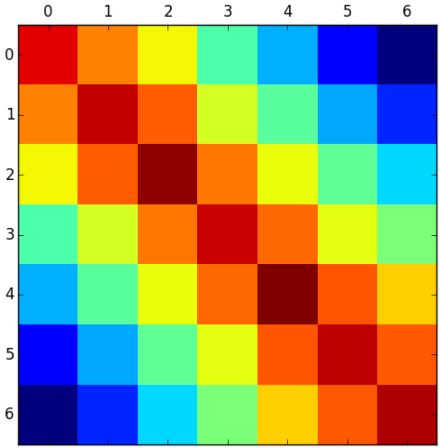
Numerical results (nearest neighbours correlations)

Distribution of the membrane potentials at $t = 10$:



Numerical results (nearest neighbours correlations)

Correlation of the membrane potentials:



Summary

- We have started the analysis of the thermodynamic limit of randomly connected networks of rate neurons with correlated synaptic weights: it generalises the work of Sompolinski and colleagues, e.g. [SCS88]
- The thermodynamic limit is described by a Gaussian process if the initial conditions are Gaussian.
- If the covariance function Λ of the synaptic weights is not a Dirac, the neurons activities remain correlated: there is no propagation of chaos unlike in previous work ([LS14, BFT15, FL16]).

Perspectives

- Analyse the limit equations and the bifurcations of their solutions
- Understand the fluctuations

References I

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- R.S. Ellis, *Entropy, large deviations and statistical mechanics*, Springer, 1985.

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References II

- Eric Luçon and Wilhelm Stannat, *Mean field limit for disordered diffusions with singular interactions*, Ann. Appl. Probab. **24** (2014), no. 5, 1946–1993.
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Metric on $\mathcal{T}^{\mathbb{Z}}$

$$d_T^{\mathbb{Z}}(u, v) = \sum_{i \in \mathbb{Z}} 2^{-|i|}(\|u^i - v^i\|_T \wedge 1)$$

Metric on $\mathcal{P}(\mathcal{T}^{\mathbb{Z}})$

Induced by the Wasserstein-1 distance:

$$D_T(\mu,\nu) = \inf_{\xi \in C(\mu,\nu)} \int d_T^{\mathbb{Z}}(u,v)\, d\xi(u,v)$$

Large deviation principle: I

For all open sets $\mathcal{O}$ of $\mathcal{P}(\mathcal{T})$

$$-\inf_{\mu \in \mathcal{O}} H(\mu) \leq \liminf_{N \to \infty} \frac{1}{N} \log \Pi^N(\mathcal{O})$$

Large deviation principle: II

The sequence Π^N is exponentially tight.

Large deviation principle: III

For every compact set F of $\mathcal{P}(\mathcal{T})$

$$\limsup_{N\to\infty} \frac{1}{N} \log \Pi^N(F) \leq - \inf_{\mu\in F} H(\mu)$$

Exponential approximation

for all $\delta > 0$

$$\lim_{m \to \infty} \overline{\lim_{n \to \infty}} \frac{1}{N} \log P^{\otimes N} \Big(D_T \big(\Psi_m (\hat{\mu}_N(B)), \hat{\mu}_N(V) \big) > \delta \Big) = -\infty$$

Definition of Ψ_m

- Note that

$$\left.\frac{dQ^N}{dP^{\otimes N}}\right|_{\mathcal{F}_t} = \exp\left(\sum_{j\in I_n}\int_0^t \theta_s^j dB_s^j - \frac{1}{2}\sum_{j\in I_n}\int_0^t \left(\theta_s^j\right)^2 ds\right)$$

where

$$\theta_t^j = \frac{1}{\sigma}\mathfrak{c}_{\hat{\mu}_c^N(V_N)}(t) + \frac{1}{\sigma^2}\mathbb{E}^{\tilde{\gamma}_t^{\hat{\mu}_c^N(V_N)}}\left[\sum_{k\in I_n} G_t^j \int_0^t G_s^k dB_s^k\right]$$

Definition of Ψ_m

- Prove that the SDE

$$Z_t^j = B_t^j + \int_0^t c_{\hat{\mu}_c^N(Z)}(s)ds + \sigma^{-2} \sum_{k \in I_n} \int_0^t \mathbb{E}^{\tilde{\gamma}_t^{\hat{\mu}_c^N(Z)}} \left[G_s^j \int_0^s G_u^k dZ_u^k \right] ds,$$

$j \in I_n$, is well-posed in $\mathcal{T}^N$ and that the law of $\hat{\mu}_c^N(Z)$ is Π_c^N.

- Construct the continuous function $\varphi_m : \mathcal{T}^{\mathbb{Z}} \times \mathcal{P}_S(\mathcal{T}^{\mathbb{Z}}) \to \mathcal{T}^{\mathbb{Z}}$ by time-discretizing this equation.
- Construct the continuous function $\Psi_m : \mathcal{P}_S(\mathcal{T}^{\mathbb{Z}}) \to \mathcal{P}_S(\mathcal{T}^{\mathbb{Z}})$ by a fixed-point argument as

$$\Psi_m(\mu) = \nu \text{ such that } \nu = \mu \circ (\varphi_m(\cdot, \nu))^{-1}$$