TD 6 : Intégrale Stochastique - Formule d'Itô

Exercice 1 :

1) $\mathbb{E}[B_t | \mathcal{F}_s] = \mathbb{E}[(B_t - B_s) | \mathcal{F}_s] + \mathbb{E}[B_s | \mathcal{F}_s]$

$= \mathbb{E}[B_t - B_s] + B_s$

$\uparrow$ indep de $\mathcal{F}_s$ $\quad$ $\uparrow$ $\mathcal{F}_s$ - mesurable

$= B_s \Rightarrow$ martingale.

2) $\mathbb{E}[B_t^2 - t | \mathcal{F}_s] = \mathbb{E}[(B_t - B_s + B_s)^2 | \mathcal{F}_s] - t$.

indep

$= \mathbb{E}[(B_t - B_s)^2 | \mathcal{F}_s] + \mathbb{E}[B_s^2 | \mathcal{F}_s] + 2\mathbb{E}[B_s(B_t - B_s)] - t$

$= (t-s) + B_s^2 + 0 - t$

$= B_s^2 - s \Rightarrow$ martingale

3) $\mathbb{E}[e^{\theta B_t - \theta^2/2\, t} | \mathcal{F}_s] = e^{-\theta^2/2\, t}\, \mathbb{E}[e^{\theta(B_t - B_s)} e^{\theta B_s} | \mathcal{F}_s]$

$= e^{-\theta^2/2\, t} e^{\theta B_s}\, \mathbb{E}[e^{\theta(B_t - B_s)}] = e^{\theta B_s} e^{-\theta^2/2\, t} e^{+\theta^2/2\,(t-s)}$

$= e^{\theta B_s - \theta^2/2\, s} \Rightarrow$ martingale.

4) $X_t = (W_t^1 \dots W_t^d)$

$$F(X_t) = F(X_0) + \sum_{i=1}^d \int_0^t \frac{\partial F}{\partial x_i}(X_s)\, dW_s^i + \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \int_0^t \frac{\partial^2 F}{\partial x_i \partial x_j}(X_s)\, dW_s^i\, dW_s^j$$

$$= \underbrace{F(X_0) + \sum_{i=1}^d \int_0^t \frac{\partial F}{\partial x_i}(X_s)\, dW_s^i}_{\text{martingale.}} + \underbrace{\frac{1}{2}\int_0^t \Delta F(X_s)\, ds}_{=0}$$

5) $X_t = (t, B_t)$.

(a) $dX_t^1 = dt$. , $dX_t^2 = dB_t$ donc ils s'écrivent bien sous la forme $dX_t^i = F^i\, dt + G^i\, dW_t$ avec des fonction $F^i \in \mathbb{L}^1([0,T])$ et $G^i \in \mathbb{L}^2([0,T])$.
Donc la formule d'Itô s'applique.

(b) Formule d'Itô $\rightarrow$

$$F(t, B_t) = F(0,0) + \int_0^t \frac{\partial F}{\partial t}\, ds + \int_0^t \frac{\partial F}{\partial x}\, dB_s + \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial t^2}\, dt\, dt + \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial x^2}\, ds$$

$$+ \int_0^t \frac{\partial F}{\partial t \partial x}\, ds\, dB_s$$

$$= F(0,0) + \int_0^t \left(\frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\right) ds + \int_0^t \frac{\partial F}{\partial x}\, dB_s$$

(c) si $\frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2} = 0$, alors $F(t, X_t) = F(0,0) + \int_0^t \frac{\partial F}{\partial x}\, dB_s$ martingale.

(d) Par la propriété de martingale,

$$\mathbb{E}[F(t, X_t)] = \mathbb{E}[\mathbb{E}[F(t, X_t) \mid \mathcal{F}_0]] = \mathbb{E}[F(0,0)] = F(0,0).$$

(e) 1er exemple : $F(t,x) = x$ $\quad \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2} = 0$.

2e — : $F(t,x) = x^2 - t$, $\quad \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2} = -1 + \frac{1}{2} \times 2 = 0$.

3e — : $F(t,x) = e^{\theta x - \theta^2/2\, t}$ $\quad \frac{\partial F}{\partial t} = -\frac{\theta^2}{2} F \quad \frac{\partial F}{\partial x} = \theta F$, $\frac{\partial^2 F}{\partial x^2} = \theta^2 F \Rightarrow$ ok.

(f) $$\frac{\partial h_n}{\partial t} = \frac{n}{2} t^{n/2 - 1} H_n\left(\frac{x}{\sqrt{t}}\right) - \frac{t^{n/2}}{2} H_n'\left(\frac{x}{\sqrt{t}}\right) \frac{x}{\sqrt{t}^3}$$

$$\frac{\partial^2 h_n}{\partial x^2} = t^{n/2-1} H_n''\left(\frac{x}{\sqrt{t}}\right) = t^{n/2-1}\left(\frac{x}{\sqrt{t}} H_n'\left(\frac{x}{\sqrt{t}}\right) - n\, H_n\left(\frac{x}{\sqrt{t}}\right)\right).$$

$$\Rightarrow \frac{\partial h_n}{\partial t} + \frac{1}{2}\frac{\partial^2 h}{\partial x^2} = 0$$

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Exo 3

$$\int_0^t \frac{W_s^{(i)}}{R_s} dW_s^{i} = B^{(i)} \quad \text{Martingale de carré } \int : \quad E\left(B^{(i)2}\right) < \infty$$

Ito $\int$ a un sens

$$\Leftarrow \quad E \int_0^t \left(\frac{1}{R_s} W_s^{(i)} \right)^2 ds \leq t \qquad 0 \leq t < \infty$$

$$\mu \{ 0 \leq s \leq t \quad R_s = 0 \} = 0 \qquad ps$$

Rappel $\quad \mathcal{Z}_\omega \equiv \{ 0 \leq t < \infty \ / \ W_t(\omega) = 0 \}$

$$\mu(\mathcal{Z}_\omega) = 0 \qquad ps$$

Alors $\quad (d-1)/R_s \quad$ définit $\quad ps/s \quad ps/\omega$

$$f_n(R_t^2) = f_n(R_0^2) + J_t(\varepsilon) + K_t(\varepsilon) + \sum_{i=1}^{d} I_t^{(i)}(\varepsilon)$$

$$I_t^{(i)}(\varepsilon) \equiv \int_0^t \left\{ 1(R_s^2 \geq \varepsilon) \frac{1}{R_s} + 1(R_s^2 < \varepsilon) \frac{1}{2\sqrt{\varepsilon}} \left(3 - \frac{R_s^2}{\varepsilon}\right) \right\} W_s^i \, dW_s^i$$

$$J_t(\varepsilon) \equiv \int_0^t 1(R_s^2 \geq \varepsilon) \frac{d-1}{2R_s} ds$$

$$K_t(\varepsilon) \equiv \int_0^t 1(R_s^2 < \varepsilon) \frac{1}{4\sqrt{\varepsilon}} \left[3d - (d+2)\frac{R_s^2}{\varepsilon} \right] ds$$

TCV monotone : $\quad \lim_{n \to \infty} J_t\left(\frac{1}{n}\right) = \int_0^t \frac{d-1}{2R_s} ds \quad ps$

$$0 \leq E K_t^2(\varepsilon) \leq \left(\frac{3d}{4\sqrt{\varepsilon}} \right)^2 \int_0^t P(R_s^2 < \varepsilon) ds$$

$$\leq \int_0^{2\pi} \int_0^{\sqrt{\varepsilon}} \frac{1}{2\pi s} e^{-r^2/2s} r \, dr \, d\theta$$

$$= 1 - e^{-\varepsilon/2s} = o(\sqrt{\varepsilon})$$

d'où $\lim_n E K_t^2(n) = 0$ et $K_t(n) \to 0$ ps

Enfin :

$$E\left(W_t^{(i)} - I_t^i(\varepsilon)\right)^2 = E\int_0^t 1(R_s^2<\varepsilon)\left[\frac{1}{R_s} - \frac{1}{2\sqrt{\varepsilon}}\left(3-\frac{R_s^2}{\varepsilon}\right)\right]^2 (W_s^i)^2\, ds$$

$$W_t^{(i)} = \int_0^t \frac{W_s^i}{R_s}\, dW_s^i$$

$$= E\int_0^t 1(R_s^2 \le \varepsilon)\left[1 - \frac{1}{2}\sqrt{\frac{R_s^2}{\varepsilon}}\left(3-\frac{R_s^2}{\varepsilon}\right)\right]^2 \times \left(\frac{W_s^i}{R_s}\right)^2 ds$$

$$\le \int_0^t P(R_s^2<\varepsilon)\, ds = o(\sqrt{\varepsilon})$$