Riemannian simplices and triangulations

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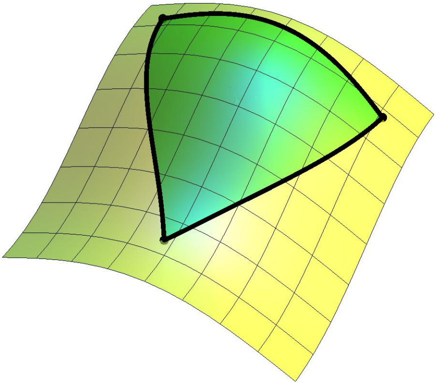
Geometric simplices in Riemannian manifolds

Motivation: Generic triangulation criteria $|\mathcal{A}| \to M$

- manifold abstract simplicial complex $\mathcal{A}$ with vertices $\mathsf{P} \subset M$
- intrinsic setting
- explicit density requirements
- Arbitrary dimension (ignore 2D pictures)



Simplices

Riemannian centres of mass

Natural way to "fill in" a simplex?

- Convex hull bad (conjectured not closed)
- Instead, barycentric coordinates on M
- Riemannian centres of mass

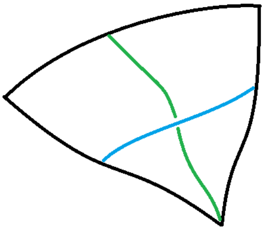
Centre of mass

$$\mathcal{E}_\lambda(x) = \frac{1}{2}\sum_i \lambda_i d_M(x, p_i)^2$$

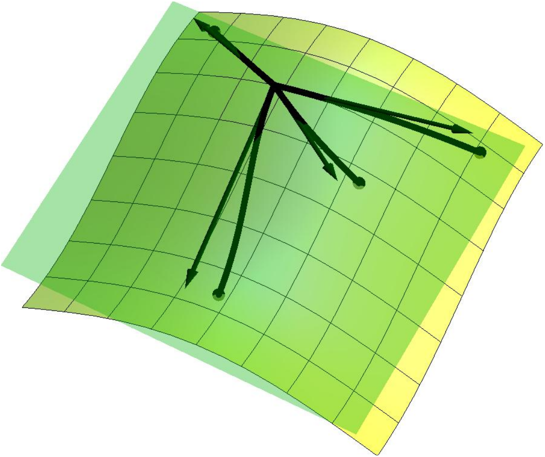
barycentric coordinates: $\lambda_i \geq 0$, $\sum \lambda_i = 1$

$$\begin{aligned} \mathcal{B}_{\sigma^j} : & \boldsymbol{\Delta}^j \to M \\ & \lambda \mapsto \operatorname*{argmin}_{x \in \overline{B}_\rho} \mathcal{E}_\lambda(x) \end{aligned}$$

$\boldsymbol{\Delta}^j$ the standard Euclidean j-simplex, $\boldsymbol{\sigma}_M$ image



Exponential map



Notation

$v_i(x) = \exp_x^{-1}(p_i)$, $\sigma(x) = \{v_0(x), \ldots, v_j(x)\} \subset T_x M$, injectivity ι_M

Non-degenerate Riemannian simplices

For fixed weights, barycenter (minimum of $\mathcal{E}_\lambda(x) = \frac{1}{2}\sum_i \lambda_i d_M(x,p_i)^2$):

$$\operatorname{grad} \mathcal{E}_\lambda(x) = -\sum_i \lambda_i \exp_x^{-1}(p_i) = -\sum_i \lambda_i v_i(x) = 0$$

Definition

A Riemannian simplex $\boldsymbol{\sigma}_M$ is *non-degenerate* if the barycentric coordinate map $\mathcal{B}_{\sigma^j} : \boldsymbol{\Delta}^j \to M$ is an embedding.

Proposition

A Riemannian simplex $\boldsymbol{\sigma}_M \subset M$ is non-degenerate if and only if $\sigma(x) \subset T_x M$ is non-degenerate for every $x \in \boldsymbol{\sigma}_M$.

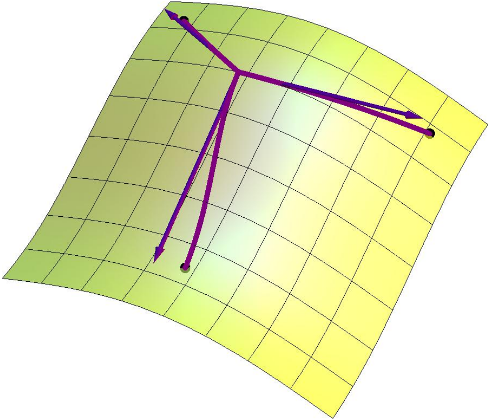
Rauch comparison theorem

Buser and Karcher (1981)

Very roughly speaking bounds the distortion in

$$\exp_x^{-1} \circ \exp_p : T_pM \to T_xM,$$

depends on bounds on sectional curvatures and distance



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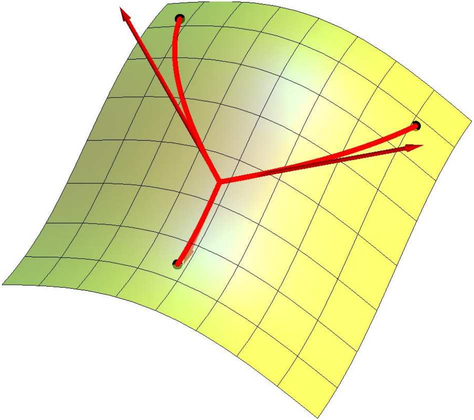

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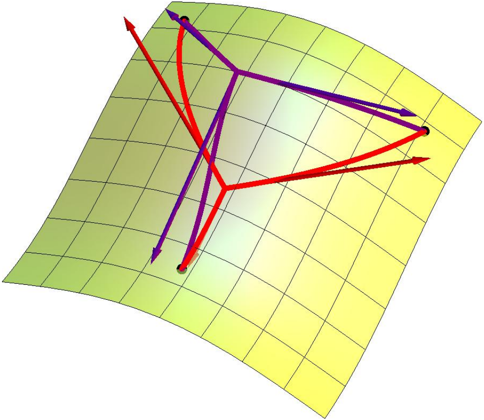

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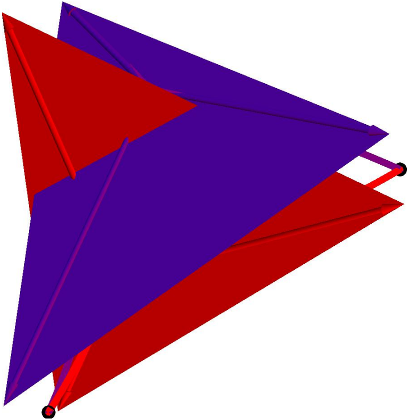
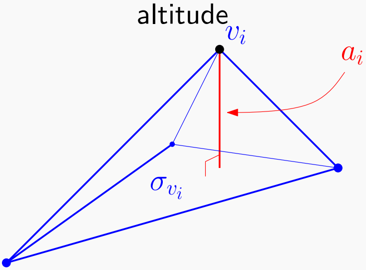


Euclidean simplex quality

Measure for how far a simplex is from degeneracy

Altitudes and thickness in $\mathbb{R}^n$



The *thickness* of a j-simplex σ with longest edge L is

$$t(\sigma) = \begin{cases} 1 & \text{if } j = 0 \\ \min_{v_i \in \sigma} \frac{a_i}{jL} & \text{otherwise.} \end{cases}$$

Another quality measure is fatness, a normalized volume.

Stability of thickness

Boissonnat, Dyer, Ghosh

Thickness is stable under distance distortions

Lemma (Thickness under distortion)

Suppose that $\sigma = \{v_0, \dots, v_k\}$ *and* $\tilde{\sigma} = \{\tilde{v}_0, \dots, \tilde{v}_k\}$ *are two* k*-simplices in* $\mathbb{R}^n$ *such that*

$$||v_i - v_j| - |\tilde{v}_i - \tilde{v}_j|| \leq C_0 L(\sigma)$$

for all $0 \leq i < j \leq k$.
If

$$C_0 = \frac{\eta t(\sigma)^2}{4} \qquad \text{with} \quad 0 \leq \eta \leq 1,$$

then

$$t(\tilde{\sigma}) \geq \frac{4}{5\sqrt{k}}(1-\eta)t(\sigma).$$

Main result: intrinsic simplex

Theorem (Non-degeneracy criteria)

Suppose M *is a Riemannian manifold with sectional curvatures* K *bounded by* $|K| \leq \Lambda$, *and* $\boldsymbol{\sigma}_M$ *is a Riemannian simplex, with* $\boldsymbol{\sigma}_M \subset B_\rho \subset M$, *where* B_ρ *is an open geodesic ball of radius* ρ *with*

$$\rho < \rho_0 = \min\left\{\frac{\iota_M}{2}, \frac{\pi}{4\sqrt{\Lambda}}\right\}. \textit{(sufficiently small neighbourhood)}$$

Then $\boldsymbol{\sigma}_M$ *is non-degenerate if there is a point* $p \in B_\rho$ *such that the lifted Euclidean simplex* $\sigma(p)$ *has thickness satisfying*

$$t(\sigma(p)) > 10\sqrt{\Lambda} L(\boldsymbol{\sigma}_M),$$

where $L(\boldsymbol{\sigma}_M)$ *is the geodesic length of the longest edge in* $\boldsymbol{\sigma}_M$.

Triangulations

Triangulation: stitching simplices together

Given:

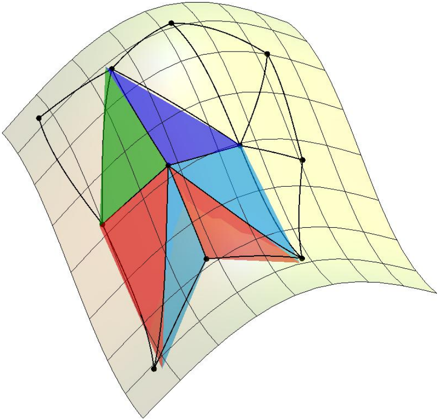
- points $\mathrm{P} \subset M$.
- (abstract) simplicial complex $\mathcal{A}$ vertices identified with P

When can we be sure that $\mathcal{A}$ triangulates M?

Avoid local conflicts by local geometric control and global coverage

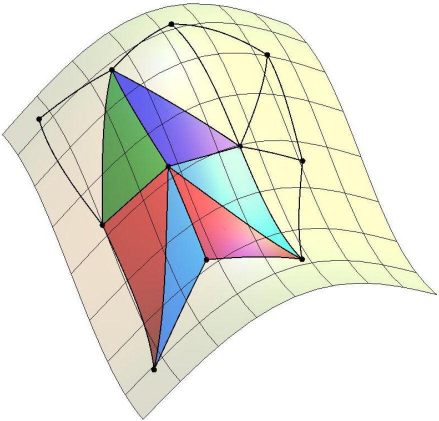
For geometric control we focus on full stars:
Closure all simplices adjacent to a vertex

Star in the tangent space

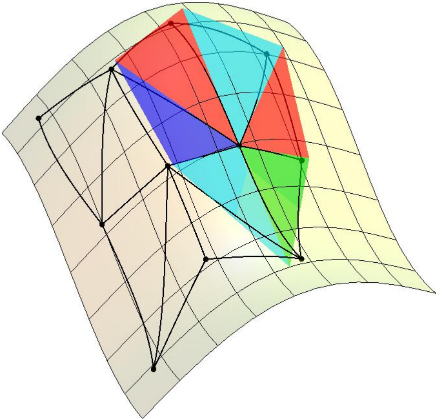


Star on the surface

Rauch comparison theorem gives geometric control: Map to manifold



Star at adjacent vertex



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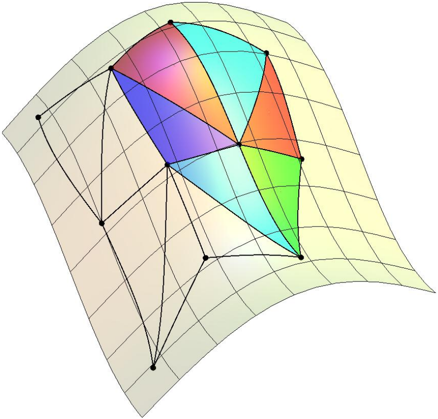
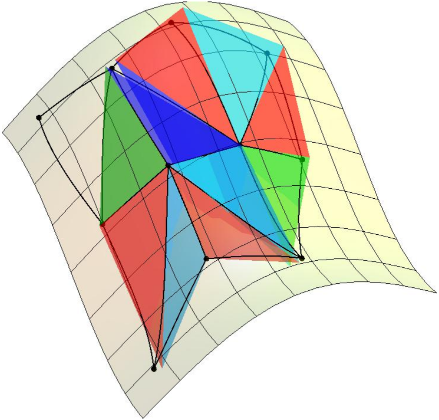


Two stars in the tangent space

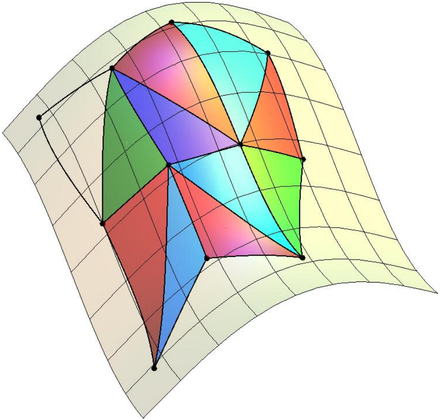
We need to cover the entire manifold



Brouwer's invariance of domain: $f : M \to N$ locally one-to-one implies local homeomorphism

Two stars on the surface

With geometric control (Rauch) we can stitch the simplices together



Theorem (Triangulation criteria)

Suppose M is a compact n-dimensional Riemannian manifold with sectional curvatures K bounded by $|K| \leq \Lambda$, and $\mathcal{A}$ is an abstract simplicial complex with finite vertex set $\mathsf{P} \subset M$. Define a quality parameter $t_0 > 0$, and let

$$h = \min \left\{ \frac{\iota_M}{4}, \frac{\sqrt{n} t_0}{6\sqrt{\Lambda}} \right\}.$$

If

- *For every $p \in \mathsf{P}$, the vertices of $\underline{\mathrm{St}}(p)$ are contained in $B_M(p; h)$, and the balls $\{B_M(p; h)\}_{p \in \mathsf{P}}$ cover M. (global coverage)*
- *For every $p \in \mathsf{P}$, the restriction of the inverse of the exponential map $\exp_p^{-1}$ to the vertices of $\underline{\mathrm{St}}(p) \subset \mathcal{A}$ defines a piecewise linear embedding of $|\underline{\mathrm{St}}(p)|$ into T_pM, realising $\underline{\mathrm{St}}(p)$ as a full star such that every simplex $\sigma(p)$ has thickness $t(\sigma(p)) \geq t_0$. (geometric controle)*

then $\mathcal{A}$ triangulates M, and the triangulation is given by the barycentric coordinate map on each simplex.

Questions?

Bibliographic notes

Karcher means

- Cartan (1928); Frechet (1948); Karcher (1977); Kendall (1990)
- Buser and Karcher (1981); Peters (1984); Chavel (2006)

Riemannian simplices

- Berger (2003)
- Rustamov (2010); Sander (2012)
- von Deylen (2014)

Theorem (Metric distortion)

If the requirements of the Triangulation Theorem, are satisfied with the scale parameter h replaced by

$$h = \min \left\{ \frac{\iota_M}{4}, \frac{t_0}{6\sqrt{\Lambda}} \right\},$$

then $\mathcal{A}$ is naturally equipped with a piecewise flat metric $d_{\mathcal{A}}$ defined by assigning to each edge the geodesic distance in M between its endpoints. If $H : |\mathcal{A}| \to M$ is the triangulation defined by the barycentric coordinate map in this case, then the metric distortion induced by H is quantified as

$$|d_M(H(x), H(y)) - d_{\mathcal{A}}(x, y)| \leq \frac{50\Lambda h^2}{t_0^2} d_{\mathcal{A}}(x, y),$$

for all $x, y \in |\mathcal{A}|$.

Theorem (Intrinsic simplex triangulation criteria)

Suppose M is a compact n-dimensional Riemannian manifold with sectional curvatures K bounded by $|K| \leq \Lambda$, and $\mathcal{A}$ is an abstract simplicial complex with finite vertex set $\mathsf{P} \subset M$. Define a quality parameter $t_0 > 0$, and let

$$h = \min\left\{\frac{\iota_M}{4}, \frac{t_0}{8\sqrt{\Lambda}}\right\}.$$

If

1. *For every simplex $\sigma = \{p_0, \ldots, p_n\} \in \mathcal{A}$, the edge lengths $\ell_{ij} = d_M(p_i, p_j)$ satisfy $\ell_{ij} < h$, and they define a Euclidean simplex $\boldsymbol{\sigma}_{\mathbb{E}}$ with $t(\boldsymbol{\sigma}_{\mathbb{E}}) \geq t_0$.*
2. *The balls $\{B_M(p; h)\}_{p \in \mathsf{P}}$ cover M, and for each $p \in \mathsf{P}$ the secant map of $\exp_p^{-1}$ realises $\underline{\mathrm{St}}(p)$ as a full star.*

then $\mathcal{A}$ triangulates M, and the triangulation is given by the barycentric coordinate map on each simplex.