Confluence of singular fibers on rational elliptic surfaces

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Outline

Results

Theorem

Of all confluences to Singular Fibers of Kodaira type II, III *and* IV, *superficially allowed by conservation of the Euler number, the following occur:*

$$\mathrm{II} \to \mathrm{I}_1 + \mathrm{I}_1 \quad \mathrm{III} \to 3\mathrm{I}_1 \quad \mathrm{III} \to \mathrm{I}_2 + \mathrm{I}_1 \quad \mathrm{III} \to \mathrm{II} + \mathrm{I}_1 \quad \mathrm{IV} \to 4\mathrm{I}_1$$
$$\mathrm{IV} \to \mathrm{I}_3 + \mathrm{I}_1 \quad \mathrm{IV} \to \mathrm{III} + \mathrm{I}_1 \quad \mathrm{IV} \to \mathrm{II} + \mathrm{I}_2 \quad \mathrm{IV} \to 2\mathrm{II} \quad \mathrm{IV} \to \mathrm{I}_2 + 2\mathrm{I}_1$$
$$\mathrm{IV} \to \mathrm{II} + 2\mathrm{I}_1 .$$

Moreover the confluence which does not occur namely $\mathrm{IV} \to 2\mathrm{I}_2$ *is obstructed by monodromy considerations.*

Theorem

Every type of confluence of singular elliptical fibers on a rational elliptical surface of type I_{b_i} *into a singular fiber of type* I_b *with* $b = \sum b_i$ *occurs.*

Elliptic curves

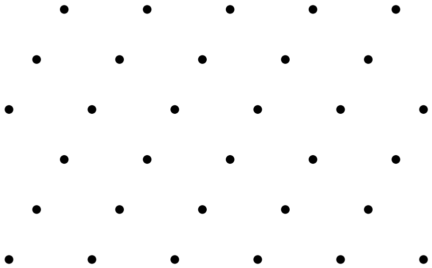
Elliptic curves

An elliptic curve C is Riemann surface of genus 1.
We shall now give two equivalent ways of describing a Riemann surface:

- $C \simeq \mathbb{C}/P$, for some Lattice P.
- C is the solution curve of the equation $y^2 = 4x^3 - g_2 x - g_3$ in affine coordinates, with ${g_2}^3 - 27{g_3}^2 \neq 0$.

$C \simeq \mathbb{C}/P$

- Pick a nonzero holomorphic vector field v.
- $(c, t) \mapsto e^{vt}(c) : C \times \mathbb{C} \to C$ is a complex analytic action of $\mathbb{C}$ on C, with one orbit
- $P = \{t \in \mathbb{C} | e^{tv}(c) = c\}$ (indep of c)
- Induces $\Phi : \mathbb{C}/P \to C$ a complex analytic diffeomorphism.
- P has a $\mathbb{Z}$-basis p_1, p_2, which is an $\mathbb{R}$-basis of $\mathbb{C}$.



C as a solution curve

Consider the Weierstrass $\wp$-function

$$\wp(t) \equiv t^{-2} + \sum_{p \in P \setminus \{0\}} ((t-p)^{-2} - p^{-2})$$

If

$$g_2 = g_2(P) = 60 \sum_{p \in P \setminus \{0\}} p^{-4} \qquad g_3 = g_3(P) = 140 \sum_{p \in P \setminus \{0\}} p^{-6},$$

we have $\wp'(t)^2 - 4\wp(t)^3 + g_2\wp(t) + g_3 = 0$.

$$\wp'(t)^2 - 4\wp(t)^3 + g_2\wp(t) + g_3 = 0$$

Let $\pi : t \mapsto [1 : x : y] = [1 : \wp(t) : \wp'(t)]$ then:

- $\pi(\mathbb{C}/P) \subset D \equiv \{[1 : x : y] | y^2 = 4x^3 - g_2 x - g_3\}$
- $\pi'(t) \neq 0 \Rightarrow \pi(\mathbb{C}/P)$ open
- $\mathbb{C}/P$ compact $\Rightarrow \pi(\mathbb{C}/P)$ compact

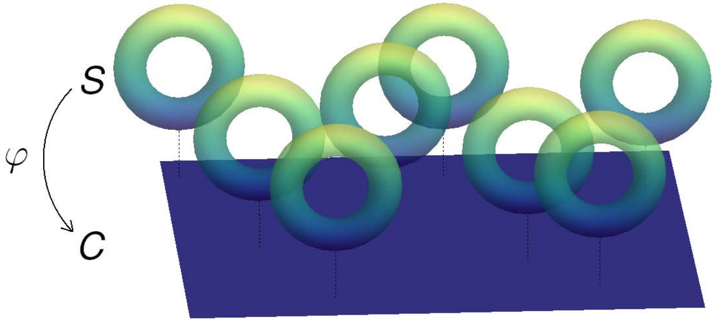
$\Rightarrow \pi$ is a holomorphic covering map.

$$\wp(t) = t^{-2} + \sum_{p \in P \setminus \{0\}} ((t-p)^{-2} - p^{-2})$$

$\wp$ has only poles in $0 + P \Rightarrow \pi$ maps only $0 + P$ to ∞
$\Longrightarrow \mathbb{C}/P$ covers D but once, π is a diffeomorphism.
Smoothness $\Rightarrow$ The geometric discriminant $\Delta \equiv {g_2}^3 - 27{g_3}^2 \neq 0$.

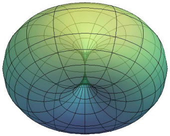
Elliptic surfaces

Definition of Elliptic surfaces



$\varphi : S \to C$ non-constant proper analytic map.

	notation	tangent map	Fiber
Regular	S^{reg}, C^{reg}	$T_s\varphi \neq 0$	Elliptic curve (donut)
Singular	S^{sing}, C^{sing}	$T_s\varphi = 0$	Element Kodaira's list (failed donut/misbaksel)

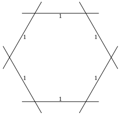


Kodaira's classification

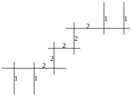
We make the following technical assumptions: S is a relatively minimal elliptical surface without multiple singular fibers.

I_1

I_b

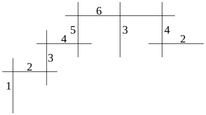


I_b^*



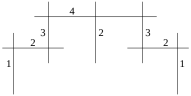
II

II^*

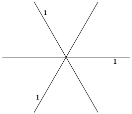


III

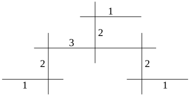
III^*



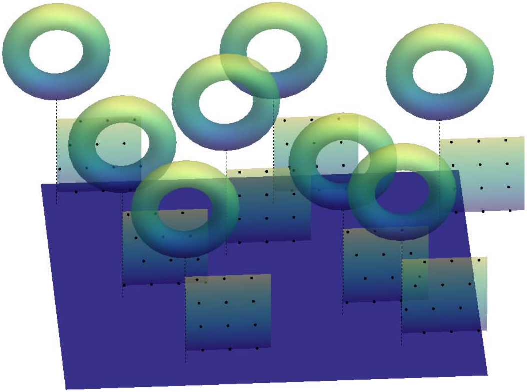
IV



IV^*



Monodromy



The monodromy is unique up to conjugation with elements of $\mathrm{SL}(2,\mathbb{Z})$, because we can pick any $\mathbb{Z}$-basis of P we like.

Type	Intersection diagram	Monodromy matrix	Euler number
I_b	$A_{b-1}^{(1)}$	$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$	b
I_b^*	$D_{b+4}^{(1)}$	$\begin{pmatrix} -1 & -b \\ 0 & -1 \end{pmatrix}$	$b+6$
II	$A_0^{(1)}$	$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$	2
II^*	$E_8^{(1)}$	$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$	10
III	$A_1^{(1)}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	3
III^*	$E_7^{(1)}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	9
IV	$A_2^{(1)}$	$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$	4
IV^*	$E_6^{(1)}$	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$	8

The Weierstrass model

We now write every regular fiber as a solution curve to the equation $y^2 = 4x^3 - g_2 x - g_3$, where g_2 and g_3 depend on c. So we can more or less describe a (relatively minimal) elliptical surface (without multiple singular fibers) by giving $g_2(c)$ and $g_3(c)$, where

Kodaira Type	Order zero of g_2	of g_3	of Δ, Euler #
I_0	≥ 0	≥ 0	0
I_b, $b \geq 1$	0	0	b
I_0^*	≥ 2	≥ 3	6
I_b^*, $b \geq 0$	2	3	$b + 6$
II	≥ 1	1	2
II^*	≥ 4	5	10
III	1	≥ 2	3
III^*	3	≥ 5	9
IV	≥ 2	2	4
IV^*	≥ 3	4	8

An elliptic surface is called rational if

$$\sum \text{Euler } \# = 12.$$

We may write

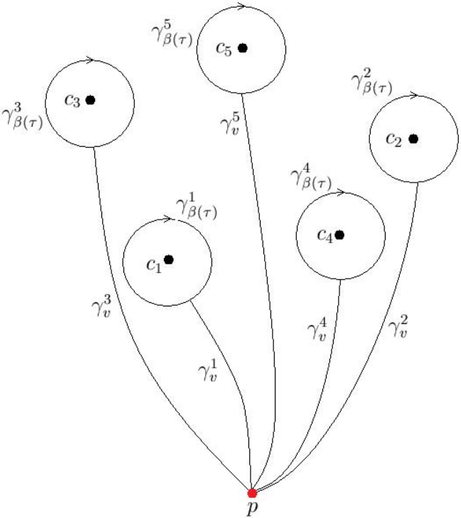
$$g_2(z) = \sum_{i=0}^{4} g_{2,i} z^i \qquad g_3(z) = \sum_{i=0}^{6} g_{3,i} z^i,$$

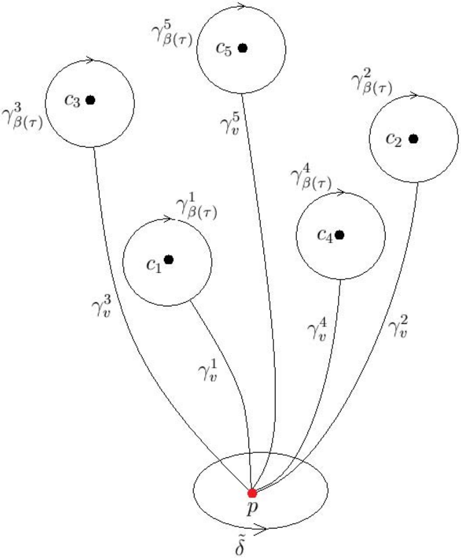
locally (over $\hat{\mathbb{C}}$, z is an affine coordinate).

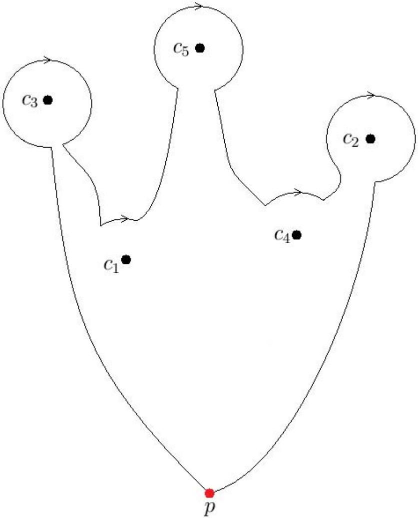
- Perturbing $g_{2,i}$ and $g_{3,i}$ gives a family of elliptic surfaces.
- $\beta(\tau) \in (g_{2,i}, g_{3,i})$ curve.
- Along β the singular fibers in the elliptic surfaces can merge: Confluence.

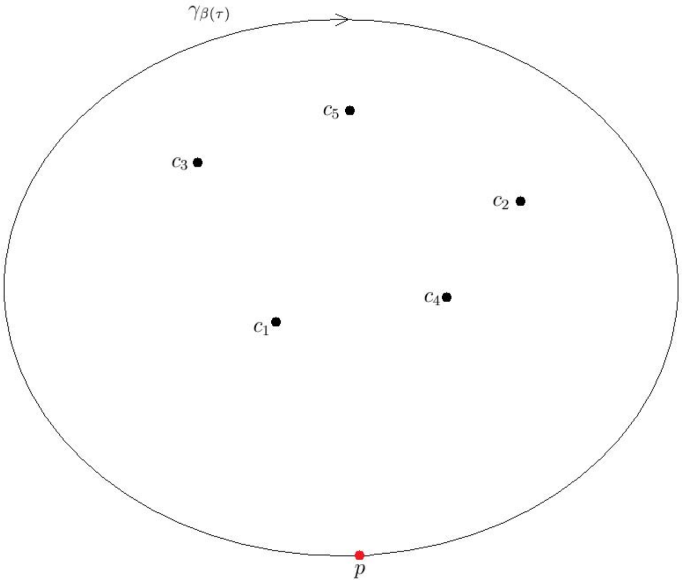
The monodromy of the singular fibers before and after confluence are related as follows

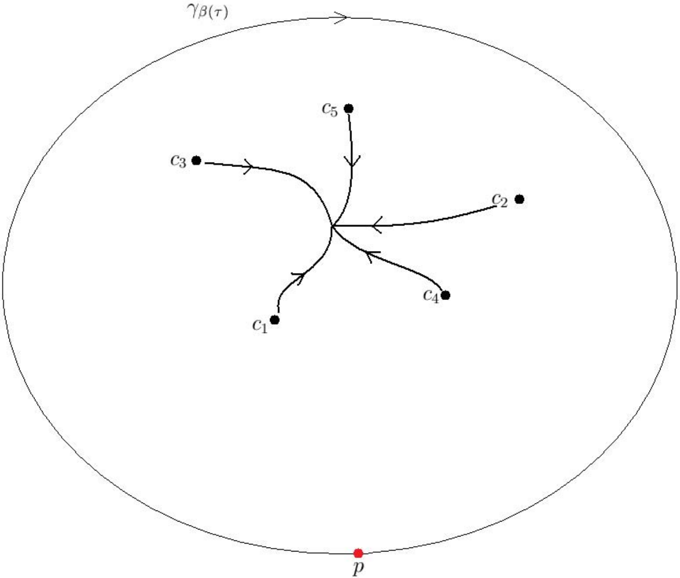
$$M_{S_c} = M_{S_{c_{\sigma(1)}}} \cdot \ldots \cdot M_{S_{c_{\sigma(N)}}}.$$


$\gamma^5_{\beta(\tau)}$
c_5
$\gamma^2_{\beta(\tau)}$
c_3
$\gamma^3_{\beta(\tau)}$
γ^5_v
c_2
$\gamma^4_{\beta(\tau)}$
$\gamma^1_{\beta(\tau)}$
c_4
c_1
γ^3_v
γ^4_v
γ^2_v
γ^1_v
p

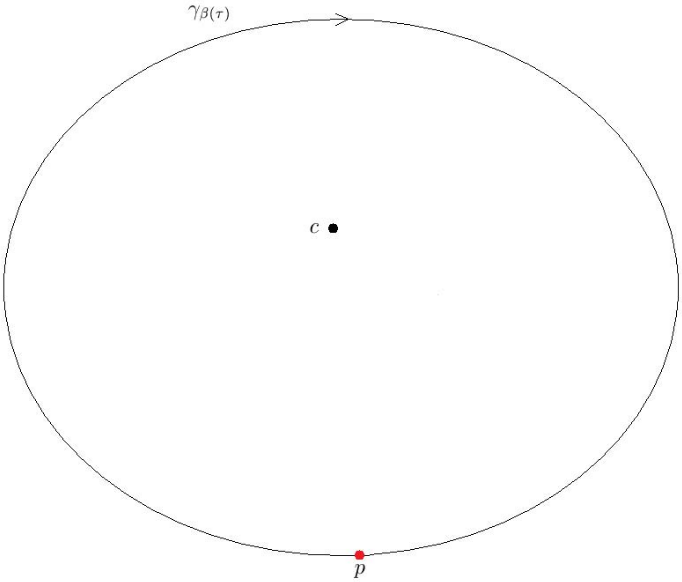

$\gamma^5_{\beta(\tau)}$
c_5
$\gamma^2_{\beta(\tau)}$
c_3
$\gamma^3_{\beta(\tau)}$
γ^5_v
c_2
$\gamma^4_{\beta(\tau)}$
$\gamma^1_{\beta(\tau)}$
c_4
c_1
γ^3_v
γ^4_v
γ^2_v
γ^1_v
p
$\tilde{\delta}$


c_5
c_3
c_2
c_4
c_1
p


$\gamma_\beta(\tau)$
c_5
c_3
c_2
c_4
c_1
p



$\gamma_{\beta(\tau)}$
c_5
c_3
c_2
c_4
c_1
p


$\gamma_\beta(\tau)$
c
p

Resultants

Resultants

Consider f and g polynomials in one variable.

$$f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n$$
$$g(x) = b_0x^m + b_1x^{m-1} + \ldots + b_m$$

f and g have (at least) N common zeros if and only if

$$h(x)f(x) = k(x)g(x),$$

for $h(x)$ and $k(x)$ polynomials of degree $m - N$ and $n - N$.

$$h(x)f(x) = k(x)g(x),$$

where

$$f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n$$
$$g(x) = b_0x^m + b_1x^{m-1} + \ldots + b_m.$$

If we write

$$h(x) = c_0x^{m-N} + c_1x^{m-N-1} + \ldots + c_{m-N},$$
$$k(x) = d_0x^{n-N} + d_1x^{n-N-1} + \ldots + d_{n-N},$$

we find equations for the coefficients of the polynomials.

This may be written in matrix form

$$
\begin{pmatrix}
a_0 & 0 & 0 & \dots & 0 & b_0 & 0 & \dots & 0 \\
a_1 & a_0 & 0 & \dots & 0 & b_1 & b_0 & \ddots & 0 \\
a_2 & a_1 & a_0 & \dots & 0 & b_2 & b_1 & \ddots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
a_n & a_{n-1} & \dots & \dots & a_0 & b_{m-2} & \dots & \dots & 0 \\
0 & a_n & \dots & \dots & a_1 & b_{m-1} & \dots & \dots & 0 \\
0 & 0 & a_n & \dots & a_2 & b_m & b_{m-1} & \dots & b_0 \\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \dots & \dots & 0 & a_n & 0 & \dots & \dots & b_m
\end{pmatrix}
\begin{pmatrix}
c_0 \\ \vdots \\ \vdots \\ \vdots \\ c_{m-N} \\ -d_0 \\ \vdots \\ \vdots \\ -d_{m-N}
\end{pmatrix}
=
\begin{pmatrix}
0 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ 0
\end{pmatrix}
$$

Linear algebra yields that this equation has solutions if and only if all determinants of $(m+n-2N+2) \times (m+n-2N+2)$ submatrices are zero. If $N = 1$ the single discriminant is called the resultant, denoted by $R(f, g)$.

The resultant of f and f' is related to the discriminant of a polynomial:

$$R(f, f') = \pm a_0 D,$$

where $f(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_n$ and D denotes the discriminant

$$D = a_0^{2n-2} \prod_{i<j} (x_i - x_j)^2,$$

where the x_i are the roots of $f(x)$.

Lemma

Let $f(x)$, $g(x)$ and $h(x)$ be three polynomials in x. Then $f(x) - yg(x)$ and $h(x)$ have at least one linear factor in common for all $y \in \mathbb{C}$ if and only if $f(x)$, $g(x)$ and $h(x)$ have a linear factor in common.

Corollary

If the resultant $R(f - yg, h)$ with respect to x as a polynomial in y is zero, then $f(x)$, $g(x)$ and $h(x)$ have a linear factor in common.

Configurations including II^*, III^* and IV^*

We consider a configuration of singular fibers where II^* is fixed in infinity.
This gives

$$g_2(z) = a$$
$$g_3(z) = bz + c,$$

By rescaling and a Tschirnhausen transformation we may write

$$g_2(z) = a$$
$$g_3(z) = z.$$

It is obvious that for $a \neq 0$ we have two singular fibers of Kodaira type I_1 and II if $a = 0$.

III^* is fixed in infinity.
We in affine coordinates (we already rescaled and transformed)

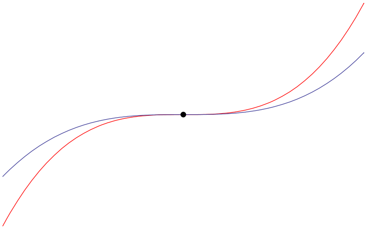
$$g_2(z) = z + 9c^3$$
$$g_3(z) = cz + d$$
$$\Delta(z) = z^3 + (243c^4 - 54cd)z + 739c^6 - 27d^2.$$

The discriminant of the geometric discriminant $\Delta(z)$ now reads

$$-19683\left(5c^3 - d\right)\left(9c^3 - d\right)^3$$

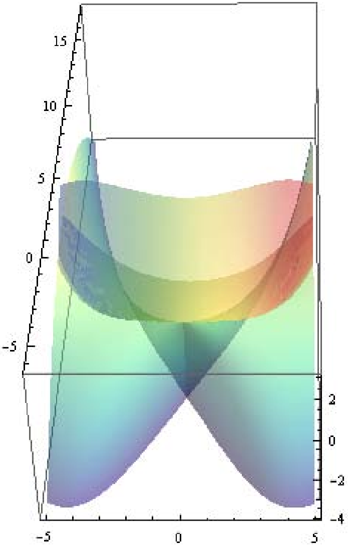
and the resultant of g_2 and g_3 $-9c^3 + d$.

Configuration	Parameter
$\mathrm{III}^* + \mathrm{III}$	$c = d = 0$
$\mathrm{III}^* + \mathrm{I}_2 + \mathrm{I}_1$	$5c^3 - d = 0$
$\mathrm{III}^* + \mathrm{II} + \mathrm{I}_1$	$9c^3 - d = 0$
$\mathrm{III}^* + 3\mathrm{I}_1$	Otherwise

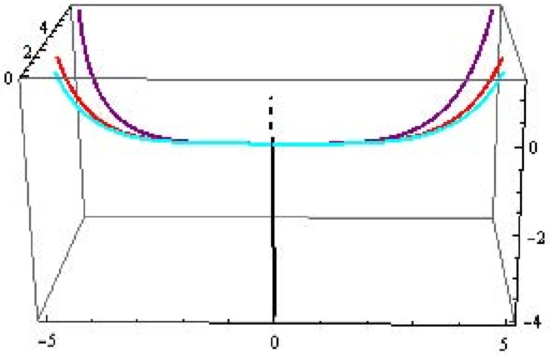


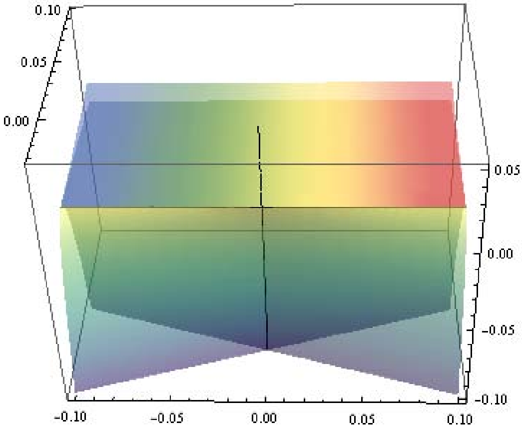
IV^* is fixed in infinity.

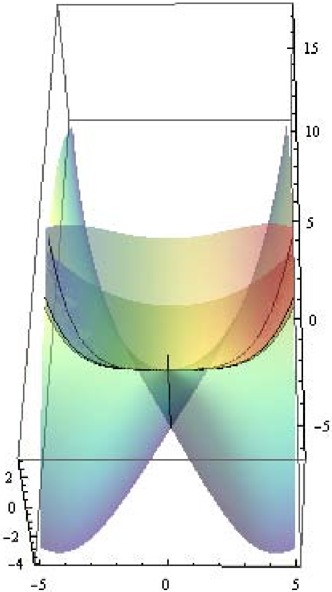
Configuration	Colour
$\mathrm{IV}^* + \mathrm{IV}$, $\mathrm{IV}^* + 2\mathrm{II}$,	
$\mathrm{IV}^* + \mathrm{II} + \mathrm{I}_2$, $\mathrm{IV}^* + \mathrm{III} + \mathrm{I}_1$,	
$\mathrm{IV}^* + \mathrm{II} + 2\mathrm{I}_1$ (generic)	Rainbow
$\mathrm{IV}^* + \mathrm{I}_3 + \mathrm{I}_1$, $\mathrm{IV}^* + \mathrm{II} + \mathrm{I}_2$,	
$\mathrm{IV}^* + \mathrm{III} + \mathrm{I}_1$,	
$\mathrm{IV}^*\ \mathrm{I}_2\ 2\mathrm{I}_1$ (generic)	Green-blue
$\mathrm{IV}^* + 4\mathrm{I}_1$	None



Configuration	Colour
$IV^* \, 2II$	black
$IV^* \, II \, I_2$	red
$IV^* \, III \, I_1$	purple
$IV^* \, I_3 \, I_1$	cyan




0.10
0.05
0.00
0.05
0.00
−0.05
−0.10
−0.10
−0.05
0.00
0.05
0.10


15
10
5
0
−5
2
0
−2
−4
−5
0
5

$\mathrm{IV} \to 2\mathrm{I}_2$

We have that the monodromy equivalence classes must satisfy

$$M_{S_c} = M_{S_{c_{\sigma(1)}}} \cdot \ldots \cdot M_{S_{c_{\sigma(N)}}}.$$

So in particular

$$\begin{aligned} \mathrm{Tr}(M_{\mathrm{IV}}) &= \mathrm{Tr}(M_{\mathrm{I}_2} A M_{\mathrm{I}_2} A^{-1}) \\ \mathrm{Tr}\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} &= \mathrm{Tr}\left(\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\right) \\ -1 &= 2(1-2c^2), \end{aligned}$$

where $A \in \mathrm{SL}(2, \mathbb{Z})$: a contradiction.

Theorem

Of all confluences to Singular Fibers of Kodaira type II, III *and* IV*, superficially allowed by conservation of the Euler number, the following occur:*

$$\mathrm{II} \to \mathrm{I}_1 + \mathrm{I}_1 \quad \mathrm{III} \to 3\mathrm{I}_1 \quad \mathrm{III} \to \mathrm{I}_2 + \mathrm{I}_1 \quad \mathrm{III} \to \mathrm{II} + \mathrm{I}_1 \quad \mathrm{IV} \to 4\mathrm{I}_1$$
$$\mathrm{IV} \to \mathrm{I}_3 + \mathrm{I}_1 \quad \mathrm{IV} \to \mathrm{III} + \mathrm{I}_1 \quad \mathrm{IV} \to \mathrm{II} + \mathrm{I}_2 \quad \mathrm{IV} \to 2\mathrm{II} \quad \mathrm{IV} \to \mathrm{I}_2 + 2\mathrm{I}_1$$
$$\mathrm{IV} \to \mathrm{II} + 2\mathrm{I}_1 .$$

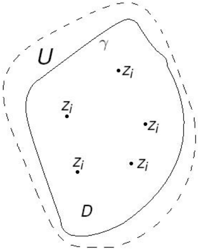
Moreover the confluence which does not occur namely $\mathrm{IV} \to 2\mathrm{I}_2$ *is obstructed by monodromy considerations.*

Weierstrass preparation theorem

Weierstrass preparation theorem

f holomorphic U neighbourhood of the origin
the zeros are given by

$$Z_f = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\mathrm{d}z = \sum_{z_i \in f^{-1}(0)\cap D} \operatorname*{Res}_{z=z_i} \frac{f'(z)}{f(z)}.$$



Theorem

Let f be as above and furthermore assume that $M = \sum m_i$. Then there exists a unique Weierstrass polynomial $W(z)$ of degree M

$$W(z) = z^M + c_1 z^{M-1} + c_2 z^{M-2} + \ldots + c_M,$$

where $W(z)$ has the same zeros as f in D or alternatively $f(z) = W(z)u(z)$ with $u(z)$ a unit in D.

Suppose we are given $\Delta_\varepsilon(z)$. We now write

$$\Delta_\varepsilon(z) = W_\varepsilon(z)u_\varepsilon(z),$$

where $W_\varepsilon(z)$ is of order b. By differentiating with respect to ε we can determine $W_\varepsilon(z)$ up to second order in ε.

$$\begin{aligned}\partial_{\varepsilon_i}\Delta_\varepsilon(z)|_{\varepsilon=0} &= u_0(z)\partial_{\varepsilon_i}W_\varepsilon(z)|_{\varepsilon=0} + W_\varepsilon(z)\partial_{\varepsilon_i}u_\varepsilon(z)|_{\varepsilon=0} \\ &= u_0(z)\partial_{\varepsilon_i}W_\varepsilon(z)|_{\varepsilon=0} + \mathcal{O}(z^b).\end{aligned}$$

In theory we can successively determine the $W^{(k)}(z)$ in the power series expansion in ε; $W_\varepsilon = \sum \varepsilon^k W^{(k)}(z)$ by this method.

Confluences to I_b

I_b arise only from I_{b_i}s.
This implies that if $\Delta_\varepsilon(z)$, $g_{2,\varepsilon}(z)$, $g_{3,\varepsilon}(z)$ and $\beta(\tau) \in \varepsilon$-space are such that

- $W_{\beta(\tau)=0} = z^b$
- $g_{2,\beta(\tau)=0}(0) \neq 0$
- $g_{3,\beta(\tau)=0}(0) \neq 0$
- $W_{\beta(\tau)} = (z - e^{i\psi_1} z_{0,\beta(\tau)})^{b_1} \ldots (z - e^{i\psi_k} z_{0,\beta(\tau)})^{b_k}$

then we have $\mathrm{I}_{b_1} + \ldots + \mathrm{I}_{b_k} \to \mathrm{I}_b$.
To impose the form of $W_{\beta(\tau)}$ we use the Weierstrass preparation theorem and the implicit function theorem (details to be found in the thesis).

Theorem

Every type of confluence of singular elliptical fibers on a rational elliptical surface of type I_{b_i} into a singular fiber of type I_b with $b = \sum b_i$ occurs.

Conclusions and outlook

Conclusions and outlook

HAVE DONE:

- We have fully discussed confluences to II, III, IV, I_b and I_0^* (not presented here).
- We have made progress on the confluences to I_1^* and II^*.
- We have in the course of doing so provided a fair number of Weierstrass model for global configurations.

TO DO:

- Find for every local confluence an example or obstruction.
- Understand how every configuration of singular elliptical fibers fits in the space of parameters of g_2 and g_3.

END