university of groningen
faculty of mathematics and natural sciences
johann bernoulli institute

On the uniqueness of the Gauss-Bonnet Theorem

M.H.M.J. Wintraecken

March 2013

Outline

The Gauss-Bonnet Theorem

Uniqueness in two dimensions

Heegaard splitting

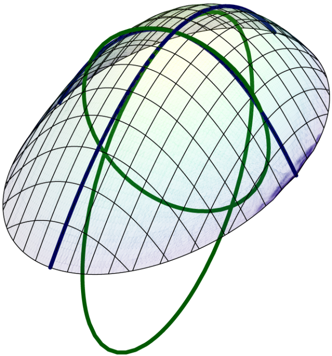
Three dimensions

The Gauss-Bonnet Theorem

Carl Friedrich Gauss

Pierre Ossian Bonnet

The Gauss-Bonnet Theorem



With local information;

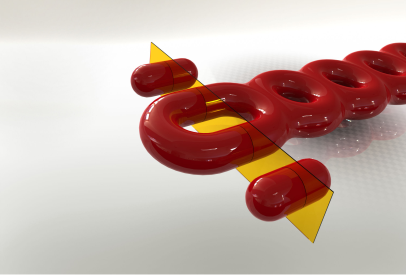
$$\int K \mathrm{d}A = 2\pi(v - e + t) = 2\pi\chi(M).$$

Gaussian curvature: $K = k_1 \cdot k_2$, with $k_i = \frac{1}{\rho_i}$, ρ_i radius of smallest/largest osculating circle.
Also

$$R_{abcd} = K(g_{ac}g_{bd} - g_{ad}g_{bc})$$

Is this the only such formula?

Uniqueness in two dimensions



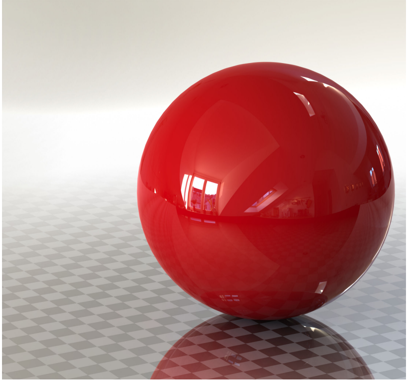
Theorem

f function on surfaces, completely determined by metric, that is locally $f(x) = F(g(x), \partial g(x), \ldots)$, independent of topology. If

$$\int_M f \, \mathrm{d}A,$$

yields a topological invariant $t_f(M)$ for all surfaces. Then $t_f(M) = c_f \, \chi(M)$, where χ is the Euler characteristic.

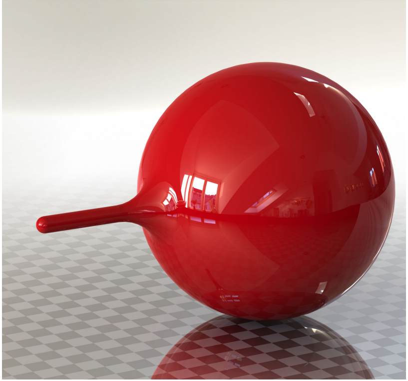
Sphere



Assume that for the sphere

$$\int_{S^2} f \, \mathrm{d}A = 2c.$$

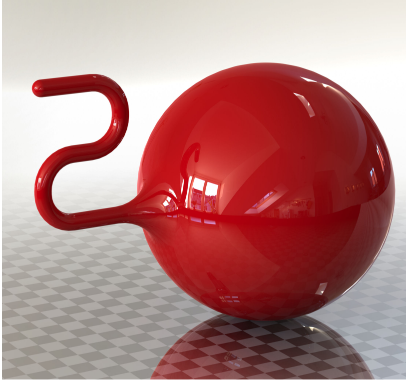
Deforming sphere



Deforming the sphere leaves the integral unaltered

$$\int_{S^2_{\text{deformed}}} f \, dA = 2c$$

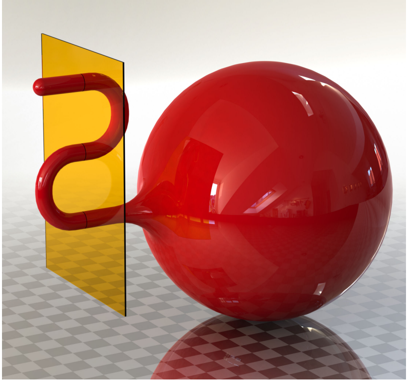
Deforming sphere



Deforming the sphere leaves the integral unaltered

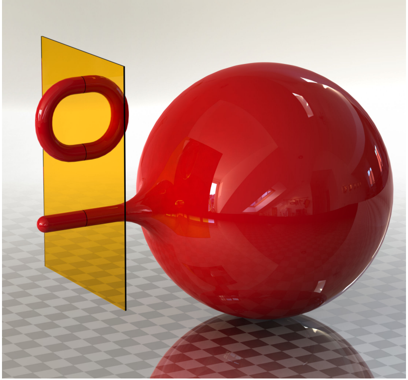
$$\int_{S^2_{\text{deformed}}} f \, \mathrm{d}A = 2c$$

Cutting sphere



Due to local isometry we can cut through the straight parts.

Reassembled surface



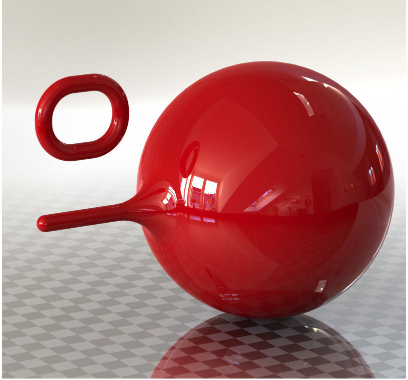
Integral additive so

$$2c = \int_{S^2_{\text{deformed}}} f \, dA$$
$$= \int_{S^2_{\text{deformed}}} f \, dA$$
$$+ \int_{(S^1 \times S^1)_{\text{deformed}}} f \, dA.$$

Which yields

$$\int_{S^1 \times S^1} f \, dA = 0.$$

Reassembled surface



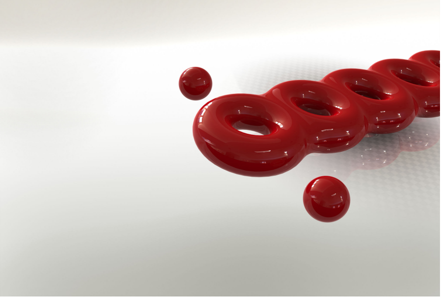
Integral additive so

$$2c = \int_{S^2_{\text{deformed}}} f \, \mathrm{d}A$$
$$= \int_{S^2_{\text{deformed}}} f \, \mathrm{d}A$$
$$+ \int_{(S^1 \times S^1)_{\text{deformed}}} f \, \mathrm{d}A.$$

Which yields

$$\int_{S^1 \times S^1} f \, \mathrm{d}A = 0.$$

Induction on genus



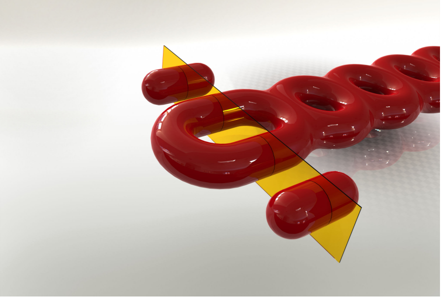
Surface of genus g (C_g) and two spheres:

$$2\int_{S^2} f\,\mathrm{d}A + \int_{C_g} f\,\mathrm{d}A = t.$$

Induction on genus

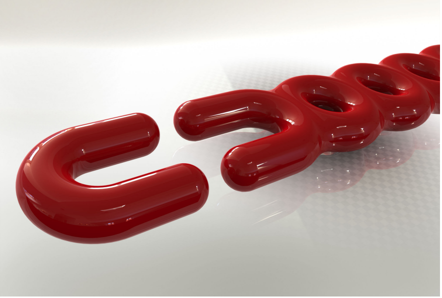
Deformation so that parts of surface are cylinders.

Induction on genus



Due to local isometry we can cut through the straight parts.

Induction on genus



We find that

$$2\int_{S^2} f\,\mathrm{d}A + \int_{C_g} f\,\mathrm{d}A$$

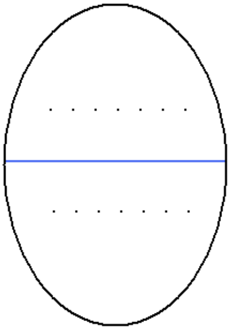
$$= \int_{S^2} f\,\mathrm{d}A + \int_{C_{g-1}} f\,\mathrm{d}A$$

Induction on genus

So

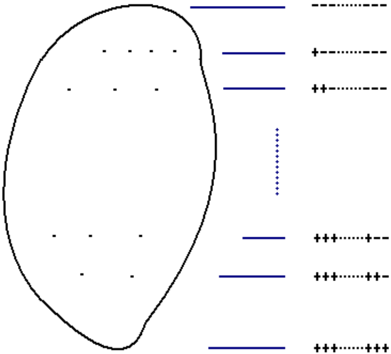
$$
\begin{aligned}
\int_{C_g} f \, \mathrm{d}A &= \int_{C_{g-1}} f \, \mathrm{d}A - \int_{S^2} f \, \mathrm{d}A \\
&= \int_{C_{g-2}} f \, \mathrm{d}A - 2 \int_{S^2} f \, \mathrm{d}A \\
&= \dots \\
&= (1-g) \int_{S^2} f \, \mathrm{d}A \\
&= 2(1-g)c = c\chi(C_g).
\end{aligned}
$$

Heegaard Splitting

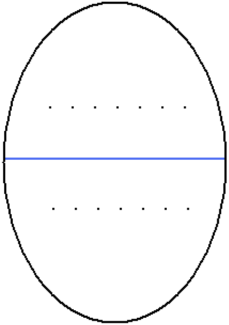


Heegaard splitting

The critical points of a Morse function can be ordered



Heegaard splitting

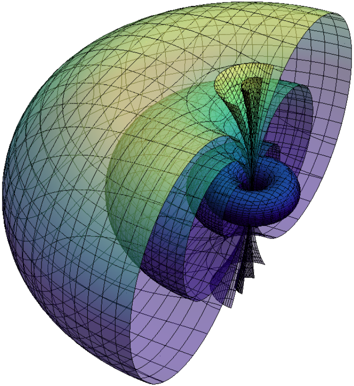


Definition A Heegard splitting is a difeomorphism of 3-dimensional compact connected manifold to a manifold formed by two 3-dimensional manifolds Π_1, Π_2 and diffeomorphism on the boundaries $\partial\Pi_1$, $\partial\Pi_2$. Here both Π_1 and Π_2 are homeomorphic to single 3-dimensional ball with g handles.

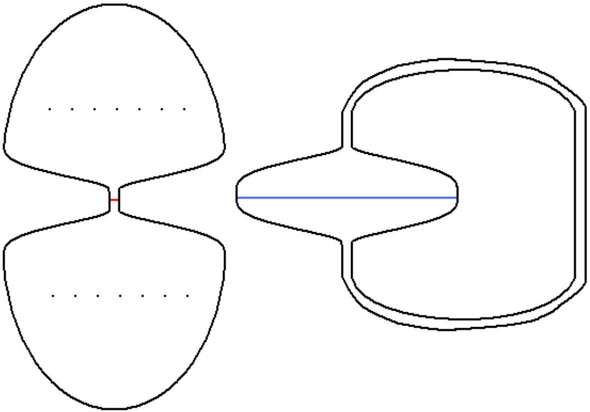
Theorem Every 3-manifold allows for a (non-unique) Heegaard splitting.

Hopf fibration

S^3 allows a Heegaard splitting for every genus g.
Genus 0 obvious, genus one: Hopf fibration.



Three dimensions

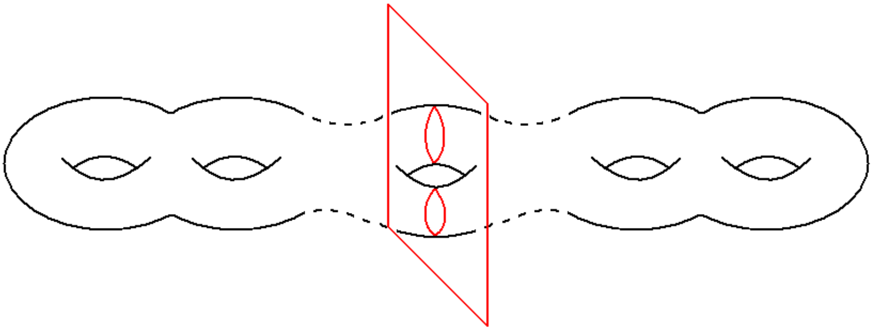


Theorem

f function on 3-manifolds, completely determined by metric, that is locally $f(x) = F(g(x), \partial g(x), \ldots)$, independent of topology. If

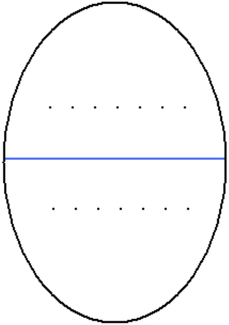
$$\int_M f \, \mathrm{d}A,$$

yields a topological invariant $t_f(M)$ for all manifolds. Then $t_f(M) = 0$.



Introduce standard metric on torus with reflection symmetry

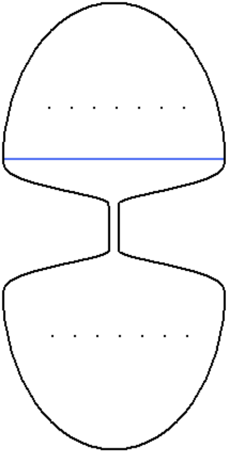
Splitting off $C_g \times S^1$



Take some manifold N which allows a Heegaard splitting of genus g.
Consider

$$\int_N f \, \mathrm{d}A = t$$

Locally to standard form



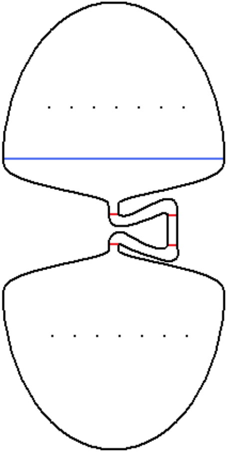
Deform some piece to $C_g \times [a, b]$ endowed with standard metric

Integral

$$\int_{N_{\mathrm{deformed}}} f \, \mathrm{d}A = t$$

invariant

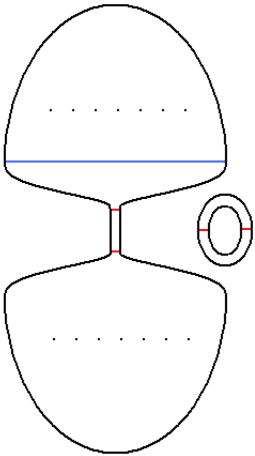
Deforming small neighbourhood



Deform $C_g \times [a, b]$ and indicate cut
Twisting does not influence topology

$$\int_{N_{\mathrm{deformed}}} f \, \mathrm{d}A = t$$

Reassemble



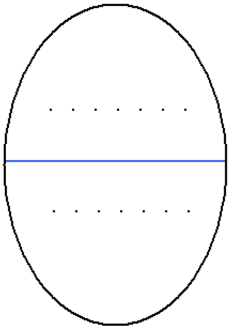
Cutting and pasting leaves an integral invariant because of local isometry

$$t = \int_N f\,\mathrm{d}A = \int_N f\,\mathrm{d}A + \int_{C_g \times S^1} f\,\mathrm{d}A$$

So

$$\int_{C_g \times S^1} f\,\mathrm{d}A = 0$$

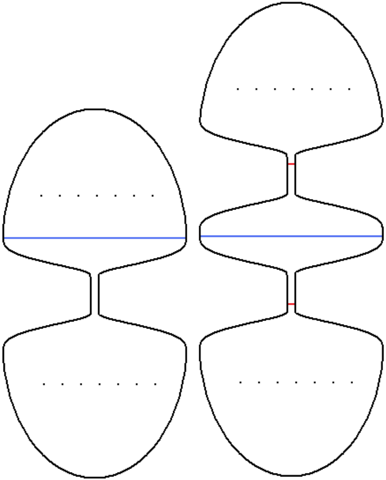
Globally to standard



Let M be a manifold which allows a Heegaard splitting of genus g.
Consider

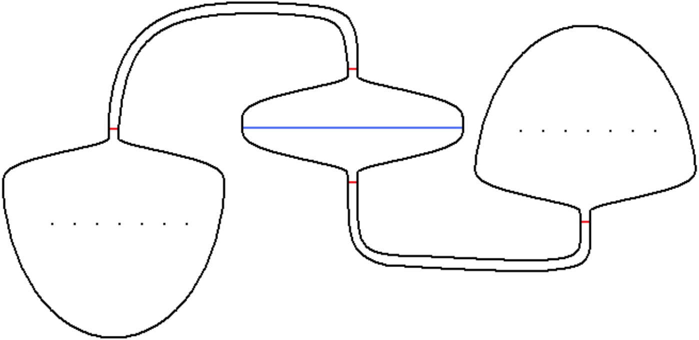
$$\int_M f\,\mathrm{d}A = t$$

Local pinching



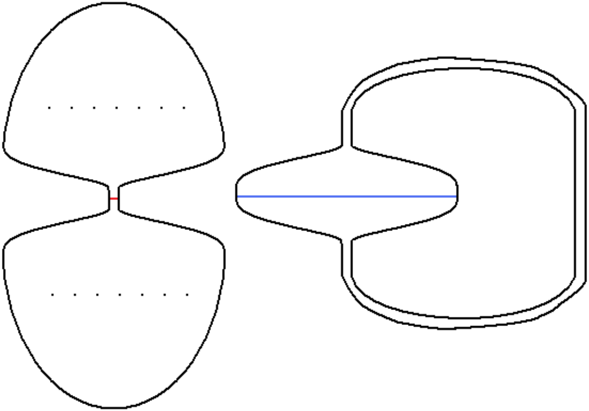
We follow the same local procedure as before

Local deforming



The ‘straightened’ pieces can again be deformed. Cutting lines indicated.

Reassemble



We reassemble into $M_g^{S(3D)}$ and $C_g \times S^1$

We have for any two manifolds M, $\tilde{M}$ admitting Heegaard splitting of genus g

$$t = \int_M f \, \mathrm{d}A = \int_{M_g^{S(3\mathrm{D})}} f \, \mathrm{d}A + \int_{C_g \times S^1} f \, \mathrm{d}A = \int_{\tilde{M}} f \, \mathrm{d}A$$

Can choose $\tilde{M} = S^3$.
Both S^3 and $S^2 \times S^1$ allow a Heegaard splitting of genus one; for any manifold

$$\int_M f \, \mathrm{d}A = \int_{S^3} f \, \mathrm{d}A = \int_{S^2 \times S^1} f \, \mathrm{d}A = 0$$

Acknowledgements

Special thanks to Erik van den Ban (UU) and Niels Bos

The End