Confluence of singular fibers on rational elliptic surfaces

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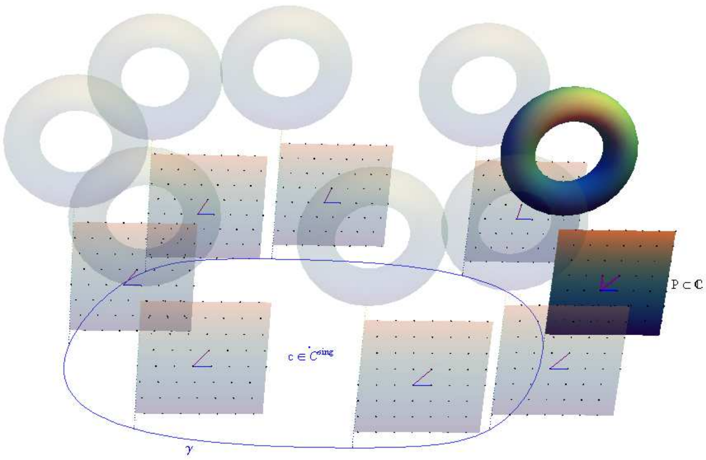

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The figure depicted on the title page sketches the monodromy of a singular fiber of Kodaira type I_1, see chapter 2 for a discussion.

Contents

Preface

An elliptic surface locally looks like a subset of the complex plane, with a fiber above each point. The fibers are generally complex tori, but are allowed to degenerate in specific points. These degenerated tori are called singular elliptical fibers. Under certain conditions these singular elliptical fibers have been classified by Kodaira [6–8]. Globally all possible configurations of singular elliptical fibers on a restrictive class of elliptic surfaces, so-called rational elliptic surfaces, have been classified by Persson [10]. The different configurations may be put into a huge continuous parameterized family of elliptic surfaces. If we move in parameter space, we see the singular fibers moving around. It can happen that several singular fibers flow together into a single singular fiber if the parameters are varied, this phenomenon is called a confluence.
Although there are publications on the subject of confluence, most notably by Naruki [9], a great number of questions remain open. We shall focus on the local case, instead of considering global configurations of singular elliptical fibers. In this thesis we will discuss all local confluences on rational elliptic surfaces which form a singular fiber of the non-starred type; that is singular fibers of type I_b, II, III and IV, according to Kodaira's classification. Some progress which we have made on other types of singular elliptical fibers will also be reported.

The outline of this thesis will be as follows: Chapter 1 discusses some techniques used; namely the Weierstrass preparation theorem, which helps to localize our problem, and generalizations of the discriminant and resultant. This chapter relies on notes by Duistermaat as well as the books by Griffiths and Harris [5] and van der Waerden [12]. The second chapter introduces elliptic surfaces, and families of elliptic surfaces. This chapter is taken from [3], with the exception of sections 2.1, which relies on [5] and 2.7, which amoung others uses results from [3, 9, 11]. One of the original articles by Kodaira [7] has had some influence on section 2.4 In the third chapter we present the results of our research. Apart from the sources mentioned above we have used the following general references [1, 2, 4, 13] and Wikipedia, the German version of which proved to be particularly useful when translating [12], since each lemma in Wikipedia contains a link to the same lemma in other languages.

The research presented in this thesis greatly relies on explicit calculations all of which have been done using Wolfram Mathematica, versions 5.0 through 7.0. The figures and diagrams in the text are of the hand of the author and have been produced using Mathematica, Microsoft Paint, Paintshop Pro and XY-pic, with the exception of the extended Dynkin diagrams in section 2.4, which were made in LaTeX and courtesy of Hans Duistermaat.

Chapter 1

Polynomials and zeros

This chapter introduces some elementary results from complex analysis and algebra. These result will be used in chapter 3. We rely among others on [5] and [12].

1.1 The Weierstrass preparation theorem

This section discusses some classical results of complex analysis, in particular the Weierstrass preparation theorem. The content is based on the first chapter of Griffiths and Harris [5] and on notes by Duistermaat.

Let f be a complex analytic function in one variable, not identically equal to zero, on a convex neighbourhood U of zero. The zeros of f form a discrete set. Now let $\gamma : [0,1] \to U \backslash \{0\}$ be a closed curve, that is $\gamma(0) = \gamma(1)$, around the origin. For simplicity we will assume that $\gamma([0,1])$ is homotopic to a circle in $U \backslash \{0\}$. Denote the two real dimensional surface in U enclosed by γ by D. Furthermore we shall assume that

$$f \mid_{\gamma([0,1])} \neq 0.$$

We have that the number of zeros M. of f in D, counted with multiplicity, is given by

$$M = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} \mathrm{d}z = \sum_{z_i \in f^{-1}(0) \cap D} \operatorname*{Res}_{z=z_i} \frac{f'(z)}{f(z)}.$$

Proof It is clear that $f'(z)/f(z)$ only has singularities in the zeros of f denoted by z_i. We now consider the Taylor expansion of f in such a point z_i

$$f(z) = c_{m_i}(z - z_i)^{m_i} + \mathcal{O}\big((z - z_i)^{m_i+1}\big).$$

This in turn yields

$$\begin{aligned}
f'(z) &= m_i c_{m_i}(z-z_i)^{m_i-1} + \mathcal{O}\big((z-z_i)^{m_i}\big) \\
\frac{f'(z)}{f(z)} &= \frac{m_i c_{m_i}(z-z_i)^{m_i-1} + \mathcal{O}\big((z-z_i)^{m_i}\big)}{c_{m_i}(z-z_i)^{m_i} + \mathcal{O}\big((z-z_i)^{m_i+1}\big)} \\
&= \frac{m_i + \mathcal{O}\big((z-z_i)\big)\big)}{(z-z_i)(1+\mathcal{O}\big((z-z_i)\big)} \\
&= \frac{m_i}{z-z_i} + \mathcal{O}(1),
\end{aligned}$$

which gives us that $f'(z)/f(z)$ has only simple poles in U, since f is supposed to be holomorphic in U. Moreover these poles are located at the same points as the zeros of f. Furthermore the residue of $f'(z)/f(z)$ in the poles equals the order of the zero of f. So we have that

$$\begin{aligned}
\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\mathrm{d}z &= \sum_{z_i\in f^{-1}(0)\cap D} \operatorname*{Res}_{z=z_i} \frac{f'(z)}{f(z)} \\
&= \sum_{z_i\in f^{-1}(0)\cap D} m_i \\
&= M,
\end{aligned}$$

where m_i denotes the multiplicity of the zero of f at z_i as before. $\square$

We now note that

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\mathrm{d}z \in \mathbb{Z}.$$

If we let $f(z)$ depend in a continuous manner on a perturbation parameter δ, denoted by a lower index[1], then there exists a $\epsilon > 0$ such that for all $\delta < \epsilon$

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\mathrm{d}z = \frac{1}{2\pi i}\int_\gamma \frac{f'_\delta(z)}{f_\delta(z)}\mathrm{d}z.$$

This is a direct consequence of the continuity of f_δ with respect to δ, the continuity of

$$\frac{1}{2\pi i}\int_\gamma \frac{f'_\delta(z)}{f_\delta(z)}\mathrm{d}z$$

and the discreteness of the set $\mathbb{Z}$. Shortly summarized we have that the number of zeros in D is invariant under small perturbations of f.

We now define

$$s_k \equiv \frac{1}{2\pi i}\int_\gamma z^k \frac{f'(z)}{f(z)}\mathrm{d}z, \qquad \text{with } k \in \mathbb{Z}.$$

[1]The function $f_\delta(z) = f_0(z)$ will be denoted simply by $f(z)$.

Note that in z_i, a zero of multiplicity m_i, one has the following

$$\operatorname*{Res}_{z=z_i}\left(z^k \frac{f'(z)}{f(z)}\right) = z_i^k m_i,$$

so that, by the same argument as before

$$s_k = \sum_{z_i \in f^{-1}(0)\cap D} z_i^k m_i.$$

Using these definitions we state the following theorem:

Theorem 1.1.1 (Weierstrass preparation theorem) *Let f be as above and furthermore assume that $M = \sum m_i$. Then there exists a unique Weierstrass polynomial $W(z)$ of degree M*

$$W(z) = z^M + c_1 z^{M-1} + c_2 z^{M-2} + \ldots + c_M,$$

with the following properties:

i) $W(z)$ has the same zeros as f in D or alternatively $f(z) = W(z)u(z)$ with $u(z)$ a unit in D.

ii) The $c_1, \ldots, c_M$ are polynomial expressions in the $s_1, \ldots, s_M$, as defined above.

Proof Note that we have

$$W(z) = \prod_{j=1}^{M}(z - z_j),$$

with again $z_j \in f^{-1}(0) \cap D$. The uniqueness of $W(z)$ as well as property *i)* are a direct consequence of this. Property *ii)* follows from writing out the above expression for $W(z)$ and comparing it to the given form of the Weierstrass polynomial

$$\begin{aligned}
W(z) &= \prod_{j=1}^{M}(z - z_j) \\
&= z^M - \left(\sum_j z_j\right) z^{M-1} + \left(\sum_{i\neq j} z_i z_j\right) z^{M-2} - \ldots + (-1)^M \left(\prod_j z_j\right) \\
&= z^M + c_1 z^{M-1} + c_2 z^{M-2} + \ldots + c_M,
\end{aligned}$$

using the expressions for s_i derived above yields

$$\begin{aligned}
c_1 &= -\sum_j z_j = -s_1 \\
c_2 &= \sum_{i \neq j} z_i z_j = \frac{s_1^2 - s_2}{2} \\
c_3 &= -\sum_{i \neq j \neq k} z_i z_j z_k = \frac{6s_4 - 8s_1 s_3 - 3s_2^2 + 6s_1^2 s_2 - s_1^4}{24} \\
\ldots &= \ldots.
\end{aligned}$$

Note that this calculation also implies that the roots and coefficients of a polynomial are related by simple polynomial expressions. □

The Weierstrass preparation theorem will facilitate a particular way of investigating the effect of a perturbation of $f(z)$ on the zeros inside D. We will assume that the function depends in a C^∞ manner on the perturbation parameters $\delta_1, \ldots, \delta_k$, denoted by δ and use the Weierstrass preparation theorem to write

$$f_\delta(z) = W_\delta(z) u_\delta(z),$$

using that for sufficiently small δ, $f_\delta(z)$ has the same number of zeros as $f_0(0)$ inside D, again denoted by M,we have that

$$W_\delta(z) = z^M + c_{1,\delta} z^{M-1} + c_{2,\delta} z^{M-2} + \ldots + c_{M,\delta},$$

as well as that $u_\delta(z)$ is a unit on D. It now suffices to study $W_\delta(z)$ if we are interested in the bifurcation of the zeros of $f_\delta(z)$.

1.2 Resultants and discriminants

This section contains a free translation and adaptation of parts of sections 33, 34 and 35 of the fifth chapter of Algebra by van der Waerden [12]. Although all adaptations must be known, we have been unable to locate the classical literature.

Let

$$\begin{aligned} f(x) &= a_0x^n + a_1x^{n-1} + \ldots + a_n \\ g(x) &= b_0x^m + b_1x^{m-1} + \ldots + b_m \end{aligned} \tag{1.1}$$

be two polynomials, where we assume that $a_0 \neq 0$ and $b_0 \neq 0$. We are looking for a necessary and sufficient condition for these two polynomial to have N or more linear factors $\varphi_1(x), \ldots, \varphi_N(x)$ in common, where of course $N \leq n, m$. We shall not exclude possibility $\varphi_i = \varphi_j$.

We shall show that $f(x)$ and $g(x)$ have common linear factors $\phi_1(x), \ldots, \phi_N(x)$ if and only if there exists an equation

$$h(x)f(x) = k(x)g(x), \tag{1.2}$$

where $h(x)$ is of degree $m - N$, $k(x)$ is of degree $n - N$ and both $k(x)$ and $h(x)$ are not identically equal to zero. Let us assume that (1.2) holds. If we now decompose both sides of the equation into prime factors then we must see the appearance of the same factors on both sides of the equation. In particular we must see all the factors of $f(x)$ on the right hand side appear as often as they do on the left. Since we assume that $k(x)$ has degree $n - N$ at most it can contain $n - N$ prime factors of $f(x)$, which implies that $g(x)$ must contain N.
Reversely let $\phi_1(x), \ldots, \phi_N(x)$ be N common linear factors of $f(x)$ and $g(x)$. Then one may simply write

$$\begin{aligned} f(x) &= \phi_1(x)\phi_2(x)\ldots\phi_N(x)k(x) \\ g(x) &= \phi_1(x)\phi_2(x)\ldots\phi_N(x)h(x) \end{aligned}$$

and equation (1.2) holds.
To investigate equation (1.2) further we write

$$\begin{aligned} h(x) &= c_0x^{m-N} + c_1x^{m-N-1} + \ldots + c_{m-N}, \\ k(x) &= d_0x^{n-N} + d_1x^{n-N-1} + \ldots + d_{n-N}. \end{aligned} \tag{1.3}$$

Writing out equation (1.2), using (1.1) and (1.3) yields

$$\begin{aligned} &(c_0x^{m-N} + c_1x^{m-N-1} + \ldots + c_{m-N})(a_0x^n + a_1x^{n-1} + \ldots + a_n) \\ &\qquad = (d_0x^{n-N} + d_1x^{n-N-1} + \ldots + d_{n-N})(b_0x^m + b_1x^{m-1} + \ldots + b_m). \end{aligned}$$

Equating the coefficients in front of the powers $x^{m+n-N}, x^{m+n-N-1}, \ldots x, 1$ on both sides yields the following system of linear equations, for the coefficients c_i and d_j

$$\begin{array}{lllll}
c_0 a_0 & & & = d_0 b_0 & & & \\
c_0 a_1 & +c_1 a_0 & & = d_0 b_1 & +d_1 b_0 & & \\
c_0 a_2 & +c_1 a_1 & +c_2 a_0 & = d_0 b_2 & +d_1 b_1 & +d_2 b_0 & \\
& \ddots & \ddots & = & \ddots & \ddots & \ddots \\
& c_{m-N-1} a_n & +c_{m-N} a_{n-1} & = & & d_{n-N-1} b_m & +d_{n-N} b_{m-1} \\
& & c_{m-N} a_n & = & & & d_{n-N} b_m
\end{array}$$

which are $n+m-N+1$ equations for $n+m-2N+2$ variables. These may be rewritten into the following matrix equation

$$\begin{pmatrix}
a_0 & 0 & 0 & \ldots & 0 & b_0 & 0 & \ldots & 0 \\
a_1 & a_0 & 0 & \ldots & 0 & b_1 & b_0 & \ddots & 0 \\
a_2 & a_1 & a_0 & \ldots & 0 & b_2 & b_1 & \ddots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
a_n & a_{n-1} & \ldots & \ldots & a_0 & b_{m-2} & \ldots & \ldots & 0 \\
0 & a_n & \ldots & \ldots & a_1 & b_{m-1} & \ldots & \ldots & 0 \\
0 & 0 & a_n & \ldots & a_2 & b_m & b_{m-1} & \ldots & b_0 \\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \ldots & \ldots & 0 & a_n & 0 & \ldots & \ldots & b_m
\end{pmatrix}
\begin{pmatrix} c_0 \\ \vdots \\ \vdots \\ \vdots \\ c_{m-N} \\ -d_0 \\ \vdots \\ \vdots \\ -d_{m-N} \end{pmatrix}
=
\begin{pmatrix} 0 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{pmatrix} \tag{1.4}$$

The matrix in (1.4) will be denoted by $M_R(f,g)$, where we will often drop the (f,g) if there is no chance for confusion to arise. To summarize we know that there exists a vector $(c_i, -d_j)$ such that the above equation is satisfied if and only if polynomials $h(x)$ and $k(x)$ exist such that equation (1.2) holds. If $N = 1$ the matrix M_R is a $(n+m) \times (n+m)$ matrix and thus equation (1.4) may be satisfied for some vector $(c_i, -d_j)$ if and only if the determinant of the matrix M_R is zero. The determinant of M_R is called the resultant of the polynomials $f(x)$ and $g(x)$, and is denoted by

$$R = R(f,g) = \det(M_R).$$

Note that the resultant R of the two polynomials is a homogeneous polynomial of degree m in the variables a_i and of degree n in the variables b_j.

Before returning to (1.4) in a general setting, we will dwell on the resultant a bit longer. We start by rewriting $f(x)$ and $g(x)$ as follows

$$\begin{aligned}
f(x) &= a_0(x-x_1)(x-x_2)\ldots(x-x_n) \\
g(x) &= b_0(x-y_1)(x-y_1)\ldots(x-y_m)
\end{aligned} \tag{1.5}$$

The coefficients a_μ of $f(x)$, if written as (1.1), are the products of a_0 and elementary symmetric functions of the roots $x_1, \ldots, x_n$. Likewise are the coefficients b_ν of $g(x)$

products of b_0 and elementary symmetric functions of the roots $y_1, \ldots, y_m$. This may be easily verified by writing out the product in equation (1.5). Because R is a polynomial of degree m in a_μ and of degree n in b_ν, R equals $a_0^m b_0^n$ times a symmetric function in x_k and y_l. We now consider the roots $x_1, \ldots, x_n, y_1, \ldots, y_m$ to be our variables. The polynomial R is identically equal to zero if and only if $x_i = y_j$, for some i, j, which is the same as having a linear factor in common. This implies hat R can be divided by $(x_i - y_j)$. The independence of the various factors $(x_i - y_j)$ implies that R is divisible by

$$S = a_0^m b_0^n \prod_i \prod_j (x_i - y_j). \tag{1.6}$$

This equation may be rewritten as

$$S = a_0^m \prod_i g(x_i), \tag{1.7}$$

where we used (1.5) to find that

$$\prod_i g(x_i) = b_0^n \prod_i \prod_j (x_i - y_j).$$

Likewise we can rewrite (1.6) using $f(x)$ as

$$S = (-1)^{nm} b_0^n \prod_j f(y_j). \tag{1.8}$$

From (1.7) one sees that S is homogeneous of degree n in b and from (1.8) one sees that S homogeneous of degree m in a. Since moreover S divides R and is of the same degree

$$S = cR,$$

with c a constant. Comparing the terms proportional to $a_0^m b_m^n$ yields $c = 1$. We may now conclude that

$$S = R = \det(M_R) = a_0^m b_0^n \prod_i \prod_j (x_i - y_j). \tag{1.9}$$

We will now apply this result to investigate the relation between the resultant of two polynomials and the so-called discriminant. The discriminant of a polynomial

$$f(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_n$$

is defined as

$$D = a_0^{2n-2} \prod_{i<j} (x_i - x_j)^2. \tag{1.10}$$

It is clear that the discriminant vanishes if and only if $x_i = x_j$ for some $i, j \in \{1, \ldots, n\}$, that is if some zero of $f(x)$ is of quadratic or higher order. It is equally obvious that $f(x)$ has a zero of quadratic or higher order if $f(x)$ and $f'(x)$ have a zero in common, which is equivalent to the resultant of $f(x)$ and $f'(x)$ being zero. This points at a relation between the discriminant of a polynomial and the resultant of $f(x)$ and $f'(x)$. To determine the exact nature of this relation we calculate $R(f, f')$. According to equation (1.8) we have

$$R(f, f') = a_0^{n-1} \prod_i f'(x_i). \tag{1.11}$$

Moreover we may easily see

$$f'(x) = \sum_i a_0(x - x_1) \ldots (x - x_{i-1})(x - x_{i+1}) \ldots (x - x_n)$$
$$f'(x_i) = a_0(x_i - x_1) \ldots (x_i - x_{i-1})(x_i - x_{i+1}) \ldots (x_i - x_n).$$

Inserting these equations into one another we get

$$R(f, f') = a_0^{2n-1} \prod_{i \neq j} (x_i - x_j).$$

Comparing this equation to (1.10) we see that

$$R(f, f') = \pm a_0 D.$$

From this equality, interesting enough by itself, we may also derive that D is a polynomial in the coefficients a_μ.

We now revert to our original setting, where N was not set to the particular value of 1. We now wish to find necessary and sufficient conditions on the $(m + n - N + 1) \times (m + n - 2N + 2)$matrix M_R so that a non-trivial vector (c_i, d_j) exists. A general result from the theory of linear algebra says that such a vector exists if and only if the matrix is not of maximal rank. This condition will not be very useful in our approach as the rank of a matrix is not a continuous function of the indices of the matrix. It is therefore convenient to turn our attention to the following lemmas, pointed out to me by Hans Duistermaat.

Lemma 1.2.1 *Let k be a field, $n < m$ and let $A : k^n \to k^m$ be a linear mapping, then A is injective if and only if there exists a projection $\pi : k^m \to k^n$ which drops $m - n$ of the m coordinates of k^m, such that $\pi \circ A$ is injective and thus bijective.*

Proof We assume that A is injective and prove that there is a projection as described above such that $\pi \circ A$ is injective. Since A is injective, $A(k^n)$ is a n-dimensional subspace of k^m. We shall denote this subspace of k^m by B. Since $n < m$ some basis vector e_{i_1} exists such that $e_{i_1} \notin B$. We now consider the $n+1$-dimensional space $B + k\, e_{i_1}$ and find some basis vector e_{i_2} which does not lie in $B + ke_{i_1}$. This process terminates after $m-n$ steps. We shall denote the space spanned by these particular basis vectors as follows

$$C = k\, e_{i_1} + k\, e_{i_2} + \ldots + k\, e_{i_{m-n}}.$$

We also have that

$$\dim(B) = n \qquad\qquad \dim(C) = m - n.$$

So that we may conclude that

$$k^m = B \oplus C.$$

We may now define a linear projection $\pi : k^m \to k^n$ by

$$\pi|_C = 0.$$

Now assume that $\pi \circ A$ is not injective, that is

$$(\pi \circ A)(v) = 0,$$

for some $v \neq 0$ and we derive a contradiction. The construction of B and the non-injectivity of A imply that

$$A(v) \in \ker \pi = C \qquad\qquad A(v) \in B$$

which implies

$$A(v) \in B \cap C = \{0\}$$

and thus by injectivity of A we get that $v = 0$, which contradicts the assumption. This gives us that $\pi \circ A$ is injective. The converse implication is obvious. $\square$

This lemma is a small variation of the Rank lemma, which can be found on page 113 and 114 of Duistermaat and Kolk [1], see also pages 313 through 315 of the same book.

The mapping $\pi \circ A : k^n \to k^n$ which is mentioned in the lemma is also a linear mapping, to which a $(n \times n)$-matrix is associated, which we shall also denote by $\pi \circ A$. We of course have that the mapping $\pi \circ A$ is injective if and only if the determinant of the matrix $\pi \circ A$ is nonzero. Combining these results we get the following:

Lemma 1.2.2 *Let $A : k^n \to k^m$ be a linear mapping then the equation $A(v) = 0$ has a nontrivial v as a solution if and only if all $n \times n$-matrices produced by dropping $m - n$ columns of the matrix associated to A have a zero determinant.*

We will now apply this lemma to equation (1.4). And we see that this equation has a nontrivial solution if and only if all $(m+n-2N+2)\times(m+n-2N+2)$-matrices produced by dropping $N-1$ columns of the matrix M_R have a zero determinant.[2]

This gives rise to the following statement; two polynomials

$$f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n$$
$$g(x) = b_0x^m + b_1x^{m-1} + \ldots + b_m$$

have N or more linear factors in common if and only if all $(m+n-2N+2)\times(m+n-2N+2)$-matrices produced by dropping $N-1$ columns of the matrix

$$M_R = \begin{pmatrix} a_0 & 0 & 0 & \ldots & 0 & b_0 & 0 & \ldots & 0 \\ a_1 & a_0 & 0 & \ldots & 0 & b_1 & b_0 & \ddots & 0 \\ a_2 & a_1 & a_0 & \ldots & 0 & b_2 & b_1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ a_n & a_{n-1} & \ldots & \ldots & a_0 & b_{m-2} & \ldots & \ldots & 0 \\ 0 & a_n & \ldots & \ldots & a_1 & b_{m-1} & \ldots & \ldots & 0 \\ 0 & 0 & a_n & \ldots & a_2 & b_m & b_{m-1} & \ldots & b_0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & \ldots & 0 & a_n & 0 & \ldots & \ldots & b_m \end{pmatrix}$$

have a zero determinant.
We shall call the determinant of such a $(m+n-2N+2)\times(m+n-2N+2)$-matrix produced by dropping $N-1$ columns of the matrix M_R a semi-resultant of order N. These semiresultants will be denoted by $V_R(N)$.[3] This also inspires the definition of a semi-discriminants of order N, $V_D(N)$ as the semi-resultants of $f(x)$ and $f'(x)$. We emphasize that not all semi-discriminants are independent as polynomials in the coefficients of the polynomial $f(x)$.

The semi-discriminants give us some information about the form of $f(x)$ but fail to discriminate between a great number of separate cases. Let us illustrate this with the following example: Let $f(x)$ be a polynomial as before and assume that all semi-discriminants of order two, $V_D(2)$, are equal to zero, then $f(x)$ may be written as

$$f(x) = a_0(x-x_1)^3(x-x_2)(x-x_3)\ldots(x-x_{n-2})$$

or

$$f(x) = a_0(x-x_1)^2(x-x_2)^2(x-x_3)\ldots(x-x_{n-2}).$$

[2]The determinants of square submatrices of a matrix are referred to as minors.

[3]There is in general no nice expression for a semi-resultant in terms the roots of $f(x)$ and $g(x)$ like for the resultant (1.9).

We shall now investigate a way to distinguish between these two possibilities.

Let $f(x)$, $g(x)$ and $h(x)$ be three polynomials in x. Then $f(x) - yg(x)$ and $h(x)$ have at least one linear factor in common for all $y \in \mathbb{C}$ if and only if $f(x)$, $g(x)$ and $h(x)$ have a linear factor in common.
Proof For convenience we write

$$\begin{aligned} f(x) &= a_0(x - x_1) \dots (x - x_n) \\ g(x) &= b_0(x - y_1) \dots (x - y_m) \\ h(x) &= c_0(x - z_1) \dots (x - z_k) \end{aligned}$$

Suppose that $f(x) - yg(x)$ and $h(x)$ have a factor in common for all $y \in \mathbb{C}$, if we choose such a non-zero y, then

$$f(x) - yg(x) = (x - z_j)p_y(x),$$

where $p_y(x)$ is a polynomial in x, for some z_j. In section 1.1 it has been derived that the roots of a polynomial depend continuously on the coefficients of the polynomial. We will now consider $f(x) - yg(x)$ to be a family of polynomials in x. It is obvious that the coefficients of $f(x) - yg(x)$ depend continuously on y, so the roots of $f(x) - yg(x)$ must depend continuously on y. Furthermore it is given that for all y, $f(x) - yg(x)$ and $h(x)$ have a common linear factor. Finally we note that the roots of $h(x)$ form a discrete set. Continuity of the roots of $f(x) - yg(x)$ and the discreetness of the rootset $h(x)$ implies that

$$f(x) - yg(x) = (x - z_j)p_y(x),$$

for all y. We now take the particular case of $y = 0$ and see that

$$f(x) = (x - z_j)p_0(x).$$

Taking this expression for $f(x)$ and letting $y \neq 0$, we see that

$$yg(x) = (x - z_j)(p_0(x) - p_y(x)).$$

This implies that $f(x)$, $g(x)$ and $h(x)$ have at least a common linear factor. The converse is obvious. $\square$

This in turn leads to the statement $f(x)$, $g(x)$ and $h(x)$ have at least one common linear factor if and only if the resultant of $f(x) - yg(x)$ and $h(x)$ with respect to x, which is a polynomial in y, is identically equal to zero. It is now also clear that $f(x)$ is of the form

$$f(x) = a_0(x - x_1)^3(x - x_2) \dots (x - x_{n-2}),$$

if and only if the resultant of $f(x) - yf'(x)$ and $f''(x)$ is identically equal to zero. Here we note that if f is of order n then the resultant of $f(x) - yf'(x)$ and $f''(x)$ seen as a polynomial in y is of order $n-2$. This means that setting the resultant to zero yields $n-1$ equations[4] for the coefficients of f. This discussion implies that we can distinguish the two possibilities for $f(x)$ in the example above, where each $V_D(2) = 0$. The procedure generalizes trivially to cases where the number of polynomials or the order of the zero is higher then three. Combining this with previous results would also lead to a way to detect for example multiple third order zeros, namely by considering the semi-resultants of $f(x) - yf'(x)$ and $f''(x)$.

[4]These equations are in turn polynomial equations in the coefficients of f.

Chapter 2

Elliptic surfaces

This chapter introduces the concepts needed in chapter 3. The first section treats intersection numbers which will play a significant role in the classification of singular elliptical fibers. This section relies of [3] and [5]. Although we shall use Kodaira's classification of singular elliptical fibers, we hope to give sufficient handles to those using the coarser classification by means of intersection or Dynkin diagrams. The second section introduces elliptic curves and the Weierstrass normal form, which is a way to describe elliptic curves. The third section briefly discusses blow ups, a technique used in the next section. Section 2.2 gives the definition of an elliptic surface and treats singular fibers, in particular the classification of singular fibers by Kodaira. The aim of the fifth section is to define the so-called monodromy of a singular fiber, which is characteristic for the type of singular fiber. Monodromy will play an important role in section 2.7 and chapter 3. Section 2.6 deals with the Weierstrass model, a manner to describe an elliptic surface. The work presented in chapter 3 will completely rely on this desciption. All these sections are taken from [3], with the exception of section 2.3 which also uses [5]. These first sections are meant as a short introduction and do not bestow the topics the attention they deserve, nor do they include all proves. The final section discusses confluences, the main topic of this thesis and bridges the gap between the second and third chapter.

In this chapter and the thesis in general we will follow the notation and terminology of [3], except for the discriminant Δ, which we shall call the geometric discriminant. We do this to avoid confusion when we discuss the discriminant of the geometric discriminant divided by some factor, as we shall do in chapter 3.

2.1 Intersection numbers

In this section we reproduce the discussion of intersection numbers of section 0.4 in Griffiths and Harris [5] and section 3.2 in Duistermaat [3].

Consider the torus T as been depicted in figure 2.1 on the left . And let A and B be two cycles[1] on T. We want to be able to say that A and B intersect one another once. Furthermore we would also like a definition to be invariant under deformation. We extend the concept of invariance under deformation of two cycles to homology invariance. We note here that homology invariance is a weaker condition then homotopy invariance and therefore deformations in the intuitive sense are included. A cycle A' shifted slightly to one side clearly should have the same intersection number. There is however a problem, as has been depicted in figure 2.1, after some deformation we may find that we have extra intersection points. We therefore need these extra intersection points to cancel out. This may be done as follows: first choose an orientation on T Then if two cycles A and B intersect transversely in a point p, we define the intersection index $A \cdot_p B$ of A and B at p to be $+1$ if the tangent vectors to A and B in turn form an oriented basis for $T_p(M)$, and -1 if not. We define the intersection number $A \cdot B$ of cycles A and B meeting transversely in smooth points to be the sum

$$A \cdot B = \sum_{p \in A \cap B} A \cdot_p B.$$

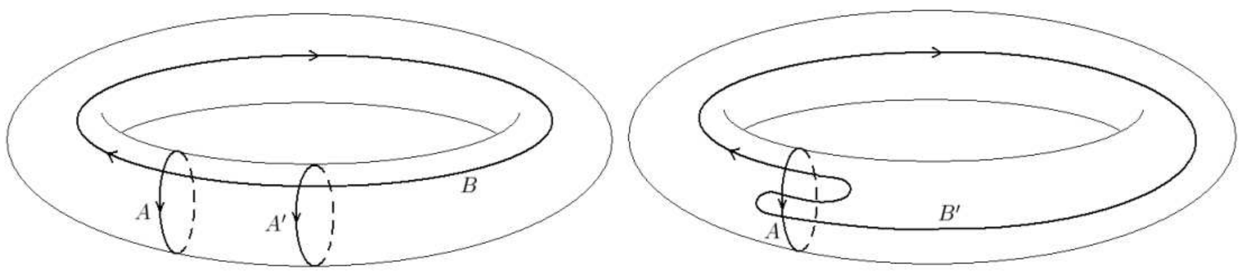


Figure 2.1: Several intersections of cycles, whose intersection number we define to be the same. Free interpretation of figures 1 and 2 of [5].

It is easy to see that this definition is homologically invariant. Let ∂C denote the boundary of a region C, and let the orientation of the boundary be outward. Take A to be some cycle then A intersects the boundary ∂C an even number of times, half of them going inward and half going outward, corresponding to intersection number -1 and 1 respectively. This implies that the intersection number of A and ∂C will be zero, since the positive and negative intersection numbers cancel out.

We shall now extend the discussion to arbitrary real dimension and to chains, formal sums of oriented simplices. Let M be a oriented smooth real d dimensional manifold and

[1] By cycle we simply mean that they have no boundary.

let A and B be oriented chains of dimension a and b respectively, such that A and B have complementary dimension. Furthermore let $|A| \cap |B|$be compact and $|\partial A| \cap |B| \neq 0$, where $|C|$ denotes the support of any chain C. ??Vragen Duistermaat niet zo duidelijk in boek blz. 34?? We shall now assume that A and B are compactly supported cycles, so that clearly these conditions hold. For each connected component I of $|A| \cap |B|$ and each open U with the following properties $I \subset U$ and $U \cap (|A| \cap |B|) \backslash I = \emptyset$, there exist oriented cycles A' and B', such that

- $A' - A = \partial\alpha$, $B' - B = \partial\beta$, where α and β are chains of respective dimension $a+1$ and $b+1$, with support in U
- $A' \cap U$ and $B' \cap U$ intersect only in their smooth parts
- every intersection of $A' \cap U$ and $B' \cap U$ is transversal, that is $T_{i'}A' \cap T_{i'}B' = 0$ for every $i' \in A' \cap B' \cap U$.

The intersection number $A' \cdot_{i'} B'$ at $i' \in A' \cap B' \cap U$ is equal to $+1$ (-1) if the basis $e_1, \ldots, e_a, f_1, \ldots f_b$ of $T_{i'}M$ is positively (negatively) oriented, where $e_1, \ldots, e_a$ and $f_1, \ldots f_b$ are positively oriented bases of $T_{i'}A$ and $T_{i'}B$ respectively, in analogy of the one dimensional case. We now define the intersection number $A \cdot_I B \in \mathbb{Z}$ of A and B along I

$$A \cdot_I B = \sum_{i' \in A' \cap B' \cap U} A' \cdot_{i'} B'.$$

We note that due to homology invariance, as discussed for the two dimensional case, the definition is independent of the exact choice of α and β, with support in U. The number

$$A \cdot B = \sum_I A \cdot_I B \in \mathbb{Z},$$

where the sum indicates the sum over each connected component I of $|A| \cap |B|$, is the intersection number of A and B.

We need to generalize this definition to a complex setting. To do so we first discuss orientations on complex spaces. Let E and F be two complex vector spaces of complex dimension n. If $A : E \to F$ and $B : E \to F$ are linear isomorphisms, then $A = B \circ C$ for a linear automorphism $C : E \to E$. Let us now denote by $E_{\mathbb{R}}$ the vector space E viewed as a vector space over $\mathbb{R}$, then the determinant of the real linear transformation $C_{\mathbb{R}} : E_{\mathbb{R}} \to E_{\mathbb{R}}$ is equal to $\det C_{\mathbb{R}} = |\det C|^2 > 0$. This implies that if we choose an orientation on one complex n-dimensional vector space, seen as a real $2n$-dimensional vector space, there is an unique orientation on all vector spaces of the same dimension such that all linear complex isomorphisms preserve orientation. This yields also an unique orientation on each complex n dimensional manifold, such that transition maps from one chart to the next are orientation preserving. The usual identification $\mathbb{R}^2 \xrightarrow{\sim} \mathbb{C} : (x, y) \mapsto x + i\,y$, defines a real linear isomorphism $\mathbb{R}^{2n} \simeq (\mathbb{R}^2)^n \xrightarrow{\sim} \mathbb{C}^n$. In this way we see that every finite dimensional complex vector space has a canonical orientation.

Let M be a complex n-dimensional manifold. M is now by the above an oriented real $2n$-dimensional manifold. If A and B are complex analytic subsets of M, the common zero sets of some holomorphic functions, of respective complex dimensions k and l, such that $k+l=n$, then A and B are oriented cycles[2] in M of complementary real dimension $2k$ and $2l$. This identification yields a well defined homologically invariant definition of intersection of two complex analytic subsets of complementary dimension.

If M is a complex analytic surface, a complex analytic manifold of complex dimension 2, then any two divisors, some formal linear combination of analytic subsets of complex dimension 1 (for a extensive general definition see section 2.4) are cycles of complementary real dimension 2 in M. If $|A| \cap |B|$ is compact this leaves us with a well-defined intersection number $A \cdot B$. For a compact curve we have a well-defined self-intersection number $A \cdot A$.[3] We stress that the intersection number of two divisors is not necessarily positive, but a negative intersection number is very restrictive as seen in the following result, lemma 3.1.8 of [3]:

Lemma 2.1.1 *Let S be a compact complex analytic surface, A an irreducible compact complex analytic curve in S and B an effective divisor in S. If $A \cdot B < 0$, then A is an irreducible component of B, with $A \cdot A < 0$. If $A \cdot A < 0$, and B is an irreducible complex analytic curve in M, which is homologous to A, then $B = A$.*

[2]The regular parts of the complex analytic subsets A and B are complex manifolds (without boundary), so the regular parts are clearly oriented cycles. The singular parts may be incorporated using the so-called triangulation theorem of Łojasiewicz, see section 3.16 of [3].

[3]Note that it is essential that A has its own codimension.

2.2 Elliptic curves and the Weierstrass normal form

In this section we copy the definitions given section 3.3 of [3] most important to our later discussion.

Let C be a compact Riemann surface of genus 1, such a Riemann surface will be called an elliptic curve. Furthermore let ω be a holomorphic complex one-form on C not identically equal to zero. The Riemann-Roch formula[4] implies that ω has no zeros because ω has no poles. Since ω is nowhere zero there exists a unique tangent vector field on C, which is holomorphic and everywhere nonzero, such that $v \cdot \omega = 1$.[5] Moreover let u be any holomorphic tangent vector field on C, then, because C is assumed compact, the maximum modulus principle holds and $u \cdot \omega = c$ is a constant and hence $u = c\,v$. Conversely if there exists a holomorphic vector field v without zeros, then the existence of ω follows.

Let us now start with such a v and denote for any $t \in \mathbb{C}$ the flow of the vector field v, the flow after a so-called complex time t, by e^{vt}. Because C is compact this flow is globally defined. For every $t \in \mathbb{C}$ we get a complex analytic diffeomorphism of C onto itself. The mapping $(t, c) \mapsto e^{vt}(c) : \mathbb{C} \times C \to C$ is a complex analytic action of the additive group $\mathbb{C}$ on C. Since we have assumed that v has no zeros, all orbits of this action are open, together with connectedness of C this implies that there is but one orbit. This may be put as follows for any initial point $c \in C$, the mapping $t \mapsto e^{tv}(c) : \mathbb{C} \to C$ is surjective and locally a complex analytic diffeomorphism, this is referred to as transitivity. We now easily see that if for a fixed $t \in \mathbb{C}$ and some $c \in C$ we have $e^{tv}(c) = c$, then $e^{tv}(c') = c'$ for every $c' \in C$. This is easy to see since we may write $c' = e^{\tau v}c$ so that $e^{\tau v}e^{tv}(c) = e^{\tau v}(c) = c' = e^{tv}e^{\tau v}(c) = e^{tv}c'$. We now define the period group P to be the set of all $t \in \mathbb{C}$ such that $e^{tv}(c) = c$. It is clear that this is a subgroup of the additive group $\mathbb{C}$, because the zero is clearly in P and if $e^{tv}(c) = c$ as well as $e^{\tau v}(c) = c$, then surely $e^{(t+\tau)v}(c) = c$, furthermore if $e^{tv}(c) = c$, then $e^{-tv}e^{tv}(c) = e^{-tv}(c) = 1\,c = c$. Note that P is independent of the initial point $c \in C$. For any choice of the initial point, the mapping $t \mapsto e^{tv}(c)$ induces a bijective mapping $\Phi : \mathbb{C}/P \to C$, which is locally a complex analytic diffeomorphism, and therefore a complex analytic diffeomorphism from $\mathbb{C}/P$ onto C. The additive group $\mathbb{C}/P$ is identified with the group of all translations on C, the automorphisms of C which preserve v, or equivalently the automorphisms which preserve every holomorphic complex one-form on C.

The period group P is a discrete subgroup of $\mathbb{C}$. Because of the diffeomorphism between C and $\mathbb{C}/P$, we have that $\mathbb{C}/P$ must also be compact. From this we may conclude that P is a so-called full lattice in $\mathbb{C}$, that is P has a $\mathbb{Z}$-basis p_1, p_2, which is at the same time an $\mathbb{R}$-basis of $\mathbb{C}$. The mapping $(x_1, x_2) \mapsto \Phi(x_1p_1 + x_2p_2) : \mathbb{R}^2 \to C$ induces a real analytic

[4] See for example page 76 of [4].

[5] We use this notation to indicate the contraction of v with ω.

diffeomorphism from the standard real two-dimensional torus $(\mathbb{R}/\mathbb{Z})^2$ onto C. We note that this real diffeomorphism forgets the complex structure the elliptic curve C has.

Let $\Phi : C \to C'$ be an complex analytic diffeomorphism and let C be an elliptic curve. It follows that C' is an elliptic curve too. Let v again be a nowhere zero tangent vector field on C and u a nowhere zero tangent vector field on C' and denote by $Q \subset \mathbb{C}$ the period lattice defined by u. Then the pushforward $\Phi_* v$ of v is a nowhere zero vector field on C'. We deduce that there is a nonzero constant $\lambda \in \mathbb{C}$ such that $\Phi_* v = \lambda u$, which implies that $Q = \lambda P$. It follows that two elliptic curves are isomorphic if and only if they have the same period lattices up to multiplication by a nonzero complex number.

We have seen that every elliptic curve C is complex analytically diffeomorphic to $\mathbb{C}/P$ for some lattice P, and thus every elliptic curve may be characterized by such a lattice P. We will now establish that every elliptic curve $\mathbb{C}/P$ is isomorphic to a cubic curve in the projective plane $\mathbb{P}^2$, namely the solution curve curve of polynomial called the Weierstrass normal form set to zero. This leads to another classification of (isomorphism classes of) elliptic curves. We shall now discuss the classical proof.

Let $\mathbb{C}/P$ be an elliptic curve and define the Weierstrass $\wp$-function on $\mathbb{C}$ as follows

$$\wp(t) \equiv t^{-2} + \sum_{\substack{p\in P\\ p\neq 0}} ((t-p)^{-2} - p^{-2}). \tag{2.1}$$

We readily see that its derivative is given by

$$\wp'(t) = -2 \sum_{p\in P} (t-p)^{-3}. \tag{2.2}$$

These series converge locally uniformly in the complement of P in $\mathbb{C}$, and therefore define holomorphic functions on the same complement of P. Moreover $\wp(t)$ is invariant under translations over elements of P. This means that $\wp$ can be viewed as as a meromorphic function on $\mathbb{C}/P$ with a pole at $0 + P \in \mathbb{C}/P$. We now define

$$\varphi(t) = \wp'(t)^2 - 4\wp(t)^3 + g_2\wp(t) + g_3$$

and consider its Laurent expansion at $t = 0$. To do so we write $t = 0 + \delta t$ and note the following

$$\wp(t) = \delta t^{-2} \sum_{\substack{p\in P\\ p\neq 0}} ((\delta t - p)^{-2} - p^{-2}) = \delta t^{-2} + 3\sum_{\substack{p\in P\\ p\neq 0}} p^{-4}\delta t^2 + 5\sum_{\substack{p\in P\\ p\neq 0}} p^{-6}\delta t^4 + \mathcal{O}(\delta t^6) \tag{2.3}$$

$$\wp(t)^3 = \delta t^{-6} + 9\sum_{\substack{p\in P\\ p\neq 0}} p^{-4}\delta t^{-3} + 15\sum_{\substack{p\in P\\ p\neq 0}} p^{-6} + \mathcal{O}(\delta t^2)$$

$$\wp'(t)^2 = 4(\delta t^{-3} + \sum_{\substack{p\in P\\ p\neq 0}} (\delta t - p)^{-3})^2 = 4\delta t^{-6} - 24\sum_{\substack{p\in P\\ p\neq 0}} p^{-4}\delta t^{-2} - 80\sum_{\substack{p\in P\\ p\neq 0}} p^{-6} + \mathcal{O}(\delta t^2),$$

where we have used that for an odd n

$$\sum_{\substack{p \in P \\ p \neq 0}} p^{-n} = 0.$$

It is now clear that the Laurent expansion of $\varphi(t)$ at $t = 0$ reads

$$\begin{aligned}\varphi(t) &= \wp'(t)^2 - 4\wp(t)^3 + g_2\wp(t) + g_3 \\ &= -60 \sum_{\substack{p \in P \\ p \neq 0}} p^{-4}\delta t^{-2} + g_2 \delta t^{-2} - 140 \sum_{\substack{p \in P \\ p \neq 0}} p^{-6} + g_3 + \mathcal{O}(\delta t^2),\end{aligned}$$

so that choosing

$$g_2 = g_2(P) = 60 \sum_{\substack{p \in P \\ p \neq 0}} p^{-4} \qquad\qquad g_3 = g_3(P) = 140 \sum_{\substack{p \in P \\ p \neq 0}} p^{-6} \tag{2.4}$$

yields $\varphi(0) = 0$. In particular we note that for this choice the function $\varphi(t)$ has no poles and therefore defines a global holomorphic function on $\mathbb{C}/P$. The maximum modulus principle now tells us that $\varphi(t)$ must be a constant, since $\varphi(0) = 0$ we have that $\varphi(t) \equiv 0$.

Therefore the mapping $t \mapsto [1 : x : y] = [1 : \wp(t) : \wp'(t)]$ induces a holomorphic mapping π from $\mathbb{C}/P$ to the curve D in $\mathbb{P}^2$ defined by the equation

$$y^2 = 4x^3 - g_2 x - g_3 \tag{2.5}$$

in affine coordinates. Here the problematic point $t = 0 + P$ is mapped to the point $[0 : 0 : 1]$, the "north pole" on the Riemann sphere, this is readily seen as $\wp(t)$ has a pole in the origin. If we use homogeneous coordinates the equation for the curve D reads

$$x_0 {x_2}^2 - 4{x_1}^3 + g_2 {x_0}^2 x_1 + g_3 {x_0}^3 = 0. \tag{2.6}$$

The polynomial in the above equation is referred to as the Weierstrass normal form of the elliptic curve $\mathbb{C}/P$. We will prove that the curve D is smooth and that π is a complex analytic diffeomorphism from $\mathbb{C}/P$ onto D.

If we now define the Hamiltonian on $\mathbb{P}^2$ by[6]

$$q(x, y) = \frac{1}{2}(y^2 - 4x^3 + g_2 x + g_3)$$

then $\pi(t) = [1 : x(t) : y(t)]$ is a solution on D, the solution set of $q = 0$, of the Hamiltonian system

$$\frac{dx}{dt} = \frac{\partial q(x, y)}{\partial y} = y, \qquad \frac{dy}{dt} = -\frac{\partial q(x, y)}{\partial x} = 6x^2 - g_2/2. \tag{2.7}$$

[6]We use the notation q as in [3] for the Hamiltonian to avoid confusion with the Hamiltonian vector field.

The first equation is equivalent to $\wp(t) = x$ and $\wp'(t) = y$ the second equation is established by differentiating (2.5). The Hamiltonian system is that of a particle of unit mass on the x-axis, with a potential equal to the cubic polynomial

$$\frac{1}{2}(-4x^3 + g_2x + g_3),$$

the canonical momentum is associated to coordinate y. We now first focus our attention on the behaviour of the solution near of the solution near $t = 0(+P)$, when $\pi(0(+P)) = [0 : 0 : 1]$, we write $[1 : x : y] = [\eta : \xi : 1]$ with $\eta = 1/y$, $\xi = x/y$. In these new affine coordinates, for a neighbourhood of infinity, the differential equations (2.7) take the form

$$\frac{d\eta}{dt} = -6\xi^2 + g_2\eta^2/s, \qquad -\frac{d\xi}{dt} = \frac{1}{2} + g_2\xi\eta + 3g_3\eta^2/2, \tag{2.8}$$

where we used (2.5) in the second differential equation. This is the Hamiltonian system on $r = 0$ for the function

$$r(\eta, \xi) = \frac{1}{2}(\eta - 4\xi^3 + g_2\eta^2\xi + g_3\eta^3).$$

We now recognize the equation $r = 0$ as (2.6), where we take $x_0 = \eta$, $x_1 = \xi$ and $x_2 = 1$. Note that $d\xi/dt \neq 0$, when $\eta = \xi = 0$. Due the the implicit function theorem we know that the point (x, y) is a singular point of the curve $\{q = 0\}$ if and only if $q(x, y) = 0$, $\partial q(x, y)/\partial x = 0$ and $\partial q(x, y)/\partial y = 0$, the latter statement is again equivalent to (x, y) is a point on $\{q = 0\}$, where the Hamiltonian vector field H_q vanishes. Because the H_q is holomorphic, we have uniqueness for the initial value problem on the Hamiltonian system. This implies that a zero (x, y) of the Hamiltonian vector field H_q is the only possible solution going through (x, y) is the constant solution $t \mapsto (x, y)$. The solution associated to the Weierstrass $\wp$-function, $t \mapsto (\wp(t), \wp'(t))$ is not constant, and therefore at least one of the equations $q(x, y) = 0$, $\partial q(x, y)/\partial x = 0$ and $\partial q(x, y)/\partial y = 0$ does not hold and we do not cross a singular point with our solution $t \mapsto (\wp(t), \wp'(t))$. We conclude that the image of π is contained in the set of smooth points of D. We now note that π is continuous and $\mathbb{C}/P$ is compact and thus $\pi(\mathbb{C}/P)$ is compact. Since $\pi(t)$ does not cross a singular point we have that

$$\frac{d\wp}{dt} \neq 0 \qquad \text{or } \frac{d^2\wp}{dt^2} \neq 0,$$

for every $t \in \mathbb{C}/P$, which is equivalent to $\pi'(t) \neq 0$, this implies that $\pi(\mathbb{C}/P)$ is open. Combining this with the fact the $\pi(\mathbb{C}/P)$ is compact yields that $\pi(\mathbb{C}/P)$ is open and closed in D and therefore $\pi(\mathbb{C}/P) = D$.

We now define the geometric discriminant of the elliptic curve to be

$$\Delta := {g_2}^3 - 27{g_3}^2,$$

which is one sixteenth of the discriminant as defined in (1.10) of the polynomial $f : x \mapsto 4x^3 - g_2x - g_3$. We have seen that $\Delta = 0$ if and only if the polynomial f has multiple

zeros. We have prove that $\pi(\mathbb{C}/P)$ is smooth and $\pi(\mathbb{C}/P) = D$, so D is smooth, which in turn implies that $\Delta \neq 0$.
The mapping $\pi : \mathbb{C}/P \to D$ is a holomorphic covering map without branch points, since $\pi'(t) \neq 0$ and $\pi(\mathbb{C}/P) = D$. Because we have also shown that the point in infinity is well behaved, we have that π is a complex analytic diffeomorphism from $\mathbb{C}/P$ to D.

We note that the coordinate t on $\mathbb{C}/P$ satisfies $dt = \pi^*(y^{-1}dx)$, and therefore the fact that π is an diffeomorphism, implies that the restriction to D of the complex one-form $y^{-1}dx$ is a holomorphic complex one-form on D without zeros.

Now assume conversely that g_2 and g_3 are complex numbers, such that $\Delta = {g_2}^3 - 27{g_3}^2 \neq 0$. This implies that setting the Weierstrass normal form equal to zero (2.6) defines a smooth curve D in $\mathbb{P}^2$. Let[7]

$$\tilde{\pi}(t) = [1 : x(t) : y(t)] = [\eta(t) : \xi(t) : 1], \qquad t \in \mathbb{C}$$

be the solution, with affine coordinates $x(t)$ and $y(t)$ on $\mathbb{P}^2\backslash\{\infty\}$ and $\eta(t)$ and $\xi(t)$ on $\mathbb{P}^2\backslash\{0\}$, of the Hamilton system defined by (2.7) and (2.8) with the initial condition $\tilde{\pi}(0) = [0 : 0 : 1]$. With this initial condition it is obvious that $\tilde{\pi}(0) \in D$. Because q and r are exactly the Weierstrass normal form in affine coordinates, the Hamiltonian vector field is tangent to D and therefore $\tilde{\pi}(t) \in D$ for every $t \in \mathbb{C}$. Due to the fact that D is smooth, $\tilde{\pi}'(t) \neq 0$ for every $t \in \mathbb{C}$, and $\tilde{\pi} : \mathbb{C} \to D$ is a holomorphic covering map. We now denote by P, the so-called period group, the set of all $t \in \mathbb{C}$ such that $\tilde{\pi}(t) = \tilde{\pi}(0)$. The mapping $\tilde{\pi}$ induces a complex analytic diffeomorphism from $\mathbb{C}/P$ onto D. Because D is compact, $\mathbb{C}/P$ is compact as well, and we see again that P must be a full lattice in $\mathbb{C}$, in the way we have discussed above. From equation (2.8) and the initial condition which reads $\eta(0) = \xi(0) = 0$ it follows that $\eta(t) = -t^3/2 + \mathcal{O}(t^6)$ and $\xi(t) = -t/2 + \mathcal{O}(t^5)$, hence $x(t) = \xi(t)/\eta(t) = t^{-2} + \mathcal{O}(t)$, as $t \to 0$. On the other hand we have seen in (2.3) that the Weierstrass $\wp$-function for the lattice P satisfies $\wp(t) = t^{-2} + \mathcal{O}(t)$ for $t \to 0$. This implies that the function $x(t) - \wp(t)$ extends to a holomorphic function on d on $\mathbb{C}/P$. If we now invoke the maximum moduli principle, we derive that d is constant. Since we have defined d such that $d(0) = 0$ we find that d=0 and thus $x(t) = \wp(t)$. By again using Hamilton's equations we see that $y(t) = x'(t) = \wp(t)$. Because $[1 : \wp(t) : \wp'(t)]$ runs through the curve defined by

$$x_0{x_2}^2 - {x_1}^3 + g_2(P){x_0}^2x_1 + g_3(P)x_0 = 0$$

as well as though the curve we started out with

$$x_0{x_2}^2 - {x_1}^3 + g_2{x_0}^2x_1 + g_3{x_0}^3 = 0,$$

[7]We denote this mapping by $\tilde{\pi}$ instead of p which is the notation used in [3] to avoid confusion with the points on the lattice P.

we may deduce that $g_2(P) = g_2$ and $g_3(P) = g_3$. We conclude that every smooth curve defined by

$$x_0{x_2}^2 - {x_1}^3 + g_2{x_0}^2 x_1 + g_3{x_0}^3 = 0,$$

is complex analytically diffeomorphic to a $\mathbb{C}/P$ for a suitable lattice P in $\mathbb{C}$ and the polynomial set to zero may therefore be seen as the Weierstrass normal form of this $\mathbb{C}/P$.

Having established the converse we now focus on the following. From (2.4) we deduce that $g_2' = g_2(\lambda P) = \lambda^{-4} g_2(P)$ and $g_3' = g_3(\lambda P) = \lambda^{-6} g_3(P)$. This implies that a D' defined by

$$x_0{x_2}^2 - {x_1}^3 + g_2'{x_0}^2 x_1 + g_3'{x_0}^3 = 0,$$

is isomorphic to the curve D, in the sense that there is a complex analytic diffeomorphism from D to D'. Since we have proven that two elliptic curves are isomorphic if and only if the period lattices are the same up to multiplication with a nonzero factor λ, we may conclude that D and D' are diffeomorphic if and only if ${g_2}^3/{g_3}^2 = {g_2'}^3/{g_3'}^2$. We now use this to define the modulus J of an elliptic curve $C \simeq \mathbb{C}/P \simeq D$, the complex number which parameters the isomorophism classes of the elliptic curves, to be

$$J(C) := {g_2}^3/\Delta = {g_2}^3/({g_2}^3 - 27{g_3}^2). \tag{2.9}$$

If $\Delta = {g_2}^3 - 27{g_3}^2 = 0$ and both $g_2 \neq 0$ and $g_3 \neq 0$, the solution curve of

$$x_0{x_2}^2 - 4{x_1}^3 + g_2{x_0}^2 x_1 + g_3{x_0}^3 = 0$$

has a singular point and is isomorphic to a Kodaira fiber of type I_1, see section 2.4. If on the other hand the discriminant equals zero and both $g_2 = 0$ and $g_3 = 0$, the solution curve has a cusp and is isomorphic to a Kodaira fiber of type II. The singular fibers of type I_1 and II are the only irreducible singular fibers in Kodaira's list. Other singular fibers of the list are found by blowing up the Weierstrass model of an arbitrary elliptic surface in a singular point in a fiber, see section 2.6.

2.3 Blow up

In this section we give definitions and some results regarding the blow up of points, once more we will not include proofs but refer the reader to section 4.2 of [3]. We have taken the customary examples from notes by Hans Duistermaat and [5]. In this section we shall denote the disk by D instead of Δ as is done in [5], since the letter Δ is already used frequently.

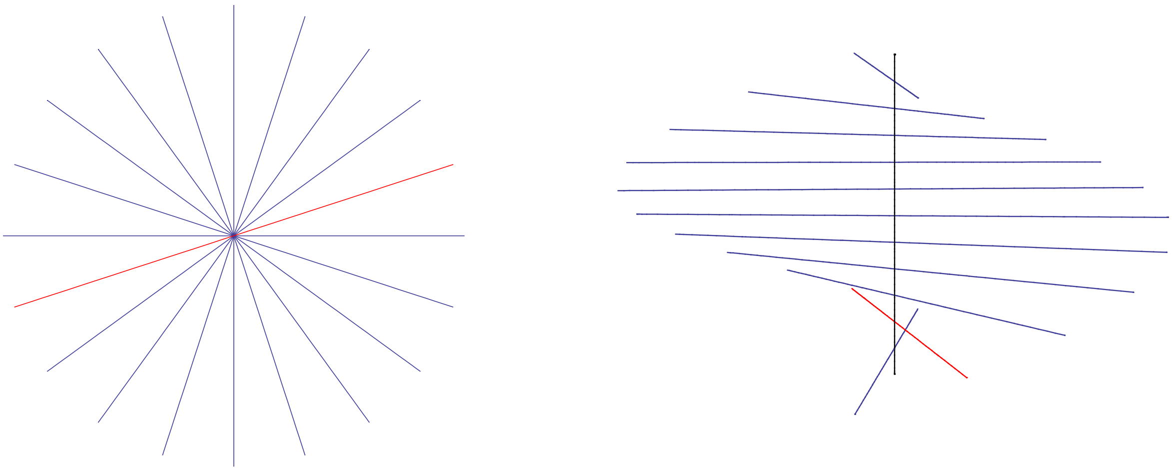


Figure 2.2: In the plane, before blowing up we see all lines through the origin, after blowing up they pass through the projective plane which has replaced the origin. The projective plane found after blowing up is drawn black. Note that the projective plane is periodic. Adaptation of figure 1 of chapter 1 of [5].

Let us first give the customary example of a blow up. Let D be the disc in $\mathbb{C}^n$ with Euclidean coordinates $z = (z_1, \ldots, z_n)$. We shall blow D up in the origin. To do so we need $\mathbb{P}^{n-1}$, on which we shall use homogeneous coordinates $l = [l_1, \ldots, l_n]$. We now define the submanifold $\tilde{D} \subset D \times \mathbb{P}^{n-1}$ given by

$$\tilde{D} = \{(z, l) : z_i l_j = z_j l_i, \text{ for all } i, j,\}.$$

If we view points l in $\mathbb{P}^{n-1}$ as lines in $\mathbb{C}^n$ then this simply mean that $z \in l$. For points outside the origin in D we have the following isomorphism

$$\pi : \tilde{D} \backslash (0 \times \mathbb{P}^{n-1}) \to D \backslash \{0\} : (z, l) \mapsto z.$$

This means that outside the origin $\tilde{D}$ and D are identical but in the origin $\tilde{D}$ "remembers" directions of lines. This method is only local and therefore extends trivially to manifolds. It can be used to desingularize the solution curve by separating several branches of the solution curve all going through one point. The inverse operation of the blow up is called a blow down.

We will illustrate how we can desingularize a curve by the following example: Consider the solution curve $f = 0$ of the following function[8]

$$f : \mathbb{C}^2 \to \mathbb{C} : (x, y) \mapsto xy.$$

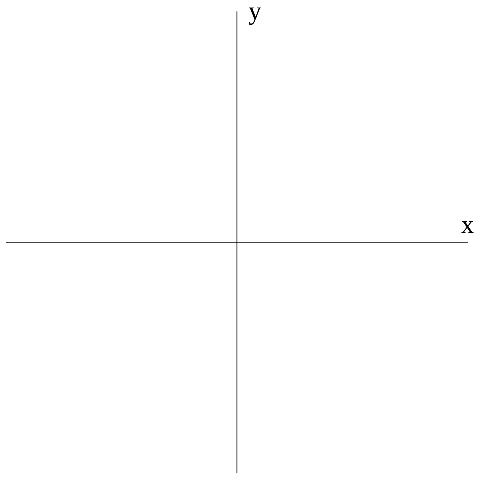


Figure 2.3: The solution curve of the equation $xy = 0$.

It is clear that the pre-image of zero under f has a singular point, namely the origin. We will blow up the plane in the origin to remove the singularity. We write $[l_1 : l_2]$ for the homogeneous coordinates on $\mathbb{P}^2$, and define

$$\tilde{D} = \{(x, y; l_1, l_2) : x\, l_2 = y\, l_1\}.$$

In affine coordinates $[1 : l_2]$ on $\mathbb{P}^2$, which describe $\mathbb{P}^2$ near the origin, we read

$$\tilde{D} = \{(x, y, l_2) : x\, l_2 = y\}.$$

If we now take $\xi = x$ as our second local coordinate on $\tilde{D}$ we find that

$$\begin{aligned} y &= l_2 \xi \\ x &= \xi, \end{aligned}$$

so that $f = 0$ reads

$$f(\xi, l_2) = l_2 \xi^2 = 0.$$

If conversely we choose $[l_1 : 1]$ as affine coordinates on $\mathbb{P}^2$, near infinity, we read

$$\tilde{D} = \{(x, y, l_1) : x = y\, l_1\}.$$

[8] We ignore infinity for the moment.

If we now take $y = \eta$ as our second coordinate we see that $f = 0$ now reads

$$f(\eta, l_1) = l_1\eta^2 = 0.$$

We now see that after blowing up one branch of the solution curve of $xy = 0$ intersects $\mathbb{P}^2$ in the origin and the other branch in infinity, and both branches of the solution no longer intersect one another. After blowing up we see something as depicted in figure 2.4.

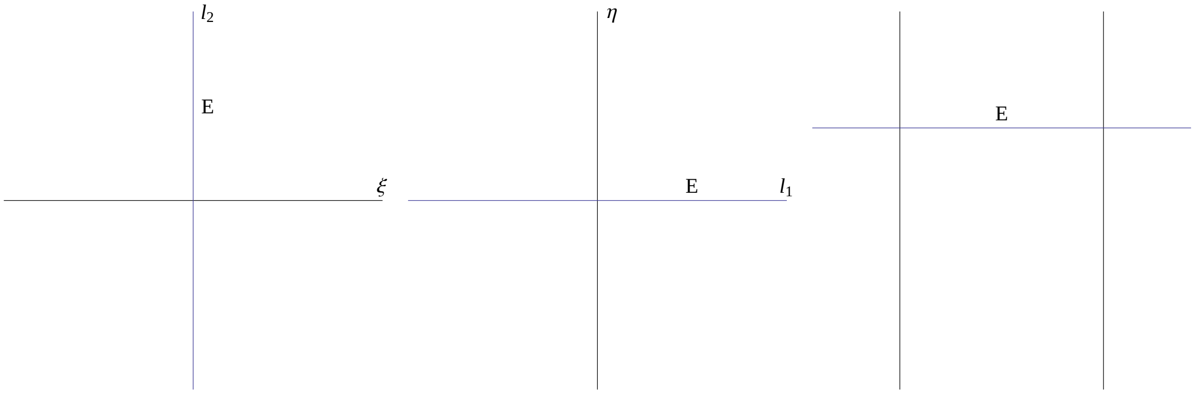


Figure 2.4: In the first two figures we have depicted $\tilde{D}$, in a manner corresponding to the local coordinates (ξ, l_2) and (η, l_1). In the third figure we have sketched what $\tilde{D}$ globally looks like. We have indicated the exceptional curve by E in all three figures.

Any point can be blown up by this procedure, however blowing up a regular point does not improve the structure of the curve or surface. We therefore wish to identify those curves in surfaces which did arise from the blowing up. To make such a distinction, the following results are useful:

Theorem 2.3.1 *Let S be a connected complex two-dimensional holomorphic manifold, and let b be a point of S. Then there exists $\tilde{S}$ and π with the following properties.*

i) $\tilde{S}$ is a complex two-dimensional complex analytic manifold, and π is a proper complex analytic mapping from $\tilde{S}$ to S.

ii) If we write $E = \pi^{-1}(\{b\})$, then the restriction to $\tilde{S}\backslash E$ of π is a complex analytic mapping from $\tilde{S}$ onto $S\backslash\{b\}$.

iii) For every $e \in E$ the tangent mapping $T_e\pi$ of π at e has rank one.

If i), ii) and iii) hold, then $\tilde{S}$ is connnected and E is an emmbedded complex projective line in $\tilde{S}$ with self-intersection number

$$E \cdot E = -1.$$

If $\tilde{S}'$ and π' satisfy i), ii) and iii) with $\tilde{S}$ and π replaced by $\tilde{S}'$ and π', respectively, then there exists a unique continuous map $\Phi : \tilde{S} \to \tilde{S}'$ such that $\pi' \circ \Phi = \pi$ and Φ us a complex analytic diffeomorphism from $\tilde{S}$ onto $\tilde{S}'$.

This inspires the following definition. Let C be any complex analytic curve in a complex analytic surface S. The curve C is called a rational curve if it is smooth and complex diffeomorhoic to $\mathbb{P}^1$. The curve C is called an exceptional curve of the first kind or a -1 curve, if it is rational and satisfies $C \cdot C = -1$. If $\pi : \tilde{S} \to S$ is a blowing up of S at the point $b \in S$, then $E = \pi^{-1}(\{b\})$ is a -1 curve in $\tilde{S}$.

The converse of the theorem 2.3.1 is the Castelnuovo-Enriques criterion:

Theorem 2.3.2 *Let C be a -1 curve in S. Then there is a smooth surface T and a blowing up $\pi : S \to T$ at a point $b \in T$ such that $C = \pi^{-1}(\{b\})$. If S is algebraic, then T is complex analytic diffeomorphic to an algebraic surface.*

We shall see that exceptional curves of the first kind or -1 curves occur in the fibers of a surface S, if we have blown up too often. These cases will generally be excluded, see the discussion regarding relatively minimal fibrations below.

2.4 Elliptic surfaces and Kodaira's classification

In this section we copy the definition of elliptic surfaces and their most important properties from [3]. The outline of the first sections of chapter 7 of this book will be adopted. Our purpose is the review all relevant results, we will therefore not give full proves of the more involved theorems and lemmas.

A complex analytic surface S is defined to be a two dimensional complex manifold. Let S and C be a connected analytic surface and curve, respectively. A non-constant proper analytic mapping $\varphi : S \to C$, that is a non-constant analytic mapping of which the inverse image of every compact set is compact, is called a fibration of the surface S over the curve C. For any $s \in S$ the tangent mapping $T_s\varphi$ of φ at s is a complex linear mapping from the complex two dimensional tangent space T_sS of S at s to the complex one-dimensional tangent space $T_{\varphi(s)}C$ of C at $\varphi(s)$. In local coordinates the tangent mapping corresponds to the total derivative of φ at s. A point $s \in S$ is called a regular point of for φ if the tangent map $T_s\varphi$ at s is surjective. Surjectivity of $T_s\varphi$ is equivalent to $T_s\varphi \neq 0$. We denote the set of all regular points for φ in S by S^{reg}, and its complement, $S \backslash S^{\text{reg}} = \{s \in S | T_s\varphi = 0\}$ the set of all singular points for φ in S, by S^{sing}. The image of the singular points, the so-called set of all singular values of φ in C, is denoted by $C^{\text{sing}} = \varphi(S^{\text{sing}})$. We write $C^{\text{reg}} = C \backslash C^{\text{sing}}$ for the set of all regular values of C. The fibers $S_c = \varphi^{-1}(\{c\})$, with $c \in C^{\text{reg}}$ and $c \in C^{\text{sing}}$ are called the regular and singular fibers of the fibration $\varphi : S \to C$ respectively. Let z be a complex analytic local coordinate on an open neighbourhood of $c \in C$, such that $z(c) = 0$. If $c \in C^{\text{sing}}$ then it can happen that the function $z \circ \varphi$ vanishes to a order strictly greater then one along some irreducible components of S_c. In this case the fiber S_c is defined as the socalled divisor $\text{Div}(z \circ \varphi)$. A divisor D on some complex manifold M of complex dimension n is a locally finite formal linear combination

$$D = \sum a_i Z_i$$

of irreducible analytic hypersurfaces of M. Z_i being an irreducible analytic hypersurface means firstly that Z_i is analytic, that is for every $m \in M$ there is an open neighbourhood U of m and a collection F of holomorphic functions $f : U \to \mathbb{C}$ such that $Z_i \cap U$ is equal to the common zeroset of all $f \in F$ on U, secondly that Z_i is irreducible which is equivalent to the fact that Z_i cannot be written as the union of two analytic subsets of M unless one of those subsets is Z_i itself, finally that Z_i is a hypersurface; the dimension of Z_i is $n - 1$. It turns out that locally $z \circ \varphi$ may be written as $ug_1^{o_1} \ldots g_k^{o_k}$, where u is a unit, $g_i(0) = 0$ and each g_i can not be written as the product of two functions each vanishing in the origin nor as the product of some unit and g_j, with $j \neq i$. We now define Z_i to be the zeroset of g_i and write

$$D(z \circ \varphi) = \sum o_i Z_i.$$

We have set $o_i = a_i$ and refer to the o_is as the orders of vanishing. For a more detailed discussion we refer the reader to section 3.1 of [3] or chapter 1 of [5]. For any $c \in C$, denote by S_c^{irr} the set of all irreducible components of the divisor S_c. We further introduce the notation

$$S_c = \sum_{\Theta \in S_c^{\text{irr}}} \mu_\Theta \Theta,$$

where Θ is an irreducible component of S_c^{irr} and $\mu_\Theta \in \mathbb{Z}_{>0}$ is the multiplicity with which $\Theta \in S_c^{\text{irr}}$ in S_c.

In the interest of brevity we now give a number of lemmas from section 7.1 of [3] without citing the proofs.

Lemma 2.4.1 *The mapping $\varphi : S \to C$ is surjective. The set C^{sign} of singular values of φ is a locally finite subset of C. For each regular value $c \in C^{reg}$ the fiber $S_c = \varphi^{-1}(\{c\})$ in S of φ over c, S_c is a non-singular compact analytic curve in S, furthermore the restriction of φ to $\varphi^{-1}(C^{reg})$ is a real analytic locally trivial fiber bundle over C^{reg}. The surface S is compact if and only if the curve C is compact. If this is the case, then φ has only finitely many singular values.*

A fibration $\varphi : S \to C$ is not necessarily a topological trivial bundle because there may exist singular fibers. The fibration $\varphi : \varphi^{-1}(C^{\text{reg}}) \to C^{\text{reg}}$, that is φ restricted to the regular points, however is a topologically trivial bundle, as we shall see. The fibration restricted to the regular point nonetheless fails to be a complex analytic locally trivial fiber bundle because in general two fibers are not complex analytically diffeomorphic.

Lemma 2.4.2 *All fibers of φ, viewed as divisors are homologous to each other. We have $S_c \cdot S_{c'} = 0$ for all $c, c' \in C$, including the case that $c = c'$.*

Lemma 2.4.3 *There exists a fibration $\psi : S \to \tilde{C}$ over a connected complex analytic curve $\tilde{C}$ and a branched covering map $\pi : \tilde{C} \to C$ such that the following diagram commutes*

$$\begin{tikzcd} S \arrow[d, "\psi"'] \arrow[dr, "\varphi"] & \\ \tilde{C} \arrow[r, "\pi"'] & C \end{tikzcd}$$

and all the fibers of ψ are connected. All regular fibers of ψ are compact Riemann surfaces of the same genus. All components of all regular fibers of φ are compact Riemann surfaces of the same genus. Furthermore the following three statements are equivalent:

- *Some regular fiber of φ is connected.*

- *Every regular fiber is connected.*

- *Any map π as above is a complex analytic diffeomorphism from $\tilde{C}$ to C.*

This inspires the following important definition:
An elliptic surface is a fibration $\varphi : S \to C$ with connected fibers such that some and hence each regular fiber of φ is a compact Riemann surface of genus one. The name elliptic surface is due to the fact that each regular fiber is an elliptic curve. The fibration $\varphi : S \to C$ in the definition is called an elliptic fibration of S.

A fibration where all exceptional curves of the first kind contained in the fibers of a fibration have been successively blown down, so that there are no longer any exceptional fibers of the first kind is referred to as a relatively minimal fibration.

Let $c \in C^{\text{reg}}$ be a regular point of the elliptic fibration $\varphi : S \to C$. Then the fiber S_c over c is an elliptic curve and one can associate a modulus as given in (2.9) to this elliptic curve. This gives us a function $J : C^{\text{reg}} \to \mathbb{C}$ which maps a point $c \in C^{\text{reg}}$ to $J(c) = J(S_c)$, called the modulus function. One can prove that the modulus function extends to a meromorphic function on C.

From this point onward we will always assume that $\varphi : S \to C$ is an relatively minimal elliptic fibration.

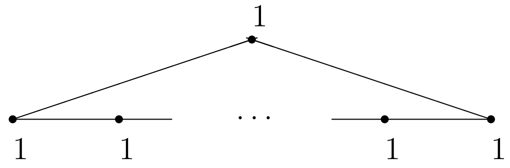
Having given a definition of relatively minimal elliptic fibration, we now turn our attention to Kodaira's classification of singular fibers. The so-called intersection diagrams play a mayor role is the classification. These intersection diagrams will be introduced first, before giving Kodaira's list of singular fibers.

Lemma 2.4.4 *If the fiber S_c has more then one irreducible component, then each irreducible component of S_c is a smoothly embedded complex projective line, with self-intersection number $\Theta \cdot \Theta = -2$. If S_c has two distinct irreducible components Θ, Θ', then $\mu_\Theta = \mu_{\Theta'}$ and $\Theta \cdot \Theta' = 2$. If S_c has more than two irreducible components, then these are disjoint, or intersect each other in only one point and transversally.*

From this point onward, let C^{red} denote the set of all $c \in C$ such that S_c has more then one irreducible component. Since each regular fiber has only one irreducible component it is clear the $C^{\text{red}} \subset C^{\text{sing}}$, C^{sing} is the finite set of all singular values of φ.
Let $r \in C^{\text{red}}$, then the matrix whose elements are $\Theta \cdot \Theta'$, with $\Theta, \Theta' \in S_r^{\text{irr}}$, is called the intersection matrix of S_r. The intersection diagram of S_r is the diagram of which the vertices are the irreducible components of S_r and two vertices are connected by an edge if the two irreducible components, represented by the vertices, intersect each other. The

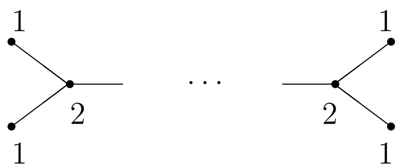
fact that S_r is connected implies that the intersection diagram is connected, which is equivalent to the fact that the intersection matrix is indecomposable. it has been proven by Kodaira that the only possibilities for the intersection diagrams are: $A_l^{(1)}$, $D_l^{(1)}$ and $E_l^{(1)}$. The intersection diagrams are depicted below and are taken from section 7.2.4 of [3]. The numbers attached to the vertices are the positive integers $\nu_\Theta = \mu_\Theta / m$, where m is the greatest common divisor of all the μ_Θ, $\Theta \in S_r^{\mathrm{irr}}$.[9]

$\mathrm{A}_l^{(1)}$



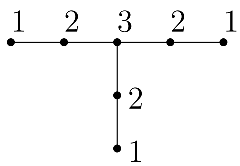
$l+1$ vertices

$\mathrm{D}_l^{(1)}$

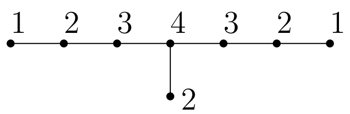


$l+1$ vertices

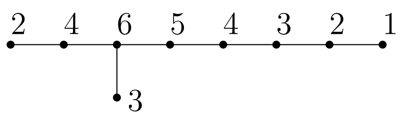
$\mathrm{E}_6^{(1)}$



$\mathrm{E}_7^{(1)}$



$\mathrm{E}_8^{(1)}$



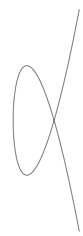
[9] See discussion of multiple singular fibers below.

These intersection diagrams are referred to as extended Dynkin diagrams, since if on deletes the vertex corresponding to the irreducible component Θ_0, of which $\nu_\Theta = 1$, as well as the edges connected to Θ_0, we find the familiar Dynkin diagrams.

We now give Kodaira's list of singular fibers including so-called multiple singular fibers. Multiple singular fibers are singular fibers of which the greatest common divisor of the multiplicities of the irreducible components is equal to m, for some $m > 1$. It can be shown that locally there exists a local m-fold cover, which is an elliptic fibration without multiple singular fibers. Globally elliptic fibrations with multiple singular fibers may be reduced to elliptical fibrations without these multiple singular fibers, as has been proven by Kodaira in theorem 6.3 of [7]. We shall always impose that our elliptic fibrations do not have multiple singular fibers. We have inserted figures for most types of singular fibers, these are inspired of the representation sketched in figure 1 of [7].

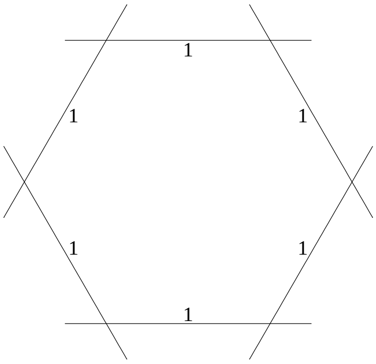
I_0 A smooth fiber, which is an elliptic curve. Its self-intersection number is zero.

I_1 A singular fiber, which is an irreducible curve with a rational curve, that is a curve which is isomorphic to a complex projective line, as its desingularisation.[10] Its self-intersection number is zero. For this singular fiber there is no real intersection diagram of Dynkin type, since there is only one irreducible component and therefore the intersection diagram would be a single point, however we will refer to this single point as the intersection diagram and refer to it as $A_0^{(1)}$.



I_b For this description we will assume that $b \in \mathbb{Z}_{\geq 2}$. A singular fiber of this type consists of a cycle of smooth rational curves Θ_i, $i \in \mathbb{Z}/b\mathbb{Z}$, its irreducible components. Each irreducible component has self-intersection number -2. The b singular points of the fiber, denoted by s_i, are the points where Θ_i intersects Θ_{i+1} transversally. There are no other singular points. Intersection diagram: $A_{b-1}^{(1)}$.

[10] The dissingularisation is a blow up as discussed in section 2.3, after which we are faced with a projective line as the image of the curve and another projective line created by blowing up.



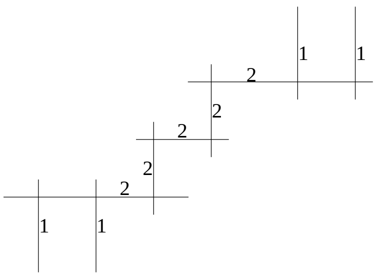
${}_m\mathrm{I}_b$ Here $m \in \mathbb{Z}_{>1}$, $b \in \mathbb{Z}_{\geq 0}$, a fiber of type I_b occurring with multiplicity m, see the discussion above.

I_0^* The fiber is the union of one smooth rational curve Θ_2 which occurs with multiplicity 2 and four smooth rational curves Θ_1, Θ_2, Θ_3 and Θ_4, each occurring with multiplicity 1. Every irreducible component has self-intersection number -2. Each of Θ_i, with $i \neq 2$, intersects Θ_2 at one point. These are all intersection points and they are distinct. Intersection diagram: $D_4^{(1)}$.

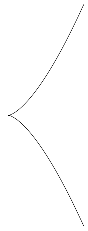
I_b^* Here we take $b \in \mathbb{Z}_{>0}$. The fiber is the union of the following:

- $b+1$ smooth rational curves, each with multiplicity one 2, denoted by Θ_i, with $2 \leq i \leq b+2$.
- four smooth rational curves of multiplicity 1 denoted by Θ_0, Θ_1, Θ_{b+3} and Θ_{b+4}.

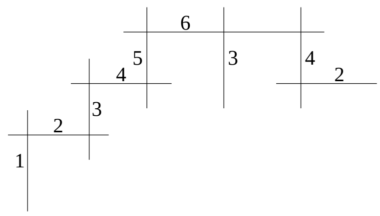
Every irreducible component Θ_i has self-intersection number -2. The Θ_i, $2 \leq i \leq b+2$ form a chain in the sense that for each $2 \leq i \leq b+2$, the curve Θ_i intersects the curve Θ_{i+1} transversally at one point. Θ_0 and Θ_1 both intersect Θ_2, each at a distinct point and transversally. Likewise Θ_{b+3} and Θ_{b+4} intersect the irreducible component Θ_{b+2}, each at a distinct point and transversally. There are no other intersection points. We stress once again that all intersection points are distinct. The intersection diagram associated to this is $D_{b+4}^{(1)}$.



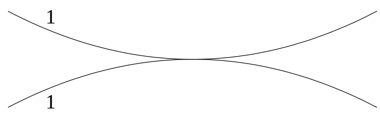
II The fiber is an irreducible curve with one singular point p, which is an ordinary cusp point of the curve. The desingularisation is a rational curve on which one point corresponds to the cusp point. The self-intersection number is 0. Once more we associate to this curve the intersection diagram $A_0^{(1)}$, see item I_1.



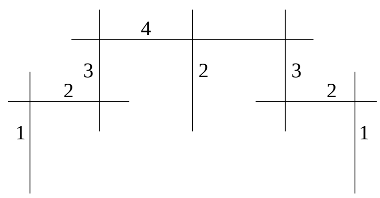
II* The fiber is the union of 9 smooth rational curves Θ_i, $0 \leq i \leq 8$, with multiplicities 1, 2, 3, 4, 6, 5, 4, 3, 2, respectively. Each irreducible component Θ_i has self-intersection number -2. For $3 \leq i \leq 7$, Θ_i intersects Θ_{i+1}, moreover Θ_8 intersects Θ_0, whereas Θ_1 intersects Θ_3 and Θ_2 intersects Θ_4. All intersections have multiplicity 1 and all intersection points are distinct. Every intersection has been mentioned. Intersection diagram: $E_8^{(1)}$.



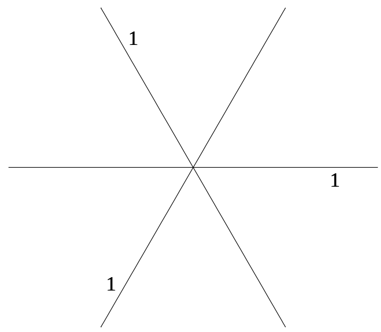
III The fiber is the union of two smooth rational curves, which intersect each other at one point, with a second order contact. The self intersection of each irreducible component is equal to -2. Intersection diagram: $A_1^{(1)}$.



III* The fiber is the union of 8 smooth rational curves Θ_i, $0 \leq i \leq 7$, with multiplicities 1, 2, 3, 4, 6, 5, 4, 3, 2, respectively. Θ_0 intersects Θ_1 and for $3 \leq i \leq 6$, Θ_i intersects Θ_{i+1}, whereas Θ_1 intersects Θ_3 and Θ_2 intersects Θ_4. Every intersection has multiplicity 1. All intersection points are distinct and there are no other intersection points then the ones discussed. Each of the irreducible component Θ_i has self-intersection number -2. Intersection diagram: $E_7^{(1)}$.



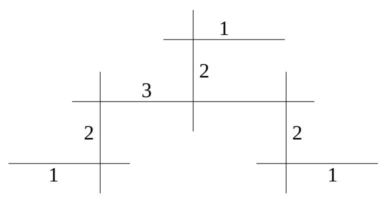
IV The fiber is the union of three rational curves each of multiplicity one, intersecting each other in a single point, with their tangent lines in general position. Each of the 3 irreducible components has self-intersection number -2, and there are no other intersections. Intersection diagram: $A_2^{(1)}$.



IV* The union of 7 smooth rational curves Θ_i, $0 \leq i \leq 6$, with multiplicities 1, 1, 2, 2, 3, 2 and 1, respectively. The intersections are as follows:

- Θ_0 intersects Θ_2, which in turn intersects Θ_4.
- Θ_1 intersects Θ_3, which in turn intersects Θ_4.
- Θ_6 intersects Θ_5, which in turn intersects Θ_4.

This list is exhaustive. All these intersections have multiplicity one. Each of the intersection points is distinct from the others. Every irreducible component has self-intersection number -2.



This completes our discussion of Kodaira's classification of singular fibers on elliptic surfaces.

2.5 Monodromy

This section treats the monodromy for relatively minimal elliptic surfaces without multiple singular fibers and is taken from subsections 3.3.3, 7.2.7, 7.2.10 and 7.2.11 of [3].

We shall revert for now to the setting where C is an elliptic curve. In section 2.2 we have discussed the period group P as a subgroup of $\mathbb{C}$. We will now regard this group in a slightly different manner. The one-dimensional space of all holomorphic vector fields on the elliptic curve C will be denoted by $\mathfrak{g}$. Its dual space $\mathfrak{g}^*$ is the space of complex analytic one-forms on C. By the same argument as in section 2.2 we have that the mapping $v \mapsto e^v(c)$ induces a complex analytic diffeomorphism from $\mathfrak{g}/P$ onto C, where P is now the set of all $v \in \mathfrak{g}$ such that $e^v(c) = c$. In section 2.2 we fixed $v \in \mathfrak{g}$, $v \neq 0$ and defined $P \subset \mathbb{C}$ as the set of all t such that $e^{tv}(c) = c$. The P defined in this manner, which we shall denote by P_v to emphasize that we fixated v, is isomorphic to the P regarded as a subset of $\mathfrak{g}$ via the linear isomorphism $t \mapsto tv$, which maps $\mathbb{C}$ to $\mathfrak{g}$ and P_v to P. The mapping $v \mapsto e^v$, which is by itself a mapping from the additive group $\mathfrak{g}$ onto the group of all translations on C denoted by G induces as isomorphism from $\mathfrak{g}/P$ onto G. The Lie group G is the identity component $\mathrm{Aut}(C)^0$ of the group $\mathrm{Aut}(C)$ of all automorphisms of C. G acts freely and transitively on C. $\mathfrak{g}$ is the Lie algebra of the Lie group G. We conclude that we have the following isomorphisms

$$\mathbb{C}/P_v \simeq \mathfrak{g}/P \simeq G \simeq C.$$

Before returning to the more general setting of elliptic fibrations we remark the following. If v_1 and v_2 as well as v_1' and v_2' are two positively oriented $\mathbb{Z}$-bases of P, then there exists a unique $M \in \mathrm{SL}(2, \mathbb{Z})$, a 2×2 matrix with integral coefficients and determinant one such that

$$v_i' = \sum_{j=1}^{2} M_i^j v_j, \qquad i = 1, 2, \text{ where } \qquad M = \begin{pmatrix} M_1^1 & M_2^1 \\ M_1^2 & M_2^2 \end{pmatrix}. \tag{2.10}$$

As already implied in the short introduction to this section we will now assume that $\varphi : S \to C$ is a relatively minimal elliptic fibration without multiple singular fibers. For every $c \in C^{\mathrm{reg}}$ we have that the fiber $\varphi^{-1}(c)$ is an elliptic curve. For such a regular point c the space $\mathfrak{g}_c$ of all holomorphic vector fields on the elliptic curve S_c is the Lie algebra of the Lie group G_c of all translations on the elliptic curve S_c. The $\mathfrak{g}_c$ form a complex line bundle over C^{reg}, in the set theoretical sense. We shall now give some results, from section 7.2.7 of [3], which endow the line bundle with a complex holomorphic structure and allow extensions to the entire C.

Lemma 2.5.1 *Let C_0 be an open subset of C^{reg}, σ a holomorphic section over C_0 of φ, and w a holomorphic section of $\sigma^*(\ker(T\,\varphi))$. That is, for every $c \in C_0$, $w(c)$ is an* *element of the tangent space $T_{\sigma(c)}(S_c) = \ker(T_{\sigma(c)}\varphi)$ of S_c at $\sigma(c)$, depending holomorphically on c. Let v denote the unique vector field on $\varphi^{-1}(C_0)$, such that for each $c \in C_0$, $v|_{S_c} \in \mathfrak{g}_c$ and $v(\sigma(c)) = w(c)$. Then v is holomorphic vector field on $\varphi^{-1}(C_0)$.*

Theorem 2.5.2 *The bundle of $\mathfrak{g}_c$, with $c \in C^{reg}$, extends to a unique holomorphic complex line bundle $\mathfrak{g}$ over C such that, for each holomorphic section $\sigma : C_0 \to S$ of φ over an open subset C_0 of C, the mapping which assigns to each $v \in \mathfrak{g}_c$, $c \in C_0 \cap C^{reg}$, its value $v(\sigma(c)) \in \ker T_{\sigma(c)}\varphi$ extends to an isomorphism from $\mathfrak{g}|_{C_0}$ onto $(\ker T\varphi)|_{\sigma(C_0)}$. For every open subset C_0 of C, the holomorphic sections v of $\mathfrak{g}$ correspond bijectively to the fiber-tangent holomorphic vector fields on $\varphi^{-1}(C_0)$, which will be denoted be the same letter v.*

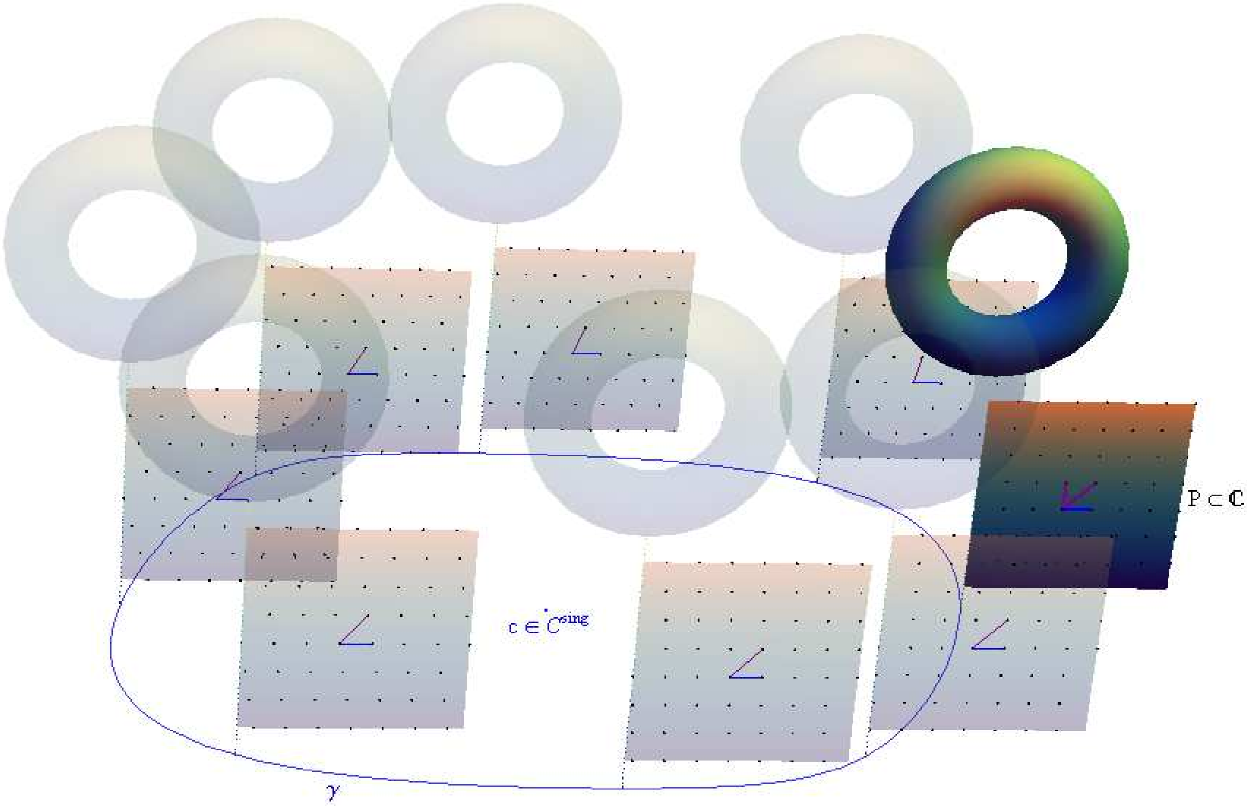


Figure 2.5: In this picture we sketch the behaviour of a $\mathbb{Z}$-basis of P_γ for a closed curve γ running around a singular point with a fiber of Kodaira type I_1. Above some points of the curve γ we see $P \subset \mathfrak{g} \simeq \mathbb{C}$ as well as the fibers, which are elliptic curves isomorphic to $\mathbb{C}/P$. One of the basis vectors is blue while the other one is purple. We have give particular attention to one fiber where we compare the original basis to basis found after running around the singular point once. We have chosen not to depict C explicitly.

The lattice P_c has a $\mathbb{Z}$-basis which is also a $\mathbb{R}$-basis of $\mathfrak{g}_c$,as has been established in section 2.2. We shall now give a lemma taken from subsection 7.2.10 of [3], which gives us a holomorphic structure on the bundle of which the fibers are P_c.

Lemma 2.5.3 *The P_c, $c \in C^{reg}$, form a holomorphic subbundle P of $\mathfrak{g}$ over C^{reg} with discrete fibers.*

We shall use the holomorphic structure on P to define the so-called monodromy, which will be used as a restriction in our investigation of confluence of singular fibers in chapter 3, as follows. For each continuous mapping $\gamma : [0,1] \to C^{\text{reg}} : t \mapsto \gamma(t)$ called a path in C^{reg}, and every $v_0 \in P_{\gamma(0)}$, there is an unique (lifted) path v is P such that $v(0) = v_0$ and $v(t) \in P_{\gamma(t)}$ for every $0 \leq t \leq 1$. The mapping $T : v_0 \mapsto v(1)$ is an orientation preserving isomorphism from $P_{\gamma(0)}$ to $P_{\gamma(1)}$. Let c_* be a fixed point in C^{reg}, called a basepoint and let γ be a loop in C^{reg} based at c_*, that is a path in C^{reg} such that $\gamma(0) = \gamma(1) = c_*$, then T as an orientation preserving automorphism of P_{c_*}. If v_1^* and v_2^* form an oriented $\mathbb{Z}$-basis of P_{c_*}, then $T(v_i^*) = v_i$ also forms an oriented $\mathbb{Z}$-basis of P_{c_*} and thus, as we have seen in formula (2.10), there exists a unique matrix $M \in \mathrm{SL}(2,\mathbb{Z})$ called the monodromy matrix defined by the loop γ, such that

$$v_i = \sum_{j=1}^{2} M_i^j v_j^*.$$

Since the fibers of P are discreet we see that the matrix M is invariant under homotopic deformations of the loop γ in C^{reg}, and therefore we can write $M = M([\gamma])$, where $[\gamma]$ denotes the homotopy class of γ. The set of all homotopy classes of loops in C^{reg} based at c_*, provided with the group structure of concatenation of loops, is called the fundamental group $\pi_1(C^{\text{reg}}, c_*)$ of C^{reg} with respect to the base point c_*. The mapping $M : [\gamma] \mapsto M([\gamma])$, is a homomorphism from $\pi_1(C^{\text{reg}}, c_*)$ to the group $\mathrm{SL}(2,\mathbb{Z})$, called the monodromy representation of the elliptic surface $\varphi : S \to C$. The subgroup $\mathfrak{M} := M(\pi_1(C^{\text{reg}}, c_*))$ of $\mathrm{SL}(2,\mathbb{Z})$ is called the monodromy group. In figures 2.5 and 2.6 we have sketched the behaviour of the period lattice and a $\mathbb{Z}$-basis of this lattice for some curve γ.

We are now ready to give the definition of the monodromy associated to a singular value of the fibration $\varphi : S \to C$. We will revert to the view of the period lattice P as a subset of $\mathbb{C}$ instead of a subset of $\mathfrak{g}$, as elucidated below. Let $c_0 \in C^{\text{sing}}$ be a singular value of the elliptic fibration $\varphi : S \to C$, with the singular fiber S_{c_0} of φ over c_0. Let $z : C_0 \to \mathbb{C}$ be a complex analytic coordinate function on an open neighbourhood C_0 of c_0 in C such that $z(c_0) = 0$. Shrinking C_0 if necessary, we may assume that z is a complex analytic diffeomorphism from C_0 onto

$$D = \{z \in \mathbb{C} \big| \, |z| < \delta\},$$

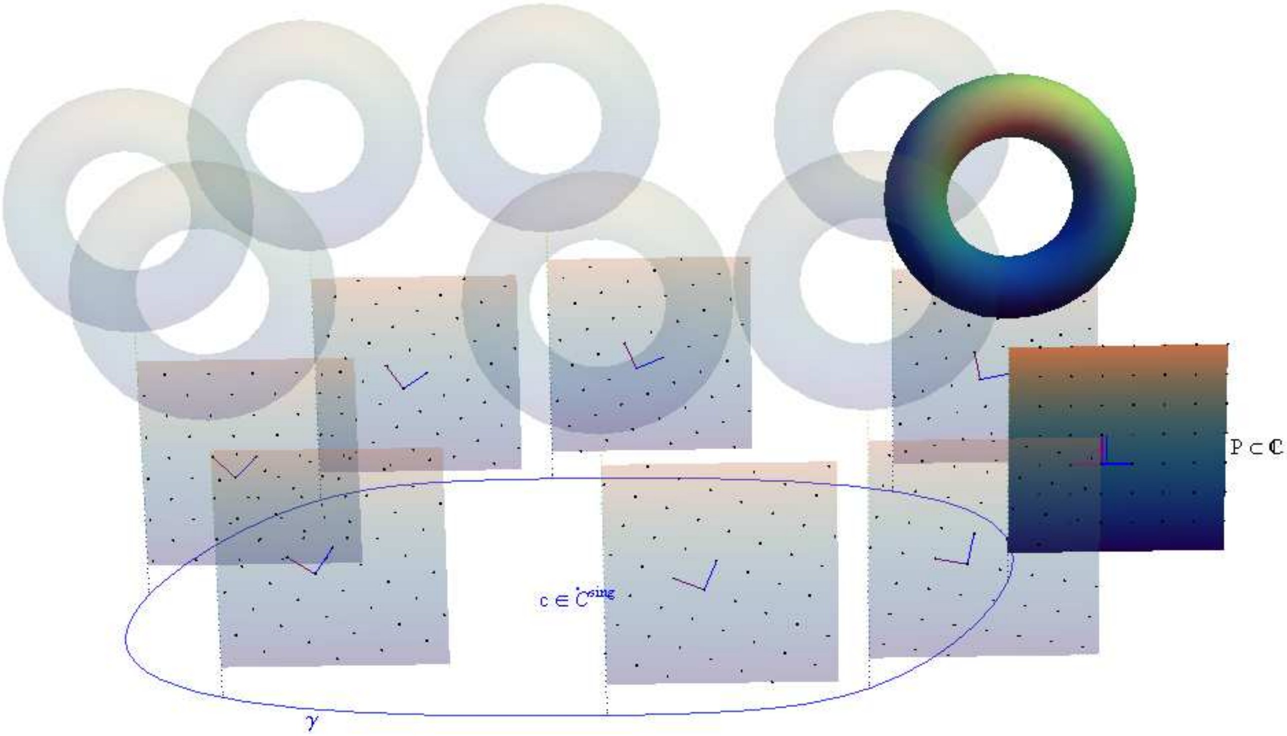


Figure 2.6: In this picture we sketch the behaviour of a $\mathbb{Z}$-basis of P_γ for a curve γ running around a singular point with a fiber of Kodaira type III. In this figure we clearly see the finite order of the monodromy of a singular fiber of Kodaira type III. The monodromy associated to a singular fiber of type I_1 on the other hand is not finite. In chapter 3 we will see that it is possible to perturb a singular fiber to which a monodromy of finite order is assocated into several singular fiber of infinite order. See the caption of figure 2.5 for further description.

where $\delta \in \mathbb{R}_{>0}$. From this point onwards we shall write φ, S and S_0 instead of $z \circ \varphi$, $\varphi^{-1}(C_0) = (z \circ \varphi)^{-1}(D)$ and S_{c_0}, respectively. Shrinking D if necessary, it can be arranged that S_0 is the only singular fiber of φ is D. Due to theorem 2.5.2 we have that there exists a holomorphic section v of $\mathfrak{g}$ over D, v is regarded as a holomorphic vector field on S. For each $z \in D$ let $P_v(z)$ denote, as before, the set of all $t \in \mathbb{C}$ such that $e^{t\,v}(s) = s$, where $\varphi(s) = z$.[11] As we have discussed in the beginning of this section, one has that for $z \in D \backslash \{0\}$ the period group $P_v(z)$ is isomorphic to $P_z \subset \mathfrak{g}_z$, via the mapping $\mathbb{C} \to \mathfrak{g} : t \mapsto t\, v_z$. We also have that the fundamental group of $D^{\mathrm{reg}} = D\backslash\{0\}$ is isomorphic to $\mathbb{Z}$ and is generated by a loop $\gamma \in D\backslash\{0\}$, with base point z_*, winding itself once around the singular point with positive orientation. The monodromy representation $M(\pi_1(D^{\mathrm{reg}}, z_*)$, is determined by the monodromy of the generating loop $M([\gamma])$, the matrix in $\mathrm{SL}(2,\mathbb{Z})$ is called the monodromy matrix around the singular value of φ. As

[11] We have that the zeroset of v is equal to $\varphi^{-1}(D) \cap S_0 = S_0$, see the proof of theorem 7.2.15 of [3], so that $e^{t\,v}(s) = s$ becomes an empty condition on t, if $\varphi(s) = 0$.

the monodromy representation depends on the choice of $\mathbb{Z}$-basis of $P(z_*)$ and every other choice of basis is related to a given basis by acting with an element of $\mathrm{SL}(2,\mathbb{Z})$ on the basis it follows that a change of basis corresponds to conjugation of the monodromy representation $M([\gamma])$ by an arbitrary element of $\mathrm{SL}(2,\mathbb{Z})$, see formula (2.10). It is only the conjugacy class of M in $\mathrm{SL}(2,\mathbb{Z})$, which is invariantly defined. Therefore we shall refer to a suitable element in the conjugation class as the entire conjugation class as the monodromy matrix.

Table 2.1: In this table we only give one monodromy matrix characterizing the conjugation class. The b is a positive integer.

Type	Intersection diagram	Monodromy matrix	Euler number
I_b	$A^{(1)}_{b-1}$	$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$	b
I^*_b	$D^{(1)}_{b+4}$	$\begin{pmatrix} -1 & -b \\ 0 & -1 \end{pmatrix}$	$b+6$
II	$A^{(1)}_{0}$	$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$	2
II^*	$E^{(1)}_{8}$	$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$	10
III	$A^{(1)}_{1}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	3
III^*	$E^{(1)}_{7}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	9
IV	$A^{(1)}_{2}$	$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$	4
IV^*	$E^{(1)}_{6}$	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$	8

Before giving the most important result of this section we remind ourselves of the definition of the topological Euler number, also referred to as the Euler characteristic. The topological Euler number of any real n-dimensional simplicial complex M is defined as the alternating sum

$$\chi(M) = \sum_{k=0}^{n} (-1)^k N_k,$$

where N_k is the number of simplices of dimension k.

Having defined the monodromy representation of a singular fiber and reviewed the definition of the Euler characteristic we can give the following result taken from lemmas

7.2.25 and 7.2.31 of [3].

Lemma 2.5.4 *The monodromy matrix and the topological Euler number of a singular fiber of given Kodaira type is as in table 2.1. We have also added the intersection diagrams as discussed in section 2.4.*

Inspection of table 2.1 yields the following corollary:

Corollary 2.5.5 *Two singular fibers of elliptic surfaces have the same Kodaira type if and only if the monodromy matrices around these fibers are conjugate by an element of* $SL(2, \mathbb{Z})$*.*

We will now give some result regarding the Euler number of elliptic surfaces, a combination of lemmas 7.2.25 and 7.2.26 of [3], see also Remark 7.3.4 of the same book

Lemma 2.5.6 *The topological Euler number* $\chi(S)$ *of a compact elliptic surface* S *is equal to the sum of the topological Euler numbers of the singular fibers. Moreover the topolocigal Euler number of every compact relatively minimal elliptic surface is a multiple of* 12*.*

Elliptic surfaces whose topological Euler number is 12 are called rational elliptic surfaces. While those with an Euler number of 24 are referred to as $K3$ surfaces.

2.6 The Weierstrass model

In this section we apply the construction which brings every elliptic curve into its Weierstrass normal form, as we have seen in section 2.2, to every regular fiber of an elliptic fibration. Extending this to every fiber gives the Weierstrass Model of an elliptic surface. The converse will also be proven. We shall assume that $\varphi : S \to C$ is a relatively minimal elliptic fibration with a holomorphic section $o : C \to S$, S and C not necessarily compact. Our discussion relies on subsection 3.1.3 and section 7.3 of [3] and notes by Hans Duistermaat.

Before applying, as announced, the construction of section 2.2, we remind ourselves about complex holomorphic line bundles. If $\pi : L \to M$ is a holomorphic complex line bundle then there is a covering of M, with non-empty open subsets U_α, such that $\pi^{-1}(U_\alpha)$ admits a trivialisation. This means that for every α there is a complex diffeomorphism τ_α from $\pi^{-1}(U_\alpha)$ onto $U_\alpha \times \mathbb{C}$, such that the restriction to $\pi^{-1}(U_\alpha)$ of π is equal to τ_α composed with the projection $U_\alpha \times \mathbb{C} \to U_\alpha : (m, c) \mapsto m$ to the first component. It follows that for each α, β such that $U_\alpha \cap U_\beta \neq \emptyset$, the retrivialisation $\tau_\alpha \circ \tau_\beta^{-1}$ is diffeomorphism of $(U_\alpha \cap U_\beta) \times \mathbb{C}$ onto itself of the form

$$(m, c) \mapsto (m, f_{\alpha\beta}(m)c),$$

for a unique nowhere zero complex holomorphic function $f_{\alpha\beta}$ on $U_\alpha \cap U_\beta$, called a transition function. The system of functions $f_{\alpha\beta}$ satisfies the cocycle condition $f_{\alpha\beta} f_{\beta\gamma} = f_{\alpha\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$.
For each $m \in M$, L_m is a one-dimensional complex vector space. The k-th tensor power of L_m, $L_m^{\otimes k}$ denoted by L_m^k, is a one-dimensional vector space, which is isomorphic to the space of all homogeneous functions of degree k on $(L_m)^* \backslash \{0\}$, where the * indicates the dual. This construction for each fiber extends to the entire bundle and yields a bundle denoted by L^k. The transition functions of the bundle L^k are the functions $(f_{\alpha\beta})^k$.

Applying the construction of section 2.2 to every fiber of the elliptic fibration $\varphi : S \to C$ is referred to as exhibiting S as a family W of Weierstrass curves. If this family has singularities, it can be viewed as obtained from S by contracting, for every reducible fiber S_c of φ, all irreducible components of S_c, which do not intersect $o(C)$.[12] We start our construction by denoting $\mathfrak{g}^*$, the dual of the Lie algebra bundle over C, by L. For every $c \in C^{\text{reg}}$ and $v \in \mathfrak{g}_c \backslash \{0\}$, the mapping $\mathbb{C} \to S_c : t \mapsto e^{tv}(o(c))$ induces an isomorphism from the elliptic curve $\mathbb{C}/P_{c,v}$ onto the fiber S_c of φ over c. Let $t_{c,v} : S_c \to \mathbb{C}/P_{c,v}$ denote the inverse of this isomorphism. Here $P_{c,v}$ denotes the period lattice in $\mathbb{C}$, which is isomorphic, via the linear mapping $\mathbb{C} \to \mathfrak{g}_c : t \mapsto tv$, to P_c the period lattice in $\mathfrak{g}_c$.
We now apply for each $c \in C^{\text{reg}}$ and each $v \in \mathfrak{g}_c \backslash \{0\}$ the construction of section 2.2, where $P_{c,v}$ plays the role of P. The Weierstrass function (2.1) and its derivative (2.2)

[12] Note that every Weierstrass curve is irreducible.

yield meromorphic functions $x_{c,v}$ and $y_{c,v}$ on S_c as follows

$$x_{c,v}(s) := \wp_{P_{c,v}}(t_{c,v}(s)) \qquad \text{and} \qquad y_{c,v}(s) := \wp'_{P_{c,v}}(t_{c,v}(s)), \qquad s \in S_c,$$

as well as the complex cumbers on S_c

$$g_2(c,v) := g_2(P_{c,v}) \qquad \text{and} \qquad g_3(c,v) := g_3(P_{c,v}).$$

For each $\lambda \in \mathbb{C}\backslash\{0\}$ we have $e^{(\lambda^{-1}t)(\lambda v)} = e^{tv}$ and $P_{c,\lambda v} = \lambda^{-1}P_{c,v}$. In analogy of our discussion of isomorphisms of elliptic curves in section 2.2 we note that due to formulae (2.1) and (2.2)

$$x_{c,\lambda v}(s) = \lambda^2 x_{c,v}(s), \qquad y_{c,\lambda v}(s) = \lambda^3 y_{c,v}(s) \tag{2.11}$$

and due to formula (2.4) that

$$g_2(c,\lambda v) = \lambda^4 g_2(c,v), \qquad g_3(c,\lambda v) = \lambda^6 g_2(c,v). \tag{2.12}$$

These functions are clearly homogeneous of degree k, with $k = 2$, 3, 4 and 6 respectively, in the variable $v \in \mathfrak{g}_c\backslash\{0\}$. As has been discussed above the space of all homogeneous function of degree k, on in this case $\mathfrak{g}\backslash\{0\}$, is isomorphic to the k-th tensor power of the dual of the space, in this case $(\mathfrak{g}_c^*)^k$.[13] This one-dimensional vector space will be denoted by L_c^k, where $L_c = (\mathfrak{g}_c)^*$. So due to (2.11) we have that $v \mapsto x_{c,v}(s)$ and $v \mapsto y_{c,v}(s)$ are elements $x(s)$ of ${L_c}^2$ and $y(s)$ of ${L_c}^3$, respectively. Likewise (2.12) implies that $v \mapsto g_2(c,v)$ and $v \mapsto g_3(c,v)$ are elements $g_2(c)$ of ${L_c}^4$ and $g_3(c)$ of ${L_c}^6$, respectively.
We now have the equation

$$y(s)^2 = 4x(s)^3 - g_2(c)x - g_3(c),$$

which is equation (2.5), with the coordinates x and y and the numbers g_2 and g_3 replaced by $x(s)$ and $y(s)$, and $g_2(c)$ and g_3 in ${L_c}^2$, ${L_c}^3$, ${L_c}^4$ and ${L_c}^6$, respectively. Because both $\wp(t)$ and $\wp'(t)$ have a pole at $t = t_{c,v}(s) = 0$ modulo $P_{c,v}$, where $s = o(c)$, of order 2 and 3, respectively, one needs to pass to the projective Weierstrass equation

$$x_0{x_2}^2 - 4{x_1}^3 + g_2(c){x_0}^2x_1 + g_3(c){x_0}^3 = 0, \tag{2.13}$$

to include a description of a neighbourhood of $t = 0$. The Weierstrass equation will now be viewed, for a given c, as a homogeneous equation of degree three in $(x_0, x_1, x_2) \in {L_c}^0 \times {L_c}^2 \times {L_c}^3$, where we define ${L_c}^0$ to be equal to $\mathbb{C}$. The solution set of the Weierstrass equation is a cubic curve in the complex projective plane $\mathbb{P}({L_c}^0 \times {L_c}^2 \times {L_c}^3) = \mathbb{P}({L_c}^0 \oplus {L_c}^2 \oplus {L_c}^3)$, the space of all one-dimensional linear subspaces of the 3-dimensional vector space ${L_c}^0 \times {L_c}^2 \times {L_c}^3$.
Because $L = \mathfrak{g}^*$ is a holomorphic line bundle over C, the ${L_c}^k$, with $c \in C$ form a

[13]We again omit the tensor product $\otimes$.

holomorphic line bundle over C as discussed above, which is denoted by L^k. We now have the corresponding holomorphic bundle $\mathbb{P}(L^0 \oplus L^2 \oplus L^3)$ over C, whose fibers are the complex projective planes

$$\mathbb{P}(L^0 \oplus L^2 \oplus L^3)_c = \mathbb{P}({L_c}^0 \times {L_c}^2 \times {L_c}^3).$$

We now have the following lemma 7.3.1. of [3]:

Lemma 2.6.1 *The mappings $x : \varphi^{-1}(C^{reg}) \to L^2$ and $y : \varphi^{-1}(C^{reg}) \to L^3$ extend to holomorphic mappings $x : S \to L^2$ and $y : S \to L^3$, respectively, with $p_2 \circ x = \varphi = p_3 \circ y$ if $p_k : L^k \to C$ denotes the canonical projection.*
The sections $g_2 : C^{reg} \to L^4$ and $g_3 : C^{reg} \to L^6$ extend to holomorphic sections $g_2 : C \to L^4$ and $g_3 : C \to L^6$ of L^4 and L^6, respectively. If $c_0 \in C^{sing}$, then the orders of zeros at $c_)$ of g_2, g_3, and the holomorphic section $\Delta = {g_2}^3 - 27{g_3}^2$ of L^{12} are given in table 2.2.

Table 2.2: The behaviour of the zeros of the defining sections g_2, g_3 and geometric discriminant Δ of a Weierstrass model for singular and non-singular fibers.

Kodaira Type	Order zero of g_2	Order zero of g_3	Order zero of Δ
I_0	≥ 0	≥ 0	0
I_b, $b \geq 1$	0	0	b
I_0^*	≥ 2	≥ 3	6
I_b^*, $b \geq 0$	2	3	$b + 6$
II	≥ 1	1	2
II^*	≥ 4	5	10
III	1	≥ 2	3
III^*	3	≥ 5	9
IV	≥ 2	2	4
IV^*	≥ 3	4	8

To exhibit the elliptic surface as a family of Weierstrass curves we have theorem 7.3.6 of [3]:

Theorem 2.6.2 *Let $\varphi : S \to C$ be a relatively minimal elliptic fibration with a holomorphic section $o : C \to S$. Let $\mathfrak{g}$ denote the Lie algebra line bundle over C, with dual bundle $L = \mathfrak{g}^*$. Then the formulas (2.4) define the sections g_2 and g_3 of the line bundles L^4 and L^6 over C. The geometric discriminant Δ, which is equal to ${g_2}^3 - 27{g_3}^2$ is a holomorphic section of the line bundle L^{12} over C and $\Delta(c) = 0$ and only if $c \in C^{sing}$, where the order of the zero of Δ at c is equal to the Euler number $\chi(S_c)$ of the singular fiber S_c. The orders of the zeros of g_2, g_3 and Δ are given in table 2.2. In particular, at* *any $c \in C$ the order of the zero of g_2 at c is < 4 or the order of the zero of g_3 at c is < 6. Let W be the complex analytic subset of the complex projective plane bundle*

$$\pi : \mathbb{P}(L^0 \oplus L^2 \oplus L^3) \to C$$

over C, defined by the projective Weierstrass equation (2.13) that is

$$W = \{(c, [x]) \in \mathbb{P}(L^0 \oplus L^2 \oplus L^3) | x_0 {x_2}^2 - 4{x_1}^3 + g_2(c){x_0}^2 x_1 + g_3(c){x_0}^3 = 0\}. \qquad (2.14)$$

Then the mapping $s \mapsto (\varphi(s), [1 : \wp(s) : \wp'(s)])$ defined by the Weierstrass $\wp$-function and its derivative extends to a proper holomorphic mapping $f : S \to W$, such that $p \circ f = \varphi$ if $p : W \to C$ is the restriction to W of π, and $\infty := f \circ o$ is the section $c \mapsto (c, \{0\} \times \{0\} \times {L_c}^3)$ at infinity of $p : W \to C$, where $\infty(C)$ is contained in the non-singular part of W. For each reducible fiber S_c of φ, f maps the union of the irreducible components of S_c, which do not intersect $o(C)$ to a point $w \in W$, and these points w form the set W^* of all singular points of W.

To appreciate the role of the function f in theorem 2.6.2 we develop the following language as used in subsection 7.2.12 of [3]. A modification of an analytic space X, locally the common zeroset of some functions,[14] is a surjective proper holomorphic map f from an analytic space Y onto X, such that there is a a nowhere dense closed analytic subset S of X, whose inverse image $f^{-1}(S)$ is nowhere dense in Y and such that the restriction of F to $Y \backslash f^{-1}(S)$ is a biholomorphic mapping from $Y \backslash f^{-1}(S)$ onto $X \backslash S$. That is, there exists a holomorphic mapping $g = g_S : X \backslash S \to Y \backslash f^{-1}(S)$ such that $f \circ g$ and $g \circ f$ are equal to the identity on $X \backslash S$ and $Y \backslash f^{-1}(S)$, respectively.
A modification $f : Y \to X$ of X with Y smooth is called a resolution of singularities of the complex analytic surface X. Resolutions of singularities are not unique, since for any resolution of singularities $f : Y \to X$, the blow-up of Y through the blow-up map $g : Z \to Y$, with Z a non-singular complex analytic surface, gives us $f \circ g : Z \to X$, which is also a resolution of singularities of X. As discussed in section 2.3, g maps a -1 curve in Z to a point in Y and therefore $f \circ g$ maps the same -1 curve in Z to a point in X. The mapping $g : Z \to Y$ is a resolution of singularities even though Y is non-singular. A resolution of singularities $f : Y \to X$ of X is called minimal if f does not map a -1 curve in Y to a point in X. We have the following result, lemmas 7.2.45 and 7.2.46 of [3], regarding existence and uniqueness

Lemma 2.6.3 *Every complex analytic surface admits a minimal resolution of singularities. Moreover let X be a complex analytic surface and $f : Y \to X$, $g : Z \to X$ resolutions of singularities of X, with $f : Y \to X$ minimal. Then there is a unique holomorphic mapping $h : Z \to Y$ such that $g = f \circ h$. If $g : Z \to X$ is also minimal, then h is a complex analytic diffeomorphism from Z to Y.*

[14]See also section 2.4.

We conclude from our discussion that the mapping $f : S \to W$ in theorem 2.6.2 is a minimal resolution of singularities. The surface W, together with its minimal resolution of singularities $f : S \to W$ is called the Weierstrass model of the elliptic fibration $\varphi : S \to C$.

We shall now prove the converse of 2.6.2, namely that suitable g_2 and g_3, that is with some restrictions on the nature of the zeroset of both functions, yield a Weierstrass model of an elliptic surface. In the following we shall not impose that the elliptic fibration is relatively minimal. Let C be a complex analytic curve, $\tilde{L}$ a holomorphic complex line bundle over C and $\tilde{g}_2$ and $\tilde{g}_3$ holomorphic sections of $\tilde{L}^4$ and $\tilde{L}^6$, respectively, such that the holomorphic section

$$\tilde{\Delta} = {\tilde{g}_2}^3 - 27{\tilde{g}_3}^2$$

of $\tilde{L}^{12}$ is not identically equal to zero. Let $\tilde{W}$ be the analytic subset of the complex projective plane bundle $\mathbb{P}(\tilde{L}^0 \oplus \tilde{L}^2 \oplus \tilde{L}^3)$ over C, which is defined as

$$\tilde{W} = \{(c, [x]) \in \mathbb{P}(\tilde{L}^0 \oplus \tilde{L}^2 \oplus \tilde{L}^3) | x_0 {x_2}^2 - 4{x_1}^3 + \tilde{g}_2(c)x_1 + g_3(c){x_0}^3 = 0\},$$

as in formula (2.14). Let $\tilde{p} : \tilde{W} \to C$ be the restriction to $\tilde{W}$ of the projection $\pi : \mathbb{P}(\tilde{L}^0 \oplus \tilde{L}^2 \oplus \tilde{L}^3) \to C$. As for W, $\tilde{W}$ has only isolated singularities and $\tilde{p} : \tilde{W} \to C$ is an elliptic fibration with the provision that $\tilde{W}$ is allowed to have singularities. The mapping $\tilde{\tau} : C \to \tilde{W}$, defined by

$$\tilde{\tau}(c) = (c, \{0\} \oplus \{0\} \oplus {\tilde{L}_c}^3),$$

for every $c \in C$ is a holomorphic section of $\tilde{p}$, such that $\tilde{\tau}(C)$ is contained in the non-singular part $\tilde{W}^0 = \tilde{W} \backslash \tilde{W}^*$.
Due to lemma 2.6.3 there exists a minimal resolution of singularities $\tilde{f} : \tilde{S} \to \tilde{W}$ of $\tilde{W}$, which is unique up to isomorphism. The mapping $\tilde{\varphi} := \tilde{p} \circ \tilde{f} : \tilde{S} \to C$ is an elliptic fibration and $\tilde{\sigma} := \tilde{f}^{-1} \circ \tilde{\tau}$ is a holomorphic section of $\tilde{\varphi}$. Successively blowing down -1 curves in fibers of the elliptic fibrations, when these occur, we arrive at a relatively minimal elliptic fibration $\varphi : S \to C$ such that $\tilde{\varphi} = \varphi \circ \tilde{b}$, where $\tilde{b} : \tilde{S} \to S$ denotes the mapping which blows the -1 curves in fibers of the elliptic fibrations down. The mapping $\sigma := \tilde{b} \circ \tilde{\sigma} : C \to S$ is holomorphic and $\varphi \circ \sigma = \varphi \circ \tilde{b} \circ \tilde{\sigma} = \tilde{\varphi} \circ \tilde{\sigma}$ is equal to the identity in C, which shows that σ is a holomorphic section of φ.
For the relatively minimal elliptic fibration $\varphi : S \to C$, with the holomorphic section $\sigma : C \to S$, we have the line bundle $L = \mathfrak{g}^*$, the holomorphic sections g_2 and g_3 of L^4 and L^6 respectively and the Weierstrass model

$$S \xrightarrow{f} W \xrightarrow{p} C \,,$$

as in theorem 2.6.2. The above discussion yields the following commutative diagram

$$\begin{tikzcd}
\tilde{S} \arrow[r, "\tilde{f}"] \arrow[dr, bend left, "\tilde{\varphi}"] \arrow[dd, "\tilde{b}"'] & \tilde{W} \arrow[d, bend left, "\tilde{p}"] \\
& C \arrow[ul, bend left, "\tilde{\sigma}"] \arrow[u, bend left, "\tilde{\tau}"] \arrow[dl, bend right, "\sigma"'] \arrow[d, bend right, "\tau"'] \\
S \arrow[r, "f"'] \arrow[ur, bend right, "\varphi"'] & W \arrow[u, bend right, "p"']
\end{tikzcd} \tag{2.15}$$

Let $\mathrm{ord}_c(m)$ denote the order at $c \in C$ of the meromorphic section m of a holomorphic line bundle K over C, where m has a zero of order $\mathrm{ord}_c(m)$ at c if $\mathrm{ord}_c(m) \in \mathbb{Z}_{>0}$, a pole of order $-\mathrm{ord}_c(m)$ at c if $\mathrm{ord}_c(m) \in \mathbb{Z}_{<0}$, and no zero or pole at c if $\mathrm{ord}_c(m) = 0$. We write $\mathrm{ord}_c(m) = \infty$ if m is identically equal to zero. With these notations we have the following result, which is the converse of theorem 2.6.2, theorem 7.3.12 of [3]:

Theorem 2.6.4 *Let Z be the set of all $c \in C$ such that $ord_c(m)(\tilde{g}_2) \geq 4$ and $ord_c(m)(\tilde{g}_3) \geq 6$. Because the geometric discriminant $\tilde{\Delta} = {\tilde{g}_2}^3 - 27{\tilde{g}_3}^2$ is not identically equal to zero, and therefore Z is a discrete subset of C, where the empty set is allowed. For every $c \in Z$, let k_c be the unique $k \in \mathbb{Z}_{>0}$ such that*

$$0 \leq ord_c(m)(\tilde{g}_2) - 4k < 4 \qquad \text{or} \qquad 0 \leq ord_c(m)(\tilde{g}_3) - 6k < 6.$$

Then the following conditions i)- iv) are equivalent

i) For every $c \in C$ we have $ord_c(m)(\tilde{g}_2) < 4$ or $ord_c(m)(\tilde{g}_3) < 6$.

ii) There is an isomorphism $\iota : \tilde{L} \to L$ of holomorphic line bundles over C, such that $g_2 = \iota^4 \circ \tilde{g}_2$ and $g_3 = \iota^6 \circ \tilde{g}_3$.

iii) The elliptic fibration $\tilde{\varphi} = \tilde{p} \circ \tilde{f} : \tilde{S} \to C$ is relatively minimal.

iv) The elliptic fibration $\tilde{\varphi} = \tilde{p} \circ \tilde{f} : \tilde{S} \to C$ is relatively minimal, $\tilde{L}$ is isomorphic to the dual of the Lie algebra bundle of $\tilde{\varphi} : \tilde{S} \to C$, and the sequence

$$\tilde{S} \xrightarrow{\tilde{f}} \tilde{W} \xrightarrow{\tilde{p}} C\,,$$

exhibits $\tilde{W}$ as the Weierstrass model of $\tilde{\varphi} : \tilde{S} \to C$.

We conclude from theorem 2.6.4 and 2.6.2 that g_2 and g_3 yield the Weierstrass model of a rational (relatively minimal) elliptic surface if and only if the total number of zeros counted with multiplicity of the geometric discriminant Δ equals 12 and the zeros of g_2, g_3 and Δ satisfy the conditions set in table 2.2.

2.7 Families and confluences

In this section we give the definition of a confluence of singular fibers. This section partially relies on remark 7.3.4 of [3] and the article of Naruki [9]. Our definition of confluence is different from that of Naruki. We rely on Tyurina [11] for one result relating families of Weierstrass models to families of elliptic surfaces.

In a deformation of elliptic surfaces, we will let the deformation depend on some deformation parameter ϵ. One will often observe several singular fibers $S_{c_1}(\epsilon), \ldots, S_{c_N}(\epsilon)$ with $c_1, \ldots c_N$ singular points of surfaces nearby some special surface flowing together into one singular fiber $S_c(0)$ at the special surface $S(0)$ of the deformation.[15] Such a phenomenon is called a confluence of singular fibers. In the next chapter we shall often adhere to the opposite view, where we start with a singular fiber $S_c(0)$ which is perturbed into several singular fibers $S_{c_1}(\epsilon), \ldots, S_{c_N}(\epsilon)$. This is of course simply a matter of point of view. We may associate monodromy matrices (or conjugacy classes thereof) $M_{S_{c_i}}$ to the singular fibers S_{c_i} in the way described in section 2.5, by defining the monodromy associated to the singular fibers as the monodromy of curve γ_i which encircles the singular fiber S_{c_i} once and counterclockwise. For a certain perturbation σ of indices, the concatenation $\gamma_{\sigma(1)} * \ldots * \gamma_{\sigma(N)}$ is homotopic to γ_0, after deformation. For this perturbations of indices σ we have the identity $M_{S_c} = M_{S_{c_{\sigma(1)}}} \cdot \ldots \cdot M_{S_{c_{\sigma(N)}}}$.

We will now make this statement more precise. Suppose that we are given a commutative diagram of the form

$$\begin{tikzcd} & \Sigma \arrow[dl, "\varphi"'] \arrow[dd, "\eta"] \\ \Gamma \arrow[dr, "\delta"'] & \\ & \mathcal{E} \end{tikzcd}$$

where Σ, Γ and $\mathcal{E}$ are complex analytic manifolds of dimension $n+2$, $n+1$ and n respectively and φ, δ and η are proper surjections. Let us set

$$S_\epsilon = \eta^{-1}(\epsilon), \qquad C_\epsilon = \delta^{-1}(\epsilon), \qquad \epsilon \in \mathcal{E}$$

and further assume that δ and η are complex analytic submersions and the fibers S_ϵ are compact.[16] The data $(\Sigma, \Gamma, \mathcal{E}, \varphi, \delta, \eta)$ above is called a C^∞ n-parameter deformation of elliptic surfaces, if the following conditions are satisfied

i) δ and η are locally trivial C^∞-fibrations.

[15]We shall generally assume that $\epsilon = 0$ is the special value of the perturbation parameter, this assumption is of course made without loss of generality.

[16]In this we differ from Naruki.

ii) The restriction $\varphi|_\epsilon : S_\epsilon \to C_\epsilon$ is an elliptic fibration.

In the work presented in chapter 3 we will often take $\mathcal{E}$ to be a closed subset of $\mathbb{C}^n$ with its origin in the boundary of $\mathcal{E}$ and allow δ and η to exhibit root-like behaviour. This means more precisely that δ and η are locally trivial C^0-fibrations in this case we will refer to the data $(\Sigma, \Gamma, \mathcal{E}, \varphi, \delta, \eta)$ as a C^0 n-parameter deformation. We are in general only interested in local confluence of singular fibers, so we may often even take δ and η to be trivial C^∞ or C^0 n-parameter deformations.
From this point onward we will always consider a C^∞ n-parameter deformation. In accordance with section 2.4 we denote by C_ϵ^{sing} the set of singular points on C_ϵ and by C_ϵ^{reg} its complement in C_ϵ. We further define

$$\Gamma^{\text{sing}} = \bigcup_{\epsilon \in \mathcal{E}} C_\epsilon^{\text{sing}}, \qquad \Gamma^{\text{reg}} = \bigcup_{\epsilon \in \mathcal{E}} C_\epsilon^{\text{reg}}.$$

We denote by $\gamma_{\tilde{\epsilon}}(t)$ the C^∞ family of loops parameterized by t, where $\gamma_{\tilde{\epsilon}}(t) \in C_{\tilde{\epsilon}}^{\text{reg}}$ for all t and $\tilde{\epsilon} \in \tilde{\mathcal{E}}$ a m-dimensional submanifold of $\mathcal{E}$ which contains the special point 0.[17]
For every $\epsilon \in \mathcal{E}$ we may associate a monodromy to the loop $\gamma_{\tilde{\epsilon}}$; $M([\gamma_{\tilde{\epsilon}}])$. Furthermore we have that the period lattice P depends continuously on the parameter ϵ. Discreetness of the period lattice P implies that $M([\gamma_{\tilde{\epsilon}}])$ is constant with respect to $\tilde{\epsilon}$. If for some fixed $\tilde{\epsilon}$, $\gamma_{\tilde{\epsilon}}(t)$ is homotopic to the concatenation $\gamma_1 * \ldots * \gamma_k$ of loops, where naturally $\gamma_j \subset C_{\tilde{\epsilon}}^{\text{reg}}$, then we have that $M([\gamma_{\tilde{\epsilon}}]) = M([\gamma_1]) \cdot \ldots \cdot M([\gamma_k])$.
We now apply this notion to the confluence of singular fibers. To do so let us first give a precise definition of a confluence of singular fibers.
We say that the singular fibers $S_{c_i,\tilde{\epsilon}}$, with $c_i(\tilde{\epsilon}) \in C_{\tilde{\epsilon}}^{\text{sing}}$, $i = 1, \ldots, N$ of S_ϵ, flow together into the same singular fiber $S_{c_0,0}$ of S_0^{sing}, if there is a curve $\beta \subset \tilde{\mathcal{E}}$, which is parameterized by τ and sends 0 to the special point 0, such that $c_i(\beta(\tau))$ are discrete for τ in a small neighbourhood of 0, but not for $\tau = 0$ itself, as well as

$$\lim_{\tau \to 0} c_i(\beta(\tau)) = c_0.$$

Notice that this definition differs greatly from the definition given in [9]. For $\tau = 0$ we have a small neighbourhood if the origin $U_0 \subset C_0$, analytically diffeomorphic to $D = \{z \in \mathbb{C} \big| |z| < \delta\}$, where $\delta \in \mathbb{R}_{>0}$, such that δ is the only singular point in U_0. We choose $\gamma_0(t)$ to be the curve which winds around c_0 once in the counterclockwise direction. Assuming U_0 and $\mathcal{E}$ to be sufficiently small, γ_0 may be extended to some family $\gamma_{\beta(\tau)} \subset C_{\beta(\tau)}^{\text{reg}}$ of loops as mentioned above by, for example, imposing a (trivial) connection of $\delta : \Gamma \to \mathcal{E}$. [18] Note that every $\gamma_{\beta(\tau)}$ is homotopic to γ_0 in Γ^{reg}. A member $\gamma_{\beta(\tau)}(t)$ of this family with $\tau \neq 0$ runs around all the c_i's into which c_0 breaks up. If $\gamma_{\beta(\tau)}^i(t)$ denotes a counterclockwise loop winding around c_i once, there is a permutation σ such that $\gamma_{\beta(\tau)}$ is homotopic to

[17] For our purpose it will not be necessary to endow $\tilde{\mathcal{E}}$ with a complex structure.
[18] The same holds for the set U_0.

$\gamma^{\sigma(1)}_{\beta(\tau)} * \gamma^{\sigma(2)}_{\beta(\tau)} * \ldots \gamma^{\sigma(N)}_{\beta(\tau)}$, as can be seen by the following argument. Let us fix some base point, which we choose to lie on the curve $\gamma_{\beta(\tau)}$, for a given τ, denoted by p. Moreover choose for each c_i a parameterized curve γ_v^i connecting p with some point on the curve $\gamma^i_{\beta(\tau)}$, such that the γ_v^i do not intersect each other nor the $\gamma^i_{\beta(\tau)}$s and $\gamma_{\beta(\tau)}$. Finally let $\tilde{\delta}$ be a loop starting and ending at a point on $\gamma_{\beta(\tau)}$ winding around p once, counterclockwise, such that $\tilde{\delta}$ intersects γ_v^i only once and $\gamma_{\beta(\tau)}$ twice, as sketched in figure 2.7. Denote by σ the permutation of indices such that $\tilde{\delta}$ intersects $\gamma_v^{\sigma(1)}$ first, $\gamma_v^{\sigma(2)}$ second et cetera. We shall focus on the concatenation $\gamma_v^i * \gamma^i_{\beta(\tau)} * (\gamma_v^i)^{-1}$, where $(\gamma_v^i)^{-1}$ denotes the curve γ_v^i inversely parameterized. The concatenation $\gamma_v^{\sigma(1)} * \gamma^{\sigma(1)}_{\beta(\tau)} * (\gamma_v^{\sigma(1)})^{-1} * \gamma_v^{\sigma(2)} * \gamma^{\sigma(2)}_{\beta(\tau)} * (\gamma_v^{\sigma(2)})^{-1}$ for example encloses $c_{\sigma(1)}$ and $c_{\sigma(2)}$, where we note that $(\gamma_v^{\sigma(1)})^{-1} * \gamma_v^{\sigma(2)}$ may be deformed such that it no longer intersects p. The concatenation $\gamma_v^{\sigma(1)} * \gamma^{\sigma(1)}_{\beta(\tau)} * (\gamma_v^{\sigma(1)})^{-1} * \ldots * \gamma_v^{\sigma(N)} * \gamma^{\sigma(N)}_{\beta(\tau)} * (\gamma_v^{\sigma(N)})^{-1}$ clearly encloses $c_1, \ldots, c_N$ and is homotopic to $\gamma_{\beta(\tau)}$.

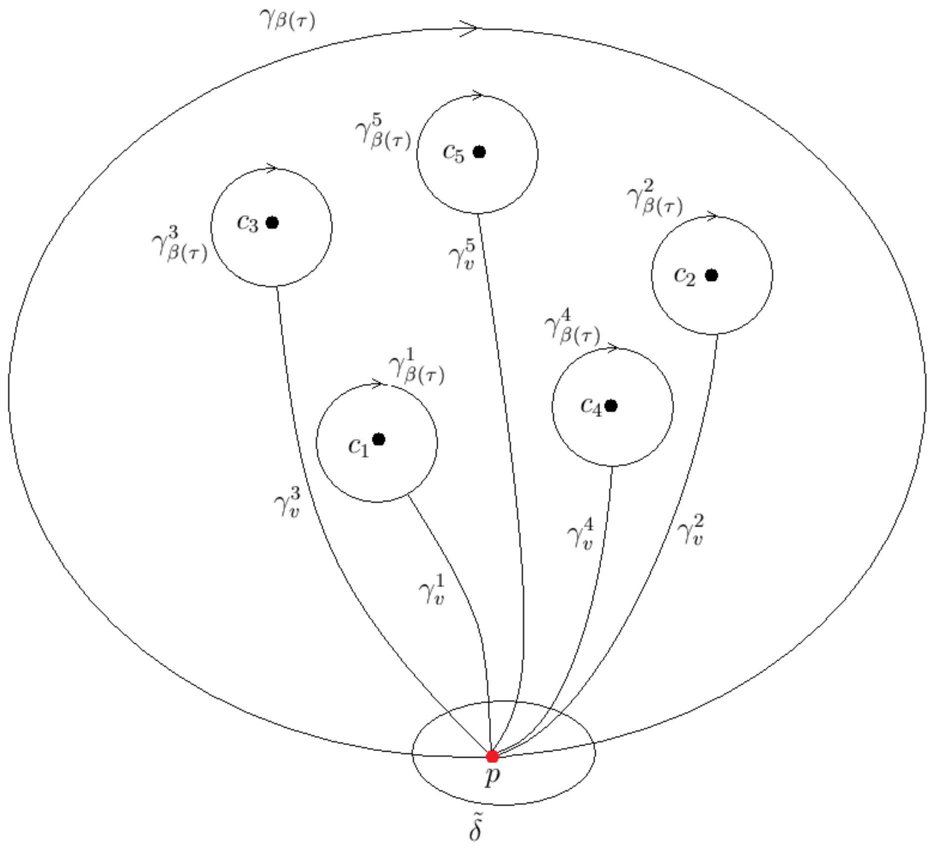


Figure 2.7: In this picture we sketch the curves $\gamma^i_{\beta(\tau)}$, γ_v^i, $\tilde{\delta}$ and $\gamma_{\beta(\tau)}$.

The consideration above yield that[19]

$$M_{S_c} = M_{S_{c_{\sigma(1)}}} \cdot \ldots \cdot M_{S_{c_{\sigma(N)}}}. \tag{2.16}$$

[19]The Ms are actually conjugacy classes in $\mathrm{SL}(2,\mathbb{Z})$

We will in chapter 3 consider deformations of Weierstrass models of elliptic surfaces, instead of the deformations of elliptic surfaces themselves, this turns out to be equivalent. We shall explain this in the following. If we choose to work in the local coordinate z (on C), for the Weierstrass model of a rational elliptic surface $\varphi : S \to C$, g_2 and g_3 are simply polynomials of fourth and sixth order in z, as has been established in section 2.6. We now write

$$\begin{aligned} g_2(z) &= \sum_{i=0}^{4} g_{2,i} z^i \\ g_3(z) &= \sum_{i=0}^{6} g_{3,i} z^i. \end{aligned} \tag{2.17}$$

We will now consider the $g_{2,i}$s and $g_{3,i}$s to be the parameters of deformation of a family of Weierstrass models of elliptic surfaces; corresponding to the coordinates of $\mathcal{E}$. This implies that we are faced with the following commutative diagram

$$\begin{tikzcd} & \Sigma \arrow[dl, "\varphi"'] \arrow[d, "f'"] \arrow[dd, bend left, "\eta"] \\ \Gamma \arrow[dr, "\delta"'] & \Omega \arrow[l, "p"'] \arrow[d, "\omega"] \\ & \mathcal{E} \end{tikzcd}$$

where Ω denotes the family of Weierstrass models W_ϵ, p is the projection so that $p|_\epsilon$ maps W_ϵ to C_ϵ, f'_ϵ is a minimal resolution of singularities, the restriction $\varphi|_\epsilon : C_\epsilon \to S_\epsilon$ is a minimal resolution of singularities and δ, η and ω are locally trivial C^∞-fibrations. The work of Tyurina [11] guarantees that the family of Weierstrass models generates a family of elliptic surfaces in a continuous manner, in particular she has proven that the resolution of singularities $f|_\epsilon$ extends continuously to a resolution of singularities of the entire family. So investigating the deformation of Weierstrass models of elliptic surfaces is equivalent to investigating the deformation of elliptic surfaces.

We end this section with some observations taken from remark 7.3.4 of [3]. As we have noticed in section 1.1 the zeros of a polynomial depend continuously on the coefficients of the polynomial and the number of zeros in D, a small neighbourhood, counted with multiplicity is invariant under small perturbations of the polynomial. Moreover we have seen in section 2.6 that singular fibers correspond to zeros of the geometric discriminant Δ and the topological Euler number of a singular fiber equals the order of the zero of the geometric discriminant. Combining these two remarks we find that the Euler number is conserved in confluences, in the sense that if several singular fibers flow together into one singular fiber that then the sum of the Euler numbers of the singular fibers before the confluence is equal to the Euler number of the singular fiber which was the product of the confluence. Likewise we have that the number zeros of g_2 and g_3 in D is invariant under perturbation, but since the zeros of g_2 and g_3 do not necessarily coincide to form the zero of the geometric discriminant, we have that the sum of the orders of the zeros of g_2 (g_3) of the merging singular fibers is less or equal to the order of the zero of g_2 (g_3) of the resulting singular fiber. This implies for example that singular fibers of type I_b may only be the result of a confluence of singular fibers of type I_{b_i}, with $\sum b_i = b$. The same argument gives that two "starred" types that is two singular fibers of the set $\{\mathrm{I}_0^*, \mathrm{I}_1^*, \ldots, \mathrm{IV}^*, \mathrm{III}^*, \mathrm{II}^*\}$, cannot merge. Finally note that if g_2 and g_3 have a linear factor in common, the discriminant of the geometric discriminant $\Delta(z) = g_2(z)^3 - 27g_3(z)^2$ is zero, implying that the resultant of $g_2(z)$ and $g_3(z)$ factors discriminant of the geometric discriminant.

Chapter 3

Confluence of singular fibres

As remarked in section 2.7 we may look upon confluences as the flowing together of several singular fibers or take the opposite view and consider a perturbation of a singular fiber from which a number of singular fibers of other type arise. We shall not distinguish between both views in this chapter. In the work presented we consider all possible singular fibers on rational elliptic surfaces, which may arise from the perturbation of a single singular fiber of type I_2, I_3, I_4, I_5, I_6, I_7, I_8, I_9, II, III, VI or I_0^*. We also remark on the perturbation of I_1^*. We always use that the topological Euler number is conserved under perturbations. For all confluences that do not arise we give one of the following arguments we have discussed in section 2.7:

- The sum of the zeros of g_2 (g_3) associated to merging singular fibers counted with multiplicity is less or equal the the zeros of g_2 (g_3) associated to the resulting singular fiber counted with multiplicity.

- The product of the conjugacy classes of the monodromy matrices associated to the merging singular fibers does not lie in the same conjugacy class as the monodromy matrix of the resulting singular fiber, see formula (2.16).

There has been previous work on the monodromy restrictions of the confluences of singular fibers, namely by Naruki [9], but this work focusses on the confluence of three singular fibers of type I_b. Moreover the article does not provide arguments to prove the existence of those confluences which are allowed by monodromy considerations. We present in the work below an explicit argument for the existence of the confluence $3\mathrm{I}_2 \to \mathrm{I}_0^*$, which according to section 5 [9] of would be disallowed.

3.1 The objective

As we have discussed in section 2.7 we can consider the coefficients of the polynomials g_2 and g_3 to be the parameters of a family of elliptic curves, which is relatively minimal if the zeros of g_2 and g_3 satisfy the conditions set in table 2.2. If we further confine g_2 and g_3 so that the geometric discriminant Δ has 12 zeros counted with multiplicity, the elliptic surface is rational.

We should be interested in the relation between the configuration of singular elliptical fibers and the values of the coefficients of g_2 and g_3. The generic configuration of singular fibers has 12 singular fibers of Kodaira type I_1. The fact that this is the generic configuration is can be seen as follows; if there would have been any other configuration the geometric discriminant Δ would have had zeros of at least second order, which would imply that the discriminant of the geometric discriminant would be zero. As noted in section 1.2 the discriminant is a polynomial in the coefficients of Δ, which in turn are polynomials in the coefficients of g_2 and g_3. This gives us that the set of coefficients of g_2 and g_3, which corresponds to non-generic configurations of singular fibers, is a complex analytic set, denoted by N_g, of codimension 1.[1] The generalized resultants and discriminants we have discussed in section 1.2 allow us to subdivide the space of coefficients of the polynomials g_2 and g_3 further. A more complicated configuration of singular elliptical fibers (in the sense that the zeros of g_2, g_3 or Δ are of higher order, see table 2.2) corresponds to complex analytic subsets of N_g of greater codimension. This subdivision of N_g into analytic sets of increasing codimension, where the set corresponding to a more complicated configuration of singular fibers may lie in the boundary of a less complicated one, is called a stratification.[2] We note that it is not clear whether the analytic set corresponding to a specific configuration is connected. The ultimate object of our line of research would be to have a complete understanding of the structure of set N_g. Should we understand the structure of N_g completely, we would know if certain singular fibers could flow together simply because (the connected components of) the strata associated to the different configurations are adjacent. Or, even stronger, we would know which (global) configurations of singular fibers can arise from the perturbation of a given configuration of singular fibers. This imposes a hierarchial structure on the list of all allowed configurations as found by Persson [10].

Completely determining the structure of this space N_g is far beyond the scope of this

[1] The fact that the dimension of N_g is one less then the dimension of the space in which it is imbedded in implies that every configuration of singular fibers, see [10], can be perturbed into a configuration with 12 singular fibers of type I_1.

[2] The codimension is not strictly increasing; for example set of configurations of singular fibers containing at least a singular fiber of type II, III, IV, II*, III*, IV* or I_b^* distinguishes itself by the condition that the resultant of g_2 and g_3 is zero, yielding a set of codimension one. The fact that the resultant of g_2 and g_3 is zero automatically implies that the discriminant of the geometric discriminant is also zero, see also section 2.7.

thesis, mainly because of the following difficulty: Section 1.2 provides us, at least in theory, with an explicit collection of sets of polynomials for a given stratum, satisfying the following condition: all polynomials in one of these sets are identically equal to zero on the stratum, while at least one of the polynomials of each other set is unequal to zero at every given point of the stratum. The polynomials in each of these sets, the generalized discriminants of Δ and the generalized resultants of g_2 and g_3, are very difficulty to calculate and not independent. It must be noted that there are computer algorithms which provide one with an independent basis, but applying these on the scale necessary to confront this problem is not feasible given the use present-day personal computers.[3] However we are provided with a collection of sets of polynomials, where each polynomial in every set is independent from any other polynomial in the same set, we are still faced with the problem that the conditions on the different sets of polynomials may be mutually exclusive. This last problem clearly arises as we can conclude from Persson [10]. Again in theory computer algebra should help to confront this problem, but fails to do so in practice.

Having considered the above difficulties we set about to work on a simpler problem; whether we can (locally) find confluences of a number of singular fibers into on singular fiber, in the sense we have discussed in section 2.7. This provides information on the relation between different strata, namely it tells us if one stratum lies in the boundary of another one.

We shall give examples of confluences by providing a family of the Weierstrass models. In section 3.2 a proof of the existence of confluences into a singular fiber of Kodaira type I_b, out of any combination of singular fibers I_{b_i}, with $\sum b_i = b$ is provided. In section 3.3 we find that apart from the obvious constriction of the Euler number before and after confluence, but one confluence of all confluence to singular fibers of type II, III and IV to be restricted by monodromy considerations, and provide explicit examples for the other confluences. In section 3.4 we do the same for the singular fiber of type I_0^*, though in this case 7 confluences are restricted by monodromy considerations.

[3]Mathematica for example can find a Gröbner basis.

3.2 Confluence to singular fibers of Kodaira type I_b.

As remarked in section 2.7 only singular fibers of type I_{b_i} may flow together to form a singular fiber of type I_b, with $b = \sum b_i$. This is easy to see because for any I_b the defining g_2 and g_3 are not equal to zero if the geometric discriminant is equal to zero. This implies that for a small perturbation of g_2 and g_3, these functions g_2 and g_3 and the geometric discriminant still do not share zeros and thus we are faced with a number of singular fibers of type I_{b_i}. In this section we will prove that every confluence of singular fibers on rational elliptic surfaces of type I_{b_i} into a singular fiber of type I_b with $b = \sum b_i$ may occur. We note that the restriction to rational elliptic surfaces implies that $b \leq 9$, as we can verify easily in the list of Persson [10]. The proof of existence given below differs from confluence to confluence:

- For a perturbation of a singular fiber of Kodaira type I_b into a singular fiber of type I_{b-e} and e singular fibers of type I_1 we will give an explicit example by giving formulae for g_2 and g_3 and verify the fact that $e\mathrm{I}_1$ singular fibers are created by using the discriminant, see section 1.2.

- For all singular fibers of Kodaira type I_b, with $b \leq 6$, not of the above type, we are able to give explicit examples, again by giving g_2 and g_3, moreover we will give the roots of the geometric discriminant Δ explicitly.

- For $\mathrm{I}_7 \to \mathrm{I}_5 + \mathrm{I}_2$, $\mathrm{I}_7 \to \mathrm{I}_4 + \mathrm{I}_2 + \mathrm{I}_1$ and $\mathrm{I}_8 \to \mathrm{I}_6 + \mathrm{I}_2$ we again use the discriminant, however we are no longer able to give g_2 and g_3 explicitly. Instead we give equations which the coefficients of the polynomials g_2 and g_3 must satisfy.

- For every other confluence we will use the combination of the Weierstrass preparation theorem and the implicit function theorem to prove existence of the confluence, this method is very implicit when it concerns the coefficients of g_2 and g_3. [4]

$\mathrm{I}_b \to \mathrm{I}_{b-e} + \mathrm{I}_1 + \ldots + \mathrm{I}_1 = \mathrm{I}_{b-e} + e\,\mathrm{I}_1$

We will give an outline of the argument for existence of all confluences of the form $\mathrm{I}_b \to \mathrm{I}_{b-e} + \mathrm{I}_1 + \ldots + \mathrm{I}_1 = \mathrm{I}_{b-e} + e\,\mathrm{I}_1$. We start out with a singular fiber of type I_b in the origin. The g_2 and g_3 yielding the singular fiber I_b are found by setting the first b coefficients of the geometric discriminant Δ to zero consecutively. Allowing the final e coefficients of the b coefficients set to zero to be perturbed into nonzero values generally yields a singular fiber of type I_{b-e} and e singular fibers of type I_1.

The actual proof of existence will be given by the construction of examples. We will for now distinguish between $b = 2, \ldots 6$ and $b = 7, 8, 9$.

[4] Method suggested by Hans Duistermaat.

$\mathbf{b = 2, \ldots, 6}$

As before we take

$$g_2(z) = \sum_{i=0}^{4} g_{2,i} z^i$$
$$g_3(z) = \sum_{i=0}^{6} g_{3,i} z^i$$

We choose $g_{2,0} = 3$ and thus to ensure that

$$\Delta(z) = g_2(z)^3 - 27 g_3(z)^2 = \mathcal{O}(z)$$

that is to study singular fibers of Kodaira type I_b we take $g_{3,0} = -1$. It is easy to see using table 2.2 that setting all other $g_{2,i}$ and $g_{3,i}$ to zero except $g_{3,b}$ we create a singular fiber of type I_b in the origin. Globally other singular fibers are created as well, generally of type I_1, but we will not concern ourselves with these global considerations.

Table 3.1: In this table the geometric discriminant in denoted by Δ and the discriminant with D.

Confluence	$g_2(z)$	$g_3(z)$	Behaviour of $\Delta(z)$
$\mathrm{I}_2 \to 2\mathrm{I}_1$	3	$-1 + \epsilon z + z^2$	$D(\Delta(z)) = 2^4 3^{18} \epsilon^2 (8 + \epsilon^2)$
$\mathrm{I}_3 \to 3\mathrm{I}_1$	3	$-1 + \epsilon z + z^3$	$D(\Delta(z)) = 2^{10} 3^{30} \epsilon^3 (27 + \epsilon^3)$
$\mathrm{I}_3 \to \mathrm{I}_2 + \mathrm{I}_1$	3	$-1 + \epsilon z^2 + z^3$	$D(\Delta(z)/z) = 2^2 3^{24} \epsilon^2 (-2^4 3^3 + 32\epsilon^3)$
$\mathrm{I}_4 \to 4\mathrm{I}_1$	3	$-1 + \epsilon z + z^4$	$D(\Delta(z)) = 2^8 3^{45} \epsilon^4 (2^{11} + 27\epsilon^3)$
$\mathrm{I}_4 \to \mathrm{I}_2 + 2\mathrm{I}_1$	3	$-1 + \epsilon z^2 + z^4$	$D(\Delta(z)/z) = 2^{13} 3^{36} \epsilon^3 (2 + \epsilon^2)^2$
$\mathrm{I}_4 \to \mathrm{I}_3 + \mathrm{I}_1$	3	$-1 + \epsilon z^3 + z^4$	$D(\Delta(z)/z^2) = 2^2 3^{30} \epsilon^2 (-2^{13} - 2^4 3^3 \epsilon^4)$
$\mathrm{I}_5 \to 5\mathrm{I}_1$	3	$-1 + \epsilon z + z^5$	$D(\Delta(z)) = 2^{22} 3^{54} \epsilon^5 (5^5 + 16\epsilon^5)$
$\mathrm{I}_5 \to \mathrm{I}_3 + 2\mathrm{I}_1$	3	$-1 + \epsilon z^3 + z^5$	$D(\Delta(z)/z^2) = 2^{12} 3^{42} \epsilon^3 (5^5 + 27\epsilon^5)$
$\mathrm{I}_5 \to \mathrm{I}_4 + \mathrm{I}_1$	3	$-1 + \epsilon z^4 + z^5$	$D(\Delta(z)/z^3) = 2^8 3^{36} \epsilon^2 (5^5 + 2^7 \epsilon^5)$
$\mathrm{I}_6 \to 6\mathrm{I}_1$	3	$-1 + \epsilon z + z^6$	$D(\Delta(z)) = 2^{12} 3^{66} 5^5 \epsilon^6 (2^{11} 3^6 + 5^5 \epsilon^6)$
$\mathrm{I}_6 \to \mathrm{I}_2 + 4\mathrm{I}_1$	3	$-1 + \epsilon z^2 + z^6$	$D(\Delta(z)/z) = 2^{29} 3^{60} \epsilon^5 (27 + \epsilon^3)^2$
$\mathrm{I}_6 \to \mathrm{I}_3 + 3\mathrm{I}_1$	3	$-1 + \epsilon z^3 + z^6$	$D(\Delta(z)/z^2) = 2^{10} 3^{63} \epsilon^5 (8 + \epsilon^2)^3$
$\mathrm{I}_6 \to \mathrm{I}_4 + 2\mathrm{I}_1$	3	$-1 + \epsilon z^4 + z^6$	$D(\Delta(z)/z^3) = 2^{19} 3^{48} \epsilon^3 (-27 + \epsilon^3)^2$
$\mathrm{I}_6 \to \mathrm{I}_5 + \mathrm{I}_1$	3	$-1 + \epsilon z^5 + z^6$	$D(\Delta(z)/z^4) = 2^8 3^{42} \epsilon^2 (2^7 3^6 + 5^5 \epsilon^6)$

If we now perturb $g_{3,b-e}$ that is we take $g_{3,b-e} = \epsilon$ we get for each $\epsilon > 0$ a singular fiber of type I_{b-e} in the origin. Furthermore we know that the roots of a polynomial are continuous functions of the coefficients of the polynomial, so this perturbation yields also e new zeros near zero. Since that $g_2(0) \neq 0$ as well as $g_3(0) \neq 0$, and therefore that $g_2(z) \neq 0$ and $g_3(z) \neq 0$ for all $z \in D$, a sufficiently small neighbourhood of the origin, these e new zeros correspond to singular fibers of type I_{b_i}. It now suffices to show that there are no singular fibers of type $\mathrm{I}_2, \mathrm{I}_3, \ldots, \mathrm{I}_e$, to prove that we have constructed a confluence of $\mathrm{I}_{b-e} + e\,\mathrm{I}_1$ to a singular fiber of type I_b. This can be done by verifying that the geometric discriminant has no zeros of order greater than 1, which is in turn equivalent to showing that the discriminant of the geometric discriminant Δ divided by z^{b-e} is not equal to zero for $\epsilon > 0$. The final step, showing to non-triviality of the $\Delta(z)/z^{b-e}$ is done by explicit calculation, see table 3.1. We note that if $e = 1$ that then this step is not necessary because we know thanks to table 2.2 that I_1 is the only singular fiber where the geometric discriminant Δ has a zero of order one.

$\mathbf{b = 7, 8, 9}$

The main difference between the case where $b = 1, \ldots, 6$ and $b = 7, 8, 9$ is that $g_{2,1}, \ldots, g_{2,4}$ are no longer set to zero and we thus are faced with more complicated perturbations, especially in the case $\mathrm{I}_8 \to \mathrm{I}_7 + \mathrm{I}_1$ and $\mathrm{I}_9 \to \mathrm{I}_8 + \mathrm{I}_1$.[5] We shall again give $g_2(z)$ and $g_3(z)$ depending on ϵ in table 3.2 as well as the discriminant of the geometrical discriminant divided by z^{b-e}. The fact that this discriminant is nonzero if the perturbation parameter ϵ is nonzero implies that there are only singular fibers of type I_1 outside of the origin, as before. To summarize the fact that the discriminant of the geometrical discriminant divided by z^{b-e} is nonzero implies that we have found a confluence of type $\mathrm{I}_b \to \mathrm{I}_{b-e} + e\,\mathrm{I}_1$.

Table 3.2: In this table the geometric discriminant in denoted by Δ and the discriminant with D.

Confluence	$g_2(z)$	$g_3(z)$	Behaviour of $\Delta(z)$
$\mathrm{I}_7 \to 7\mathrm{I}_1$	$3+z$	$-1+\epsilon z-\frac{z}{2}-\frac{z^2}{2^3 3}+\frac{z^3}{2^4 3^3}$ $-\frac{z^4}{2^7 3^3}+\frac{z^5}{2^8 3^4}-\frac{7z^6}{2^{10} 3^6}$	$D(\Delta(z)) = \frac{7^{13}}{2^{179} 3^{31}}\epsilon^7+\mathcal{O}(\epsilon^8)$
$\mathrm{I}_7 \to \mathrm{I}_2 + 5\mathrm{I}_1$	$3+z$	$-1-\frac{z}{2}-\frac{z^2}{2^3 3}+\epsilon z^2+\frac{z^3}{2^4 3^3}$ $-\frac{z^4}{2^7 3^3}+\frac{z^5}{2^8 3^4}-\frac{7z^6}{2^{10} 3^6}$	$D(\frac{\Delta(z)}{z}) = \frac{7^{13} 5^5}{2^{174} 3^{37}}\epsilon^6+\mathcal{O}(\epsilon^7)$
$\mathrm{I}_7 \to \mathrm{I}_3 + 4\mathrm{I}_1$	$3+z$	$-1-\frac{z}{2}-\frac{z^2}{2^3 3}+\frac{z^3}{2^4 3^3}+\epsilon z^3$ $-\frac{z^4}{2^7 3^3}+\frac{z^5}{2^8 3^4}-\frac{7z^6}{2^{10} 3^6}$	$D(\frac{\Delta(z)}{z^2}) = -\frac{7^{13}}{2^{157} 3^{38}}\epsilon^5+\mathcal{O}(\epsilon^6)$
$\mathrm{I}_7 \to \mathrm{I}_4 + 3\mathrm{I}_1$	$3+z$	$-1-\frac{z}{2}-\frac{z^2}{2^3 3}+\frac{z^3}{2^4 3^3}$ $-\frac{z^4}{2^7 3^3}+\epsilon z^4+\frac{z^5}{2^8 3^4}-\frac{7z^6}{2^{10} 3^6}$	$D(\frac{\Delta(z)}{z^3}) = -\frac{7^{13}}{2^{156} 3^{36}}\epsilon^4+\mathcal{O}(\epsilon^5)$
$\mathrm{I}_7 \to \mathrm{I}_5 + 2\mathrm{I}_1$	$3+z$	$-1-\frac{z}{2}-\frac{z^2}{2^3 3}+\frac{z^3}{2^4 3^3}$ $-\frac{z^4}{2^7 3^3}+\frac{z^5}{2^8 3^4}+\epsilon z^5-\frac{7z^6}{2^{10} 3^6}$	$D(\frac{\Delta(z)}{z^4}) = \frac{7^{13}}{2^{145} 3^{40}}\epsilon^3+\mathcal{O}(\epsilon^4)$
$\mathrm{I}_7 \to \mathrm{I}_6 + \mathrm{I}_1$	$3+z$	$-1-\frac{z}{2}-\frac{z^2}{2^3 3}+\frac{z^3}{2^4 3^3}$ $-\frac{z^4}{2^7 3^3}+\frac{z^5}{2^8 3^4}-\frac{7z^6}{2^{10} 3^6}+\epsilon z^6$	$D(\frac{\Delta(z)}{z^5}) = \frac{7^{13}}{2^{138} 3^{41}}\epsilon^2+\mathcal{O}(\epsilon^3)$
$\mathrm{I}_8 \to 8\mathrm{I}_1$	$3+z+\frac{73z^2}{12}$ $+z^3+z^4$	$-1-\frac{z}{2}+\epsilon z-\frac{37z^2}{2^2 3}$ $-\frac{7\cdot 31 z^3}{2^3 3^3}-\frac{5^2 z^4}{2^2 3}-\frac{z^5}{3}$	$D(\Delta(z)) = -2^{13} 3^{44} 7^7 2017\ \epsilon^8$ $+\mathcal{O}(\epsilon^9)$
$\mathrm{I}_8 \to \mathrm{I}_2 + 6\mathrm{I}_1$	$3+z+\frac{73z^2}{12}$ $+z^3+z^4$	$-1-\frac{z}{2}-\frac{37z^2}{2^2 3}+\epsilon z^2$ $-\frac{7\cdot 31 z^3}{2^3 3^3}-\frac{5^2 z^4}{2^2 3}-\frac{z^5}{3}$	$D(\frac{\Delta(z)}{z}) = -2^{19} 3^{45} 2017\ \epsilon^7$ $+\mathcal{O}(\epsilon^8)$
$\mathrm{I}_8 \to \mathrm{I}_3 + 5\mathrm{I}_1$	$3+z+\frac{73z^2}{12}$ $+z^3+z^4$	$-1-\frac{z}{2}-\frac{37z^2}{2^2 3}$ $-\frac{7\cdot 31 z^3}{2^3 3^3}+\epsilon z^3-\frac{5^2 z^4}{2^2 3}-\frac{z^5}{3}$	$D(\frac{\Delta(z)}{z^2}) = 2^{12} 3^{34} 5^5 2017\ \epsilon^6$ $+\mathcal{O}(\epsilon^7)$
$\mathrm{I}_8 \to \mathrm{I}_4 + 4\mathrm{I}_1$	$3+z+\frac{73z^2}{12}$ $+z^3+z^4$	$-1-\frac{z}{2}-\frac{37z^2}{2^2 3}$ $-\frac{7\cdot 31 z^3}{2^3 3^3}-\frac{5^2 z^4}{2^2 3}+\epsilon z^4-\frac{z^5}{3}$	$D(\frac{\Delta(z)}{z^3}) = 2^{19} 3^{29} 2017\ \epsilon^5$ $+\mathcal{O}(\epsilon^6)$
$\mathrm{I}_8 \to \mathrm{I}_5 + 3\mathrm{I}_1$	$3+z+\frac{73z^2}{12}$ $+z^3+z^4$	$-1-\frac{z}{2}-\frac{37z^2}{2^2 3}$ $-\frac{7\cdot 31 z^3}{2^3 3^3}-\frac{5^2 z^4}{2^2 3}-\frac{z^5}{3}+\epsilon z^5$	$D(\frac{\Delta(z)}{z^4}) = -2^{10} 3^{27} 2017\ \epsilon^4$ $+\mathcal{O}(\epsilon^5)$
$\mathrm{I}_8 \to \mathrm{I}_6 + 2\mathrm{I}_1$	$3+z+\frac{73z^2}{12}$ $+z^3+z^4$	$-1-\frac{z}{2}-\frac{37z^2}{2^2 3}$ $-\frac{7\cdot 31 z^3}{2^3 3^3}-\frac{5^2 z^4}{2^2 3}-\frac{z^5}{3}+\epsilon z^6$	$D(\frac{\Delta(z)}{z^5}) = -2^{11} 3^{19} 2017\ \epsilon^3$ $+\mathcal{O}(\epsilon^4)$

[5] The Weierstrass model for global configuration $\mathrm{I}_9\, 3\mathrm{I}_1$ used in the construction of the confluences to a singular fiber of type I_9 was pointed out to me by Hans Duistermaat.

Table 3.2 – continued from previous page

Confluence	$g_2(z)$	$g_3(z)$	Behaviour of $\Delta(z)$
$\mathrm{I}_8 \to \mathrm{I}_7 + \mathrm{I}_1$	$3 + z + \frac{73z^2}{12} + \epsilon z^2 + z^3 + z^4$	$-1 - \frac{z}{2} - \frac{37+2\cdot 3\epsilon}{2^2 3} z^2 - \frac{7\cdot 31+2\cdot 3^2\epsilon}{2^3 3^3} z^3 - \frac{2\cdot 19+2^2 3\epsilon+\epsilon^2}{2^3 3} z^4 - \frac{2^2 3^2-\epsilon^2}{2^4 3^2} z^5 + \frac{2^4 3^3+2^3 3^3\epsilon+5\cdot 7\epsilon^2+2\epsilon^3}{2^5 3^3} z^6$	$D(\frac{\Delta(z)}{z^6}) = \frac{3^{30} 23\cdot 313}{2^{16}}\epsilon^2 + \mathcal{O}(\epsilon^3)$
$\mathrm{I}_9 \to 9\mathrm{I}_1$	$\frac{1}{12} + \frac{z}{3} + \epsilon z + \frac{z^2}{2} - \frac{5\cdot 23 z^3}{3^2} - \frac{7\cdot 67 z^4}{2^2 3^2}$	$-\frac{1}{2^3 3^3} - \frac{z}{2^2 3^2} - \frac{5z^2}{2^3 3^2} + z^3 + \frac{7\cdot 11 z^4}{2^3 3} + \frac{13 z^5}{2^2} - \frac{27143 z^6}{2^3 3^4}$	$D(\frac{\Delta(z)}{z}) = \frac{59^{41}}{2^4 3^{80}}\epsilon^7 + \mathcal{O}(\epsilon^8)$
$\mathrm{I}_9 \to \mathrm{I}_2 + 7\mathrm{I}_1$	$\frac{1}{12} + \frac{z}{3} + \frac{z^2}{2} - \frac{5\cdot 23 z^3}{3^2} - \frac{7\cdot 67 z^4}{2^2 3^2}$	$-\frac{1}{2^3 3^3} - \frac{z}{2^2 3^2} - \frac{5z^2}{2^3 3^2} + \epsilon z^2 + z^3 + \frac{7\cdot 11 z^4}{2^3 3} + \frac{13 z^5}{2^2} - \frac{27143 z^6}{2^3 3^4}$	$D(\frac{\Delta(z)}{z^2}) = \frac{7^7 59^{38}}{2^{12} 3^{67}}\epsilon^6 + \mathcal{O}(\epsilon^7)$
$\mathrm{I}_9 \to \mathrm{I}_3 + 6\mathrm{I}_1$	$\frac{1}{12} + \frac{z}{3} + \frac{z^2}{2} + \epsilon z^2 - \frac{5\cdot 23 z^3}{3^2} - \frac{7\cdot 67 z^4}{2^2 3^2}$	$-\frac{1}{2^3 3^3} - \frac{z}{2^2 3^2} - \frac{5z^2}{2^3 3^2} - \frac{\epsilon z^2}{2^2 3} + z^3 + \frac{7\cdot 11 z^4}{2^3 3} + \frac{13 z^5}{2^2} - \frac{27143 z^6}{2^3 3^4}$	$D(\frac{\Delta(z)}{z^3}) = \frac{59^{35}}{2^9 3^{60}}\epsilon^6 + \mathcal{O}(\epsilon^7)$
$\mathrm{I}_9 \to \mathrm{I}_4 + 5\mathrm{I}_1$	$\frac{1}{12} + \frac{z}{3} + \frac{z^2}{2} - \frac{5\cdot 23 z^3}{3^2} - \frac{7\cdot 67 z^4}{2^2 3^2} + \epsilon z^4$	$-\frac{1}{2^3 3^3} - \frac{z}{2^2 3^2} - \frac{5z^2}{2^3 3^2} + z^3 + \frac{7\cdot 11 z^4}{2^3 3} + \frac{13 z^5}{2^2} - \frac{27143 z^6}{2^3 3^4}$	$D(\frac{\Delta(z)}{z^4}) = \frac{5^5 59^{32}}{2^{16} 3^{59}}\epsilon^4 + \mathcal{O}(\epsilon^5)$
$\mathrm{I}_9 \to \mathrm{I}_5 + 4\mathrm{I}_1$	$\frac{1}{12} + \frac{z}{3} + \frac{z^2}{2} - \frac{5\cdot 23 z^3}{3^2} - \frac{7\cdot 67 z^4}{2^2 3^2} - 12\epsilon z^4$	$-\frac{1}{2^3 3^3} - \frac{z}{2^2 3^2} - \frac{5z^2}{2^3 3^2} + z^3 + \frac{7\cdot 11 z^4}{2^3 3} + \epsilon z^4 + \frac{13 z^5}{2^2} - \frac{27143 z^6}{2^3 3^4}$	$D(\frac{\Delta(z)}{z^5}) = \frac{2^5 59^{29}}{3^{49}}\epsilon^3 + \mathcal{O}(\epsilon^4)$
$\mathrm{I}_9 \to \mathrm{I}_6 + 3\mathrm{I}_1$	$\frac{1}{12} + \frac{z}{3} + \frac{z^2}{2} - \frac{5\cdot 23 z^3}{3^2} - \frac{7\cdot 67 z^4}{2^2 3^2} - 6\epsilon z^4$	$-\frac{1}{2^3 3^3} - \frac{z}{2^2 3^2} - \frac{5z^2}{2^3 3^2} + z^3 + \frac{7\cdot 11 z^4}{2^3 3} + \frac{\epsilon z^4}{2} + \frac{13 z^5}{2^2} + \epsilon z^5 - \frac{27143 z^6}{2^3 3^4}$	$D(\frac{\Delta(z)}{z^6}) = \frac{59^{26}}{2^6 3^{40}}\epsilon^2 + \mathcal{O}(\epsilon^3)$
$\mathrm{I}_9 \to \mathrm{I}_7 + 2\mathrm{I}_1$	$\frac{1}{12} + \frac{z}{3} + \frac{z^2}{2} - \frac{5\cdot 23 z^3}{3^2} - \frac{7\cdot 67 z^4}{2^2 3^2} - 12\epsilon z^4$	$-\frac{1}{2^3 3^3} - \frac{z}{2^2 3^2} - \frac{5z^2}{2^3 3^2} + z^3 + \frac{7\cdot 11 z^4}{2^3 3} + \epsilon z^4 + \frac{13 z^5}{2^2} + 2\epsilon z^5 - \frac{27143 z^6}{2^3 3^4} + \epsilon z^6$	$D(\frac{\Delta(z)}{z^7}) = \frac{2^2 59^{24}}{3^{38}}\epsilon + \mathcal{O}(\epsilon^2)$
$\mathrm{I}_9 \to \mathrm{I}_8 + \mathrm{I}_1$	$\frac{1}{12} + \frac{z}{3} + \frac{(1-\epsilon)z^2}{2} - \frac{5\cdot 23 z^3}{3^2} + \epsilon z^3 - \frac{7\cdot 67 z^4}{2^2 3^2} + \frac{3\epsilon(2+\epsilon)z^4}{2^2}$	$-\frac{1}{2^3 3^3} - \frac{z}{2^2 3^2} - \frac{5z^2}{2^3 3^2} + \frac{\epsilon z^2}{2^3 3} + z^3 + \frac{7\cdot 11 z^4}{2^3 3} + \epsilon z^4 - \frac{\epsilon(2+\epsilon)z^4}{8} + \frac{13 z^5}{2^2} - \frac{\epsilon(2\cdot 5\cdot 13-3^2\epsilon)z^5}{2^2 3^2} - \frac{27143 z^6}{2^3 3^4} + \frac{\epsilon(233-3\cdot 5\epsilon+3\epsilon^2)z^6}{2^3 3}$	$D(\frac{\Delta(z)}{z^8}) = \frac{59^{20}}{3^{31}} + \mathcal{O}(\epsilon)$

Confluences found by explicit calculation

In some very rare cases we are able to find a very elegant geometric discriminant which factors. In these cases the zeros of the geometric discriminant are easily found by explicit computation. From the fact that only I_{b_i} may merge into a I_b, as mentioned above, we derive that zeros of order b_i correspond to singular fibers of type I_{b_i}. These examples are listed in table 3.3.

Table 3.3: In this table the geometric discriminant in denoted by Δ

Confluence	$g_2(z)$	$g_3(z)$	$\Delta(z)$	Roots near $z=0$
$\mathrm{I}_4 \to 2\mathrm{I}_2$	3	$-1+\frac{\epsilon^2 z^2}{4}$	$-\frac{3^3}{2^4}z^2(2z+\epsilon)^2\times$	0, 0,
		$+\epsilon z^3+z^4$	$(-8+z^2\epsilon^2+4\epsilon z^3+4z^4)$	$\epsilon/2$, $\epsilon/2$
$\mathrm{I}_5 \to \mathrm{I}_3+\mathrm{I}_2$	3	$-1+\frac{\epsilon^2 z^3}{4}$	$-\frac{3^3}{2^4}z^3(2z+\epsilon)^2\times$	0, 0, 0
		$+\epsilon z^4+z^5$	$(-8+z^3\epsilon^2+4\epsilon z^4+4z^5)$	$\epsilon/2$, $\epsilon/2$
$\mathrm{I}_5 \to 2\mathrm{I}_2+\mathrm{I}_1$	3	$-1+\epsilon z^2$	$-\frac{3^3}{2^2}z^2\times$	0, 0
		$-\frac{3\epsilon^{2/3}}{2^{2/3}}z^3+z^5$	$(2\epsilon-3\cdot 2^{1/3}\epsilon^{2/3}z+2z^2)\times$	$(\epsilon/2)^{1/3}$, $(\epsilon/2)^{1/3}$,
			$(-4+2\epsilon z^2-3\cdot 2^{1/3}\epsilon^{2/3}z^3+2z^5)$	$-2^{2/3}\epsilon^{1/3}$
$\mathrm{I}_6 \to \mathrm{I}_4+\mathrm{I}_2$	3	$-1+\frac{\epsilon^2 z^4}{4}$	$-\frac{3^3}{2^4}z^4(2z+\epsilon)^2\times$	0, 0, 0, 0
		$+\epsilon z^5+z^6$	$(-8+\epsilon^2 z^4+4\epsilon z^5+4z^6)$	$\epsilon/2$, $\epsilon/2$
$\mathrm{I}_6 \to \mathrm{I}_3+\mathrm{I}_2$	3	$-1+\epsilon z^3$	$-\frac{3^3}{2^2}z^3\times$	0, 0, 0
$+\mathrm{I}_1$		$-\frac{3\epsilon^{2/3}}{2^{2/3}}z^4+z^6$	$(2\epsilon-3\cdot 2^{1/3}\epsilon^{2/3}z+2z^3)\times$	$(\epsilon/2)^{1/3}$, $(\epsilon/2)^{1/3}$,
			$(-4+2\epsilon z^3-3\cdot 2^{1/3}\epsilon^{2/3}z^4+2z^6)$	$-2^{2/3}\epsilon^{1/3}$
$\mathrm{I}_6 \to 2\mathrm{I}_2+2\mathrm{I}_1$	3	$-1+\frac{3\epsilon^{4/3}}{2^{2+2/3}}z^2$	$-\frac{3^3}{2^6}z^2\times$	0, 0,
		$+\epsilon z^3+z^6$	$(3\cdot 3^{1/3}\epsilon^{4/3}-2^3\epsilon z+2^3 z^4)\times$	$\frac{1}{2}(2^{1/3}\epsilon^{1/3}+i2^{5/6}\epsilon^{1/3})$,
			$(-2^4+2^{1/3}3\epsilon^{4/3}z^2$	$\frac{1}{2}(2^{1/3}\epsilon^{1/3}-i2^{5/6}\epsilon^{1/3})$,
			$+2^3\epsilon z^3+2^3 z^6)$	$-\frac{\epsilon^{1/3}}{2^{2/3}}$, $-\frac{\epsilon^{1/3}}{2^{2/3}}$
$\mathrm{I}_6 \to 3\mathrm{I}_2$	3	$-1+\frac{\epsilon^2}{2^2}z^2$	$-\frac{3^3}{2^4}z^2(2z^2-\epsilon)^2\times$	0, 0,
		$-\epsilon z^4+z^6$	$(-2^3+\epsilon^2 z^2-2^2\epsilon z^4+2^2 z^6)$	$\sqrt{\frac{\epsilon}{2}}$, $\sqrt{\frac{\epsilon}{2}}$,
				$-\sqrt{\frac{\epsilon}{2}}$, $-\sqrt{\frac{\epsilon}{2}}$
$\mathrm{I}_6 \to 2\mathrm{I}_3$	3	$-1+\epsilon^3 z^3$	$-27z^3(z+\epsilon)^3\times$	0, 0,0,
		$+3\epsilon^2 z^4$	$(-2+z^6+3z^5\eta+3z^4\eta^2+z^3\eta^3)$	$-\epsilon$, $-\epsilon$, $-\epsilon$
		$+3\epsilon z^5+z^6$		

Confluences and the discriminant

The following confluence are not unlike the previous set given, however the perturbation parameter will now generally be a solution to some polynomial equation. Moreover since the polynomial expressions are so complicated the roots can no longer be given explicitly.

$\mathrm{I}_7 \to \mathrm{I}_5 + \mathrm{I}_2$
We consider a Weierstrass model determined by

$$g_2(z) = 3 + z^3 + z^4$$
$$g_3(z) = -1 - \frac{z^3}{2} - \frac{z^4}{2} + \eta z^5 - \frac{z^6}{2^3 3} + \epsilon z^6.$$

We now note that for $\epsilon, \eta = 0$ the geometric discriminant behaves as follows

$$D\left(\frac{\Delta(z)}{z^7}\right) = \frac{3^7 2207^3}{2^{26}}.$$

This implies that the singular fibers outside the origin are of type I_1. From continuity we may conclude that these singular fibers do not merge for sufficiently small ϵ, η. We note that

$$\Delta(z) = 2 \cdot 3^3 \eta z^5 + \mathcal{O}(z^6)$$

and that

$$\begin{aligned} D\left(\frac{\Delta(z)}{z^5}\right) = \frac{3^{14}}{2^{30}}&(2207 - 2^6 3 \cdot 7 \cdot 31\epsilon + 2^7 3^5 5\epsilon^2 + 2^{12} 3^7 \epsilon^4 \\ &+ 2^5 3 \cdot 257\eta + 2^9 3^3 5\epsilon\eta - 2^{11} 3^5 7\epsilon^2\eta + 2^{11} 3^3 \eta^2 + 2^{13} 3^5 \epsilon\eta^2 \\ &+ 2^9 3^5 \eta^3 - 2^{12} 3^6 \epsilon\eta^3 + 2^{12} 3^6 \eta^4)^3 (2^4 3^3 \epsilon^2 - 2^7 3^4 \epsilon^3 - 2^4 3^6 \epsilon^4 \\ &+ 2^7 3^7 \epsilon^5 - 2^4 3^2 \eta + 2^4 3^5 \epsilon\eta \\ &+ 2^6 3^4 \epsilon^2\eta - 2^4 3^5 5 \cdot 7\epsilon^3\eta - 2^2 3 \cdot 5^2 11\eta^2 + 2^5 3^2 5 \cdot 7\epsilon\eta^2 \\ &+ 2^2 3^4 5^2 11\epsilon^2\eta^2 - 2^2 2089\eta^3 + 2^7 3 \cdot 5^2 \epsilon\eta^3 - 2^3 3^4 5^3 \epsilon^2\eta^3 \\ &- 2^4 3 \cdot 5^3 \eta^4 + 2^2 3^3 5^4 \epsilon\eta^4 + 3 \cdot 5^5 \eta^6). \end{aligned}$$

We wish to set $D(\Delta(z)/z^5)$ equal to zero because this would imply that Δ is of the following form

$$\Delta(z) = z^5 (z - z_0)^2 u(z),$$

with z_0 and $u(z)$ is a unit in a small neighbourhood of the origin and depends on ϵ and η, z_0 is zero for $\epsilon, \eta = 0$ but not identically equal to zero. This in turn would imply that we have a singular fiber of type I_2 and a singular fiber of type I_5 in the origin, apart from some singular fiber of type I_1 some distance from the origin. To find a solution curve running through the origin in the ϵ, η-plane such that $D(\Delta(z)/z^5) = 0$ it is sufficient to find a solution curve running through the origin in the ϵ, η-plane of

$$\begin{aligned}2^4 3^3 \epsilon^2 - 2^7 3^4 \epsilon^3 - 2^4 3^6 \epsilon^4 + 2^7 3^7 \epsilon^5 - 2^4 3^2 \eta + 2^4 3^5 \epsilon\eta + 2^6 3^4 \epsilon^2 \eta - 2^4 3^5 5 \cdot 7 \epsilon^3 \eta \\ -2^2 3 \cdot 5^2 11 \eta^2 + 2^5 3^2 5 \cdot 7 \epsilon \eta^2 + 2^2 3^4 5^2 11 \epsilon^2 \eta^2 - 2^2 2089 \eta^3 \\ +2^7 3 \cdot 5^2 \epsilon \eta^3 - 2^3 3^4 5^3 \epsilon^2 \eta^3 - 2^4 3 \cdot 5^3 \eta^4 + 2^2 3^3 5^4 \epsilon \eta^4 + 3 \cdot 5^5 \eta^6 = 0.\end{aligned}$$

Such a solution curve obviously exists.

$\mathrm{I}_8 \to \mathrm{I}_6 + \mathrm{I}_2$

We consider a Weierstrass model determined by

$$\begin{aligned}g_2(z) &= 3 + \frac{z^2}{12} + \epsilon z^2 + z^3 + z^4 \\ g_3(z) &= -1 - \frac{z}{2} - \frac{1 + 2 \cdot 3\epsilon}{2^2 3} z^2 - \frac{1 + 2 \cdot 3^2 \epsilon}{2^3 3^3} z^3 - \frac{2^4 3 + \epsilon^2}{2^3 3} z^4 + \frac{-2^4 3 + \epsilon^2}{2^4 3^2} z^5 + \eta z^6.\end{aligned}$$

We note that for $\epsilon, \eta = 0$

$$D\left(\frac{\Delta(z)}{z^8}\right) = 2^{20} 3^4 5 \cdot 73 \cdot 101$$

and in general

$$\Delta(z) = \frac{2^5 3^2 \epsilon + \epsilon^2 - 2\epsilon^3 + 2^5 3^3 \eta}{2^4} z^6 + \mathcal{O}(z^7).$$

Furthermore we have that

$$\begin{aligned}D\left(\frac{\Delta(z)}{z^5}\right) = -\frac{1}{2^{52} 3^{12}} (-2^5 3^2 \epsilon - \epsilon^2 + 2\epsilon^3 \eta)^2 \times \\ (-2^4 3^2 \epsilon + \epsilon^3 - 2^4 3^3 \eta)\big(-2^9 3^4 5 \cdot 73 \cdot 101 + \mathcal{O}(\epsilon, \eta)\big).\end{aligned}$$

By the same argument as before choosing a solution to either

$$-2^5 3^2 \epsilon - \epsilon^2 + 2\epsilon^3 \eta = 0$$

or

$$-2^4 3^2 \epsilon + \epsilon^3 - 2^4 3^3 \eta = 0$$

gives us a confluence of type $\mathrm{I}_8 \to \mathrm{I}_6 + \mathrm{I}_2$. It is remarkable that we are able to chose between two solutions, even in this restricted setting.

$I_7 \rightarrow I_4 + I_2 + I_1$

We now consider a Weierstrass model detemined by

$$g_2(z) = 3 + z^3 + z^4$$
$$g_3(z) = -1 - \frac{z^3}{2} - \frac{1-\epsilon}{2} z^4 + \eta z^5 + \frac{z^6}{2^3 3}.$$

We note that for $\epsilon, \eta = 0$

$$D\left(\frac{\Delta(z)}{z^7}\right) = \frac{3^7 2207^3}{2^{26}}$$

and in general

$$\Delta(z) = 3^3 z^4 + \mathcal{O}(z^5).$$

Since we are interested in a confluence of type $I_7 \rightarrow I_4 + I_2 + I_1$ we have to prove that Δ behaves as follows for a certain curve in ϵ, η-space, going through the origin,

$$\Delta(z) = z^4 (z - z_1)^2 (z - z_2) u(z),$$

where z_1 and z_2 are unequal to zero and to each other for a point on the curve not equal to the origin and $u(z)$ is a unit. To achieve this, it is sufficient to show that there exists a curve γ in ϵ, η-space going through the origin such that

$$D\left(\frac{\Delta(z)}{z^4}\right)\Bigg|_\gamma = 0,$$

but

$$D\left(\frac{\Delta'(z)}{z^3}\right)\Bigg|_\gamma \neq 0.$$

We note that if

$$D\left(\frac{\Delta(z)}{z^4}\right)$$

and

$$D\left(\frac{\Delta'(z)}{z^3}\right)$$

have only $\epsilon, \eta = 0$ as a common zero in a neighbourhood of the origin in the ϵ, η-plane, that is if the resultant of these two discriminants as a function of η as well the resultant of these two discriminants as a function of ϵ is zero, it is sufficient to find a curve γ in the ϵ, η-plane such that

$$D\left(\frac{\Delta(z)}{z^4}\right)\Bigg|_\gamma = 0.$$

We first fond the discriminant of the geometric discriminant divided by z^4

$$
\begin{aligned}
D\left(\frac{\Delta(z)}{z^4}\right) =& \frac{3^{17}}{2^{36}}\big(2207 - 2^{10}3\cdot 5\epsilon + 2^5 3^3 59\epsilon^2 - 2^6 3^4 17\epsilon^3 + 2^8 3^5\epsilon^4 + 2^5 3\cdot 257\eta \\
& - 2^5 3^3 179\epsilon\eta + 2^9 3^4 7\epsilon^2\eta - 2^9 3^5\epsilon^3\eta + 2^{11}3^3\eta^2 + 2^9 3^5\eta^3 + 2^{12}3^6\eta^4\big)\times \\
& \big(- 2^6 3^7\epsilon^2 + 2^7 3^5 37\epsilon^3 - 7\cdot 89\cdot 2213\epsilon^4 + 2^6 3\cdot 3361\epsilon^5 - 2^7 3^3 41\epsilon^6 + 2^6 3\cdot 61\epsilon^7 \\
& - 2^6 3^5 7\cdot 31\epsilon^2\eta + 2^6 3^2 5\cdot 23\cdot 103\epsilon^3\eta - 2^6 3\cdot 19259\epsilon^4\eta + 2^6 3^2 5\cdot 251\epsilon^5\eta \\
& - 2^{12}3^2\epsilon^6\eta - 2^9 3^6\epsilon\eta^2 - 2^8 3^2 7\cdot 17\cdot 31\epsilon^2\eta^2 + 2^4 3^3 18077\epsilon^3\eta^2 - 2^9 3^2 383\epsilon^4\eta^2 \\
& + 2^{10}3^2 19\epsilon^5\eta^2 - 2^9 3^3\epsilon^6\eta^2 - 2^{12}3^5\eta^3 - 2^9 3^4\epsilon\eta^3 - 2^9 3^3 491\epsilon^2\eta^3 + 2^9 3^4 61\epsilon^3\eta^3 \\
& - 2^9 3\cdot 233\epsilon^4\eta^3 - 2^{10}3^4 5^2 11\eta^4 + 2^{14}3^4 17\epsilon\eta^4 - 2^5 3^4 3001\epsilon^2\eta^4 - 2^{10}3^3 5\cdot 19\epsilon^3\eta^4 \\
& + 2^{11}3^4 5\epsilon^4\eta^4 - 2^{10}3^4\epsilon^5\eta^4 - 2^{10}3^3 2089\eta^5 + 2^{10}3^4 5^2 13\epsilon\eta^5 + 2^{10}3^3 5^2 13\epsilon^2\eta^5 \\
& - 2^{10}3^4 5^2\epsilon^3\eta^5 - 2^{12}3^4 5^3\eta^6 + 2^8 3^5 5^3\epsilon\eta^6 + 2^8 3^4 5^5\eta^8\big)
\end{aligned}
$$

and then the resultants

$$
\begin{aligned}
R_\epsilon\left(D\left(\frac{\Delta(z)}{z^4}\right), D\left(\frac{\Delta'(z)}{z^3}\right)\right) &= c_\epsilon\eta^6 \\
R_\eta\left(D\left(\frac{\Delta(z)}{z^4}\right), D\left(\frac{\Delta'(z)}{z^3}\right)\right) &= c_\eta\epsilon^6,
\end{aligned}
$$

where R_ϵ indicates the resultant with respect to ϵ, R_η the resultant with respect to η and c_ϵ and c_η are constants. We may now conclude that a solution curve γ to the equation

$$
\begin{aligned}
0 =& - 2^6 3^7\epsilon^2 + 2^7 3^5 37\epsilon^3 - 7\cdot 89\cdot 2213\epsilon^4 + 2^6 3\cdot 3361\epsilon^5 - 2^7 3^3 41\epsilon^6 + 2^6 3\cdot 61\epsilon^7 \\
& - 2^6 3^5 7\cdot 31\epsilon^2\eta + 2^6 3^2 5\cdot 23\cdot 103\epsilon^3\eta - 2^6 3\cdot 19259\epsilon^4\eta + 2^6 3^2 5\cdot 251\epsilon^5\eta \\
& - 2^{12}3^2\epsilon^6\eta - 2^9 3^6\epsilon\eta^2 - 2^8 3^2 7\cdot 17\cdot 31\epsilon^2\eta^2 + 2^4 3^3 18077\epsilon^3\eta^2 - 2^9 3^2 383\epsilon^4\eta^2 \\
& + 2^{10}3^2 19\epsilon^5\eta^2 - 2^9 3^3\epsilon^6\eta^2 - 2^{12}3^5\eta^3 - 2^9 3^4\epsilon\eta^3 - 2^9 3^3 491\epsilon^2\eta^3 + 2^9 3^4 61\epsilon^3\eta^3 \\
& - 2^9 3\cdot 233\epsilon^4\eta^3 - 2^{10}3^4 5^2 11\eta^4 + 2^{14}3^4 17\epsilon\eta^4 - 2^5 3^4 3001\epsilon^2\eta^4 - 2^{10}3^3 5\cdot 19\epsilon^3\eta^4 \\
& + 2^{11}3^4 5\epsilon^4\eta^4 - 2^{10}3^4\epsilon^5\eta^4 - 2^{10}3^3 2089\eta^5 + 2^{10}3^4 5^2 13\epsilon\eta^5 + 2^{10}3^3 5^2 13\epsilon^2\eta^5 \\
& - 2^{10}3^4 5^2\epsilon^3\eta^5 - 2^{12}3^4 5^3\eta^6 + 2^8 3^5 5^3\epsilon\eta^6 + 2^8 3^4 5^5\eta^8
\end{aligned}
$$

has all desired properties.

Confluences and the Weierstrass preparation theorem

All confluences other then the ones mentioned above will not be given explicitly, but existence will be proven by making use of the Weierstrass preparation theorem and the implicit function theorem. As mentioned in section 1.1 it is sufficient to investigate the behaviour of the Weierstrass polynomial of a function $f(z)$ to deduce the local information about the zeros of $f(z)$. In the following the geometric discriminant $\Delta(z)$, determined by

$$\Delta(z) = g_2(z)^2 - 27g_3(z)^2$$

will play the role of $f(z)$. As in section 1.1 we allow the perturbation parameter, denoted by δ, to be higher dimensional. In explicit examples we will denote the perturbation of each coefficient of the polynomial g_2 and g_3 by a different Greek letter. We shall take $g_2(0), g_3(0) \neq 0$ and $\Delta(z)$ to have a zero of order b in the origin for $\delta = 0$, corresponding to a singular fiber of type I_b in the origin. We will prove the existence of a (one-dimensional) curve in the perturbation parameter space δ, denoted by δ', and thus the existence of $c_{1,\delta'}, c_{2,\delta'}, \ldots c_{b,\delta'}$ in

$$W_{\delta'}(z) = z^b + c_{1,\delta'}z^{b-1} + c_{2,\delta'}z^{b-2} + \ldots + c_{b,\delta'},$$

such that the $W(z)$, for $\delta' \neq 0$, has zeros of the desired orders $b_1, \ldots, b_j$. The fact that $W(z)$ has zeros of order $b_1, \ldots, b_j$ is sufficient to conclude that we are faced with a confluence $\mathrm{I}_b \to \mathrm{I}_{b_1} + \ldots + \mathrm{I}_{b_j}$, since a singular fiber of type I_b can only be perturbed into some $\mathrm{I}_{b_1}, \ldots \mathrm{I}_{b_j}$, with $b_1 + \ldots + b_j = b$. The existence will be proven by giving a set of equations on $c_{1,\delta} \ldots c_{b,\delta}$, seen as equations in δ, such that if they are satisfied $W_\delta(z)$ has zeros of order $b_1, \ldots b_j$. The existence of a solution curve δ' of these equations will be proven by applying the implicit function theorem. Below we shall give the implicit function theorem as formulated in [1]:

Theorem 3.2.1 *Assume W to be open in $\mathbb{C}^n \times \mathbb{C}^p$ and $f : W \to \mathbb{C}^n$ complex-differentiable. Let*

$$(x^0, y^0) \in W, \qquad f(x^0; y^0) = 0, \qquad D_x f(x^0; y^0) \in \mathrm{Aut}(\mathbb{C}^n).$$

Then there exists open neighbourhoods U of x^0 in $\mathbb{C}^n$ and V of y^0 in $\mathbb{C}^p$ with the following properties:

$$\textit{for every } y \in V \qquad \textit{there exists a unique } x \in U, \qquad \textit{with } f(x; y) = 0.$$

In this way we obtain a complex-differentiable mapping: $\mathbb{C}^p \to \mathbb{C}^n$ satisfying

$$\psi : V \to U, \qquad \textit{with } \psi(y) = x \qquad \textit{and } f(x; y) = 0,$$

which is uniquely determined by these properties. Furthermore, the derivative $D\psi(y) \in \mathrm{Lin}(\mathbb{C}^p, \mathbb{C}^n)$ of ψ at y is given by

$$D\psi(y) = -D_x f(\psi(y); y)^{-1} \circ D_y f(\psi(y); y) \qquad (y \in V).$$

We will give the full proof for the first confluence but we shall not repeat the argument in full for the following cases.

$\mathrm{I}_7 \to \mathrm{I}_4 + \mathrm{I}_3$

We start with a Weierstrass model defined by

$$\begin{aligned} g_2(z) &= 3 + z^3 + z^4 \\ g_3(z) &= -1 - \frac{z^3}{2} - \frac{z^4}{2} + \epsilon z^4 + \eta z^5 - \frac{z^6}{2^3 3} + \delta z^6, \end{aligned}$$

so that

$$\begin{aligned} \Delta(z) =& 2 \cdot 3^3 \epsilon z^4 + 2 \cdot 3^3 \eta z^5 + 2 \cdot 3^3 \delta z^6 + \left(\frac{3^2}{2} + 3^3 \epsilon \right) z^7 + \left(\frac{3^2}{2^2} + 3^3 \epsilon - 3^3 \epsilon^2 + 3^3 \eta \right) z^8 \\ &+ \left(-\frac{1}{8} + 3^3 \delta + (3^3 - 2 \cdot 3^3 \epsilon) \eta \right) z^9 + 3 \left(\frac{5}{8} + 3^2 (1 - 2\epsilon) \delta + \frac{3\epsilon}{2^2} - 3^2 \eta^2 \right) z^{10} \\ &+ \left(3 + \left(\frac{3^2}{2^2} - 2 \cdot 3^3 \delta \right) \right) z^{11} + \left(1 - 3^3 \left(\frac{1}{2^3 3} + \delta \right)^2 \right) z^{12}. \end{aligned}$$

It is clear that for $\epsilon, \eta, \delta = 0$ we find a zero of order seven in the origin, corresponding to a singular fiber of type I_7, but for $\epsilon \neq 0$ we find a zero of order four, corresponding to a singular fiber of type I_4. From the Weierstrass preparation theorem we deduce that the geometric discriminant must be of the form

$$\Delta(z) = z^4 \big(z^3 + c_{1,\epsilon,\eta,\delta} z^2 + c_{2,\epsilon,\eta,\delta} z + c_{3,\epsilon,\eta,\delta} \big) u_{\epsilon,\eta,\delta}(z)$$

where we made the dependence on the perturbation parameters obvious by the underindices and where u is again a unit in a neighbourhood of the origin. We note that the product of z^4 and the third order polynomial has been called the Weierstrass polynomial[6] and we therefore use the notation

$$W_{\epsilon,\eta,\delta}(z) = z^3 + c_{1,\epsilon,\eta,\delta} z^2 + c_{2,\epsilon,\eta,\delta} z + c_{3,\epsilon,\eta,\delta}$$

and refer to $W_{\epsilon,\eta,\delta}$ as he reduced Weierstrass polynomial. Our aim is the prove that there exists a curve δ' in the ϵ, η, δ-space so that

$$W_{\epsilon,\eta,\delta}(z) = (z - z_{0,\epsilon,\eta,\delta})^3, \tag{3.1}$$

where $z_{0,\epsilon,\eta,\delta}|_{\delta'}$ is equal to zero if and only if δ' is at the origin. Equation (3.1) is equivalent to

$$\begin{aligned} c_{1,\epsilon,\eta,\delta} &= -3 z_{0,\epsilon,\eta,\delta} \\ c_{2,\epsilon,\eta,\delta} &= 3 (z_{0,\epsilon,\eta,\delta})^2 \\ c_{3,\epsilon,\eta,\delta} &= -(z_{0,\epsilon,\eta,\delta})^3, \end{aligned}$$

[6] See also section 1.1.

which in turn yields

$$\begin{aligned} c_{2,\epsilon,\eta,\delta} - \frac{1}{3}(c_{1,\epsilon,\eta,\delta})^2 &= 0 \\ c_{3,\epsilon,\eta,\delta} - \frac{1}{3^3}(c_{1,\epsilon,\eta,\delta})^3 &= 0. \end{aligned} \tag{3.2}$$

We shall now view this equation as an equation in the variables ϵ, η, δ. To prove the existence of a solution curve δ' in a neighbourhood of the origin, it is sufficient to prove that

$$D(c_{2,\epsilon,\eta,\delta} - \frac{1}{3}(c_{1,\epsilon,\eta,\delta})^2, c_{3,\epsilon,\eta,\delta} - \frac{1}{3^3}(c_{1,\epsilon,\eta,\delta})^3)|_0 \in \mathrm{Aut}(\mathbb{C}^2),$$

where D indicates the total derivative with respect to two variables, any combination of ϵ, η and δ will do. In the following we will not indicated these variables explicitly. Since the value of $c_{1,\epsilon,\eta,\delta}$ is zero at $\epsilon = \eta = \delta = 0$ we need to establish that

$$D(c_{2,\epsilon,\eta,\delta}, c_{3,\epsilon,\eta,\delta})|_0 \in \mathrm{Aut}(\mathbb{C}^2).$$

Alternatively we may also prove that the rank of

$$D(c_{2,\epsilon,\eta,\delta}, c_{3,\epsilon,\eta,\delta})$$

is maximal, in this case D indicates the total derivative with respect to ϵ, η and δ. We shall always assume we take the derivative at zero, we shall make this explicit no longer.

We now determine $c_{1,\epsilon,\eta,\delta}, c_{2,\epsilon,\eta,\delta}$ and $c_{3,\epsilon,\eta,\delta}$ perturbatively by making use of the Weierstrass preparation theorem. We have that

$$\Delta_{\epsilon,\eta,\delta}(z) = z^4 W_{\epsilon,\eta,\delta} u_{\epsilon,\eta,\delta}(z)$$

so that

$$\begin{aligned} \frac{\partial}{\partial\epsilon}\Delta_{\epsilon,\eta,\delta}(z) = \frac{\partial}{\partial\epsilon}\big(z^4 W_{\epsilon,\eta,\delta}(z) u_{\epsilon,\eta,\delta}(z)\big) &= z^4\left(\frac{\partial}{\partial\epsilon}W_{\epsilon,\eta,\delta}(z)\right)u_{\epsilon,\eta,\delta}(z) + z^4 W_{\epsilon,\eta,\delta}(z)\frac{\partial}{\partial\epsilon}u_{\epsilon,\eta,\delta}(z) \\ \frac{\partial}{\partial\eta}\Delta_{\epsilon,\eta,\delta}(z) = \frac{\partial}{\partial\eta}\big(z^4 W_{\epsilon,\eta,\delta}(z) u_{\epsilon,\eta,\delta}(z)\big) &= z^4\left(\frac{\partial}{\partial\eta}W_{\epsilon,\eta,\delta}(z)\right)u_{\epsilon,\eta,\delta}(z) + z^4 W_{\epsilon,\eta,\delta}(z)\frac{\partial}{\partial\eta}u_{\epsilon,\eta,\delta}(z) \\ \frac{\partial}{\partial\delta}\Delta_{\epsilon,\eta,\delta}(z) = \frac{\partial}{\partial\delta}\big(z^4 W_{\epsilon,\eta,\delta}(z) u_{\epsilon,\eta,\delta}(z)\big) &= z^4\left(\frac{\partial}{\partial\delta}W_{\epsilon,\eta,\delta}(z)\right)u_{\epsilon,\eta,\delta}(z) + z^4 W_{\epsilon,\eta,\delta}(z)\frac{\partial}{\partial\delta}u_{\epsilon,\eta,\delta}(z). \end{aligned}$$

We are only interested in a neighbourhood of the origin in ϵ, η, δ-space, so we consider

$$\begin{aligned} \frac{\partial}{\partial\epsilon}\Delta_{\epsilon,\eta,\delta}(z)\big|_0 &= z^4\left(\frac{\partial}{\partial\epsilon}W_{\epsilon,\eta,\delta}(z)\right)u_{\epsilon,\eta,\delta}(z)\big|_0 + z^4 W_{\epsilon,\eta,\delta}(z)\frac{\partial}{\partial\epsilon}u_{\epsilon,\eta,\delta}(z)\big|_0 \\ \frac{\partial}{\partial\eta}\Delta_{\epsilon,\eta,\delta}(z)\big|_0 &= z^4\left(\frac{\partial}{\partial\eta}W_{\epsilon,\eta,\delta}(z)\right)u_{\epsilon,\eta,\delta}(z)\big|_0 + z^4 W_{\epsilon,\eta,\delta}(z)\frac{\partial}{\partial\eta}u_{\epsilon,\eta,\delta}(z)\big|_0 \\ \frac{\partial}{\partial\delta}\Delta_{\epsilon,\eta,\delta}(z)\big|_0 &= z^4\left(\frac{\partial}{\partial\delta}W_{\epsilon,\eta,\delta}(z)\right)u_{\epsilon,\eta,\delta}(z)\big|_0 + z^4 W_{\epsilon,\eta,\delta}(z)\frac{\partial}{\partial\delta}u_{\epsilon,\eta,\delta}(z)\big|_0. \end{aligned}$$

If we now also note that $W_{\epsilon,\eta,\delta}(z)|_0 = z^3$, this equation reduces to

$$\begin{aligned}
\frac{\partial}{\partial\epsilon}\Delta_{\epsilon,\eta,\delta}(z)\big|_0 &= z^4\left(\frac{\partial}{\partial\epsilon}W_{\epsilon,\eta,\delta}(z)\right)u_{\epsilon,\eta,\delta}(z)\big|_0 + z^7\frac{\partial}{\partial\epsilon}u_{\epsilon,\eta,\delta}(z)\big|_0\\
\frac{\partial}{\partial\eta}\Delta_{\epsilon,\eta,\delta}(z)\big|_0 &= z^4\left(\frac{\partial}{\partial\eta}W_{\epsilon,\eta,\delta}(z)\right)u_{\epsilon,\eta,\delta}(z)\big|_0 + z^7\frac{\partial}{\partial\eta}u_{\epsilon,\eta,\delta}(z)\big|_0\\
\frac{\partial}{\partial\delta}\Delta_{\epsilon,\eta,\delta}(z)\big|_0 &= z^4\left(\frac{\partial}{\partial\delta}W_{\epsilon,\eta,\delta}(z)\right)u_{\epsilon,\eta,\delta}(z)\big|_0 + z^7\frac{\partial}{\partial\delta}u_{\epsilon,\eta,\delta}(z)\big|_0.
\end{aligned}$$

The unit around zero is determined by

$$\left.\frac{\Delta_{\epsilon,\eta,\delta}(z)}{z^7}\right|_0 = u_{\epsilon,\eta,\delta}(z)|_0,$$

which together with the previous formulae implies that the first derivatives of $W_{\epsilon,\eta,\delta}(z)$ at the origin can be determined from the fourth through seventh coefficient of the polynomial $\Delta(z)$. From this we may deduce that

$$W(z) = z^3 + \left(\frac{10\epsilon}{3} - 6\eta + 12\delta\right)z^2 + (-6\epsilon + 12\eta)z + 12\epsilon + \mathcal{O}(|(\epsilon,\eta,\delta)|^2).$$

This implies that the rank of

$$D(c_{2,\epsilon,\eta,\delta}, c_{3,\epsilon,\eta,\delta})$$

is maximal, which means that we have proven the existence of a curve δ' such that

$$\Delta_{\delta'}(z) = z^4(z - z_{0,\delta'})^3 u_{\delta'}(z)$$

and thus the existence of a confluence of type $\mathrm{I}_7 \to \mathrm{I}_4 + \mathrm{I}_3$.

Let us now reflect on this proof. We did focus on a confluence of type $\mathrm{I}_7 \to \mathrm{I}_4 + \mathrm{I}_3$ and thus imposed the equation (3.1). If we would have been interested in a confluence of type $\mathrm{I}_7 \to \mathrm{I}_4 + \mathrm{I}_2 + \mathrm{I}_1$ we would have imposed for example[7]

$$W_{\epsilon,\eta,\delta}(z) = (z - z_{0,\epsilon,\eta,\delta})^2(z + z_{0,\epsilon,\eta,\delta})$$

This would have altered the coefficients in (3.2), but it still would have been sufficient to prove that the rank of

$$D(c_{2,\epsilon,\eta,\delta}, c_{3,\epsilon,\eta,\delta})$$

is maximal. So in fact we have also found an alternative method to prove the existence of confluences of type $\mathrm{I}_7 \to \mathrm{I}_4 + \mathrm{I}_2 + \mathrm{I}_1$ and $\mathrm{I}_7 \to \mathrm{I}_4 + \mathrm{I}_1 + \mathrm{I}_1 + \mathrm{I}_1$. More generally it indicates that in general the existence of a confluence of type $\mathrm{I}_b \to \mathrm{I}_{b-e} + \mathrm{I}_e$ implies the existence

[7] It is practical to let the position of all roots depend on one parameter to make sure we can distinguish roots.

of a confluence to type $\mathrm{I}_b \longrightarrow \mathrm{I}_{b-e} + \mathrm{I}_{e_1} + \ldots + \mathrm{I}_{e_j}$, with $e_1 + \ldots + e_j = e$. Moreover we may conclude that the existence of a confluence of type $\mathrm{I}_b \longrightarrow \mathrm{I}_{b-e} + \mathrm{I}_e$ seems to imply the existence of a confluence of type $\mathrm{I}_b \longrightarrow \mathrm{I}_{b-e'} + \mathrm{I}_{e_1} + \ldots + \mathrm{I}_{e_j}$, with $e_1 + \ldots + e_j = e' \leq e$, in this case we simply impose that $z^{e-e'}$ factors $W_{\epsilon,\eta,\delta}$. We shall for now continue to write out all individual cases for $\mathrm{I}_b \longrightarrow \mathrm{I}_{b-e} + \mathrm{I}_e$. The method to find $W_{\epsilon,\eta,\delta}(z)$ perturbatively in ϵ, η and δ clearly applies generally and we will not repeat our discussion in every example.

$\mathrm{I}_7 \longrightarrow \mathrm{I}_3 + \mathrm{I}_2 + \mathrm{I}_2$

We consider a Weierstrass model defined by

$$\begin{aligned} g_2(z) &= 3 + z^3 + z^4 \\ g_3(z) &= -1 - \Big(\frac{1}{2} - \varphi\Big) z^3 - \Big(\frac{1}{2} - \epsilon\Big) z^4 + \eta z^5 - \Big(\frac{1}{24} - \delta\Big) z^6 \end{aligned}$$

This gives us a geometric discriminant of the form

$$\Delta(z) = z^3 W(z) u(z),$$

where, by the same method as above, we can derive that

$$\begin{aligned} W(z) =& z^4 + c_1 z^3 + c_2 z^2 + c_3 z + c_4 \\ =& z^4 + (12\delta + \frac{10\epsilon}{3} - 6\eta - \frac{5\varphi}{6}) z^3 + (-6\epsilon + 12\eta + \frac{10\varphi}{3}) z^2 \\ &+ (12\epsilon - 6\varphi) z + 12\varphi + \mathcal{O}(|(\epsilon, \eta, \delta)|^2). \end{aligned}$$

We will impose that

$$W(z) = (z - z_0)^2 (z + z_0)^2,$$

where z_0 depends on the perturbation parameters, which is equivalent to imposing

$$c_1 = 0 \qquad c_2^2 - 4c_4 = 0 \qquad c_3 = 0.$$

Because the rank of

$$D(c_1, c_3, -4c_4)$$

is maximal, there exists a solution curve in $\epsilon, \eta, \delta, \varphi$-space such that $W(z)$ is of the said form for this curve.

$\mathrm{I}_7 \longrightarrow \mathrm{I}_2 + \mathrm{I}_2 + \mathrm{I}_2 + \mathrm{I}_1$

We consider the Weierstrass model defined by

$$\begin{aligned} g_2(z) &= 3 + z^3 + z^4 \\ g_3(z) &= -1 + \sigma z + \psi z^2 - \Big(\frac{1}{2} - \phi\Big) z^3 - \Big(\frac{1}{2} - \epsilon\Big) z^4 + \eta z^5 - \Big(\frac{1}{24} - \delta\Big) z^6. \end{aligned}$$

This gives us a geometric discriminant of the form

$$\Delta(z) = zW(z)u(z),$$

where, by the same method as above, we can derive that

$$\begin{aligned}
W(z) =& z^6 + c_1 z^5 + c_2 z^4 + c_3 z^3 + c_4 z^2 + c_5 z + c_6 \\
=& z^6 + (12\delta + \frac{10\epsilon}{3} - 6\eta - \frac{11\sigma}{24} - \frac{5\phi}{6} + \frac{109\psi}{108})z^5 + (-6\epsilon + 12\eta + \frac{109\sigma}{108}\frac{10\phi}{3} - \frac{5\psi}{6})z^4 \\
&+ (12\epsilon - \frac{5\sigma}{6} - 6\phi + \frac{10\psi}{3})z^3 + (\frac{10\sigma}{3} + 12\phi - 6\psi)z^2 + (-6\sigma + 12\psi)z + 12\sigma.
\end{aligned}$$

We will impose that

$$W(z) = (z - z_0)^2 (z - e^{\frac{2\pi i}{3}} z_0)^2 (z - e^{-\frac{2\pi i}{3}} z_0)^2,$$

where z_0 depends on the perturbation parameters, which is equivalent to imposing

$$c_1 = 0 \qquad c_2 = 0 \qquad c_3^2 - 4c_6 = 0 \qquad c_4 = 0 \qquad c_5 = 0.$$

Because the rank of

$$D(c_1, c_2, c_4, c_5, -4c_6)$$

is maximal, there exists a solution curve in $\epsilon, \eta, \delta, \phi, \psi, \sigma$-space such that $W(z)$ is of the said form for this curve.

$\mathrm{I}_7 \to \mathrm{I}_3 + \mathrm{I}_3 + \mathrm{I}_1$

We consider the same Weierstrass model as the Weierstrass model for $\mathrm{I}_7 \to \mathrm{I}_2 + \mathrm{I}_2 + \mathrm{I}_2 + \mathrm{I}_1$ but impose that

$$W(z) = (z - z_0)^3 (z + z_0)^3,$$

which is equivalent to

$$c_1 = 0 \qquad c_3 = 0 \qquad 3c_4 - c_2^2 = 0 \qquad c_5 = 0 \qquad 9c_6 - c_2 c_4 = 0.$$

Again we may derive that the

$$D(c_1, c_3, c_4, c_5, 9c_6)$$

is of maximal rank and thus a confluence of type $\mathrm{I}_7 \to \mathrm{I}_3 + \mathrm{I}_3 + \mathrm{I}_1$ exists.

$\mathrm{I}_7 \to \mathrm{I}_3 + \mathrm{I}_2 + \mathrm{I}_1 + \mathrm{I}_1$

We consider the same Weierstrass model as for $\mathrm{I}_7 \to \mathrm{I}_3 + \mathrm{I}_2 + \mathrm{I}_2$, but impose

$$W(z) = (z - z_0)^2 (z^2 + z_0^2),$$

which is equivalent to

$$2c_2 - c_1^2 = 0 \qquad c_3 - 2^2 c_1^3 = 0 \qquad 2^4 c_4 - c_1^4 = 0.$$

Again we may derive that the rank of

$$D(c_2, c_3, c_4)$$

is of maximal rank and thus a confluence of type $\mathrm{I}_7 \longrightarrow \mathrm{I}_3 + \mathrm{I}_2 + \mathrm{I}_1 + \mathrm{I}_1$ exists.

$\mathrm{I}_7 \longrightarrow \mathrm{I}_2 + \mathrm{I}_2 + \mathrm{I}_1 + \mathrm{I}_1 + \mathrm{I}_1$

We consider the same Weierstrass model as for $\mathrm{I}_7 \longrightarrow \mathrm{I}_3 + \mathrm{I}_3 + \mathrm{I}_1$, but impose

$$W(z) = (z + z_0)^2 (z^3 - z_0^3),$$

which is equivalent to

$$4c_2 - c_1^2 = 0 \qquad 2^3 c_3 - c_1^3 = 0 \qquad 2^3 c_4 + c_1^4 = 0 \qquad 2^5 c_5 + c_1^5 = 0.$$

Again we may derive that the rank of

$$D(c_2, c_3, c_4, c_5)$$

is of maximal rank and thus a confluence of type $\mathrm{I}_7 \longrightarrow \mathrm{I}_2 + \mathrm{I}_2 + \mathrm{I}_1 + \mathrm{I}_1 + \mathrm{I}_1$ exists.

This completes our discussion of the possible perturbations of singular fibers of Kodaira type I_7.

In the following we will no longer give the explicit expression which we will impose on $W(z)$ but we suffice by giving the defining equations of the Weierstrass model, $g_2(z)$ and $g_3(z)$ depending on some perturbation parameters. The g_2 and g_3 are chosen such that for all perturbation parameters zero of the discriminant $\Delta(z)$ of order $b - e$ is fixed in the origin, yielding a singular fiber of type I_{b_e}, but if the perturbation parameters are set to zero we find a singular fiber of type I_b in the origin. Furthermore we will give

$$W(z) = z^e + c_1 z^{e-1} + \ldots + c_e.$$

In our examples we have always made sure that dimension of the space of perturbations is greater then e and the rank of

$$D(c_1, \ldots, c_e)$$

is maximal. As we remarked before this is sufficient to guarantee the existence of any confluence of type $\mathrm{I}_b \longrightarrow \mathrm{I}_{b-e} + \mathrm{I}_{e_1} + \ldots + \mathrm{I}_{e_j}$, with $e_1 + \ldots + e_j = e$

$\mathrm{I}_8 \longrightarrow \mathrm{I}_5 + \mathrm{I}_{e_1} + \ldots + \mathrm{I}_{e_j}$, with $e_1 + \ldots + e_j = 3$

We choose our Weierstrass model to be defined by

$$
\begin{aligned}
g_2(z) =&3 + (1+\epsilon)z + \Big(\frac{73}{12} + \theta + 6\psi\Big)z^2 + (1+\psi)z^3 + (1+\phi)z^4 \\
g_3(z) =& -1 - \frac{1}{2}(1+\epsilon)z - \frac{1}{24}\Big((1+\epsilon)^2 + 12\Big(\theta + \frac{1}{12}(1+72(1+\psi))\Big)\Big)z^2 \\
&+ \frac{1}{432}\Big((1+\epsilon)^3 - 216(1+\psi) - 36(1+\epsilon)\Big(\theta + \frac{1}{12}(1+72(1+\psi))\Big)\Big)z^3 \\
&- \frac{1}{3456}\Big((1+\epsilon)^4 + 1728(1+\phi) + 288(1+\epsilon)(1+\psi) \\
&- 24(1+\epsilon)^2\Big(\theta + \frac{1}{12}(1+72(1+\psi))\Big) - 144\Big(\theta + \frac{1}{12}(1+72(1+\psi))\Big)^2\Big)z^4 \\
&- (\frac{1}{3} - \eta)z^5 + z^6
\end{aligned}
$$

so that the geometric discriminant is of the form

$$\Delta(z) = z^5 W(z) u(z),$$

where we may derive that

$$
\begin{aligned}
W(z) =& z^3 + (\delta + 6\eta + \phi + 2\psi)z^2 + \Big(6\delta + \eta - \theta + \frac{37\phi}{12} - \frac{11\psi}{2}\Big)z\Big)z \\
&6\eta - \epsilon + \frac{\phi}{2} + 3\psi + \mathcal{O}(|(\epsilon, \eta, \delta, \phi, \psi, \theta)|^2).
\end{aligned}
$$

$\mathrm{I}_8 \to \mathrm{I}_4 + \mathrm{I}_{e_1} + \ldots + \mathrm{I}_{e_j}$, with $e_1 + \ldots + e_j = 4$

We choose our Weierstrass model to be defined by

$$
\begin{aligned}
g_2(z) =&3 + z + (\frac{73}{12} + \theta + 6\psi)z^2 + (1+\psi)z^3 + (1+\phi)z^4 \\
g_3(z) =& -1 - \frac{z}{2} - \frac{37 + 6\theta + 36\psi}{12}z^2 - \frac{217 + 18\theta + 216\psi}{216}z^3 - (\frac{25}{12} - \epsilon)z^4 \\
&- (\frac{1}{3} - \eta)z^5 + \delta z^6,
\end{aligned}
$$

so that the geometric discriminant is of the form

$$\Delta(z) = z^4 W(z) u(z),$$

where we may derive that

$$
\begin{aligned}
W(z) =& z^4 + (\delta + 6\eta + \phi + 2\psi)z^3 + \Big(6\delta + 6\epsilon + \eta + 2\theta + \frac{73\phi}{12} + 13\psi\Big)z^2 \\
&+ \Big(\epsilon + 6\eta + \frac{\theta}{2} + \phi + \frac{73\psi}{12}\Big)z + 6\epsilon + 3\theta + 3\phi + \frac{37\psi}{2} + \mathcal{O}(|(\epsilon, \eta, \delta, \phi, \psi, \theta)|^2).
\end{aligned}
$$

$\mathrm{I}_8 \to \mathrm{I}_3 + \mathrm{I}_{e_1} + \ldots + \mathrm{I}_{e_j}$, with $e_1 + \ldots + e_j = 5$

We choose our Weierstrass model to be defined by

$$
\begin{aligned}
g_2(z) =& 3 + z + (\frac{73}{12} + \theta + 6\psi)z^2 + (1+\psi)z^3 + (1+\phi)z^4 \\
g_3(z) =& -1 - \frac{z}{2} - \frac{37 + 6\theta + 36\psi}{12}z^2 - \Big(\frac{217}{216} - \alpha\Big)z^3 - \Big(\frac{25}{12} - \epsilon\Big)z^4 \\
& - \Big(\frac{1}{3} - \eta\Big)z^5 + \delta z^6,
\end{aligned}
$$

so that the geometric discriminant is of the form

$$\Delta(z) = z^3 W(z) u(z),$$

where we may derive that

$$
\begin{aligned}
W(z) =& z^5 + \Big(-\frac{2\alpha}{3} + \delta + 6\eta - \frac{\theta}{18} + \phi + \frac{4\psi}{3}\Big)z^4 + \Big(6\delta + 6\epsilon + \eta + 2\theta + \frac{73\phi}{12} + 13\psi\Big)z^3 \\
& + \Big(6\alpha + \epsilon + 6\eta + \theta + \phi + \frac{145\psi}{12}\Big)z^2 + \Big(\alpha + 6\epsilon + \frac{37\theta}{12} + 3\phi + \frac{39\psi}{2}\Big)z \\
& + 6\alpha + \frac{\theta}{2} + 6\psi + \mathcal{O}(|(\alpha, \epsilon, \eta, \delta, \phi, \psi, \theta)|^2).
\end{aligned}
$$

$\mathrm{I}_8 \to \mathrm{I}_2 + \mathrm{I}_{e_1} + \ldots + \mathrm{I}_{e_j}$, with $e_1 + \ldots + e_j = 6$

We choose our Weierstrass model to be defined by

$$
\begin{aligned}
g_2(z) =& 3 + z + (\frac{73}{12} + \theta + 6\psi)z^2 + (1+\psi)z^3 + (1+\phi)z^4 \\
g_3(z) =& -1 - \frac{z}{2} - \Big(\frac{37}{12} - \beta\Big)z^2 - \Big(\frac{217}{216} - \alpha\Big)z^3 - \Big(\frac{25}{12} - \epsilon\Big)z^4 \\
& - \Big(\frac{1}{3} - \eta\Big)z^5 + \delta z^6,
\end{aligned}
$$

so that the geometric discriminant is of the form

$$\Delta(z) = z^2 W(z) u(z),$$

where we may derive that

$$
\begin{aligned}
W(z) =& z^6 + \Big(-\frac{2\alpha}{3} + \frac{\beta}{9} + \delta + 6\eta + \phi + \frac{5\psi}{3}\Big)z^5 + \Big(-\frac{2\beta}{3} + 6\delta + 6\epsilon + \eta + \frac{5\theta}{3} + \frac{73\phi}{12} + 11\psi\Big)z^4 \\
& + \Big(6\alpha + \epsilon + 6\eta + \theta + \phi + \frac{145\psi}{12}\Big)z^3 + \Big(\alpha + 6\beta + 6\epsilon + \frac{73\theta}{12} + 3\phi + \frac{75\psi}{2}\Big)z^2 \\
& + (6\alpha + \beta + \theta + 9\psi)z + 6\beta + 3\theta + 18\psi + \mathcal{O}(|(\alpha, \beta, \epsilon, \eta, \delta, \phi, \psi, \theta)|^2).
\end{aligned}
$$

Two remarks are now useful. First if we consider the parameters of g_2 and g_3 as our variables[8] we may consider the sets in this space such that g_2 and g_3 define a Weierstrass model with singular fibers of some given type for example $\mathrm{I}_8 + 5\mathrm{I}_1$ (here we switched from the local to the global point of view). These sets are algebraic subvarieties with other algebraic subvarieties cut out. We might be interested in knowing how the set corresponding to $\mathrm{I}_8 + 5\mathrm{I}_1$ is attached to the set corresponding to lets say $\mathrm{I}_5 + \mathrm{I}_3 + 5\mathrm{I}_1$. We now note that in our examples we have chosen the space of perturbations bigger then necessary, this should give us a lower bound on the dimension of the set corresponding to for example $\mathrm{I}_5 + \mathrm{I}_3 + 5\mathrm{I}_1$ attached to $\mathrm{I}_8 + 5\mathrm{I}_1$, namely the dimension of the space of perturbations minus the number of variables necessary to fix the type of the confluence, in this case 3.
Secondly, as remarked before, the existence of the confluence of type $\mathrm{I}_8 \to \mathrm{I}_2 + \mathrm{I}_{e_1} + \ldots + \mathrm{I}_{e_j}$, with $e_1 + \ldots + e_j = 6$, implies the existence of any confluence of type $\mathrm{I}_8 \to \mathrm{I}_{8-e} + \mathrm{I}_{e_1} + \ldots + \mathrm{I}_{e_j}$, with $e_1 + \ldots + e_j = e \leq 6$. Let us formalize this point of view in the following lemma:

Lemma 3.2.2 *Let the geometric discriminant be of the form*

$$\Delta(z) = z^2 W(z) u(z),$$

where $u(z)$ is a unit, $W(z)$ a polynomial, which we shall refer to as the reduced Weierstrass polynomial, and both $W(z)$ and $u(z)$ depend of some perturbation parameters $\delta_1, \ldots, \delta_m$. Furthermore if we write

$$W(z) = z^{b-2} + c_1 z^{b-3} + \ldots + c_b,$$

the maximality of the rank of the jacobian

$$D(c_1, c_2, \ldots, c_b)|_0$$

implies that every confluence of type $I_b \to I_{b-e} + I_{e_1} + \ldots + I_{e_j}$, with $e_1 + \ldots + e_j = e \leq b-2$ exists.

Proof To prove the existence of any confluence of type $\mathrm{I}_b \to \mathrm{I}_{b-e} + \mathrm{I}_{e_1} + \ldots + \mathrm{I}_{e_j}$, with $e_1 + \ldots + e_j = e \leq b-2$ it is sufficient to prove that there is a curve δ' in $\delta_1, \delta_2, \ldots, \delta_m$-space such that $W(z)$ is of the form

$$W(z) = z^{b-e-2}(z - e^{i\theta_0} z_{0,\delta_1,\delta_2,\ldots,\delta_m})(z - e^{i\theta_1} z_{0,\delta_1,\delta_2,\ldots,\delta_m}) \ldots (z - e^{i\theta_e} z_{0,\delta_1,\delta_2,\ldots,\delta_m}),$$

[8] We focus on the projected g_2 and g_3 and ignore the equivalence of several projected g_2 and g_3 due to the fact that they originate from the same polynomials in homogenous coordinates.

where we impose that the $\theta_i \in [-\pi, \pi)$ are fixed and $\theta_1 = \ldots = \theta_{e_1}$, $\theta_{e1+1} = \ldots = \theta_{e_2}$, $\ldots$, $\theta_{e_{j-1}+1} = \ldots = \theta_{e_j}$ and no other equalities occur. Imposing this means that

$$\begin{aligned}
W(z) &= z^{b-e-2}(z - e^{i\theta_0} z_{0,\,\delta_1,\delta_2,\ldots,\delta_m})(z - e^{i\theta_1} z_{0,\,\delta_1,\delta_2,\ldots,\delta_m}) \ldots (z - e^{i\theta_e} z_{0,\,\delta_1,\delta_2,\ldots,\delta_m}) \\
&= z^{b-e-2}\big(z^e - z^{e-1}(e^{i\theta_0} + \ldots + e^{i\theta_e}) z_{0,\,\delta_1,\delta_2,\ldots,\delta_m} \\
&\quad + z^{e-2}(e^{i\theta_0} e^{i\theta_1} + \ldots + e^{i\theta_{e-1}} e^{i\theta_e}) z^2_{0,\,\delta_1,\delta_2,\ldots,\delta_m} \\
&\quad + \ldots + (-1)^e z^e_{0,\,\delta_1,\delta_2,\ldots,\delta_m}(e^{i\theta_0} \ldots e^{i\theta_e}\big) \\
&= z^{b-2} + c_1 z^{b-3} + \ldots + c_{b-2},
\end{aligned}$$

in particular we have that

$$\begin{aligned}
c_1 &= -(e^{i\theta_0} + \ldots + e^{i\theta_e}) z_{0,\,\delta_1,\delta_2,\ldots,\delta_m} \\
c_2 &= (e^{i\theta_0} e^{i\theta_1} + \ldots + e^{i\theta_{e-1}} e^{i\theta_e}) z^2_{0,\,\delta_1,\delta_2,\ldots,\delta_m} \\
&\ \ \vdots \\
c_e &= (-1)^e z^e_{0,\,\delta_1,\delta_2,\ldots,\delta_m}(e^{i\theta_0} \ldots e^{i\theta_e}) \\
c_{e+1} &= 0 \\
&\ \ \vdots \\
c_{b-2} &= 0.
\end{aligned}$$

If we pick[9] the θ_is such that $(e^{i\theta_0} + \ldots + e^{i\theta_e}) \neq 0$ we may use the first equation above to substitute the $z_{0,\,\delta_1,\delta_2,\ldots,\delta_m}$ if all of the above equations, where the equation does not read $c_i = 0$. This implies that it suffices to impose equations of the sort $c_j = 0$ or $c_j - \alpha_j c_1^j = 0$, where $\alpha_j = (e^{i\theta_0} \ldots e^{i\theta_{j-1}} + \ldots + e^{i\theta_{e-j+1}} \ldots e^{i\theta_e})(e^{i\theta_0} + \ldots + e^{i\theta_e})^{-j}$. Like in our extensive discussion of the confluence $\mathrm{I}_7 \to \mathrm{I}_4 + \mathrm{I}_3$, we have that the maximality of the rank of the Jacobian[10]

$$D(c_1, c_2, \ldots, c_b)|_0$$

implies that

$$D(c_2 - \alpha_2 c_1^2, \ldots, c_{b-2})|_0$$

is an automorphism of $\mathbb{C}^{b-3}$, which via the implicit function theorem yields the existence of a solution curve δ' and thus the confluence itself. □

This lemma renders all proofs above superfluous, with the exception of the confluence $\mathrm{I}_7 \to 3\mathrm{I}_2 + \mathrm{I}_1$ and $\mathrm{I}_8 \to \mathrm{I}_2 + \mathrm{I}_{e_1} + \ldots + \mathrm{I}_{e_j}$, but writing out the first examples explicitly improves our intuition.

Taking the result of lemma 3.2.2 to heart, we will confine ourselves to discussing the confluence $\mathrm{I}_9 \to \mathrm{I}_2 + \mathrm{I}_{e_1} + \ldots + \mathrm{I}_{e_j}$, with $e_1 + \ldots + e_j = 7$, in the same manner as above, since discussing all possibilities for the first singular fiber is too cumbersome.

[9]This is not strictly necessary but this simplifies the argument somewhat.

[10]We assume we derive with respect to the right number of coordinates.

$\mathrm{I}_9 \to \mathrm{I}_2 + \mathrm{I}_{e_1} + \ldots + \mathrm{I}_{e_j}$, with $e_1 + \ldots + e_j = 7$

We choose our Weierstrass model to be defined by

$$
\begin{aligned}
g_2(z) =& \frac{(1+\beta)^4}{2^2 3} + \frac{(1+\alpha)(1+\beta)^3}{3} z + \left(\frac{1}{2} + \eta\right) z^2 - \left(\frac{5 \cdot 23}{3^2} - \lambda\right) z^3 - \left(\frac{7 \cdot 67}{2^2 3^2} - \delta\right) z^4 \\
g_3(z) =& - \frac{(1+\beta)^6}{2^3 3^3} - \frac{(1+\alpha)(1+\beta)^5}{2^2 3^2} z - \left(\frac{5}{2^3 3^2} - \epsilon\right) z^2 + (1+\gamma)(1+\beta)^3 z^3 \\
&+ \left(\frac{7 \cdot 11}{2^3 3} + \phi\right) z^4 + \left(\frac{13}{2^2} + \iota\right) z^5 - \left(\frac{27143}{2^3 3^4} - \kappa\right) z^6,
\end{aligned}
$$

so that the geometric discriminant is of the form

$$\Delta(z) = z^2 W(z) u(z),$$

where we may derive that

$$
\begin{aligned}
W(z) =& z^7 \\
&- \frac{3^2}{2^4 59^3} (3^2 811\delta + 2 \cdot 5 \cdot 59^2 \alpha + 29 \cdot 59^2 \beta + 2^2 3^3 59^2 \epsilon - 3 \cdot 59^2 \eta + 2^2 3^4 29 \iota \\
&\qquad - 2^2 3^5 \kappa + 2 \cdot 3^2 5 \cdot 41 \lambda) z^6 \\
&- \frac{3^2}{2^4 59^3} (3^2 383\delta - 2^2 59^2 \alpha - 2^3 59^2 \beta + 2 \cdot 3^2 5 \cdot 41 \eta - 2^2 3^5 \iota - 2^2 3^5 \kappa + 3^2 811 \lambda \\
&\qquad - 3^2 383 \phi) z^5 \\
&- \frac{3^2}{2^4 59^3} (2^3 3^4 29 \gamma - 2 \cdot 3^3 5 \delta + 2 \cdot 3 \cdot 5 \cdot 41 \alpha + 2 \cdot 23071 \beta - 3^2 \cdot 811 \eta - 2^2 3^5 \iota \\
&\qquad - 2^2 3^4 \kappa + 3^2 383 \lambda - 2^2 3^5 \phi) z^4 \\
&+ \frac{3^3}{2^4 59^3} (2^2 3^4 \gamma + 3^2 5 \delta - 811 \alpha - 1871 \beta - 2^2 3^3 29 \epsilon - 3 \cdot 383 \eta + 2^2 3^3 \iota + 2 \cdot 3^2 5 \lambda \\
&\qquad + 2^2 3^4 \phi) z^3 \\
&+ \frac{3^3}{2^4 59^3} (2^2 3^4 \gamma + 3^2 \delta - 11 \cdot 19 \alpha - 2 \cdot 59 \beta + 2^2 3^4 \epsilon + 2 \cdot 3^2 5 \eta + 3^2 5 \lambda + 2^2 3^3 \phi) z^2 \\
&+ \frac{3^3}{2^4 59^3} (2^2 3^3 \gamma + 3 \cdot 7 \alpha + 2^5 5 \beta + 2^2 3^4 \epsilon + 3^2 5 \eta + 3^2 \lambda) z \\
&+ \frac{3^4}{2^4 59^3} (2\alpha + 7\beta + 2^2 3^2 \epsilon + 3\eta) + \mathcal{O}(|(\alpha, \beta, \gamma, \delta, \epsilon, \eta, \phi, \iota, \kappa, \lambda)|^2).
\end{aligned}
$$

This completes our discussion of confluences to a singular fiber of Kodaira type I_b, we shall summarize our result in the following theorem

Theorem 3.2.3 *Every type of confluence of singular elliptical fibers on a rational elliptic surface of type I_{b_i} into a singular fiber of type I_b with $b = \sum b_i$ occurs.*

Our method does not rely in any sense on the rationality of the elliptic surface. We therefore conjecture that theorem 3.2.3 also holds for $K3$-surfaces. Using the method above this conjecture should not be very hard to prove.

3.3 Confluence to singular fibers of Kodaira type II, III and IV.

In this section we shall give examples of every imaginary confluence to a singular fiber of type II, III or IV, except IV $\to$ 2I_2. The properties of these examples will be verified by explicit calculation, as can be seen in table 3.4, with the exception of the confluences IV $\to$ II$+2\mathrm{I}_1$ and IV $\to$ $\mathrm{I}_2+2\mathrm{I}_1$. The confluences of type IV $\to$ II$+2\mathrm{I}_1$ and IV $\to$ $\mathrm{I}_2+2\mathrm{I}_1$ are treated separately and rely on the same discriminant argument as used before. We will prove that there exists no confluence of the type IV $\to$ 2I_2, by using monodromy considerations, not unlike the considerations seen in [9].

Table 3.4: In this table the zeros of the geometric discriminant are denoted by z_0.

Confluence	$g_2(z)$	$g_3(z)$	zeros z_0 of $\Delta(z)$	$g_2(z_0)$	$g_3(z_0)$
II $\to$ $\mathrm{I}_1+\mathrm{I}_1$	ϵ	z	$\pm\frac{\epsilon^{3/2}}{3^{3/2}}$	ϵ	$\pm\frac{\epsilon^{3/2}}{3^{3/2}}$
III $\to$ $3\mathrm{I}_1$	z	ϵ	$3\epsilon^{3/2}, 3\epsilon^{3/2}e^{\pm\frac{2\pi}{3}}$	$3\epsilon^{3/2}, 3\epsilon^{3/2}e^{\pm\frac{2\pi}{3}}$	ϵ, ϵ
III $\to$ $\mathrm{I}_2+\mathrm{I}_1$	$z+3\epsilon^2$	$\frac{\epsilon z}{2}+\epsilon^3$	$0,\ 0,\ -\frac{9\epsilon^2}{4}$	$3\epsilon^2,\ 3\epsilon^2,\ \frac{3\epsilon}{4}$	$\epsilon^3,\ \epsilon^3,\ -\frac{\epsilon^3}{8}$
III $\to$ II $+\mathrm{I}_1$	z	ϵz	$0,\ 0,\ 27\epsilon^2$	$0,\ 0,\ 27\epsilon^2$	$0,\ 0,\ 27\epsilon^3$
IV $\to$ $4\mathrm{I}_1$	ϵ	z^2	$\pm\frac{\epsilon^{3/4}}{3^{3/4}},\ \pm\frac{i\epsilon^{3/4}}{3^{3/4}}$	ϵ, ϵ	$\frac{\epsilon^{3/2}}{3^{3/2}},\ -\frac{\epsilon^{3/2}}{3^{3/2}}$
IV $\to$ $\mathrm{I}_3+\mathrm{I}_1$	$-2(6\epsilon)^{1/2}z$ $+3\epsilon^2$	$z^2+\epsilon^3$ $-6^{1/2}\epsilon^{3/2}z$	$0,\ 0,\ 0,\left(\frac{2\epsilon}{3}\right)^{3/2}$	$3\epsilon^2,\ 3\epsilon^2,\ 3\epsilon^2,$ $\frac{\epsilon^2}{3}$	$\epsilon^3,\ \epsilon^3,\ \epsilon^3,$ $-\left(\frac{\epsilon}{3}\right)^3$
IV $\to$ III $+\mathrm{I}_1$	ϵz	z^2	$0,\ 0,\ 0,\ \frac{\epsilon^3}{3^3}$	$0,\ 0,\ 0,\ \frac{\epsilon^4}{3^3}$	$0,\ 0,\ 0,\ \frac{\epsilon^6}{3^6}$
IV $\to$ II $+\mathrm{I}_2$	ϵz	$z^2+\frac{\epsilon^3 z}{2^2 3^3}$	$0,\ 0,\ \frac{\epsilon^3}{2^2 3^3},\ \frac{\epsilon^3}{2^2 3^3}$	$0,\ 0,\ \frac{\epsilon^4}{2^2 3^3},\ \frac{\epsilon^4}{2^2 3^3}$	$0,\ 0,\ \frac{\epsilon^6}{2^3 3^6},\ \frac{\epsilon^6}{2^3 3^6}$
IV $\to$ 2II	0	$z(z-\epsilon)$	$0,\ 0,\ \epsilon,\ \epsilon$	$0,\ 0,\ 0,\ 0$	$0,\ 0,\ 0,\ 0$

IV $\to$ $\mathrm{I}_2+2\mathrm{I}_1$

We choose

$$g_2(z) = \frac{z\epsilon z}{2} + 3\epsilon^2$$
$$g_3(z) = z^2 + \frac{\epsilon^2 z}{4} + \epsilon^3,$$

so that

$$\Delta(z) = \frac{9}{16}(\epsilon-96)\epsilon^3 z^2 + \frac{1}{8}(\epsilon-108)\epsilon z^3 - 27z^4.$$

From the explicit form of the geometric discriminant we see that there is a second order zero in the origin, by construction we have that $g_2(0) \neq 0$ and $g_3(0) \neq 0$ for $\epsilon \neq 0$, so for $\epsilon \neq 0$ we are faced with a singular fiber of type I_2 in the origin. We further note that the discriminant of the geometric discriminant divided by z^2 is

$$D\left(\frac{\Delta(z)}{z^2}\right) = \frac{1}{64}\epsilon^3(\epsilon-72)^3.$$

This implies that there are only singular fibers of type I_1 outside the origin, from which we conclude that we have a confluence of type $\mathrm{IV} \to \mathrm{I}_2 + 2\mathrm{I}_1$.

$\mathrm{IV} \to \mathrm{II} + 2\mathrm{I}_1$

We choose

$$\begin{aligned} g_2(z) &= \epsilon z \\ g_3(z) &= z^2 + \epsilon z, \end{aligned}$$

so that

$$\Delta(z) = -27\epsilon^2 z^2 + \epsilon(\epsilon^2 - 54)z^3 - 27z^4.$$

From the explicit form of the geometric discriminant we again see that there is a second order zero in the origin, but in this case we have by construction that $g_2(0) = 0$ and $g_3(0) = 0$ for $\epsilon \neq 0$, so for $\epsilon \neq 0$ we are faced with a singular fiber of type II in the origin. We again note that the discriminant of the geometric discriminant divided by z^2 is

$$D\left(\frac{\Delta(z)}{z^2}\right) = \epsilon^4(\epsilon^2 - 108).$$

This implies that there are only singular fibers of type I_1 outside the origin, from which we conclude that we have a confluence of type $\mathrm{IV} \to \mathrm{II} + 2\mathrm{I}_1$.

$\mathrm{IV} \to 2\mathrm{I}_2$

As discussed in section 2.5 we may associate a conjugacy class of monodromy matrices in $\mathrm{SL}(2,\mathbb{Z})$ to each type of singular fiber. The conjugacy class of monodromy matrices is associated to curves running around a singular fiber, and the class is invariant under homotopic deformation of the curve. In this case we consider a curve running around a singular fiber of type IV, which splits into two singular fibers of type I_2. For suitably chosen paths we have that the monodromy matrix associated to a path around IV is, after deformation, the product of the monodromy matrix associated to a path around the first singular fiber of type I_2 and the monodromy matrix associated to a path around the second singular fiber of type I_2 (see also section 2.7). We shall denote a monodromy matrix in the conjugacy class associated to IV by M_{IV} and a monodromy matrix in the conjugacy class associated to I_2 by M_{I_2}. So we have that

$$A_1 M_{\mathrm{IV}} A_1^{-1} = A_2 M_{\mathrm{I}_2} A_2^{-1} A_3 M_{\mathrm{I}_2} A_3^{-1},$$

with A_1, A_2, $A_3 \in \mathrm{SL}(2,\mathbb{Z})$ from which we can conclude that the eigenvalues of M_{IV} are equal to the eigenvalues of $M_{\mathrm{I}_2} A M_{\mathrm{I}_2} A^{-1}$, where again $A \in \mathrm{SL}(2,\mathbb{Z})$. In table 2.1 we have been given that

$$\begin{aligned} M_{\mathrm{IV}} &= \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \\ M_{\mathrm{I}_2} &= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

If we further write

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{3.3}$$

where a, b, c, $d \in \mathbb{Z}$ and $ad - bc = 1$, we may easily derive that the eigenvalues $\lambda_\pm$ of $M_{\mathrm{I}_2} A M_{\mathrm{I}_2} A^{-1}$ are

$$\lambda_\pm = 1 - 2c^2 \pm \sqrt{1 + 4c^4 - 8c^2 - 8ac^2 - 4a^2c^2}.$$

On the other hand we have that the eigenvalues of M_{IV} are $-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. Equating both expressions for the eigenvalues yields that $c = \sqrt{3}/2$ which contradicts the fact that $c \in \mathbb{Z}$.

This completes our discussion of the confluence to singular fibers of Kodaira type II, III and IV. We summarize the result of this section in the following theorem

Theorem 3.3.1 *Of all confluences to Singular Fibers of Kodaira type* II, III *and* IV*, superficially allowed by conservation of the Euler number, the following occur:*

$\mathrm{II} \to \mathrm{I}_1 + \mathrm{I}_1$ $\quad \mathrm{III} \to 3\mathrm{I}_1$ $\quad \mathrm{III} \to \mathrm{I}_2 + \mathrm{I}_1$ $\quad \mathrm{III} \to \mathrm{II} + \mathrm{I}_1$ $\quad \mathrm{IV} \to 4\mathrm{I}_1$ $\quad \mathrm{IV} \to \mathrm{I}_3 + \mathrm{I}_1$

$\mathrm{IV} \to \mathrm{III} + \mathrm{I}_1$ $\quad \mathrm{IV} \to \mathrm{II} + \mathrm{I}_2$ $\quad \mathrm{IV} \to 2\mathrm{II}$ $\quad \mathrm{IV} \to \mathrm{I}_2 + 2\mathrm{I}_1$ $\quad \mathrm{IV} \to \mathrm{II} + 2\mathrm{I}_1$.

Moreover the confluence which does not occur namely $\mathrm{IV} \to 2\mathrm{I}_2$ *is obstructed by monodromy considerations.*

Note that in a confluence to a singular fiber of type II the complementary singular fiber II^* remains fixed in infinity. In our examples of confluences to III and IV, III^* and IV^*, respectively, remain fixed in infinity. This means that we have provided an explicit Weierstrass normal form for the following configurations in the list of Persson [10]:

$\mathrm{II}^*\,\mathrm{II}$ $\quad \mathrm{II}^*\,2\mathrm{I}_1$ $\quad \mathrm{III}^*\,\mathrm{III}$ $\quad \mathrm{III}^*\,3\mathrm{I}_1$ $\quad \mathrm{III}^*\,\mathrm{I}_2\,\mathrm{I}_1$ $\quad \mathrm{III}^*\,\mathrm{II}\,\mathrm{I}_1$ $\quad \mathrm{IV}^*\,\mathrm{IV}$ $\quad \mathrm{IV}^*\,4\mathrm{I}_1$ $\quad \mathrm{IV}^*\,\mathrm{I}_3\,\mathrm{I}_1$

$\mathrm{IV}^*\,\mathrm{III}\,\mathrm{I}_1$ $\quad \mathrm{IV}^*\,\mathrm{II}\,\mathrm{I}_2$ $\quad \mathrm{IV}^*\,2\mathrm{II}$ $\quad \mathrm{IV}^*\,\mathrm{I}_2\,2\mathrm{I}_1$ $\quad \mathrm{IV}^*\,\mathrm{II}\,2\mathrm{I}_1$.

With respect to our ultimate goal of understanding the stratification of the space N_g we are now in a position to give more insight into the intricate structure of the set of strata corresponding to a configuration containing a singular fiber of Kodaira type II^*, III^* or IV^*.[11]

We consider a configuration of singular fibers where II^* is fixed in infinity. This gives that in affine coordinates in a neighbourhood of the origin

$$\begin{aligned} g_2(z) &= a \\ g_3(z) &= bz + c, \end{aligned}$$

[11] Suggested by Hans Duistermaat.

where a, $b \neq 0$ and c are constants. By rescaling and a so-called Tschirnhausen transformation, that is a coordinate transformation which sends z to z plus a constant to remove the next to leading term of a polynomial, we may write

$$g_2(z) = a$$
$$g_3(z) = z.$$

It is obvious that for $a \neq 0$ we have two singular fibers of Kodaira type I_1. We find, dividing out symmetries, that the family of configurations containing at least a singular fiber of type II^* is one-dimensional, which degenerates to a configuration $\mathrm{II}^*\,\mathrm{II}$ for one particular value.

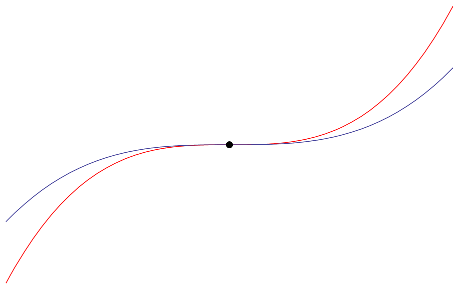


Figure 3.1: Sketch of the curves in parameter space corresponding to different configurations of singular fibers including a singular fiber of type III^*.

We now consider a configuration of singular fibers where III^* is fixed in infinity. We now have in affine coordinates that

$$\begin{aligned} g_2(z) &= az + b \\ g_3(z) &= cz + d \\ \Delta(z) &= a^3z^3 + (3a^2b - 27c^2)z^2 + (2ab^2 - 54cd)z + b^3 - 27d^2, \end{aligned}$$

where $a \neq 0$, b, c and d are constants. By rescaling and a Tschirnhausen transformation on the geometric discriminant we may set $a = 1$ and $b = 9c^3$, so that

$$\Delta(z) = z^3 + (243c^4 - 54cd)z + 739c^6 - 27d^2.$$

The discriminant of the geometric discriminant $\Delta(z)$ now reads

$$-19683\left(5c^3 - d\right)\left(9c^3 - d\right)^3$$

and the resultant of g_2 and g_3 with respect to z

$$-9c^3 + d.$$

We notice that the resultant of g_2 and g_3 divides the discriminant of the geometric discriminant as was noted in section 2.7. The occurrence of the third power is typical.

We have a configuration of type $\mathrm{III}^*\,\mathrm{III}$ if $c = d = 0$, a configuration of type $\mathrm{III}^*\,\mathrm{I}_2\,\mathrm{I}_1$ if $5c^3 - d = 0$, a configuration of type $\mathrm{III}^*\,\mathrm{II}\,\mathrm{I}_1$ if $9c^3 - d = 0$ and a configuration of type $\mathrm{III}^*\,3\mathrm{I}_1$ otherwise. A sketch of the situation is given in figure 3.1.

For a configuration including a singular fiber of type IV^* we have according to Persson [10] and our examples above the following possibilities:

$$\mathrm{IV}^*\,\mathrm{IV} \quad \mathrm{IV}^*\,4\mathrm{I}_1 \quad \mathrm{IV}^*\,\mathrm{I}_3\,\mathrm{I}_1 \quad \mathrm{IV}^*\,\mathrm{III}\,\mathrm{I}_1 \quad \mathrm{IV}^*\,\mathrm{II}\,\mathrm{I}_2 \quad \mathrm{IV}^*\,2\mathrm{II} \quad \mathrm{IV}^*\,\mathrm{I}_2\,2\mathrm{I}_1 \quad \mathrm{IV}^*\,\mathrm{II}\,2\mathrm{I}_1.$$

The configuration which includes two singular fibers of type I_2 is excluded on monodromy grounds. If we incorporate the Tschrirnhausen transformation and rescaling, we have

$$\begin{aligned}
g_2(z) &= az + b \\
g_3(z) &= d + \frac{a^3 z}{54} + z^2 \\
\Delta(z) &= -27z^4 + \left(-\frac{a^6}{108} + 3a^2 b - 54d \right) z^2 + (3ab^2 - a^3 d)z + b^3 - 27d^2.
\end{aligned}$$

In this case the discriminant of the geometric discriminant $\Delta(z)$ reads

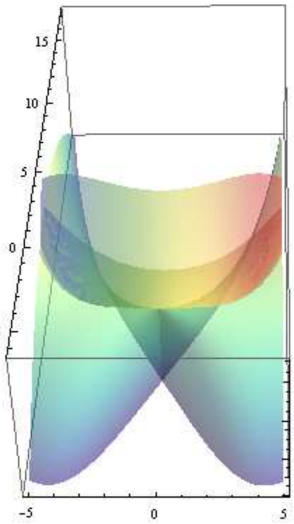
$$\begin{aligned}
-\frac{1}{314928} \left(a^4 b - 54b^2 - 54a^2 d\right)^3 \big(a^{12} &- 891a^8 b + 240570a^4 b^2 - 10077696b^3 + 18954a^6 d \\
&- 11337408a^2 bd + 272097792d^2\big)
\end{aligned}$$

and the resultant of g_2 and g_3 equals

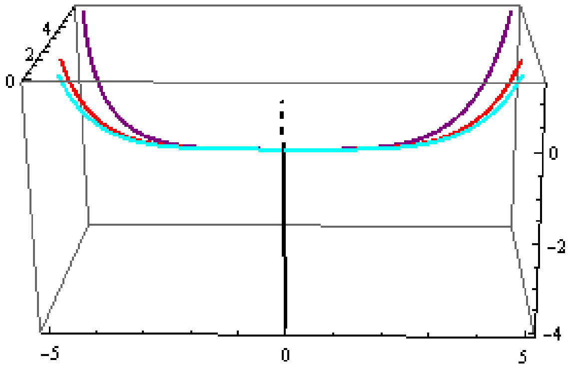
$$1/54(-a^4 b + 54b^2 + 54a^2 d).$$

We thus have that the solution surface to the equation $-a^4 b + 54b^2 + 54a^2 d = 0$ encompasses the configurations $\mathrm{IV}^*\,\mathrm{IV}$, $\mathrm{IV}^*\,\mathrm{III}\,\mathrm{I}_1$, $\mathrm{IV}^*\,\mathrm{II}\,\mathrm{I}_2$, $\mathrm{IV}^*\,2\mathrm{II}$ and $\mathrm{IV}^*\,\mathrm{II}\,2\mathrm{I}_1$, while its complement intersected with the solution surface of $a^{12} - 891a^8 b + 240570a^4 b^2 - 10077696b^3 + 18954a^6 d - 11337408a^2 bd + 272097792d^2 = 0$ corresponds to the configurations $\mathrm{IV}^*\,\mathrm{I}_3\,\mathrm{I}_1$ and $\mathrm{IV}^*\,\mathrm{I}_2\,2\mathrm{I}_1$. The configuration $\mathrm{IV}^*\,4\mathrm{I}_1$ is characterized by the fact that discriminant of the geometric discriminant, in affine coordinates, is unequal to zero. Having made the distinction between the three cases, we focus of the first. Clearly the $a = b = d = 0$ solution of the equation $-a^4 b + 54b^2 + 54a^2 d = 0$ corresponds to the configuration $\mathrm{IV}^*\,\mathrm{IV}$, while the solution $a = b = 0$ and $d \neq 0$ corresponds to the configuration $\mathrm{IV}^*\,2\mathrm{II}$. If on the other hand $a \neq 0$ we may solve the equation with respect to the variable d. Using this solution yields

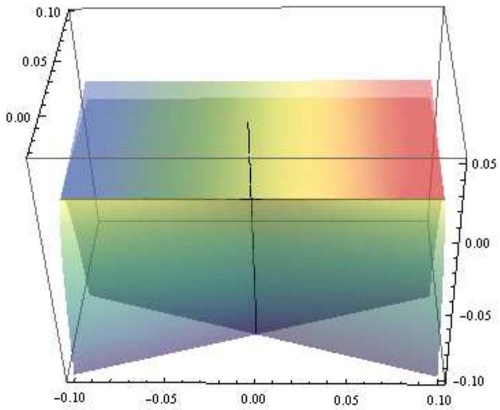
$$\begin{aligned}
g_2(z) &= b + az \\
g_3(z) &= \frac{1}{54a^2}(b + az) \left(a^4 - 54b + 54az\right) \\
\Delta(z) &= -\frac{1}{108a^4}(b + az)^2 \left(a^8 - 216a^4 b + 2916b^2 - 5832abz + 2916a^2 z^2\right).
\end{aligned}$$



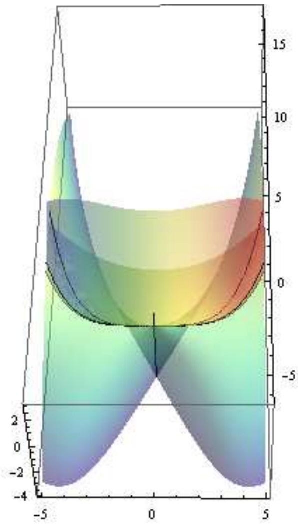
(a) The surface on which the resultant of g_2 and g_3 is zero is indicated in green-blue. The surface of which the discriminant is zero but not the resultant of g_2 and g_3 is indicated in rainbow colours.



(b) The line corresponding to the configuration $\mathrm{IV}^*\,2\mathrm{II}$ is indicated in black, for positive d the line has been dashed to indicate that the roots are no longer real. The line corresponding to the configuration $\mathrm{IV}^*\,\mathrm{II}\,\mathrm{I}_2$ is indicated in red,$\mathrm{IV}^*\,\mathrm{III}\,\mathrm{I}_1$ in purple, $\mathrm{IV}^*\,\mathrm{I}_3\,\mathrm{I}_1$ in cyan.



(c) A zoom in of the area around the origin, the different lines as discussed in 3.2(b) are all indicated in black but can not be distinguished on this scale.



(d) Overview of the stratification, again with all lines indicated in black.

Figure 3.2: Sketches of the stratification of the (a, b, d)-space as discussed above. The upper left figure sketches the different surfaces, the upper right the different one-dimensional components. The lower figures give a more general overview, on the left in a small neighbourhood of zero, on the right on a bigger scale.

The resultant of $b + az$ and $a^4 - 54b + 54az$, with respect to the variable z, equals $a(a^4 - 108b)$. Setting this resultant equal to zero as well as using the solution for d yields the

solution curve $(a, a^4/108, a^6/11664)$ in (a, b, d)-space, with $a \neq 0$, this clearly corresponds to the configuration $\mathrm{IV}^*\,\mathrm{III}\,\mathrm{I}_1$. Setting the discriminant of $a^8 - 216a^4b + 2916b^2 - 5832abz + 2916a^2z^2$ equal to zero yields $b = a^4 216$. Again using our previous solution of the equation $-a^4b + 54b^2 + 54a^2d = 0$, with respect to d, we find a solution curve $(a, a^4/216, a^6/15552)$ in (a, b, d)-space, with $a \neq 0$. This solution corresponds to the configuration $\mathrm{IV}^*\,\mathrm{II}\,\mathrm{I}_2$. The solutions curves $(a, a^4/108, a^6/11664)$ and $(a, a^4/216, a^6/15552)$ are exactly the common zeros of $-a^4b + 54b^2 + 54a^2d$ and $a^{12} - 891a^8b + 240570a^4b^2 - 10077696b^3 + 18954a^6d - 11337408a^2bd + 272097792d^2$. The complement of these solution curves in the solution surface to $-a^4b + 54b^2 + 54a^2d = 0$, corresponds to the configuration $\mathrm{IV}^*\,\mathrm{II}\,2\mathrm{I}_1$.
We now shall focus on the distinction between the configurations $\mathrm{IV}^*\,\mathrm{I}_3\,\mathrm{I}_1$ and $\mathrm{IV}^*\,\mathrm{I}_2\,2\mathrm{I}_1$. We now may employ the method discussed in section 1.2 and calculate the resultant of $\Delta(z)$ and $\Delta'(z) - y\Delta''(z)$ and set each of the coefficients of the resulting polynomial in y equal to zero. This results in two solution curves namely the curve $(a, a^4/108, a^6/11664)$ in (a, b, d)-space corresponding to the configuration $\mathrm{IV}^*\,\mathrm{III}\,\mathrm{I}_1$,which we already discussed and the solution curve $(a, 7a^4/1728, 37a^6/746496)$ in (a, b, d)-space corresponding to the configuration $\mathrm{IV}^*\,\mathrm{I}_3\,\mathrm{I}_1$. Sketches of the stratification of the parameter space are included in figure 3.2. It concerns a sketch of the real curves, the complex part is ignored. We note that the choice $a = b = 0$ and $d > 0$ corresponding to the configuration $\mathrm{IV}^*\,2\mathrm{II}$ does encompass imaginary roots in the z-plane, given our discussion above these are included anyway.

3.4 Confluence to singular fibers of Kodaira type I_0^*.

In this section we discuss the confluences of to singular fibers of type I_0^*. As always we have that the zeros of the geometric discriminant $\Delta(z)$ are conserved in a confluence, which is equivalent to conservation of the Euler number. Of all 26 confluences allowed by this constraint a further 8 are excluded by monodromy considerations. We will give precise arguments for the 8 excluded cases. Furthermore we will construct examples of the remaining 18 confluences, verify their properties by explicit calculation and thus prove their existence.[12] In the case of a singular fiber of type I_0^* we know thanks to Persson [10], that it is possible to consider a very interesting configuration of singular fibers: two singular fibers of type I_0^*. In our examples of confluences one of these singular fibers will be placed at infinity and one at zero. The singular fiber of type I_0^* at zero will be perturbed into several singular fibers. Since the product of all monodromy matrices of the singular fibers, including the singular fiber at infinity, must be the identity, we see that if a configuration of a singular fiber of type I_0^* and several other singular fibers of type say $A_1, \ldots, A_n$ exists, the confluence of the singular fibers of type $A_1, \ldots, A_n$ to a singular fiber of type I_0^* is not disallowed by monodromy considerations. The following configurations which contain at least a singular fiber of type I_0^* are included in Persson's list:[13]

$\mathrm{I}_0^*\,\mathrm{I}_4\,2\mathrm{I}_1$	$\mathrm{I}_0^*\,\mathrm{IV}\,\mathrm{II}$	$\mathrm{I}_0^*\,\mathrm{IV}\,2\mathrm{I}_1$	$\mathrm{I}_0^*\,\mathrm{I}_3\,\mathrm{II}\,\mathrm{I}_1$	$\mathrm{I}_0^*\,\mathrm{I}_3\,3\mathrm{I}_1$	$\mathrm{I}_0^*\,2\mathrm{III}$	$\mathrm{I}_0^*\,\mathrm{III}\,\mathrm{I}_2\,\mathrm{I}_1$	$\mathrm{I}_0^*\,\mathrm{III}\,\mathrm{II}\,\mathrm{I}_1$
$\mathrm{I}_0^*\,\mathrm{III}\,3\mathrm{I}_1$	$\mathrm{I}_0^*\,3\mathrm{I}_2$	$\mathrm{I}_0^*\,2\mathrm{I}_2\,2\mathrm{I}_1$	$\mathrm{I}_0^*\,\mathrm{I}_2\,2\mathrm{II}$	$\mathrm{I}_0^*\,\mathrm{I}_2\,\mathrm{II}\,2\mathrm{I}_1$	$\mathrm{I}_0^*\,\mathrm{I}_2\,4\mathrm{I}_1$	$\mathrm{I}_0^*\,3\mathrm{II}$	$\mathrm{I}_0^*\,2\mathrm{II}\,2\mathrm{I}_1$
$\mathrm{I}_0^*\,\mathrm{II}\,4\mathrm{I}_1$	$\mathrm{I}_0^*\,6\mathrm{I}_1$	$2\mathrm{I}_0^*$					

We shall give an explicit example of the transition of each of these configurations to a configuration $2\mathrm{I}_0^*$, where one of these two singular fibers is placed at infinity and the other one at zero. In the construction of the examples we generally try to fix one of the singular fibers arising after perturbation of the singular fiber of type I_0^* in the origin, as this simplifies calculations considerably. We shall often content ourselves with describing the situation for a non-zero perturbation parameter, because we start out in exactly the same manner each time.

[12] The set of all allowed confluences of does contain a confluence of type $\mathrm{I}_0^* \to \mathrm{I}_2 + \mathrm{I}_2 + \mathrm{I}_2$ despite of the conclusion of Naruki [9], see section 5. The construction of an example of this confluence suggested by Hans Duistermaat will be given below.

[13] Note the appearance of the configuration $\mathrm{I}_0^*\,3\mathrm{I}_2$, which is therefore allowed by monodromy.

Examples

$\mathrm{I}_0^* \rightarrow \mathrm{I}_4 + 2\mathrm{I}_1$

For this confluence we choose

$$\begin{aligned} g_2(z) &= 3\epsilon^2 - 6(-2)^{1/3}\epsilon z \\ g_3(z) &= -\epsilon^3 + 3(-2)^{1/3}\epsilon^2 z - \frac{3(-1)^{2/3}\epsilon}{2^{1/3}} z^2 + z^3 \\ \Delta(z) &= \frac{27}{2} z^4 (2z^2 - 6(-2)^{2/3} z\epsilon + 3(-2)^{1/3}\epsilon^2). \end{aligned}$$

From these formulae we clearly see a zero of order four in the origin. It is easy to see that $g_2(0) = 3\epsilon^2$, so that the zero in the origin corresponds to a singular fiber of type I_4. The other zeros of the geometric discriminant Δ are

$$\frac{(-1)^{2/3}(3 - 2\sqrt{3})\epsilon}{2^{1/3}} \qquad \frac{(-1)^{2/3}(3 + 2\sqrt{3})\epsilon}{2^{1/3}},$$

which obviously correspond to singular fibers of type I_1.

$\mathrm{I}_0^* \rightarrow \mathrm{IV} + \mathrm{II}$

For this confluence we choose

$$\begin{aligned} g_2(z) &= 0 \\ g_3(z) &= z^2(z - \epsilon) \\ \Delta(z) &= -27 z^4 (z - \epsilon)^2. \end{aligned}$$

Obviously we are now faced with a zero of fourth order and a zero of second order of the geometric discriminant $\Delta(z)$. Since $g_2(z) = 0$, they correspond to singular fibers of type IV and II.

$\mathrm{I}_0^* \rightarrow \mathrm{IV} + 2\mathrm{I}_1$

For this confluence we choose

$$\begin{aligned} g_2(z) &= z^2 \\ g_3(z) &= z^2(z + \epsilon) \\ \Delta(z) &= z^6 - 27 z^4 (z + \epsilon)^2. \end{aligned}$$

Again we clearly see a singular fiber of type IV in the origin. Furthermore we have that the discriminant of $\Delta(z)/z^4$ equals $2^2 3^3 \epsilon^2$, which implies that the two other zeros of $\Delta(z)$ correspond to singular fibers of type I_1.

$\mathrm{I}_0^* \to \mathrm{I}_3 + \mathrm{II} + \mathrm{I}_1$

For this confluence we choose

$$\begin{aligned} g_2(z) &= 3\epsilon(2z+\epsilon) \\ g_3(z) &= \frac{1}{2}(2z+\epsilon)(z^2 - 2\epsilon z - 2\epsilon^2) \\ \Delta(z) &= -\frac{3^3}{2^2} z^3 (z - 2^2\epsilon)(2z+\epsilon)^2. \end{aligned}$$

There is obviously a zero of third order in the origin, one of second order in $z = -\epsilon/2$ and one of first order in $z = 4\epsilon$. It is also clear that the zero in the origin corresponds to a singular fiber of type I_3. Since $(2z+\epsilon)$ is a factor of both g_2 and g_3, we identify the zero in $z = -\epsilon/2$ as belonging to a singular fiber of type II. The remaining singular fiber is necessarily of type I_1.

$\mathrm{I}_0^* \to \mathrm{I}_3 + 3\mathrm{I}_1$

For this confluence we choose

$$\begin{aligned} g_2(z) &= z^2 + 3\epsilon^2 \\ g_3(z) &= z^3 + \frac{1}{2}\epsilon z^2 + \epsilon^3 \\ \Delta(z) &= -\frac{1}{4} z^3 (2^3 13 z^3 + 2^2 3^3 \epsilon z^2 - 3^2 \epsilon^2 z + 2^3 3^3 \epsilon^3). \end{aligned}$$

We are clearly faced with a singular fiber of type I_3 in the origin since z^3 factors $\Delta(z)$ but $g_2(0) \neq 0$. We may further derive that

$$D\left(\frac{\Delta(z)}{z^3}\right) = -\frac{3^6 109^3 \epsilon^6}{2^4},$$

where D denotes the discriminant. This implies that there are three singular fibers of type I_1 outside the origin.

$\mathrm{I}_0^* \to 2\mathrm{III}$

For this confluence we choose

$$\begin{aligned} g_2(z) &= z^2 - \epsilon^2 \\ g_3(z) &= 0 \\ \Delta(z) &= (z^2 - \epsilon^2)^3. \end{aligned}$$

The third order zeros of $\Delta(z)$, namely $\pm\epsilon$, must correspond to singular fibers of type III since $g_3 = 0$.

$\mathrm{I}_0^* \to \mathrm{III} + \mathrm{I}_2 + \mathrm{I}_1$

We choose

$$\begin{aligned} g_2(z) &= -3(z-\epsilon)(z+\epsilon) \\ g_3(z) &= (z-\epsilon)^2(2z+\epsilon) \\ \Delta(z) &= -27z^2(z-\epsilon)^3(5z+3\epsilon). \end{aligned}$$

The geometric discriminant has a second order zero in the origin, a third order zero in ϵ and finally a first order zero in $-3\epsilon/5$. By construction both g_2 and g_3 are not equal to zero in the origin, which implies we have a singular fiber of type I_2. Clearly $(z-\epsilon)$ factors both g_2 and g_3, which gives us that the zero in ϵ corresponds to a singular fiber of type III. The remaining zero corresponds to a singular fiber of type I_1.

$\mathrm{I}_0^* \to \mathrm{III} + \mathrm{II} + \mathrm{I}_1$

To construct such a confluence of this type we need a common linear factor of both g_2 and g_3 as well as a linear factor which divides g_2 and whose square divides g_3. The following choice will do

$$\begin{aligned} g_2(z) &= z(z-\epsilon) \\ g_3(z) &= z^2(z-\epsilon) \\ \Delta(z) &= -z^3(z-\epsilon)^2(26z+\epsilon). \end{aligned}$$

It is obvious that there is a singular fiber of type III placed in the origin and a singular fiber of type II in ϵ. The remaining zero is of type I_1.

$\mathrm{I}_0^* \to \mathrm{III} + 3\mathrm{I}_1$

In this case we need a linear factor which divides g_2 and whose square divides g_3, but g_2 and g_3 should not have further common factors then this. The following choice will suffice

$$\begin{aligned} g_2(z) &= z(z+\epsilon) \\ g_3(z) &= z^3 \\ \Delta(z) &= -z^3(z-\epsilon)(13z^2+5\epsilon z+\epsilon^2). \end{aligned}$$

It is obvious that we find a singular fiber of type III in the origin. We may further derive that

$$D\left(\frac{\Delta(z)}{z^3}\right) = -3^9\epsilon^6,$$

where as usual D denotes the discriminant. From this calculation we obtain that the singular fibers for a non-zero perturbation parameter outside the origin are of type I_1.

$\mathrm{I}_0^* \to 3\,\mathrm{I}_2$

For this confluence we need the discriminant to be a square of a third order polynomial in z. We therefore write $\Delta(z) = -27 f(z)^2$. Since we assume that there is a singular fiber of type I_0^* at infinity, g_3 is of degree 2 and may be written as follows $g_3(z) = 3p(z)q(z)$, where p and q are linear functions. Moreover neither p nor q may divide g_2, since this would yield a singular fiber of type II. From the definition of the geometric discriminant we deduce that

$$p(z)^3 q(z)^3 = (g_3(z) - f(z))(g_3(z) + f(z)).$$

Combining this with the fact that neither p nor q divides g_2 yields

$$p(z)^3 = C_1(g_3(z) - f(z)) \qquad\qquad q(z)^3 = C_2(g_3(z) + f(z)),$$

with C_1 and C_2 constants, which in turn may be absorbed in $p(z)$ and $q(z)$. We now have that

$$g_2(z) = 3p(z)q(z) \qquad g_3(z) = \frac{p(z)^3 + q(z)^3}{2} \qquad \Delta(z) = -\frac{27}{4}(p(z)^3 - q(z)^3).$$

By rescaling, a Tschirnhauser transformation and taking into account that g_2 and g_3 have no common factor we may set

$$p(z) = az + b \qquad\qquad q(z) = z + a^2 b,$$

where $a^3 \neq 1$.
We therefore choose

$$\begin{aligned} g_2(z) &= 3(2z+\epsilon)(z+4\epsilon) \\ g_3(z) &= \frac{(2z+\epsilon)^3 + (z+4\epsilon)^3}{2} \\ \Delta(z) &= -\frac{3^3 7^2}{4}(z-3\epsilon)^2(z^2+3\epsilon z+3\epsilon^2)^2. \end{aligned}$$

The zeros of $\Delta(z)$ are therefore 3ϵ, $-\frac{1}{2}i(\sqrt{3}-3i)\epsilon$ and $-\frac{1}{2}i(\sqrt{3}+3i)\epsilon$, at which the value of g_2 is $3\cdot 7^2\epsilon^2$, $\frac{3}{2}(-13-3\sqrt{3}i)\epsilon^2$ and $\frac{3}{2}(-13+3\sqrt{3}i)\epsilon^2$ respectively. This implies that we indeed have $3\,\mathrm{I}_2$ for $\epsilon \neq 0$.

$\mathrm{I}_0^* \to 2\mathrm{I}_2 + 2\mathrm{I}_1$

Here we choose

$$\begin{aligned} g_2(z) =& \epsilon(z+3\epsilon) \\ g_3(z) =& \frac{1}{2}(2z^3 + (3 2^{2/3} - 2^{1/3})\epsilon z^2 - \epsilon^2 z - 2\epsilon^3) \\ \Delta(z) =& -\frac{1}{4}z^2(-2^2 3^3 z^4 + 2^2 3^3(2^{1/3} - 3 2^{2/3})\epsilon z^3 - 27(-16 + 18\,2^{1/3} + 2^{2/3})\epsilon^2 z^2 \\ & + 2(110 - 27\,2^{1/3} + 81\,2^{2/3})\epsilon^3 z + 9(1 - 6\,2^{1/3})^2\epsilon^4). \end{aligned}$$

The geometric discriminant $\Delta(z)$ has a two zeros of order two, one in the origin and another in $z = \frac{1}{6}(2^{1/3} - 6\,2^{2/3})\epsilon$ and two zeros of first order in $z = \frac{1}{6}(2\,2^{1/3}\epsilon - 3\,2^{2/3}\epsilon - 2\sqrt{-6\epsilon^2 + 18\,2^{1/3}\epsilon^2 + 2^{2/3}\epsilon^2})$ and $z = \frac{1}{6}(2\,2^{1/3}\epsilon - 3\,2^{2/3}\epsilon + 2\sqrt{-6\epsilon^2 + 18\,2^{1/3}\epsilon^2 + 2^{2/3}\epsilon^2})$. The value of g_2 in the two zeros of second order is $3\epsilon^2$ and $\frac{1}{6}(18 + 2^{1/3} - 6\,2^{2/3})\epsilon^2$ respectively, which implies we have indeed constructed a confluence of $\mathrm{I}_0^* \to 2\mathrm{I}_2 + 2\mathrm{I}_1$.

$\mathrm{I}_0^* \to \mathrm{I}_2 + 2\mathrm{II}$

To construct two singular fibers of type II, we must impose that g_2 and g_3 have two linear factors in common. If we also take rescaling into account we deduce that we may write

$$\begin{aligned}
g_2(z) &= c(z-\zeta_1)(z-\zeta_2)\\
g_3(z) &= (z-\zeta_1)(z-\zeta_2)(z-\zeta_3)\\
\Delta(z) &= (z-\zeta_1)^2(z-\zeta_2)^2((-27+c^3)z^2 + (54\zeta_3 - c^3\zeta_1 - z^3\zeta_2)z + c^3\zeta_1\zeta_2 - 27\zeta_3^2).
\end{aligned}$$

It is now sufficient to impose that $\zeta_1 \neq \zeta_2 \neq \zeta_3$ and the discriminant of $(-27+c^3)z^2 + (54\zeta_3 - c^3\zeta_1 - z^3\zeta_2)z + c^3\zeta_1\zeta_2 - 27\zeta_3^2$ is equal to zero. We consequently choose

$$\begin{aligned}
g_2(z) &= \left(z - 3\sqrt{\frac{3}{26}}\epsilon\right)\left(z + 3\sqrt{\frac{3}{26}}\epsilon\right)\\
g_3(z) &= (z-\epsilon)\left(z - 3\sqrt{\frac{3}{26}}\epsilon\right)\left(z + 3\sqrt{\frac{3}{26}}\epsilon\right)\\
\Delta(z) &= -\frac{1}{26}(26z - 27\epsilon)^2\left(z - 3\sqrt{\frac{3}{26}}\epsilon\right)^2\left(z + 3\sqrt{\frac{3}{26}}\epsilon\right)^2.
\end{aligned}$$

This gives us a confluence of type $\mathrm{I}_0^* \to \mathrm{I}_2 + 2\mathrm{II}$.

$\mathrm{I}_0^* \to \mathrm{I}_2 + \mathrm{II} + 2\mathrm{I}_1$

For this construction we will again impose that a singular fiber of type I_2 lies in the origin and that g_2 and g_3 have one common factor. This leads to the following example

$$\begin{aligned}
g_2(z) &= \epsilon(z+3\epsilon)\\
g_3(z) &= \frac{1}{18}(z+3\epsilon)(18z^2 - \epsilon z - 6\epsilon^2)\\
\Delta(z) &= -\frac{1}{12}z^2(z+3\epsilon)^2(2^2 3^4 z^2 - 2^2 3^2 \epsilon z - 5\,43\epsilon^2).
\end{aligned}$$

Here we find two second order zeros of the geometric discriminant namely the origin and -3ϵ and two of first order namely $\frac{1}{18}(1 \pm 6\sqrt{6})\epsilon$. We find that the value of g_2 in the second order zeros is 0 and $3\epsilon^2$, respectively, which establishes that we have constructed a confluence of type $\mathrm{I}_0^* \to \mathrm{I}_2 + \mathrm{II} + 2\mathrm{I}_1$.

$I_0^* \to I_2 + 4I_1$

For this confluence we impose that for a non-zero perturbation parameter we have a singular fiber of type I_2 in the origin. This leads to the following Weierstrass model

$$\begin{aligned} g_2(z) &= z^2 + 3\epsilon^2 \\ g_3(z) &= z^3 + \epsilon^3 \\ \Delta(z) &= -z^2(26z^4 - 9\epsilon^2 z^2 + 54\epsilon^3 z - 27\epsilon^4). \end{aligned}$$

The second order zero of the geometric discriminant in the origin corresponds to a singular fiber of type I_2, since the value of g_2 in the origin is $3\epsilon^2$. We may further derive that

$$D\left(\frac{\Delta(z)}{z^2}\right) = -2^{11}3^9 7^3 13\,\epsilon^{12},$$

so that the other singular fibers must be of type I_1.

$I_0^* \to 3II$

For this confluence the following choice is the obvious one

$$\begin{aligned} g_2(z) &= 0 \\ g_3(z) &= z^3 + \epsilon^3 \\ \Delta(z) &= -27(z^3 + \epsilon^2)^3. \end{aligned}$$

We clearly see that every zero is of second order and the $g_2 = 0$ and that these zeros thus correspond to singular fibers of type II.

$I_0^* \to 2\,II + 2I_1$

To find a Weierstrass model for this confluence we must impose two common factors for g_2 and g_3, the least complicated choice seems to be

$$\begin{aligned} g_2(z) &= (z - \epsilon)(z + \epsilon) \\ g_3(z) &= z(z - \epsilon)(z + \epsilon) \\ \Delta(z) &= -(z - \epsilon)^2(z + \epsilon)^2(26z^2 + \epsilon^2). \end{aligned}$$

The geometric discriminant has clearly two second order zeros corresponding to singular fibers of type II, the first order zeros of the geometric discriminant automatically correspond to singular fibers of type I_1.

$I_0^* \to II + 4\,I_1$

For this confluence we impose that g_2 and g_3 have one common factor, which for convenience will be placed in the origin, therefore our Weierstrass model will be

$$\begin{aligned} g_2(z) &= z^2 \\ g_3(z) &= z(z^2 + \epsilon) \\ \Delta(z) &= -z^2(26z^4 + 54\epsilon z^2 + 27\epsilon^2). \end{aligned}$$

We have clearly constructed a singular fiber of type II in the origin. We can verify that other singular fibers are of type I_1 by proving that the discriminant of the geometric discriminant divided by z^2 is nonzero for a nonzero perturbation parameter, indeed

$$D\left(\frac{\Delta(z)}{z^2}\right) = 2^9 3^9 13\,\epsilon^6.$$

$\mathrm{I}_0^* \to 6\,\mathrm{I}_1$

A generic perturbation of a singular fiber other then a singular fiber of type I_1 yields χ singular fibers of type I_1, where χ is the Euler number of the singular fiber before perturbation. This implies that almost any perturbation will do, however the following model will be convenient

$$\begin{aligned}
g_2(z) &= z^2 + \epsilon \\
g_3(z) &= z^3 \\
\Delta(z) &= -(2z^2 - \epsilon)(13z^4 + 5\epsilon z^2 + \epsilon^2).
\end{aligned}$$

Again we use the discriminant to verify that the singular fibers are of type I_1

$$D(\Delta(z)) = 2^7 3^{18} 13\,\epsilon^{15}.$$

This completes our discussion of examples of confluences to singular fibers of type I_0^*

Confluences obstructed by monodromy

We shall now discuss the obstructions. Some of these obstructions are found by explicit calculation of the product of the conjugacy classes of matrices. We note that the monodromy matrix associated to the singular fiber I_0^* is minus the identity and will therefore be invariant under conjugation. From this we may conclude that it suffices, in the case of n singular fibers joining into a singular fiber of type I_1^*, to conjugate only $n-1$ of the monodromy matrices (by monodromy matrix we mean in this setting the element of the conjugation class as given in table 2.1) in the product which yields the monodromy matrix of I_0^*. However most obstructions are found by calculating the eigenvalues of the product n monodromy matrices, $n-1$ of which are conjugated, and setting these equal to -1, the eigenvalue of the monodrony matrix associated to the unperturbed singular fiber.

$\mathrm{I}_0^* \to \mathrm{I}_5 + \mathrm{I}_1$

In this case it suffices to calculate the eigenvalues of the following matrices

$$
\begin{aligned}
M_{\mathrm{I}_1}AM_{\mathrm{I}_5}A^{-1} &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\
M_{\mathrm{I}_5}AM_{\mathrm{I}_1}A^{-1} &= \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix},
\end{aligned}
$$

where A is an element of $SL(2,\mathbb{Z})$. The eigenvalues of these matrices are given by

$$
\lambda_{\pm} = \frac{1}{2}(2 - 5c^2 \pm \sqrt{5}\sqrt{5c^4 - 4c^2}).
$$

Clearly the eigenvalues of

$$
\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}
$$

are both equal to -1. If we therefore impose that $\lambda_{\pm} = -1$ we find that $c = \pm 2/\sqrt{5}$ which contradicts that $A \in SL(2,\mathbb{Z})$.

$\mathrm{I}_0^* \to \mathrm{I}_4 + \mathrm{I}_2$

We again calculate the eigenvalues in this case of

$$
\begin{aligned}
M_{\mathrm{I}_2}AM_{\mathrm{I}_4}A^{-1} &= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\
M_{\mathrm{I}_4}AM_{\mathrm{I}_2}A^{-1} &= \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.
\end{aligned}
$$

The eigenvalues of these matrices are given by

$$
\lambda_{\pm} = 1 - 4c^2 \pm 2\sqrt{2}\sqrt{2c^4 - c^2},
$$

equating these to -1 yields $c = \pm 1/\sqrt{2}$, which again contradicts that $A \in SL(2,\mathbb{Z})$.

$\mathrm{I}_0^* \to \mathrm{I}_4 + \mathrm{II}$

We again use the same approach and calculate the eigenvalues in this case of

$$
\begin{aligned}
M_{\mathrm{II}}AM_{\mathrm{I}_4}A^{-1} &= \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\
M_{\mathrm{I}_4}AM_{\mathrm{II}}A^{-1} &= \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.
\end{aligned}
$$

The eigenvalues of these matrices are given by

$$
\lambda_{\pm} = \frac{1}{2}(1 - 4a^2 - 4ac - 4c^2 \pm \sqrt{(1 - 4a^2 - 4ac - 4c^2)^2 - 4}),
$$

equating these to -1 yields $c = \frac{1}{2}(-a \pm \sqrt{3}\sqrt{1-a^2})$, which again contradicts that $A \in SL(2,\mathbb{Z})$.

$\mathrm{I}_0^* \to \mathrm{IV} + \mathrm{I}_2$

We again calculate the eigenvalues in this case of

$$M_{\mathrm{IV}} A M_{\mathrm{I}_2} A^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$
$$M_{\mathrm{I}_2} A M_{\mathrm{IV}} A^{-1} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

The eigenvalues of these matrices are given by

$$\lambda_\pm = \frac{1}{2}(-1 - 2a^2 - 2ac - 2c^2 \pm \sqrt{(1 + 2a^2 + 2ac + 2c^2)^2 - 4}),$$

equating these to -1 yields, $c = \frac{1}{2}(-a \pm \sqrt{2 - 3a^2})$ which again contradicts that $A \in SL(2,\mathbb{Z})$.

$\mathrm{I}_0^* \to 2\mathrm{I}_3$

We again calculate the eigenvalues in this case of

$$M_{\mathrm{I}_3} A M_{\mathrm{I}_3} A^{-1} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

The eigenvalues of this matrix are given by

$$\lambda_\pm = \frac{1}{2}(2 - 9c^2 \pm 3\sqrt{9c^4 - 4c^2}),$$

equating these to -1 yields, $c = \pm 2/3$, which again contradicts that $A \in SL(2,\mathbb{Z})$.

$\mathrm{I}_0^* \to \mathrm{I}_3 + \mathrm{III}$

We again calculate the eigenvalues in this case of

$$M_{\mathrm{III}} A M_{\mathrm{I}_3} A^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$
$$M_{\mathrm{I}_3} A M_{\mathrm{III}} A^{-1} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

The eigenvalues of these matrices are given by

$$\lambda_\pm = \frac{1}{2}(-3a^2 - 3c^2 \pm \sqrt{(3a^2 + 3c^2)^2 - 4}),$$

equating these to -1 yields $c = \pm\sqrt{2 - 3a^2}/\sqrt{3}$, which again contradicts that $A \in SL(2,\mathbb{Z})$.

$\mathrm{I}_0^* \to \mathrm{I}_3 + \mathrm{I}_2 + \mathrm{I}_1$

For this confluence we shall consider not the eigenvalues but the full matrix product, so we verify that there are no $A_1,\, A_2 \in SL(2,\, \mathbb{Z})$ such that

$$M_{\mathrm{I}_3} A_1 M_{\mathrm{I}_2} A_1^{-1} A_2 M_{\mathrm{I}_1} A_2^{-1} = M_{\mathrm{I}_0^*},$$

nor for any permutation of the monodromy matrices M_{I_1}, M_{I_2} and M_{I_3}. Writing out first of the above equations, where we denote

$$A_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \qquad\qquad A_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix},$$

yields

$$\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d_1 & -b_1 \\ -c_1 & a_1 \end{pmatrix} \cdot \\ \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d_2 & -b_2 \\ -c_2 & a_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Solving this equation with respect to a_1, c_1 and c_2 yields among others $c_1 = \pm\sqrt{2/3}$, which contradicts the assumption that $A_1,\, A_2 \in SL(2,\, \mathbb{Z})$. Solving the equations for a permutation of the monodromy matrices M_{I_1}, M_{I_2} and M_{I_3} yields $c = \pm 2/\sqrt{3}$, $c = \pm\sqrt{2/3}$ or $c = \pm\sqrt{2}$, again contradiction $A_1,\, A_2 \in SL(2,\, \mathbb{Z})$. This is sufficient to prove that this confluence can not be realized.

$\mathrm{I}_0^* \to 2\mathrm{I}_2 + \mathrm{II}$

We use the same method as for the confluence $\mathrm{I}_0^* \to \mathrm{I}_3 + \mathrm{I}_2 + \mathrm{I}_1$, that is we solve

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d_1 & -b_1 \\ -c_1 & a_1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \cdot \\ \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d_2 & -b_2 \\ -c_2 & a_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

with respect to a_1, c_1 and c_2. This yields the following solutions

a_1	c_1	a_2	c_2
0	$-\sqrt{\frac{3}{2}}$	$\frac{1}{\sqrt{2}}$	$-\sqrt{2}$
0	$\sqrt{\frac{3}{2}}$	$\frac{1}{\sqrt{2}}$	$-\sqrt{2}$
0	$-\sqrt{\frac{3}{2}}$	$-\frac{1}{\sqrt{2}}$	$\sqrt{2}$
0	$\sqrt{\frac{3}{2}}$	$-\frac{1}{\sqrt{2}}$	$\sqrt{2}$
$-\sqrt{\frac{3}{2}}$	0	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$
$-\sqrt{\frac{3}{2}}$	0	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
$\sqrt{\frac{3}{2}}$	0	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$
$\sqrt{\frac{3}{2}}$	0	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$

a_1	c_1	a_2	c_2
$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	0	$-\sqrt{\frac{3}{2}}$
$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	0	$\sqrt{\frac{3}{2}}$
$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0	$-\sqrt{\frac{3}{2}}$
$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0	$\sqrt{\frac{3}{2}}$
$\frac{2a_2^2-1}{\sqrt{2+2a_2^2-2\sqrt{a_2^2}\sqrt{6-3a_2^2}}}$	$-\sqrt{\frac{1+a_2^2-\sqrt{a_2^2}\sqrt{6-3a_2^2}}{2}}$	a_2	$-\frac{a_2}{2}+\frac{\sqrt{a_2^2}\sqrt{6-3a_2^2}}{2a_2}$
$\frac{1-2a_2^2}{\sqrt{2+2a_2^2-2\sqrt{a_2^2}\sqrt{6-3a_2^2}}}$	$\sqrt{\frac{1+a_2^2-\sqrt{a_2^2}\sqrt{6-3a_2^2}}{2}}$	a_2	$-\frac{a_2}{2}+\frac{\sqrt{a_2^2}\sqrt{6-3a_2^2}}{2a_2}$
$\frac{1-2a_2^2}{\sqrt{2+2a_2^2-2\sqrt{a_2^2}\sqrt{6-3a_2^2}}}$	$\sqrt{\frac{1+a_2^2-\sqrt{a_2^2}\sqrt{6-3a_2^2}}{2}}$	a_2	$-\frac{a_2}{2}-\frac{\sqrt{a_2^2}\sqrt{6-3a_2^2}}{2a_2}$
$\frac{2a_2^2-1}{\sqrt{2+2a_2^2-2\sqrt{a_2^2}\sqrt{6-3a_2^2}}}$	$-\sqrt{\frac{1+a_2^2-\sqrt{a_2^2}\sqrt{6-3a_2^2}}{2}}$	a_2	$-\frac{a_2}{2}-\frac{\sqrt{a_2^2}\sqrt{6-3a_2^2}}{2a_2}$

The first twelve solutions lie clearly not in $\mathbb{Z}$. For the four last solution we note the following

$$\sqrt{6-3a_2^2} \in \mathbb{R},$$

implies that $a_2 = -1, 0$ or 1, where 0 may be discarded immediately. The remaining option $a_2 = \pm 1$ may be excluded because this would imply that

$$\frac{\sqrt{a_2^2}\sqrt{6-3a_2^2}}{2a_2} \notin \mathbb{Q}.$$

For the two other perturbations we find very similar solutions, which maybe excluded on the same grounds.

This concludes the discussion of perturbations of a singular fiber of Kodaira type I_0^*. We shall summarize our result in the following theorem

Theorem 3.4.1 *Of all confluences to Singular Fibers of Kodaira type* I_0^**, supperfically allowed by conservation of Euler number, the following occur:*

$$\begin{array}{lllll}
\mathrm{I}_0^* \to \mathrm{I}_4 + 2\mathrm{I}_1 & \mathrm{I}_0^* \to \mathrm{IV} + \mathrm{II} & \mathrm{I}_0^* \to \mathrm{IV} + 2\mathrm{I}_1 & \mathrm{I}_0^* \to \mathrm{I}_3 + \mathrm{II} + \mathrm{I}_1 & \mathrm{I}_0^* \to \mathrm{I}_3 + 3\mathrm{I}_1 \\
\mathrm{I}_0^* \to 2\mathrm{III} & \mathrm{I}_0^* \to \mathrm{III} + \mathrm{I}_2 + \mathrm{I}_1 & \mathrm{I}_0^* \to \mathrm{III} + \mathrm{II} + \mathrm{I}_1 & \mathrm{I}_0^* \to \mathrm{III} + 3\mathrm{I}_1 & \mathrm{I}_0^* \to 3\mathrm{I}_2 \\
\mathrm{I}_0^* \to 2\mathrm{I}_2 + 2\mathrm{I}_1 & \mathrm{I}_0^* \to \mathrm{I}_2 + 2\mathrm{II} & \mathrm{I}_0^* \to \mathrm{I}_2 + \mathrm{II} + 2\mathrm{I}_1 & \mathrm{I}_0^* \to \mathrm{I}_2 + 4\mathrm{I}_1 & \mathrm{I}_0^* \to 3\mathrm{II} \\
\mathrm{I}_0^* \to 2\mathrm{II} + 2\mathrm{I}_1 & \mathrm{I}_0^* \to \mathrm{II} + 4\mathrm{I}_1 & \mathrm{I}_0^* \to 6\mathrm{I}_1. & &
\end{array}$$

Moreover the confluences which do not occur

$$\begin{array}{lllll}
\mathrm{I}_0^* \to \mathrm{I}_5 + \mathrm{I}_1 & \mathrm{I}_0^* \to \mathrm{I}_4 + \mathrm{I}_2 & \mathrm{I}_0^* \to \mathrm{I}_4 + \mathrm{II} & \mathrm{I}_0^* \to \mathrm{IV} + \mathrm{I}_2 & \mathrm{I}_0^* \to 2\mathrm{I}_3 \\
\mathrm{I}_0^* \to \mathrm{I}_3 + \mathrm{III} & \mathrm{I}_0^* \to \mathrm{I}_3 + \mathrm{I}_2 + \mathrm{I}_1, & & &
\end{array}$$

are obstructed by monodromy considerations.

3.5 Confluence to singular fibers of Kodaira type I_1^*.

In this section we discuss our limited success in providing examples of confluences to singular fibers of Kodaira type I_1^*, much in the same way as did in section 3.4.

From the confluences not disallowed by the conservation of the Euler number, five are excluded on grounds of the order zeros of g_2 or g_3 before and after confluence, as discussed in the introduction of the chapter, namely

$$\mathrm{I}_1^* \to 3\mathrm{II} + \mathrm{I}_1 \quad \mathrm{I}_1^* \to 2\mathrm{III} + \mathrm{I}_1 \quad \mathrm{I}_1^* \to \mathrm{III} + 2\mathrm{II} \quad \mathrm{I}_1^* \to \mathrm{IV} + \mathrm{II} + \mathrm{I}_1 \quad \mathrm{I}_1^* \to \mathrm{IV} + \mathrm{III}.$$

Of the remaining confluences a further 7 are excluded by monodromy considerations. We are able to give examples of 23 different types of confluences, but some 4 still elude us although we can make some remarks on the possibilities of finding the appropriate examples, or verifying their properties.

As in the previous sections we shall only discuss the situation after the perturbation of the singular fiber of type I_1^* in our examples. The configuration before perturbation always consists of a singular fiber of Kodaira type I_1^* in the origin and a number of singular fibers of type I_1 and a singular fiber of type II, III or IV placed at infinity when convenient.

Examples

$\mathrm{I}_1^* \to 7\mathrm{I}_1$

We choose

$$\begin{aligned} g_2(z) &= z^2 + z^3 + \epsilon \\ g_3(z) &= -\frac{z^3}{3\sqrt{3}} \\ \Delta(z) &= \left(z^3 + \epsilon\right)\left(3z^4 + 3z^5 + z^6 + 3z^2\epsilon + 2z^3\epsilon + \epsilon^2\right), \end{aligned}$$

so that the discriminant of the geometric discriminant Δ equals

$$14348907\epsilon^{16}\left(16 + 27\epsilon^2\right),$$

which implies that after perturbation we are faced with 7 singular fibers of type I_1.

$\mathrm{I}_1^* \to \mathrm{I}_0^* + \mathrm{I}_1$

We choose

$$\begin{aligned} g_2(z) &= z^2(3 + z) \\ g_3(z) &= z^3(-1 + \epsilon) \\ \Delta(z) &= z^6\left((3 + z)^3 - 27(-1 + \epsilon)^2\right). \end{aligned}$$

In this case the discriminant of the geometric discriminant divided by z^6 equals $-19683(-1+\epsilon)^4$ implying that only one singular fiber of type I_1 arise apart from the singular fiber in the origin.

$\mathrm{I}_1^* \to \mathrm{I}_2 + 5\mathrm{I}_1$

We choose

$$\begin{aligned} g_2(z) &= 3z^2 + z^3 + 3\epsilon^2 \\ g_3(z) &= -z^3 + \epsilon^3 \\ \Delta(z) &= z^2\left(27z^5 + 9z^6 + z^7 + 81z^2\epsilon^2 + 54z^3\epsilon^2 + 9z^4\epsilon^2 + 54z\epsilon^3 + 81\epsilon^4 + 27z\epsilon^4\right). \end{aligned}$$

In this case the discriminant of the geometric discriminant divided by z^2 equals

$$-282429536481\epsilon^{14}\left(54 + 27\epsilon + 9\epsilon^2 + \epsilon^3\right)^3\left(8748 - 2360\epsilon + 594\epsilon^2 + 117\epsilon^3 + 135\epsilon^4 + 27\epsilon^5\right)$$

implying that only singular fibers of type I_1 arise apart from the singular fiber in the origin.

$\mathrm{I}_1^* \to \mathrm{I}_3 + 4\mathrm{I}_1$

Likewise, we choose

$$\begin{aligned} g_2(z) &= 3z^2 + z^3 + 3\epsilon^2 \\ g_3(z) &= -z^3 + \frac{3z^2\epsilon}{2} + \epsilon^3 \\ \Delta(z) &= \frac{1}{4}z^3\left(108z^4 + 36z^5 + 4z^6 + 324z^2\epsilon + 81z\epsilon^2 + 216z^2\epsilon^2 + 36z^3\epsilon^2 + 216\epsilon^3 + 108\epsilon^4\right). \end{aligned}$$

In this case the discriminant of the geometric discriminant divided by z^3 equals

$$-\frac{10460353203\epsilon^7\left(270 + 243\epsilon + 72\epsilon^2 + 8\epsilon^3\right)^3\left(384 + 262\epsilon + 19\epsilon^2 + 8\epsilon^3 + 8\epsilon^4\right)}{4096}$$

implying that only singular fibers of type I_1 arise apart from the singular fiber in the origin.

$\mathrm{I}_1^* \to \mathrm{I}_4 + 3\mathrm{I}_1$

Again, we choose

$$\begin{aligned} g_2(z) &= 3\left(z^2 - 4z\epsilon + \epsilon^2\right) \\ g_3(z) &= -z^3 + z^4 - \frac{15z^2\epsilon}{2} + 6z\epsilon^2 - \epsilon^3 \\ \Delta(z) &= -\frac{27}{4}z^4\left(-8z^3 + 4z^4 - 60z^2\epsilon + 12z\epsilon(9 + 4\epsilon) - \epsilon^2(27 + 8\epsilon)\right). \end{aligned}$$

In this case the discriminant of the geometric discriminant divided by z^4 equals

$$-12397455648\epsilon^3\left(-27 + 36\epsilon + 4\epsilon^2\right)^3$$

implying that only singular fibers of type I_1 arise apart from the singular fiber in the origin.

$\mathrm{I}_1^* \to \mathrm{I}_4 + \mathrm{I}_2 + \mathrm{I}_1$

We now choose

$$
\begin{aligned}
g_2(z) &= -5z^3 + z^2(3+\epsilon) + \frac{1}{45} z \left(9 + 36\epsilon - \sqrt{3}\sqrt{27 - 108\epsilon + 144\epsilon^2 - 64\epsilon^3}\right) \\
&\quad + \frac{\left(9 + 36\epsilon - \sqrt{3}\sqrt{27 - 108\epsilon + 144\epsilon^2 - 64\epsilon^3}\right)^2}{24300} \\
g_3(z) &= 5z^4 + \frac{1}{540} z^2 (6+\epsilon) \left(9 + 36\epsilon - \sqrt{3}\sqrt{27 - 108\epsilon + 144\epsilon^2 - 64\epsilon^3}\right) \\
&\quad + \frac{z\left(9 + 36\epsilon - \sqrt{3}\sqrt{27 - 108\epsilon + 144\epsilon^2 - 64\epsilon^3}\right)^2}{24300} \\
&\quad + \frac{\left(9 + 36\epsilon - \sqrt{3}\sqrt{27 - 108\epsilon + 144\epsilon^2 - 64\epsilon^3}\right)^3}{19683000} \\
&\quad + \frac{1}{2} z^3 \left(2 + \epsilon + \frac{1}{54}\left(-9 - 36\epsilon + \sqrt{3}\sqrt{27 - 108\epsilon + 144\epsilon^2 - 64\epsilon^3}\right)\right).
\end{aligned}
$$

It can be verified by explicit calculation that z^4, but not z^5, factors the geometric discriminant and that the discriminant of the geometric discriminant divided by z^4 is identically equal to zero, while the resultant of g_2 and g_3 is not. The fact that the confluence $\mathrm{I}_1^* \to \mathrm{I}_4 + \mathrm{I}_3$ is excluded on monodromy grounds completes the argument.

$\mathrm{I}_1^* \to \mathrm{I}_5 + 2\mathrm{I}_1$

We now choose

$$
g_2(z) = 3z^2 + z^3 + 3\epsilon^2 - \frac{3z\left(2 \cdot 2^{1/3}\epsilon^2 + 2^{2/3}\left(-2\epsilon^3 - \epsilon^4 + \sqrt{\epsilon^7(4+\epsilon)}\right)^{2/3}\right)}{\left(-2\epsilon^3 - \epsilon^4 + \sqrt{\epsilon^7(4+\epsilon)}\right)^{1/3}}
$$

$$
\begin{aligned}
g_3(z) = z^3 - \epsilon^3 + z^2 \Bigg(&- \frac{9\epsilon}{2} - \frac{3\epsilon^3}{2^{1/3}\big(-2\epsilon^3 - \epsilon^4 + \sqrt{\epsilon^7(4+\epsilon)}\big)^{2/3}} \\
&- \frac{3\big(-2\epsilon^3 - \epsilon^4 + \sqrt{\epsilon^7(4+\epsilon)}\big)^{2/3}}{22^{2/3}\epsilon}\Bigg) \\
+ \frac{3z\big(22^{1/3}\epsilon^3 + 2^{2/3}\epsilon\big(-2\epsilon^3 - \epsilon^4 + \sqrt{\epsilon^7(4+\epsilon)}\big)^{2/3}\big)}{2\big(-2\epsilon^3 - \epsilon^4 + \sqrt{\epsilon^7(4+\epsilon)}\big)^{1/3}} \\
+ \frac{1}{32} z^4 \Bigg(&\frac{72 \cdot 2^{1/3}\epsilon^3}{\big(-2\epsilon^3 - \epsilon^4 + \sqrt{\epsilon^7(4+\epsilon)}\big)^{4/3}} \\
&+ \frac{96 \cdot 2^{1/3}}{\big(-2\epsilon^3 - \epsilon^4 + \sqrt{\epsilon^7(4+\epsilon)}\big)^{1/3}} \\
&+ \frac{30 \cdot 2^{2/3}\big(-2\epsilon^3 - \epsilon^4 + \sqrt{\epsilon^7(4+\epsilon)}\big)^{1/3}}{\epsilon^2} \\
&+ \frac{9 \cdot 2^{2/3}\sqrt{\epsilon^7(4+\epsilon)}\big(-2\epsilon^3 - \epsilon^4 + \sqrt{\epsilon^7(4+\epsilon)}\big)^{1/3}}{\epsilon^5} \\
&+ \frac{12 \cdot 2^{1/3}\big(-2\epsilon^3 - \epsilon^4 + \sqrt{\epsilon^7(4+\epsilon)}\big)^{2/3}}{\epsilon^3} \\
&+ \frac{8\epsilon\big(3 \cdot 2^{2/3} + 8 \cdot 2^{1/3}\big(-2\epsilon^3 - \epsilon^4 + \sqrt{\epsilon^7(4+\epsilon)}\big)^{1/3}\big)}{\big(-2\epsilon^3 - \epsilon^4 + \sqrt{\epsilon^7(4+\epsilon)}\big)^{2/3}} \\
&+ \frac{-36 + 23 \cdot 2^{2/3}\big(-2\epsilon^3 - \epsilon^4 + \sqrt{\epsilon^7(4+\epsilon)}\big)^{1/3}}{\epsilon}\Bigg).
\end{aligned}
$$

It can be verified by explicit calculation that z^5 factors the geometric discriminant and $g_2(0)$ is not equal to zero. The fact that the confluences $\mathrm{I}_1^* \to \mathrm{I}_5 + \mathrm{I}_2$, $\mathrm{I}_1^* \to \mathrm{I}_5 + \mathrm{II}$ and $\mathrm{I}_1^* \to \mathrm{I}_6 + \mathrm{I}_1$ are excluded on monodromy grounds completes the argument.

$\mathrm{I}_1^* \to \mathrm{II} + 5\mathrm{I}_1$

We now choose

$$
\begin{aligned}
&g_2(z) = z\left(z + z^2 + \epsilon\right) \\
&g_3(z) = -\frac{1}{9} z\left(\sqrt{3}z^2 - 9\epsilon\right) \\
&\Delta(z) = z^2\left(3z^5 + 3z^6 + z^7 + 6\sqrt{3}z^2\epsilon + 3z^3\epsilon + 6z^4\epsilon + 3z^5\epsilon - 27\epsilon^2 + 3z^2\epsilon^2 + 3z^3\epsilon^2 + z\epsilon^3\right).
\end{aligned}
$$

It is clear that z factors both g_2 and g_3, implying that we find for nonzero values of ϵ a singular fiber of type II in the origin. We can easily verify that the discriminant of the geometric discriminant divided by z^2 is not identically equal to zero, thus yielding 5 singular fibers of type I_1.

$\mathrm{I}_1^* \to 2\mathrm{II} + 3\mathrm{I}_1$

Likewise, we now choose

$$\begin{aligned}
g_2(z) &= 3(1+z)(z-\epsilon)(z+\epsilon)\\
g_3(z) &= z(z-\epsilon)(z+\epsilon)\\
\Delta(z) &= 27(z-\epsilon)^2(z+\epsilon)^2\left(3z^3+3z^4+z^5-\epsilon^2-3z\epsilon^2-3z^2\epsilon^2-z^3\epsilon^2\right).
\end{aligned}$$

It is clear that $z+\epsilon$ and $z-\epsilon$ factor both g_2 and g_3, implying that we find for nonzero values of ϵ singular fibers of type II at $\pm\epsilon$. We can easily verify that the discriminant of the geometric discriminant divided by $(z-\epsilon)^2(z+\epsilon)^2$ is not identically equal to zero, thus yielding 3 singular fibers of type I_1 originating from the singular fiber of type I_1^*.

$\mathrm{I}_1^* \to \mathrm{I}_2 + \mathrm{II} + 3\mathrm{I}_1$

We let our Weierstrass model be defined by

$$\begin{aligned}
g_2(z) &= -3-3\big(z-\epsilon^2\big)\big(z+3z^2+\epsilon^2\big)\\
g_3(z) &= \big(-z+\epsilon^2\big)\big(z\epsilon^2+\epsilon^4+z^2\sqrt{-1-18\epsilon^2}\big)\\
\Delta(z) &= -27z^2\big(z-\epsilon^2\big)^2\times\\
&\qquad\big(9z^3+27z^4+27z^5+2z\epsilon^2-9z^2\epsilon^2-27z^4\epsilon^2+\epsilon^4-9z\epsilon^4\\
&\qquad-27z^2\epsilon^4-9\epsilon^6+2z\epsilon^2\sqrt{-1-18\epsilon^2}+2\epsilon^4\sqrt{-1-18\epsilon^2}\big).
\end{aligned}$$

We have arranged it such that there is a singular fiber of type I_2 in the origin and one of type II at ϵ^2. The fact that the discriminant of $\Delta/(z^2(z-\epsilon^2)^2)$ is nonzero gives that the other singular fibers originating from I_1^* are singular fibers of type I_1.

$\mathrm{I}_1^* \to \mathrm{I}_3 + \mathrm{II} + 2\mathrm{I}_1$

Identically, we let our Weierstrass model be defined by

$$\begin{aligned}
g_2(z) &= \big(3z+z^2-3\epsilon\big)(z-\epsilon)\\
g_3(z) &= \frac{(z-\epsilon)\big(2z^5\sqrt{-\epsilon}+2iz^2\sqrt{\epsilon}-4iz\epsilon^{3/2}-iz^2\epsilon^{3/2}+2i\epsilon^{5/2}\big)}{2\sqrt{-\epsilon}}\\
\Delta(z) &= \frac{1}{4\epsilon^{3/2}}z^3(z-\epsilon)^2\big(-108iz^3(-\epsilon)^{5/2}+216iz^4\sqrt{-\epsilon}\epsilon\\
&\qquad+324iz^3(-\epsilon)^{3/2}\epsilon+216iz^2(-\epsilon)^{5/2}\epsilon+108z^2\epsilon^{3/2}\\
&\qquad+36z^3\epsilon^{3/2}+4z^4\epsilon^{3/2}-108z^7\epsilon^{3/2}-108iz^4\sqrt{-\epsilon}\epsilon^2\\
&\qquad-216z\epsilon^{5/2}-72z^2\epsilon^{5/2}-4z^3\epsilon^{5/2}+108\epsilon^{7/2}+9z\epsilon^{7/2}\big).
\end{aligned}$$

We have arranged things so that there is a singular fiber of type I_3 in the origin and one of type II at ϵ^2. The fact that the discriminant of $\Delta/(z^3(z-\epsilon^2)^2)$ is nonzero gives that the other singular fibers originating from I_1^* are singular fibers of type I_1.

$\mathrm{I}_1^* \to \mathrm{I}_4 + \mathrm{II} + \mathrm{I}_1$

Identically, we let our Weierstrass model be defined by

$$\begin{aligned} g_2(z) &= 3\big(z-\epsilon^2\big)\big(z-\epsilon^2-2\epsilon^{8/3}-\epsilon^{10/3}\big) \\ g_3(z) &= -\frac{\big(z-\epsilon^2\big)}{16\big(1+\epsilon^{2/3}\big)^3}\big(-16z^2+8z^3-48z^2\epsilon^{2/3}+12z^3\epsilon^{2/3}-48z^2\epsilon^{4/3}+6z^3\epsilon^{4/3} \\ &\qquad +32z\epsilon^2-8z^2\epsilon^2+z^3\epsilon^2+144z\epsilon^{8/3}+18z^2\epsilon^{8/3}+264z\epsilon^{10/3} \\ &\qquad +12z^2\epsilon^{10/3}-16\epsilon^4+256z\epsilon^4+2z^2\epsilon^4-96\epsilon^{14/3}+144z\epsilon^{14/3} \\ &\qquad -240\epsilon^{16/3}+48z\epsilon^{16/3}-320\epsilon^6+8z\epsilon^6-240\epsilon^{20/3}-96\epsilon^{22/3}-16\epsilon^8\big) \\ \Delta(z) &= -\frac{27z^4}{256\big(1+\epsilon^{2/3}\big)^6}\big(2+\epsilon^{2/3}\big)^3\big(z-\epsilon^2\big)^2\times \\ &\qquad \big(-32z+8z^2-96z\epsilon^{2/3}+12z^2\epsilon^{2/3}-96z\epsilon^{4/3}+6z^2\epsilon^{4/3} \\ &\qquad +32\epsilon^2-16z\epsilon^2+z^2\epsilon^2+168\epsilon^{8/3}+36z\epsilon^{8/3}+372\epsilon^{10/3} \\ &\qquad +24z\epsilon^{10/3}+448\epsilon^4+4z\epsilon^4+312\epsilon^{14/3}+120\epsilon^{16/3}+20\epsilon^6\big) \end{aligned}$$

In this case we are able the calculate the roots of the geometric discriminant explicitly, they are

$$\begin{aligned} z =&0 \\ z =&\epsilon^2 \\ z =&\frac{2}{8+12\epsilon^{2/3}+6\epsilon^{4/3}+\epsilon^2}\times \\ &\Big(8+24\epsilon^{2/3}+24\epsilon^{4/3}+4\epsilon^2-9\epsilon^{8/3}-6\epsilon^{10/3}-\epsilon^4 \\ &\pm\big(64+384\epsilon^{2/3}+960\epsilon^{4/3}+1152\epsilon^2+192\epsilon^{8/3}-1632\epsilon^{10/3}-2992\epsilon^4 \\ &-2976\epsilon^{14/3}-1956\epsilon^{16/3}-880\epsilon^6-264\epsilon^{20/3}-48\epsilon^{22/3}-4\epsilon^8\big)^{1/2}\Big) \end{aligned}$$

where the first root occurs with multiplicity four and the second with multiplicity two. The first root corresponds to the singular fiber of type I_4, the second to the singular fiber of type II, the final two roots correspond to singular fibers of type I_1 but only the minus solution originates from the singular fiber of type I_1^*.

$\mathrm{I}_1^* \to \mathrm{III} + 4\mathrm{I}_1$

We choose to fix the singular fiber of type III in the origin which yields

$$\begin{aligned} g_2(z) &= z\left(3z+z^2+\epsilon\right) \\ g_3(z) &= -z^2(z-\epsilon) \\ \Delta(z) &= z^3\left(27z^4+9z^5+z^6+81z^2\epsilon+18z^3\epsilon+3z^4\epsilon-18z\epsilon^2+3z^2\epsilon^2+\epsilon^3\right). \end{aligned}$$

The fact that the discriminant of $\Delta(z)/z^3$ is nonzero gives that all other singular fibers are of type I_1.

$I_1^* \to \mathrm{III} + \mathrm{II} + 2\mathrm{I}_1$

As previously we take

$$g_2(z) = z(z-\epsilon)(3+z+\epsilon)$$
$$g_3(z) = -z^2(z-\epsilon)$$
$$\Delta(z) = z^3(z-\epsilon)^2\left(27z^2+9z^3+z^4-27\epsilon+9z^2\epsilon+2z^3\epsilon-27\epsilon^2-9z\epsilon^2-9\epsilon^3-2z\epsilon^3-\epsilon^4\right).$$

The fact that the discriminant of $\Delta(z)/(z^3(z-\epsilon)^2)$ is nonzero gives that all other singular fibers are of type I_1.

$I_1^* \to \mathrm{III} + \mathrm{II} + \mathrm{I}_2$

As before we take

$$g_2(z) = \frac{(3+z+3\epsilon)\left(z+3\epsilon+4z\epsilon+3\epsilon^2+5z\epsilon^2+\epsilon^3+2z\epsilon^3\right)\left(z+3\epsilon+2z\epsilon+6\epsilon^2+4\epsilon^3+\epsilon^4\right)}{(1+\epsilon)^2(1+2\epsilon)^2}$$
$$g_3(z) = \frac{\left(1+z+z^2\right)\left(z+3\epsilon+4z\epsilon+3\epsilon^2+5z\epsilon^2+\epsilon^3+2z\epsilon^3\right)\left(z+3\epsilon+2z\epsilon+6\epsilon^2+4\epsilon^3+\epsilon^4\right)^2}{(1+\epsilon)^2(1+2\epsilon)^3}$$
$$\begin{aligned}\Delta(z) = -&\frac{z^2}{(1+\epsilon)^6(1+2\epsilon)^6}\left(z+3\epsilon+4z\epsilon+3\epsilon^2+5z\epsilon^2+\epsilon^3+2z\epsilon^3\right)^2\times\\ &\left(z+3\epsilon+2z\epsilon+6\epsilon^2+4\epsilon^3+\epsilon^4\right)^3\times\\ &\big(27+72z+53z^2+27z^3+270\epsilon+438z\epsilon+293z^2\epsilon+108z^3\epsilon+810\epsilon^2+969z\epsilon^2\\ &\quad+589z^2\epsilon^2+135z^3\epsilon^2+1179\epsilon^3+1124z\epsilon^3+619z^2\epsilon^3+54z^3\epsilon^3+963\epsilon^4+792z\epsilon^4\\ &\quad+405z^2\epsilon^4+432\epsilon^5+324z\epsilon^5+162z^2\epsilon^5+81\epsilon^6+54z\epsilon^6+27z^2\epsilon^6\big).\end{aligned}$$

The interesting roots of the geometric discriminant $\Delta(z)$ are

$$\begin{aligned}z &= 0\\ z &= \frac{-3\epsilon-3\epsilon^2-\epsilon^3}{(1+\epsilon)^2(1+2\epsilon)}\\ z &= \frac{-3\epsilon-6\epsilon^2-4\epsilon^3-\epsilon^4}{1+2\epsilon},\end{aligned}$$

the first of which corresponds to the singular fiber of type I_2, the second to the singular fiber of type II and the final to the singular fiber of type III, this can be verified by explicit calculation.

$\mathrm{I}_1^* \to \mathrm{I}_3 + 2\mathrm{II}$

Here we take

$$
\begin{aligned}
g_2(z) &= \frac{(3+z)(z-\epsilon)\big(z-2z\sqrt{\epsilon}-\epsilon+z\epsilon\big)}{(-1+\sqrt{\epsilon})^2}\\
g_3(z) &= \frac{(z-\epsilon)\big(z-2z\sqrt{\epsilon}-\epsilon+z\epsilon\big)\big(-3z-3z^2-3z^4+3z\sqrt{\epsilon}+3z^2\sqrt{\epsilon}+3z^4\sqrt{\epsilon}+3\epsilon-z^2\epsilon\big)}{3\big(-1+\sqrt{\epsilon}\big)^3}\\
\Delta(z) &= -\frac{1}{\big(-1+\sqrt{\epsilon}\big)^6} z^3(z-\epsilon)^2\big(z-2z\sqrt{\epsilon}-\epsilon+z\epsilon\big)^2\times\\
&\quad \big(27+18z+53z^2+54z^3+27z^5-54\sqrt{\epsilon}-36z\sqrt{\epsilon}-106z^2\sqrt{\epsilon}-108z^3\sqrt{\epsilon}-54z^5\sqrt{\epsilon}\\
&\quad +63\epsilon-16z\epsilon+53z^2\epsilon+72z^3\epsilon+27z^5\epsilon-36\epsilon^{3/2}+34z\epsilon^{3/2}-18z^3\epsilon^{3/2}+8\epsilon^2+4z\epsilon^2\big).
\end{aligned}
$$

From the factorisation of the geometric discriminant and the functions g_2 and g_3 we can easily see the roots corresponding to the singular fibers of type II, the singular fiber of type I_3 can be found in the origin.

$\mathrm{I}_1^* \to \mathrm{III} + 2\mathrm{I}_2$

For this example we take a somewhat different approach.

$$
\begin{aligned}
g_2(z) &= (z-\epsilon)\big(3z+z^3-3\epsilon\big)\\
g_3(z) &= (z-\epsilon)^2\big(z+\delta z^2+z^3+z^4-\epsilon\big)\\
\Delta(z) &= -z^2(z-\epsilon)^3\big(54\delta z^2+27z^3+27\delta^2z^3+54z^4+54\delta z^4+18z^5+54\delta z^5+54z^6\\
&\qquad +26z^7-108\delta z\epsilon-54z^2\epsilon-27\delta^2z^2\epsilon-108z^3\epsilon-54\delta z^3\epsilon-18z^4\epsilon\\
&\qquad -54\delta z^4\epsilon-54z^5\epsilon-27z^6\epsilon+54\delta\epsilon^2+27z\epsilon^2+54z^2\epsilon^2\big).
\end{aligned}
$$

Again we impose a singular fiber of type I_2 in the origin and a singular fiber of type III in ϵ. The discriminant of $\Delta/(z^2(z-\epsilon)^3)$ again factors into a part proportional to the resultant of $g_2/(z-\epsilon)$ and $g_3/(z-\epsilon)^2$ to the third power and another factor. The resultant of this factor and the resultant of $g_2/(z-\epsilon)$ and $g_3/(z-\epsilon)^2$ with respect to both δ and ϵ are nonzero and it is therefore sufficient to find a solution curve, going through the origin in δ,ϵ-space, to the polynomial equation which sets the second factor of the discriminant equal to zero to ensure that we have two singular of type I_2.

$\mathrm{I}_1^* \to \mathrm{III} + \mathrm{I}_2 + 2\mathrm{I}_1$

This example is a simpler version of the previous one, we take

$$
\begin{aligned}
g_2(z) &= \big(3z+z^3-3\epsilon\big)\,(z-\epsilon)\\
g_3(z) &= (z-\epsilon)^2\,\big(z+2z^2+z^3+z^4-\epsilon\big)\\
\Delta(z) &= -z^2(z-\epsilon)^3\big(108z^2+135z^3+162z^4+126z^5+54z^6+26z^7\\
&\qquad -216z\epsilon-162z^2\epsilon-216z^3\epsilon-126z^4\epsilon-54z^5\epsilon-27z^6\epsilon\\
&\qquad -27z^6\epsilon+108\epsilon^2+27z\epsilon^2+54z^2\epsilon^2\big).
\end{aligned}
$$

Here we have imposed a singular fiber of type I_2 in the origin and one of type III in ϵ. The fact that the discriminant of $\Delta(z)/(z^2(z-\epsilon)^3)$ is not equal to zero for $\epsilon \neq 0$ yields that all other singular fibers arising from I_1^* are of type I_1.

$\mathrm{I}_1^* \to \mathrm{III} + \mathrm{I}_3 + \mathrm{I}_1$

Again a small variation in comparison to the previous example, we take

$$\begin{aligned}
g_2(z) &= \left(3z + z^2 + z^3 - 3\epsilon\right)(z-\epsilon) \\
g_3(z) &= \frac{1}{2}\left(2z + z^2 + 2z^3 + 2z^4 - 2\epsilon\right)(z-\epsilon)^2 \\
\Delta(z) &= -\frac{1}{4}z^3(z-\epsilon)^3\big(99z^2 + 248z^3 + 168z^4 + 204z^5 + 104z^6 - 207z\epsilon \\
&\qquad - 468z^2\epsilon - 180z^3\epsilon - 216z^4\epsilon - 108z^5\epsilon + 108\epsilon^2 + 216z\epsilon^2\big).
\end{aligned}$$

Here we have imposed a singular fiber of type I_3 in the origin and one of type III in ϵ. The fact that the discriminant of $\Delta(z)/(z^3(z-\epsilon)^3)$ is not equal to zero for $\epsilon \neq 0$ yields that the other singular fiber arising from I_1^* is of type I_1.

$\mathrm{I}_1^* \to \mathrm{IV} + 3\mathrm{I}_1$

We take

$$\begin{aligned}
g_2(z) &= z^2(3 + z + \epsilon) \\
g_3(z) &= -z^2(z-\epsilon) \\
\Delta(z) &= z^4\left(27z^3 + 9z^4 + z^5 + 54z\epsilon + 27z^2\epsilon + 18z^3\epsilon + 3z^4\epsilon - 27\epsilon^2 + 9z^2\epsilon^2 + 3z^3\epsilon^2 + z^2\epsilon^3\right).
\end{aligned}$$

We clearly find a singular fiber of type IV in the origin and discriminant considerations as above yield that the only other singular fibers are of type I_1.

$\mathrm{I}_1^* \to \mathrm{IV} + \mathrm{I}_2 + \mathrm{I}_1$

We now take

$$\begin{aligned}
g_2(z) &= 3(z-\epsilon)^2 \\
g_3(z) &= (z-\epsilon)^2\left(z + z^2 + z^4 - \epsilon\right) \\
\Delta(z) &= -27z^2\left(1 + z^2\right)\left(2z + z^2 + z^4 - 2\epsilon\right)(z-\epsilon)^4
\end{aligned}$$

We clearly find a singular fiber of type I_2 in the origin and one of type IV in ϵ. By discriminant considerations as above or explicit calculations we find all other singular fibers to be of type I_1, one of which originates from the origin.

$\mathrm{I}_1^* \to 2\mathrm{I}_2 + \mathrm{II} + \mathrm{I}_1$

For this confluence we take

$$\begin{aligned}
g_2(z) &= \left(z-\epsilon^6\right)\left(3z+\delta z^2-3\epsilon^4\right)\\
g_3(z) &= \frac{1}{2}\left(z-\epsilon^6\right)\left(2z^2+2z^5-z\epsilon^3-3z\epsilon^5+2\epsilon^9\right)\\
\Delta(z) &= -\frac{1}{4}z^2\left(z-\epsilon^6\right)^2\times\\
&\qquad\qquad\big(-108\delta z^3-36\delta^2z^4+216z^5-4\delta^3z^5+108z^8-108z\epsilon^3-108z^4\epsilon^3\\
&\qquad\qquad+324z\epsilon^4+216\delta z^2\epsilon^4+36\delta^2z^3\epsilon^4-324z\epsilon^5-324z^4\epsilon^5+27\epsilon^6+108z\epsilon^6\\
&\qquad\qquad+108\delta z^2\epsilon^6+36\delta^2z^3\epsilon^6+4\delta^3z^4\epsilon^6-162\epsilon^8-108\delta z\epsilon^8+216\epsilon^9+216z^3\epsilon^9\\
&\qquad\qquad-81\epsilon^{10}-216\delta z\epsilon^{10}-36\delta^2z^2\epsilon^{10}+108\delta\epsilon^{14}\big).
\end{aligned}$$

This choice of $g_2/\left(z-\epsilon^6\right)$ and $g_3/\left(z-\epsilon^6\right)$ ensures that there is a singular fiber of type I_2 in the origin and one of type II at ϵ^6. The rest of the discussion will follow the same lines as the example of the confluence $\mathrm{I}_1^* \to \mathrm{III}+2\mathrm{I}_2$. The discriminant factors into a part proportional to the resultant of g_2 and g_3 to the third power and a second part. Since the resultant of the first factor and the second factor is nontrivial with respect to the variable ϵ as well as the variable δ, we know that setting the second factor to zero yields an extra singular fiber of type I_2 or I_3. Verifying that the resultant of the discriminant of the derivative of the geometric discriminant with respect to z and the second factor of the discriminant with respect to δ and ϵ are nontrivial, excludes the possibility of a singular fiber of type I_3. This concludes the discussion.

$\mathrm{I}_1^* \to \mathrm{I}_3 + \mathrm{I}_2 + \mathrm{II}$

For this confluence we take

$$\begin{aligned}
g_2(z) &= \left(z-\epsilon^2\right)\left(3z-3\delta^2\epsilon^2\right)\\
g_3(z) &= \left(z-\epsilon^2\right)\left(z^3-\frac{z^2\left(-3-6\delta^2+\delta^4\right)}{8\delta}-\frac{1}{2}z\delta\left(3+\delta^2\right)\epsilon^2+\delta^3\epsilon^4\right)\\
\Delta(z) &= -\frac{27z^3\left(z-\epsilon^2\right)^2}{64\delta^2}\big(9z+48z^2\delta-28z\delta^2+64z^3\delta^2+96z^2\delta^3+30z\delta^4\\
&\qquad\qquad-16z^2\delta^5-12z\delta^6+z\delta^8-8\delta^2\epsilon^2-192z\delta^3\epsilon^2\\
&\qquad\qquad+24\delta^4\epsilon^2-64z\delta^5\epsilon^2-24\delta^6\epsilon^2+8\delta^8\epsilon^2+128\delta^5\epsilon^4\big)
\end{aligned}$$

This ensures that a singular fiber of type I_3 is present in the origin and one of type II in ϵ^2. The discriminant of $\Delta(z)/(z^3(z-\epsilon^2)^2)$ now reads

$$\frac{531441}{1024\delta^4}\left(81-18\delta^2+\delta^4+216\delta\epsilon^2\right)\left(1-2\delta^2+\delta^4-8\delta^3\epsilon^2\right)^3,$$

where the factor popping up to the third power is again proportional to the resultant of $g_2/\left(z-\epsilon^2\right)$ and $g_3/\left(z-\epsilon^2\right)$. Setting the first factor equal to zero gives us another singular fiber of type I_2. This equation is solved with respect to δ. We can easily verify that the limit of ϵ to zero yields a singular fiber of type I_1^* in the origin and that the only other singular fiber apart from the singular fiber in infinity is of type I_1.

Conjectures regarding examples

$\mathrm{I}_1^* \to 2\mathrm{I}_2 + 3\mathrm{I}_1$

We choose

$$
\begin{aligned}
g_2(z) &= 3z^2 + z^3 + 3\epsilon^2\\
g_3(z) &= \delta z^2 - z^3 + \epsilon^3\\
\Delta(z) &= z^2\big(-27\delta^2 z^2 + 54\delta z^3 + 27z^5 + 9z^6 + z^7 + 81z^2\epsilon^2 + 54z^3\epsilon^2 + 9z^4\epsilon^2 - 54\delta\epsilon^3\\
&\qquad + 54z\epsilon^3 + 81\epsilon^4 + 27z\epsilon^4\big).
\end{aligned}
$$

In this case the discriminant of the geometric discriminant divided by z^2 equals

$$
\begin{aligned}
-387420489\epsilon^3 &\left(27\delta^2 + 9\delta^3 - 54\delta\epsilon - 18\delta^2\epsilon + 54\epsilon^2 + 9\delta\epsilon^2 + 27\epsilon^3 + 9\epsilon^4 + \epsilon^5\right)^3 \times\\
&\big(186624\delta^4 + 135000\delta^5 + 25000\delta^6 + 839808\delta^3\epsilon + 607500\delta^4\epsilon + 112500\delta^5\epsilon\\
&- 58320\delta^3\epsilon^2 - 27000\delta^4\epsilon^2 - 169128\delta^2\epsilon^3 - 83700\delta^3\epsilon^3 - 2361960\delta\epsilon^4\\
&- 1805976\delta^2\epsilon^4 - 364500\delta^3\epsilon^4 + 6377292\epsilon^5 + 5849496\delta\epsilon^5 + 1694925\delta^2\epsilon^5\\
&+ 151875\delta^3\epsilon^5 - 1720440\epsilon^6 - 1404054\delta\epsilon^6 - 328050\delta^2\epsilon^6 + 433026\epsilon^7\\
&+ 229635\delta\epsilon^7 + 85293\epsilon^8 + 98415\epsilon^9 + 19683\epsilon^{10}\big),
\end{aligned}
$$

where the factor appearing to the third power is a multiple of the resultant of g_2 and g_3. Since the resultant of both factors of the geometric discriminant is non-trivial for both parameters we find that we have at least two singular fibers of type I_2, if we set the second factor equal to zero using the parameter δ. Verifying that further discriminants are not equal to zero would exclude more complicated cases, for this we have strong indications but no proof.

$\mathrm{I}_1^* \to \mathrm{I}_3 + \mathrm{I}_2 + 2\mathrm{I}_1$

Along the lines of the previous examples we choose

$$
\begin{aligned}
g_2(z) &= 3z^2 + z^3 + 3\epsilon^2\\
g_3(z) &= (-1+\delta)z^3 + \frac{3z^2\epsilon}{2} + \epsilon^3\\
\Delta(z) &= \frac{1}{4}z^3\big(216\delta z^3 - 108\delta^2 z^3 + 108z^4 + 36z^5 + 4z^6 + 324z^2\epsilon - 324\delta z^2\epsilon + 81z\epsilon^2 + 216z^2\epsilon^2\\
&\qquad + 36z^3\epsilon^2 + 216\epsilon^3 - 216\delta\epsilon^3 + 108\epsilon^4\big).
\end{aligned}
$$

Now the discriminant of the geometric discriminant divided by z^3 reads

$$\begin{aligned}\frac{10460353203}{4096}&(-2+2\delta-\epsilon)\epsilon^6\times\\&\big(-270+702\delta-648\delta^2+216\delta^3-243\epsilon+432\delta\epsilon-216\delta^2\epsilon-72\epsilon^2+72\delta\epsilon^2-8\epsilon^3\big)^3\times\\&\big(64\delta^2-64\delta^3+16\delta^4-192\epsilon+272\delta\epsilon-120\delta^2\epsilon+40\delta^3\epsilon-35\epsilon^2-48\delta\epsilon^2\\&\quad+24\delta^2\epsilon^2+8\epsilon^3-8\delta\epsilon^3-8\epsilon^4\big),\end{aligned}$$

where the factor appearing to the third power is a multiple of the resultant of g_2 and g_3. Since the resultant of both factors of the geometric discriminant is non-trivial for both parameters we find that we have at least one singular fiber of type I_3 and one of type I_2, if we set the third factor equal to zero using the parameter δ. Again, verifying that further discriminants are not equal to zero would exclude more complicated cases, for this we have strong indications but no proof.

We can not find examples of the confluences $\mathrm{I}_1^* \to 3\mathrm{I}_2+1\mathrm{I}_1$ and $\mathrm{I}_1^* \to \mathrm{I}_3+2\mathrm{I}_2$, for the same reasons that we can not verify the exact nature of the above examples. We can verify that there are no obstructions by monodromy considerations, this makes us believe that such confluences indeed exist.

Monodromy obstructions

We use the same methods as for the monodromy obstructions found for confluences to singular fibers of type I_0^*.

$\mathrm{I}_1^* \to \mathrm{IV} + \mathrm{I}_3$

In this case it suffices to calculate the eigenvalues of the following matrices

$$\begin{aligned}M_{\mathrm{I}_3}AM_{\mathrm{IV}}A^{-1} &= \begin{pmatrix}1&3\\0&1\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}0&1\\-1&-1\end{pmatrix}\begin{pmatrix}d&-b\\-c&a\end{pmatrix}\\ M_{\mathrm{IV}}AM_{\mathrm{I}_3}A^{-1} &= \begin{pmatrix}0&1\\-1&-1\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}1&3\\0&1\end{pmatrix}\begin{pmatrix}d&-b\\-c&a\end{pmatrix},\end{aligned}$$

where A is an element of $SL(2,\mathbb{Z})$. The eigenvalues of these matrices are given by

$$\lambda_\pm = \frac{1}{2}\big(-1-3c^2+3cd-3d^2 \pm \sqrt{-4+(1+3c^2-3cd+3d^2)^2}\big).$$

Clearly the eigenvalues of

$$\begin{pmatrix}-1&-1\\0&-1\end{pmatrix}$$

are both equal to -1. If we therefore impose that $\lambda_\pm = -1$ we find that $3c^2-3cd+3d^2 = 1$. Since 3 does not divide 1 this contradicts that $A \in SL(2,\mathbb{Z})$.

$\mathrm{I}_1^* \to \mathrm{I}_5 + \mathrm{II}$

Again we calculate the eigenvalues of the following matrices

$$\begin{aligned} M_{\mathrm{I}_5} A M_{\mathrm{II}} A^{-1} &= \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\ M_{\mathrm{II}} A M_{\mathrm{I}_5} A^{-1} &= \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \end{aligned}$$

where A is an element of $SL(2,\mathbb{Z})$. The eigenvalues of these matrices are given by

$$\lambda_\pm = \frac{1}{2}\left(1 - 5a^2 - 5ac - 5c^2 \pm \sqrt{-4 + \left(-1 + 5a^2 + 5ac + 5c^2\right)^2}\right).$$

Imposing that $\lambda_\pm = -1$ we find that $\left(-1 + 5a^2 + 5ac + 5c^2\right)^2 = 4$ which yields $5a^2 + 5ac + 5c^2 = 1$. Since 5 does not divide 1 this contradicts that $A \in SL(2,\mathbb{Z})$.

$\mathrm{I}_1^* \to \mathrm{I}_4 + \mathrm{III}$

Again we calculate the eigenvalues of the following matrices

$$\begin{aligned} M_{\mathrm{I}_5} A M_{\mathrm{II}} A^{-1} &= \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\ M_{\mathrm{III}} A M_{\mathrm{I}_4} A^{-1} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \end{aligned}$$

where A is an element of $SL(2,\mathbb{Z})$. The eigenvalues of these matrices are given by

$$\lambda_\pm = \frac{1}{2}\left(-4a^2 - 4c^2 \pm \sqrt{-4 + \left(4a^2 + 4c^2\right)^2}\right).$$

Imposing that $\lambda_\pm = -1$ we find that $\left(4a^2 + 4c^2\right)^2 = 4$ which yields $4a^2 + 4c^2 = 2$. Since 4 does not divide 2 this contradicts that $A \in SL(2,\mathbb{Z})$.

$\mathrm{I}_1^* \to \mathrm{I}_6 + \mathrm{I}_1$

Again it suffices to calculate the eigenvalues of the following matrices

$$\begin{aligned} M_{\mathrm{I}_6} A M_{\mathrm{I}_1} A^{-1} &= \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\ M_{\mathrm{I}_1} A M_{\mathrm{I}_6} A^{-1} &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \end{aligned}$$

where A is an element of $SL(2,\mathbb{Z})$. The eigenvalues of these matrices are given by

$$\lambda_\pm = 1 - 3c^2 \pm \sqrt{3}\sqrt{-2c^2 + 3c^4}.$$

Imposing that $\lambda_\pm = -1$ we find that $c = \pm\sqrt{2/3}$ which contradicts that $A \in SL(2,\mathbb{Z})$.

$\mathrm{I}_1^* \to \mathrm{I}_5 + \mathrm{I}_2$

As before we calculate the eigenvalues of the following matrices

$$\begin{aligned} M_{\mathrm{I}_5} A M_{\mathrm{I}_2} A^{-1} &= \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\ M_{\mathrm{I}_2} A M_{\mathrm{I}_5} A^{-1} &= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \end{aligned}$$

where A is an element of $SL(2,\mathbb{Z})$. The eigenvalues of these matrices are given by

$$\lambda_\pm = 1 - 5c^2 \pm \sqrt{5}\sqrt{-2c^2 + 5c^4}.$$

Imposing that $\lambda_\pm = -1$ we find that $c = \pm\sqrt{2/5}$ which contradicts that $A \in SL(2,\mathbb{Z})$.

$\mathrm{I}_1^* \to \mathrm{I}_4 + \mathrm{I}_3$

Yet again it suffices to calculate the eigenvalues of the following matrices

$$\begin{aligned} M_{\mathrm{I}_4} A M_{\mathrm{I}_3} A^{-1} &= \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\ M_{\mathrm{I}_3} A M_{\mathrm{I}_4} A^{-1} &= \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \end{aligned}$$

where A is an element of $SL(2,\mathbb{Z})$. The eigenvalues of these matrices are given by

$$\lambda_\pm = 1 - 6c^2 \pm 2\sqrt{3}\sqrt{-c^2 + 3c^4}.$$

Imposing that $\lambda_\pm = -1$ we find that $c = \pm 1/\sqrt{3}$ which contradicts that $A \in SL(2,\mathbb{Z})$.

$\mathrm{I}_1^* \to 2\mathrm{I}_3 + \mathrm{I}_1$

In this case we prefer to examine the trace of the matrices much like Naruki [9]. We calculate the trace T of

$$\begin{aligned} M_{\mathrm{I}_1} A M_{\mathrm{I}_3} A^{-1} B M_{\mathrm{I}_3} B^{-1} = &\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \cdot \\ &\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d_1 & -b_1 \\ -c_1 & a_1 \end{pmatrix}, \end{aligned}$$

and all possible permutations of M_{I_1}, M_{I_3} and M_{I_3}. This yields

$$\begin{aligned} T &= 2 - 9c^2 - 3a_1^2c^2 + 6aa_1cc_1 - 9a_1c^2c_1 - 3c_1^2 - 3a^2c_1^2 + 9acc_1^2 \\ T &= 2 - 3c^2 - 9a_1^2c^2 + 18aa_1cc_1 - 9a_1c^2c_1 - 3c_1^2 - 9a^2c_1^2 + 9acc_1^2 \\ T &= 2 - 3c^2 - 3a_1^2c^2 + 6aa_1cc_1 - 9a_1c^2c_1 - 9c_1^2 - 3a^2c_1^2 + 9acc_1^2, \end{aligned}$$

for each permutation, respectively. Equating these traces to the trace of

$$\begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix},$$

namely 2 and bringing the 2 to the other side of the equation yields that 4 must equal a multiple of 3, a clear contradiction.

3.6 Outlook

In this thesis the confluences to singular fibers of the non-starred type, II, III, IV and I_b on rational elliptic surfaces have been fully discussed. A lot of work remains to be done on the confluences to singular fibers of the starred types[14] as well on the elucidation of the stratification of the space on coefficients of g_2 and g_3 as discussed in section 3.1. With regard to the first objective we will now discuss a method which might help future investigations.

As described in section 3.1 we should we able to use, at least in theory, the generalized discriminants and resultants from section 1.2 to unravel all information regarding the structure of the space N_g. We can on the other hand combine these methods with the Weierstrass preparation theorem to localize the approach. This combination faces the same complications as discussed in section 3.1, but is hopefully interesting from a theoretical point of view.

Let us assume that we start out with a Weierstrass model defined by some g_2 and g_3, for a singular fiber of a given type in the origin. To study perturbations of the singular fiber we let the coefficients of g_2 and g_3 depend on perturbation parameters, denoted by ϵ_i, which are zero if the singular fiber is the unperturbed singular fiber in the origin. We will now apply the Weierstrass preparation theorem to g_2, g_3 and Δ, that is we write

$$\begin{aligned}\Delta(z) &= W_\Delta(z)u_\Delta(z)\\ g_2(z) &= W_{g_2}(z)u_{g_2}(z)\\ g_3(z) &= W_{g_3}(z)u_{g_3}(z),\end{aligned}$$

where u_Δ, u_{g_2} and u_{g_3} are units. Imposing that a perturbation of the singular fiber in the origin yields a certain set of singular fibers is equivalent to imposing the order of the zeros of W_Δ in combination with the order of the zeros of W_{g_2} and W_{g_3} at the given zero of W_Δ. Information about these common zeros can be deduced from the discriminants and semi-discriminant as described in section 1.2 of W_Δ and the resultants and semi-resultants of W_{g_2}, W_{g_3} and W_Δ. For a given set of singular fibers a number of these resultants and discriminants must be set to zero while others are must definitely be nonzero. To refer to the discussion in section 1.2, if we need a polynomial $f(x)$ to have two zeros of order two, but no zero of order three we must set all semi-discriminants of order 2 of f equal to zero, but not all the semi-discriminants of higher order nor all coefficients of y of the resultant $R(f - yf', f'')$ may be equal to zero. This gives us a number of polynomials in the perturbation parameters ϵ_i which must be set to zero, as well as an assortment of sets of polynomials where at least one of the polynomials in each set is not equal to zero. A set of points is called a semi-algebraic set, if it is defined by such a set of equalities and inequalities. We shall now return to the general setting and denote the polynomials

[14] More work on this has been done then presented in this thesis.

which must be set to zero by[15] $\boldsymbol{P}(\boldsymbol{\epsilon})$ and each set of polynomials which must not all be zero by $\boldsymbol{Q}_j(\boldsymbol{\epsilon})$. This gives us one algebraic set for $\boldsymbol{P}$ denoted by M, embedded in the parameter space. Every set of polynomials $\mathbf{Q}_j$ also induces an algebraic set denoted by L_j. Since the sub-discriminants are not all independent, if seen as polynomials, the dimension of these sets is not obvious. It would be sufficient, though not necessary, to prove that there exists a curve a $\delta \in \mathbb{R}_+$ and curve

$$\gamma(t) \in M \backslash (\cup_j L_j), \qquad \text{for } t \in (0, \epsilon),$$

such that $\gamma(0) = 0$. If we can reduce the sets of polynomials $\boldsymbol{P}(\boldsymbol{\epsilon})$ and $\boldsymbol{Q}_j(\boldsymbol{\epsilon})$ to sets of independent polynomials, we could use the implicit function theorem (3.2.1) to describe the tangent spaces of M and L_j in $\boldsymbol{\epsilon}=\mathbf{0}$. To prove the existence of the said curve γ it is now sufficient to verify that there exist linear subspaces of the tangent space of M in $\boldsymbol{\epsilon}=\mathbf{0}$ which do not lie in any of the tangent spaces of L_j.

Having discussed this method we remark that if the current progress in calculating power of computers and algorithms will continue in the future, our methods could be employed to greater ends in a few decades. It would be especially interesting to see whether further investigations would find a confluence which is obstructed by other considerations then the order of the zeros of g_2 and g_3 or monodromy. Until now no such exception has been found.

[15] We use boldface notation to emphasize the multidimensional character.

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