On the optimal triangulation of convex hypersurfaces, whose vertices lie in ambient space

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Abstract. Let Σ be a strictly convex (hyper-)surface, S_m an optimal triangulation (piecewise linear in ambient space) of Σ whose m vertices lie on Σ and $\tilde{S}_m$ an optimal triangulation of Σ with m vertices. Here we use optimal in the sense of minimizing $d_H(S_m, \Sigma)$, where d_H denotes the Hausdorff distance. In 'Lagerungen in der Ebene, auf der Kugel und im Raum' Fejes Tóth conjectured that the leading term in the asymptotic development of $d_H(S_m, \Sigma)$ in m is twice that of $d_H(\tilde{S}_m, \Sigma)$. This statement is proven.

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1. Introduction

In the seminal book 'Lagerungen in der Ebene, auf der Kugel und im Raum' [2] by Fejes Tóth, inscribed and circumscribed optimal triangulations approximating convex surfaces in $\mathbb{R}^3$ and the 'Approximierbarkeit' (approximation parameter A_2) are introduced. By a triangulation we shall always mean ageometric realization of a simplicial complex in Euclidean space, that is piecewise linear in ambient space. Furthermore, unless stated otherwise we take a simplicial complex to mean the geometric realization. Optimal triangulations with m vertices are triangulations or polytopes[1] which minimize the Hausdorff distance between the surface and the polytope in the space of triangulations with m vertices. Depending on the setting these vertices lie on the surface (inscribed), the faces touch the surface (circumscribed), or the vertices are are in general position. The Hausdorff distance between two sets X and Y in Euclidean space of arbitrary but fixed dimension d is defined as:

$$d_H(X, Y) = \max\{\sup_{x \in X} \inf_{y \in Y} \|x - y\|, \sup_{y \in Y} \inf_{x \in X} \|x - y\|\},$$

where $\|x - y\|$ denotes the standard Euclidean distance of x and y. The inverse of the asymptotic value of the product of the number of vertices and the Hausdorff distance (or more generally the product of $m^{2/(d-1)}$ and the Hausdorff distance, where d is the dimension of the Euclidean space) is referred to as the *approximation parameter* A_2 (A_d in dimension d). Fejes Tóth gave a lower bound on the inverse of the approximation parameter for inscribed triangulations depending on the Gaussian curvature. His reasoning was roughly speaking as follows: Given a point p on the surface there is a plane, intersecting the surface, with distance η from this point whose normal is equal to

[1] In the inscribed convex case there is no essential difference.

the normal to the surface at that point. Because the surface is convex the intersection approaches an ellipse as η becomes smaller, whose semi-axes are $\sqrt{2\eta/k_1(p)}$ and $\sqrt{2\eta/k_2(p)}$, with k_i the principal curvatures at the point on the surface. The area of the largest inscribed triangle in this ellipse is $\sqrt{27}\eta/(2\sqrt{K(p)})$. The (one-sided) distance of this triangle to the surface is η. Because we consider the largest triangle this gives us a lower bound, for any triangle t and hence for a triangulation we find

$$d_H m_t \gtrsim \sum_{m_t} \frac{2}{\sqrt{27}} \sqrt{K_t(p)} \mathrm{Area}_t,$$

with m_t the number of triangles, $K_t(p)$ the gaussian curvature at the point[2] p (depending on t), Area_t the area and $\gtrsim$ refers to the leading term in the asymptotic expansion of $d_H m_t$ as η tends to zero. Fejes Tóth then argued somewhat heuristically that this lower bound can be obtained. Schneider [7] generalized the discussion of Fejes Tóth to convex hypersurfaces in Euclidean space arbitrary dimension and gave a solid proof of the formula for the approximation parameter A_d of inscribed polytopes of convex C^3 hypersurfaces, which reads

$$\frac{1}{A_d} = \lim_{m\to\infty} m^{2/(d-1)} d_H(\Sigma, T_m) = \frac{1}{2}\left(\frac{\theta_{d-1}}{\kappa_{d-1}} \int_\Sigma \sqrt{K(x)} \mathrm{d}\mu\right)^{2/(d-1)},$$

where $\kappa_d = \pi^{d/2}/\Gamma(1 + d/2)$ is the volume of the d-dimensional unit ball, θ_d the covering density of the ball in d-dimensional space, $\mathrm{d}\mu$ the volume form and K the Gaussian curvature. The covering density is defined as the infimum of the density over all coverings of, in this case, Euclidean space by the Euclidean unit ball, see for example [6]. Moreover, a fairly explicit description of the optimal triangulations in terms of optimal coverings was given. Gruber [3, 4] then slightly extended the work by Schneider by considering hypersurfaces which are of lower differentiability class (C^2). Furthermore he discussed the complexity of circumscribing polytopes resulting in the same approximation parameter A_d, a topic which fell outside the scope of the article by Schneider. Fejes Tóth also conjectured that the complexity of triangulations whose vertices are in general position, that is neither in- nor circumscribed, is half of the complexity in the inscribed setting. None of the previously mentioned papers gave a proof of this. Below we provide a proof of this statement for not strictly convex C^1 hypersurfaces with positive reach. Using the previously mentioned results this implies:

Theorem 1.1. *Let Σ be a strictly convex C^2 hypersurface embedded in $\mathbb{R}^d$ and for every m let S_m be an optimally approximating simplicial complex with m vertices having Hausdorff distance $d_H(S_m, \Sigma)$. Then we have*

$$\lim_{m\to\infty} m^{2/(d-1)} d_H(S_m, \Sigma) = \frac{1}{2A_d} = \frac{1}{4}\left(\frac{\theta_{d-1}}{\kappa_{d-1}} \int_\Sigma \sqrt{K(x)} \mathrm{d}\mu\right)^{2/(d-1)}.$$

2. The sphere

To illustrate the problem we first consider the standard circle S^1 in $\mathbb{R}^2$, with radius 1 and centred at the origin. We approximate the circle by a regular polygon with m vertices, P_m. Due to symmetry this is the optimal manner, because the Hausdorff distance must be attained in every edge. Suppose that for an optimal polygon the Hausdorff distance is not attained in one of the edges, then we can perturb one of its vertices so that the Hausdorff distance is not attained in this edge nor in its neighbours, via induction we find that the polygon is not optimal. Naturally the centre of the regular polygon is the origin. The circumradius of the regular polygon will be denoted by R. A sketch is provided in figure 1. Clearly the points on the polygon furthest from or closest to the centre are the vertices and the centres of the edges. In these points are the only points where the Hausdorff distance

[2] Above p was introduced as the point in the neighbourhood of t such that the normal to t equals the normal of the surface at p.

can be attained. The distance between the circle and the vertex or the centre of the edge is given by $R-1$ and $1-R\cos(\pi/m)$, respectively, which yields that the Hausdorff distance between the circle and the regular polygon is

$$\begin{aligned} d_H(P_m, S^1) &= \max\left\{R-1, 1-R(1-d_H(P_m^{\text{in}}, S^1))\right\} \\ &= \max\left\{R-1, 1-R\cos\left(\frac{\pi}{m}\right)\right\}, \end{aligned}$$

where P_m^{in} denotes the inscribed polygon, that is the polygon with $R=1$.

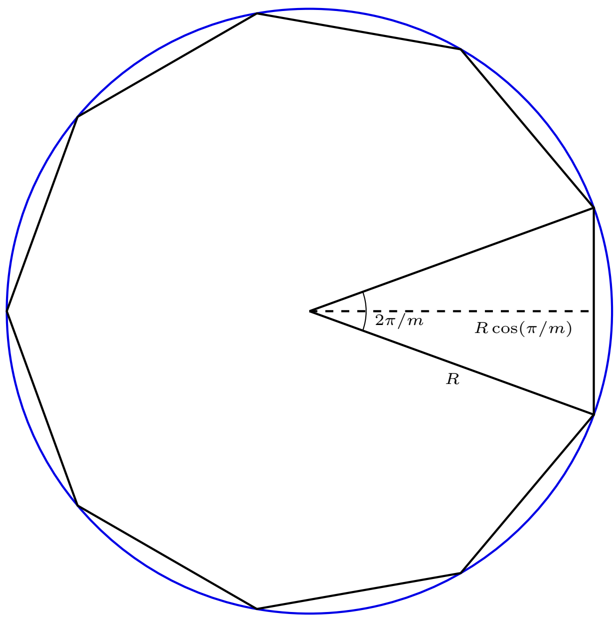

FIGURE 1. A polygon P_m and a circle, both with the same centre. R denotes the circumradius of P_m. We depict the inscribed case.

We can minimize $d_H(P_m, S^1)$ with respect to R by the following choice

$$R = \frac{2}{1+\cos\left(\frac{\pi}{m}\right)} = \frac{2}{2-d_H(P_m^{\text{in}}, S^1)}.$$

So that

$$d_H(P_m, S^1) = \frac{2}{1+\cos\left(\frac{\pi}{m}\right)} - 1 = \frac{2}{2-d_H(P_m^{\text{in}}, S^1)} - 1,$$

for sufficiently large m or rather sufficiently low $d_H(P_m^{\text{in}}, S^1)$ we can develop this expression

$$\begin{aligned} d_H(P_m, S^1) &= \frac{2}{1+\cos\left(\frac{\pi}{m}\right)} - 1 = \frac{1}{4}\left(\frac{\pi}{m}\right)^2 + \mathcal{O}\left(\frac{1}{m^4}\right) \\ d_H(P_m, S^1) &= \frac{2}{2-d_H(P_m^{\text{in}}, S^1)} - 1 = \tfrac{1}{2} d_H(P_m^{\text{in}}, S^1) + \mathcal{O}((d_H(P_m^{\text{in}}, S^1)^2). \end{aligned}$$

Remark 2.1. One may wonder about the equivalent statement for the Banach-Mazur distance (δ^{BM}). The Banach-Mazur distance on convex bodies which are symmetric in the origin ($\mathcal{C}_0$), have been treated extensively by Gruber [3, 4]. For two convex bodies $C, D \in \mathcal{C}_0$ the Banach-Mazur distance is defined [3] as

$$\delta^{\mathrm{BM}}(C, D) = \inf\{\lambda > 1 : C \subset \ell(D) \subset \lambda C, \ell : \mathbb{R}^d \to \mathbb{R}^d \text{ linear}\}$$

In this setting it is again clear that the optimal approximating body of S^1 is a regular polygon P_m (in this case with its interior). Because the definition includes linear maps ℓ that work on P_m, rescaling P_m by R does not influence the result, that is $\delta^{\mathrm{BM}}(S^1, P_m) = \delta^{\mathrm{BM}}(S^1, RP_m)$, with the interiors having been left implicit.

For general S^d we are not able to give an explicit description of the inscribed polytope which approximates S^d optimally, as there are only so many Platonic solids. However, suppose that we are given such a polytope P_m^{in}, then by definition the vertices ($\{v_i\}$) of P_m^{in} lie on S^d and there are points ($\{q_i\}$) on the faces which attain a distance $d_H(P_m^{\text{in}}, S^d)$ to the sphere. We can now consider the polytope P_m^{in} rescaled by a factor R, denoted by $R\,P_m^{in}$. For $R\,P_m^{\text{in}}$, the points v_i, q_i, the vertices and the points where the maximum distance is attained in the inscribed setting, are the points closest to and furthest from the origin. This means that in these points the Hausdorff distance can be attained. Which in turn implies that

$$d_H(R\,P_m^{in}, S^d) = \max\left\{R - 1, 1 - R(1 - d_H(P_m^{\text{in}}, S^d))\right\},$$

which is minimized with respect to R by

$$R^* = \frac{2}{2 - d_H(P_m^{\text{in}}, S^d)}.$$

Therefore,

$$d_H(R^*\,P_m^{in}, S^d) = \frac{2}{2 - d_H(P_m^{\text{in}}, S^d)} - 1 = \frac{1}{2} d_H(P_m^{\text{in}}, S^d) + \mathcal{O}((d_H(P_m^{\text{in}}, S^d)^2).$$

In this manner we have constructed a simplicial complex ($R^*\,P_m^{in}$) such that the Hausdorff distance of this complex is half of the Hausdorff distance in the inscribed case, up to leading order. On the other hand let us now assume that we have a simplicial complex S_m with m vertices which minimizes $d_H(S_m, S^d)$, we can now make the following construction; we take the vertices v_i of S_m and project these on the sphere along the normal $\pi(v_i)$. The simplicial complex with these vertices and the simplexes corresponding to the original simplicial complex S_m we will denote by $\tilde{P}_m$. By the corresponding we mean that the convex hull of $v_{i_1}, \ldots, v_{i_l}$, denotes by s_j, lies in S_m, if and only if the convex hull of $\pi(v_{i_1}), \ldots, \pi(v_{i_l})$, denotes by s_j^π, lies in $\tilde{P}_m$. We have for individual simplexes in the complex $d_H(s_i, s_i^\pi) \leq d_H(S_m, S^d)$, which is in this case not very hard to see, this will be proven in a general setting in lemma 3.4. From which we conclude that $d_H(S_m, \tilde{P}_m) \leq d_H(S_m, S^d)$. This in turn, in combination with the triangle inequality, yields

$$d_H(\tilde{P}_m, S^d) \leq d_H(S_m, \tilde{P}_m) + d_H(S_m, S^d) \leq 2d_H(S_m, S^d).$$

This means that we have proven theorem 1.1 in the case of the sphere, if we assume existence. In the next section we prove that there is indeed an inscribed polytope P_n^{in} which approximates S^d optimally.

3. Result

In this section we first discuss the continuity of the Hausdorff distance and prove some results on existence of optimal triangulations. By an optimal triangulation we mean that that there is no complex with the same number of vertices (on the surface or in general position, depending on the context) that achieves a smaller Hausdorff distance. We then give one rather simple lemma that says that if we have an optimally approximating simplicial complex S_m with m vertices in ambient space of a hypersurface Σ, then the optimally approximating inscribed polytope P_m satisfies $d_H(P_m, \Sigma) \leq 2d_H(S_m, \Sigma)$. Subsequently, given an inscribed polytope P_m^{in}, we construct a simplicial complex S_m such that

$$d_H(S_m, \Sigma) \leq \frac{1}{2} d_H(P_m^{\text{in}}, \Sigma) + o(d_H(P_m^{\text{in}}, \Sigma)).$$

The proof consists of two steps. Roughly speaking, we first push every point on the polytope outwards by $\frac{1}{2} d_H(P_m^{\text{in}}, \Sigma)$ using the normal to the surface to create a new hypersurface. Note that this is no longer a simplicial complex. The second step considers the Hausdorff distance between this surface and the simplicial complex found by 'pushing the vertices outwards'.

Suppose that we are given a combinatorial simplicial structure on the set $\{1, \ldots, m\}$ and two (possibly degenerate) geometric realizations $S_m, \tilde{S}_m$ in $\mathbb{R}^d$ with (ordered) vertex sets $\{v_1, \ldots, v_m\} = V$ and $\{\tilde{v}_1, \ldots, \tilde{v}_m\} = \tilde{V}$. If we now interpret V, $\tilde{V}$ as elements in $(\mathbb{R}^d)^m$ and assume that $|V - \tilde{V}| \leq \delta$, then for each combinatorial simplex $\{i_1, \ldots, i_k\}$ we have $d_H(\text{CH}(v_{i_1}, \ldots, v_{i_k}), \text{CH}(\tilde{v}_{i_1}, \ldots, \tilde{v}_{i_k})) \leq \delta$, by linearity and, thus, $d_H(S_m, \tilde{S}_m) \leq \delta$, here CH denotes the convex hull.

Now we can prove that for a combinatorial simplicial complex and manifold Σ

$$d_H(\cdot, \Sigma) : (\mathbb{R}^d)^m \to \mathbb{R}_{\geq 0} : V \to d_H(V, \Sigma),$$

where we identify the vertices with the geometric realization of the complex, is continuous. Let $V, \tilde{V} \in (\mathbb{R}^d)^m$, with $\|V - \tilde{V}\| \leq \delta$, then the triangle inequality yields

$$d_H(V, \Sigma) - d_H(V, \tilde{V}) \leq d_H(\tilde{V}, \Sigma) \leq d_H(V, \Sigma) + d_H(V, \tilde{V}).$$

$|d_H(V, \Sigma) - d_H(\tilde{V}, \Sigma)| \leq \delta$, which shows continuity in the setting for a fixed simplicial structure. We now note that there are but a finite number of simplicial structures on a finite set. This together with the fact that the minimum of a finite number of continuous functions is again continuous gives us that $d_H(\cdot, \Sigma)$ is continuous for any set of simplicial structures, in particular those corresponding to topological $d-1$-manifolds or topological $d-1$-spheres. This means that we have continuity in a very broad setting.

In the following we shall sometimes refer to the reach. Let X be a hypersurface in $\mathbb{R}^d$. The reach $R(X)$ of X is the largest distance to X such that if the distance between a point in $\mathbb{R}^d$ and X is smaller than $R(X)$, there is a unique closest point on X. Federer [1] has shown that a C^2 manifold has strictly positive reach. This result is not instrumental in the proofs of the following lemmas, but is used in Theorem 3.6.

Lemma 3.1. *For each m there exists an inscribed polytope P_m^{in} which approximates a given compact convex hypersurface Σ optimally, where we assume that the vertices of P_m^{in} lie on Σ. Here we specifically allow the vertices to coincide, so that we can use compactness arguments.*

Proof. It suffices to note that $d_H(\cdot, \Sigma)$ is continuous if we restrict the domain to $\Sigma^m = \Sigma \times \cdots \times \Sigma$ and topological $d-1$ manifolds, using the same identifications as above. Because Σ and therefore Σ^m is compact and a continuous function attains its minimum on a compact set. Note that if we approximate a convex surface by a simplicial complex which is a topological $d-1$-manifold, then the optimally approximating triangulation is itself a convex surface, because any simplicial complex lies in the convex hull of its vertex set. This implies that the inward normal to the hypersurface intersects the convex hull first and then the simplicial complex, so that the Hausdorff distance to the convex surface is smaller. So P_m^{in} is indeed an inscribed polytope. ☐

A similar statement is true for triangulations whose vertices do not lie on the hypersurface.

Lemma 3.2. *Let Σ be a compact hypersurface. There exists a simplicial complex S_m, with m vertices which approximates Σ optimally.*

Proof. Σ is compact, so it is bounded and thus contained in some ball $B(q, \rho)$ with radius ρ and centre q. Because $(B(q, 2\rho))^m$ is compact $d_H(\cdot, \Sigma)$ attains its minimum on this set. This is also a global minimum because we have that $d_H(S_m, \Sigma) \leq d_H(S_1, \Sigma) \leq \rho$. □

Remark 3.3. In the lemma above we use the obvious statement that for two optimal simplicial complexes (or polytopes) S_n and S_m, with n and m vertices respectively, where $n > m$ we have that $d_H(S_n, \Sigma) \leq d_H(S_m, \Sigma)$. However a strict inequality does not hold in the most general setting. An example of this is the following; consider the circle S^1 and its optimal approximating simplicial complexes for $m = 1$ and $m = 2$. These optimal approximating simplicial complexes are a point in the centre and any line segment which contains the centre and does not extend beyond twice the radius of the circle. It is easy to see that $d_H(S_1, S^1) = d_H(S_2, S^1)$.

We now focus on the lemmas that discuss the relation between the Hausdorff distance of triangulations whose vertices are restricted to a given convex hypersurface and those that are not restricted to this hypersurface.

Lemma 3.4. *For a given m suppose that the simplicial complex S_m (in general, that is not necessarily inscribed) optimally approximates a compact convex hypersurface Σ, that is, there is no $\tilde{S}_m$ such that $d_H(\tilde{S}_m, \Sigma) < d_H(S_m, \Sigma)$. Then the optimally approximating inscribed polytope P_m with m vertices satisfies $d_H(P_m, \Sigma) \leq 2d_H(S_m, \Sigma)$.*

Proof. For each vertex v_i of S_m choose a point v_i^{on} on Σ closest[3] to v_i. We define π to be the mapping $\pi : (v_1, \ldots, v_m) \mapsto (v_1^{\text{on}}, \ldots, v_m^{\text{on}})$. We endow $\{v_1^{\text{on}}, \ldots, v_m^{\text{on}}\}$ with the same simplicial structure as on $\{v_1, \ldots, v_m\}$. The resulting simplicial complex will be denoted by S_m^{on}. So π can be viewed as a simplicial map. Due to linearity we have that for every simplex $\{v_{i_1} \ldots v_{i_k}\}$, we have $d_H(\text{CH}(v_{i_1}, \ldots, v_{i_k}), \text{CH}(v_{i_1}^{\text{on}}, \ldots, v_{i_k}^{\text{on}})) \leq d_H(S_m, \Sigma)$ and therefore $d_H(S_m, S_m^{\text{on}}) \leq d_H(S_m, \Sigma)$, which in turn, using the triangle inequality, yields

$$d_H(S_m^{\text{on}}, \Sigma) \leq d_H(S_m^{\text{on}}, S_m) + d_H(S_m, \Sigma) \leq 2d_H(S_m, \Sigma).$$

Note that by the argument we have given in lemma 3.1 the optimal approximating simplicial complex is indeed a polytope. So by definition of optimality on enclosed polytopes we have that $d_H(P_m, \Sigma) \leq d_H(S^{\text{on}}, \Sigma)$ which yields that $d_H(P_m, \Sigma) \leq 2d_H(S_m, \Sigma)$. □

For the following lemma we need two observations: Let Σ be a convex (but not necessarily strictly convex) hypersurface and S_m a sequence of optimally approximating triangulations of Σ with m vertices. Suppose that $T_{m(i)}$ is a convergent (sub-) sequence of simplices with $T_{m(i)} \subset S_{m(i)}$ of which the lengths of the edges does not go to zero, then the sequence converges to a subset of a hyperplane that is also contained in Σ, because the Hausdorff distance between Σ and $S_{m(i)}$ tends to zero.

Secondly, if $\Sigma = \partial C$ is a convex C^1 hypersurface with C a convex body and L a line segment that is contained in Σ then the normal to Σ is constant along L. This can be seen by projecting the tangent spaces along L on a hyperplane orthogonal to L: If the normal is not constant then we can pick two points (p,q) where the normals are not the same. Let us denote by OL_p, OL_q the hyperplane that contain p and q respectively and are orthogonal to L. Let us write $C_p = OL_p \cap C$ and $C_q = OL_q \cap C$. We translate OL_q along L such that OL_p and OL_q coincide. If the normals are not the same there is a point r in the translated C_q that does not lie within the half space marked by the normal at p, which contradicts convexity, because the line that connects the (untranslated version) of that point r and p does not lie within C.

[3]If $d_h(S_m, \Sigma) \leq R(\Sigma)$, with R the reach, the point is unique.

Lemma 3.5. *Let P_m^{in} be an optimally approximating inscribed polytope with Hausdorff distance $d_H(P_m^{in}, \Sigma)$ to a (not necessarily strictly) convex C^1 hypersurface with positive reach, such that $d_H(P_m^{in}, \Sigma)$ is smaller than reach of Σ. Then we can construct a simplicial complex S_m such that*

$$d_H(S_m, \Sigma) \leq \frac{1}{2} d_H(P_m^{in}, \Sigma) + o(d_H(P_m^{in}, \Sigma)).$$

Proof. Let v_i be the vertices of P_m^{in} then we choose the vertices $\tilde{v}_i$ to be $v_i + \frac{1}{2} d_H(P_m^{\text{in}}, \Sigma)\nu(v_i)$, where ν denotes the normal to the hypersurface. We endow the vertex set $\{\tilde{v}_i\}$ with the same simplicial structure as $\{v_i\}$ has. The complex which arises will be denoted by S_m. This also corresponds to the boundary of the convex hull of $\{\tilde{v}\}$. We can see this as follows, P_m^{in} is the boundary of the convex hull of $\{v_i\}$. It suffices to prove that a $(d-1)$-dimensional simplex $v_{i_1}, \ldots, v_{i_d}$ in P_m^{in} corresponds to a simplex in $\partial(\text{CH}(\tilde{v}_1, \ldots, \tilde{v}_m))$. Suppose that is does not, then there exists a vertex $\tilde{v}_j$ which lies outside the plane spanned by $\tilde{v}_{i_1}, \ldots \tilde{v}_{i_d}$, this is impossible because the v_j does not lie on P_m. To see why the mapping defined above reduces the Hausdorff distance by half, up to higher order, we turn to the alternative definition of the Hausdorff distance, see for example Munkres [5]:

$$d_H(X, Y) = \inf\{\epsilon | X \subset U(Y, \epsilon) \text{ and } Y \subset U(X, \epsilon)\},$$

where $U(X, \epsilon)$ denotes the ϵ-neighbourhood of X. Let C be the compact convex body such that $\Sigma = \partial C$. From this we see that P_m^{in} is contained in an inner rim inside the convex hypersurface Σ that is

$$P_m^{\text{on}} \subset U(\Sigma, d_H(P_m^{\text{on}}, \Sigma)) \cap C.$$

We also have that

$$\Sigma \subset U(P_m^{\text{on}}, d_H(P_m^{\text{on}}, \Sigma)).$$

This yields that for every point $x \in P_m^{\text{on}}$ we have a unique point $y \in \Sigma$ which is closest to x, moreover the vector $(x - y)$ is normal to the hypersurface. We may now define the mapping Π by

$$\Pi : x \mapsto x + \frac{1}{2} d_H(P_m^{\text{on}}, \Sigma) \frac{x - y}{|x - y|}$$

and consider $\Pi(P_m^{\text{on}})$. By definition we have that

$$\Pi(P_m^{\text{on}}) \subset U(\Sigma, 1/2 d_H(P_m^{\text{on}}, \Sigma)).$$

We shall now show that

$$\Sigma \subset U(\Pi(P_m^{\text{on}}), \frac{1}{2} d_H(P_m^{\text{on}}, \Sigma)).$$

For every $y \in \Sigma$ there is a $x \in P_m^{\text{on}}$ such that $y - x$ is normal to Σ and $|y - x| \leq d_H(P_m^{\text{on}}, \Sigma)$. The first intersection point of $\{c - \lambda\nu(x) | \lambda \in \mathbb{R}\}$ and P_m^{on} will do, where by the first we mean the point with the smallest λ associated. Such an intersection point exists because of the following; suppose there exists a $y \in \Sigma$ such that

$$\{y - \lambda\nu(y) | \lambda \in \mathbb{R}\} \cap P_m^{\text{on}} = \emptyset,$$

then the line $\{y - \lambda\nu(y) | \lambda \in \mathbb{R}\}$ intersects Σ at some other point $\tilde{y} = y - \tilde{\lambda}\nu(y)$, without first intersecting P_m^{on}. This also means that there is a point $y_e = y - \tilde{\lambda}\nu(y)/2$, which has equal distance to y and $\tilde{y}$ so y_e lies further from Σ than the reach, but this contradicts the assumption that $d_H(P_m^{\text{on}}, \Sigma) \leq R(\Sigma)$. We will now show that $|y - x| \leq d_H(P_m^{\text{on}}, \Sigma)$. Suppose that there is a y' such that $|y' - x| \leq |y - x|$, then we can find a point along $y - x$ with equal distance to two points of Σ, namely y and y', again contradicting the assumption that $d_H(P_m^{\text{on}}, \Sigma) \leq R(\Sigma)$. Therefore y is the point on Σ which is closest to x and thus $|y-x| \leq d_H(P_m^{\text{in}}, \Sigma)$. Given the special role we have thrust on the normal ν it is clear that

$$\Sigma \subset U(\Pi(P_m^{\text{in}}), \frac{1}{2} d_H(P_m^{\text{in}}, \Sigma)).$$

This implies that

$$d_H(\Pi(P_m^{\text{in}}), \Sigma) \leq \frac{1}{2} d_H(P_m^{\text{in}}, \Sigma).$$

Finally we argue that $d_H(\Pi(P_m^{\text{in}}), S_m)$ tends to zero as m tends to infinity, faster than $d_H(P_m^{\text{in}}, \Sigma)$ tends to zero, that is $d_H(\Pi(P_m^{\text{in}}), S_m) = o(d_H(P_m^{\text{in}}, \Sigma))$. The normals $(x-y)/|x-y|$, where x is an element of T and y the point of Σ closest to x, line up with $\nu(v_i)$, where v_i is some vertex of T. Because the surface is continuously differentiable the normal is continuous, so if $T \subset S_m$ is an element of a convergent sequence of triangles and the edge lengths tend to zero $(x-y)/|x-y|$ converges trivially to the normal at any vertex because of continuity, if the edge lengths do not tend to zero we use the observation above that the normal along a line segment contained in the hypersurface is constant to conclude that the normals converge. This implies that $(x-y)/|x-y| - \nu(v_i)$ tends to zero so $d_H(P_m^{\text{in}}, \Sigma)((x-y)/|x-y| - \nu(v_i))$ tends to zero faster than $d_H(P_m^{\text{in}}, \Sigma)$. Because $d_H(P_m^{\text{on}}, \Sigma) \leq d_H(P_m^{\text{in}}, \Sigma)|(x-y/|x-y| - \nu(v_i)|$, by definition of the mapping Π, we have that

$$d_H(S_m, \Sigma) \leq d_H(\Pi(P_m^{\text{in}}), S_m) + d_H(\Pi(P_m^{\text{in}}), \Sigma) \leq \frac{1}{2} d_H(P_m^{\text{in}}, \Sigma) + o(d_H(P_m^{\text{in}}, \Sigma)).$$

We also used the triangle inequality for the first inequality. So S_m is a simplicial complex sufficiently close to Σ. □

We are now able by combining Lemmas 3.1, 3.2, 3.4 and 3.5 to prove Theorem 1.1, which we shall display in full.

Theorem 3.6. *Let Σ be a strictly convex C^2 hypersurface in $\mathbb{R}^d$. For every m let S_m be an optimally approximating simplicial complex with m vertices having Hausdorff distance $d_H(S_m, \Sigma)$ to the convex hypersurface Σ. Then we have*

$$\lim_{m\to\infty} m^{2/(d-1)} d_H(S_m, \Sigma) = \frac{1}{4} \left(\frac{\theta_{d-1}}{\kappa_{d-1}} \int_\Sigma \sqrt{K(x)} \mathrm{d}\mu \right)^{2/(d-1)},$$

where κ_d is the volume of the d-dimensional ball $\pi^{d/2}/\Gamma(1+d/2)$, θ_d is the covering density of the ball in d-dimensional space and K the Gaussian curvature.

Proof. By Gruber and Schneider [3, 4, 7] we have that

$$\lim_{m\to\infty} m^{2/(d-1)} d_H(P_m^{\text{in}}, \Sigma) = \frac{1}{2} \left(\frac{\theta_{d-1}}{\kappa_{d-1}} \int_\Sigma \sqrt{K(x)} \mathrm{d}\mu \right)^{2/(d-1)},$$

where P_m^{in} an optimally approximating inscribed polytope simplicial complex, which is automatically also a polytope. Lemmas 3.4 and 3.5 give us

$$\frac{1}{2} d_H(P_m, \Sigma) \leq d_H(S_m, \Sigma)$$

and the existence of a simplicial complex $\tilde{S}_m$, for sufficiently large m, satisfying

$$d_H(\tilde{S}_m, \Sigma) \leq \frac{1}{2} d_H(P_m^{\text{in}}, \Sigma) + o(d_H(P_m^{\text{in}}, \Sigma)).$$

By optimality the latter equation implies

$$d_H(S_m, \Sigma) \leq \frac{1}{2} d_H(P_m^{\text{in}}, \Sigma) + o(d_H(P_m^{\text{in}}, \Sigma)).$$

Furthermore, lemmas 3.1 and 3.2 give us the existence of the simplicial complexes involved. So that

$$\begin{aligned}\frac{1}{2}\lim_{m\to\infty} m^{2/(d-1)} d_H(P_m^{\text{in}},\Sigma) &\leq \lim_{m\to\infty} m^{2/(d-1)} d_H(S_m,\Sigma)\\ &\leq \lim_{m\to\infty} m^{2/(d-1)}\left(\frac{1}{2}d_H(P_m^{\text{in}},\Sigma)+o(d_H(P_m^{\text{in}},\Sigma))\right)\\ &= \frac{1}{2}\lim_{m\to\infty} m^{2/(d-1)} d_H(P_m^{\text{in}},\Sigma).\end{aligned}$$

Using the result of Gruber and Schneider yields

$$\lim_{m\to\infty} m^{2/(d-1)} d_H(S_m,\Sigma) = \frac{1}{4}\left(\frac{\theta_{d-1}}{\kappa_{d-1}}\int_\Sigma \sqrt{K(x)}\mathrm{d}\mu\right)^{2/(d-1)},$$

the desired result. □

Discussion. The fact that the surface is convex is essential to our line of reasoning, because this insures that the simplices whose vertices lie on the surface do not intersect the surface. This is the reverse of the case of negative curvature, where in general the simplex and the surface will intersect. Optimal triangulations in the non-convex setting are part of ongoing research.

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List of symbols

Symbol	Meaning
A_d	Fejes Tóth's approximation parameter
CH	Convex hull of a subset of $\mathbb{R}^d$
C	Convex body
d_H	Hausdorff distance
κ_d	Volume of the d-dimensional ball, that is $\pi^{d/2}/\Gamma(1+d/2)$.
K	Gaussian curvature
L	Line segment contained in a hypersurface
m	Number of vertices
m_t	Number of triangles (only 2 dimensional case)
OL_p	Hyperplane orthogonal to the line segment L going through $p \in L$
P_m	Polygon/polytope with m vertices
P_m^{in}	Inscribed polygon/polytope with m vertices
S_m	Simplicial complex with m vertices
S_m^{on}	Simplicial complex whose m vertices lie on the hypersurface (often at a stage where the complex is not yet prover to be convex)
Σ	Hypersurface
θ_d	The optimal covering density of Euclidean space by unit balls
$U(X,\epsilon)$	ϵ neighbourhood of X
v_i, $\tilde{v}_i$	Vertices
V_i	Set of vertices

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