Confluence of singular fibers on rational elliptic surfaces

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1 preliminaries

1.1 Resultants and discriminants

This subsection contains some obvious adaptation of parts of sections 33, 34 and 35 of the fifth chapter of Algebra by van der Waerden [7], concerning resultants. Although all adaptations must be known, we have been unable to locate the classical literature.

Let

$$f(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_n, \qquad g(x) = b_0 x^m + b_1 x^{m-1} + \ldots + b_m$$

be two polynomials, where we assume that $a_0 \neq 0$ and $b_0 \neq 0$.

Lemma 1.1 *Suppose that $N \leq n, m$. Then $f(x)$ and $g(x)$ have N or more linear factors $\varphi_1(x), \ldots, \varphi_N(x)$ in common if and only if there exist non-zero polynomials $h(x)$ and $k(x)$ of degree $m - N$ and $n - N$ respectively, such that*

$$h(x)f(x) = k(x)g(x). \tag{1}$$

We shall not exclude possibility $\varphi_i = \varphi_j$.

Proof Let us assume that (1) holds for some given $h(x)$ and $k(x)$. If we now decompose both sides of the equation into prime factors then we must see the appearance of the same factors on both sides of the equation. In particular we must see all the factors of $f(x)$ on the right hand side appear as often as they do on the left. Since we assume that $k(x)$ has degree $n - N$ at most it can contain $n - N$ prime factors of $f(x)$, which implies that $g(x)$ must contain N.
Conversely let $\phi_1(x), \ldots, \phi_N(x)$ be N common linear factors of $f(x)$ and $g(x)$. Then one may simply write

$$f(x) = \phi_1(x)\phi_2(x)\ldots\phi_N(x)k(x), \qquad g(x) = \phi_1(x)\phi_2(x)\ldots\phi_N(x)h(x)$$

and equation (1) holds. □

By writing

$$h(x) = c_0 x^{m-N} + c_1 x^{m-N-1} + \ldots + c_{m-N}, \qquad k(x) = d_0 x^{n-N} + d_1 x^{n-N-1} + \ldots + d_{n-N}.$$

and comparing the coefficients of the left and right hand side of equation (1) we find that the existence of $h(x)$ and $k(x)$ in lemma 1.1 is equivalent to the existence of a vector $(c_1, -d_j)$ such that

$$\left(\begin{array}{ccccccccc} a_0 & 0 & 0 & \dots & 0 & b_0 & 0 & \dots & 0 \\ a_1 & a_0 & 0 & \dots & 0 & b_1 & b_0 & \ddots & 0 \\ a_2 & a_1 & a_0 & \dots & 0 & b_2 & b_1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ a_n & a_{n-1} & \dots & \dots & a_0 & b_{m-2} & \dots & \dots & 0 \\ 0 & a_n & \dots & \dots & a_1 & b_{m-1} & \dots & \dots & 0 \\ 0 & 0 & a_n & \dots & a_2 & b_m & b_{m-1} & \dots & b_0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 0 & a_n & 0 & \dots & \dots & b_m \end{array}\right) \left(\begin{array}{c} c_0 \\ \vdots \\ \vdots \\ \vdots \\ c_{m-N} \\ -d_0 \\ \vdots \\ \vdots \\ -d_{m-N} \end{array}\right) = \left(\begin{array}{c} 0 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{array}\right). \tag{2}$$

The $(n+m-N+1) \times (n+m-2N+2)$ matrix in (2) will be denoted by S_R.

Let us now turn our attention to the following result of linear algebra

Lemma 1.2 *Let $A : k^n \to k^m$ be a linear mapping then the equation $A(v) = 0$ has a nontrivial v as a solution if and only if all $n \times n$-matrices produced by dropping $m - n$ columns of the matrix associated to A have a zero determinant.*

This lemma is a consequence of the Rank lemma, which can for example be found on pages 113 and 114 of Duistermaat and Kolk [1]. This gives us the following result

Lemma 1.3 *$f(x)$ and $g(x)$ have N or more linear factors in common if and only if all $(m+n-2N+2) \times (m+n-2N+2)$-matrices produced by dropping $N-1$ columns of the matrix S_R associated to f and g have zero determinant.*

If $N = 1$, S_R is a square matrix and is called the Sylvester matrix, in this case the determinant of S_R is called the resultant and is denoted by $R(f,g)$. The resultant is related to the discriminant $D = a_0^{2n-2} \prod_{i<j} (x_i - x_j)^2$ of a polynomial $f(x)$ via the equation $R(f, f') = \pm a_0 D$, see for example section 35 of [7]. We emphasize the determinants of $(m+n-2N+2) \times (m+n-2N+2)$-submatrices of S_R are polynomials in the coefficients of f and g, but not all these determinants are independent as polynomials in the coefficients of the polynomial $f(x)$.

In lemma 1.1 we emphasized that we did not exclude possibility $\varphi_i = \varphi_j$, to further investigate this we turn our attention on the following example.
Let $f(x)$, $g(x)$ and $h(x)$ be three polynomials in x. Then $f(x) - yg(x)$ and $h(x)$ have at least one linear factor in common for all $y \in \mathbb{C}$ if and only if $f(x)$, $g(x)$ and $h(x)$ have a linear factor in common.
Proof Suppose that $f(x) - yg(x)$ and $h(x)$ have a factor in common for all $y \in \mathbb{C}$, if we choose such a non-zero y, then $f(x) - yg(x) = (x - x_j)p_y(x)$, where $p_y(x)$ is a polynomial in x and x_j is some root of $h(x)$. Since the roots of a polynomial depend continuously on the coefficients of the polynomial the roots of $f(x) - yg(x)$ are continuous with respect to y. Moreover the roots of $h(x)$ form a discreet set this implies that $f(x) - yg(x) = (x - x_j)p_y(x)$ for all y. We now take the particular case of $y = 0$ and see that $f(x) = (x - z_j)p_0(x)$. Taking this expression for $f(x)$ and letting $y \neq 0$, we see that $yg(x) = (x - z_j)(p_0(x) - p_y(x))$. This implies that $f(x)$, $g(x)$ and $h(x)$ have at least a common linear factor. The converse is obvious. □

This in turn leads to the statement $f(x)$, $g(x)$ and $h(x)$ have at least one common linear factor if and only if the resultant of $f(x) - yg(x)$ and $h(x)$ with respect to x, which is a polynomial in y, is identically equal to zero. It is now also clear that $f(x)$ has a third order zero if and only if the resultant of $f(x) - yf'(x)$ and $f''(x)$ is identically equal to zero.

Using these techniques (and the obvious generalizations) we are able to verify whether and how often $\varphi_i = \varphi_j$ as discussed in lemma 1.1, for $i \neq j$.

1.2 Families of elliptic surfaces and confluences

Let $\varphi : S \to C$ be a relatively minimal elliptic fibration without multiple singular fibers. We shall denote the set of singular points of the elliptic surface by $S^{\text{sing}} = \{s \in S | T_s\varphi = 0\}$ and the set of regular points of C by $C^{\text{reg}} = C \backslash \varphi(S^{\text{sing}})$. In the following we shall use the monodromy associated to a singular fiber and the Weierstrass model of an elliptic surface.

Roughly speaking monodromy can be understood as follows. Every regular fiber of the elliptic fibration is an elliptic curve. Of course each elliptic curve can be described by $\mathfrak{g}/P$, where $\mathfrak{g}$ is the Lie algebra of the G, the identity component of the group of automorphisms of the elliptic curve and P is a lattice in $\mathfrak{g}$. It can be shown that the fiberwise construction extents to a unique holomorphic complex line bundle $\mathfrak{g}$ and holomorphic subbundle P with discreet fibers, for a precise discussion see section ????? of [2]. For each continuous mapping $\gamma : [0, 1] \to C^{\text{reg}} : t \mapsto \gamma(t)$ called a path in C^{reg}, and every $v_0 \in P_{\gamma(0)}$, there is an unique (lifted) path v is P such that $v(0) = v_0$ and $v(t) \in P_{\gamma(t)}$ for every $0 \leq t \leq 1$. The mapping $T : v_0 \mapsto v(1)$ is an orientation preserving isomorphism from $P_{\gamma(0)}$ to $P_{\gamma(1)}$. Let c_* be a fixed point in C^{reg}, called a base point and let γ be a loop in C^{reg} based at c_*, that is a path in C^{reg} such that $\gamma(0) = \gamma(1) = c_*$, then T as an orientation preserving automorphism of P_{c_*}. If v_1^* and v_2^* form an oriented $\mathbb{Z}$-basis of P_{c_*}, then $T(v_i^*) = v_i$ also forms an oriented $\mathbb{Z}$-basis of P_{c_*} and thus there exists a unique matrix $M \in \text{SL}(2, \mathbb{Z})$ called the monodromy matrix defined by the loop γ, such that $v_i = M_i^j v_j^*$. The monodromy matrix defined as such clearly depends on the choice of v_1^* and v_2^*, it can be verified that the conjugation class of M within $\text{SL}(2, \mathbb{Z})$ is defined invariantly. We shall often use the word monodromy matrix to indicate the conjugation class. Discreetness of the fibers of P implies that the definition of monodromy is invariant under homotopic deformation of γ. If γ runs around one singular point $c \in \varphi(S^{\text{sing}})$ in C once with counterclockwise orientation we find a monodromy matrix. This monodromy matrix is indicative of the Kodaira type of the singular fiber around which singular point we run. The monodromy matrix for each type is given in table 1. For a full discussion of monodromy see section ????? of [2].

Table 1: In this table we only give one monodromy matrix characterizing the conjugation class, b is a positive integer.

Type	Intersection diagram	Monodromy matrix	Order zero of g_2	Order zero of g_3	Order zero of Δ/ Euler number
I_0		$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	≥ 0	≥ 0	0
I_b	$A_{b-1}^{(1)}$	$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$	0	0	b
I_0^*	$D_4^{(1)}$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	≥ 2	≥ 3	6
I_b^*	$D_{b+4}^{(1)}$	$\begin{pmatrix} -1 & -b \\ 0 & -1 \end{pmatrix}$	2	3	$b + 6$
II	$A_0^{(1)}$	$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$	≥ 1	1	2
II^*	$E_8^{(1)}$	$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$	≥ 4	5	10
III	$A_1^{(1)}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	1	≥ 2	3
III^*	$E_7^{(1)}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	3	≥ 5	9
IV	$A_2^{(1)}$	$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$	≥ 2	2	4
IV^*	$E_6^{(1)}$	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$	≥ 3	4	8

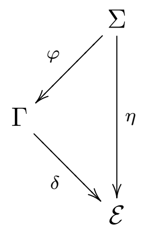
Let us now focus our attention to the Weierstrass model. The main idee behind the Weierstrass model is the following. Every elliptic curve may be described by the projective Weierstrass equation $x_0{x_2}^2 - 4{x_1}^3 + g_2{x_0}^2x_1 + g_3{x_0}^3 = 0$, in $\mathbb{P}^2$, where the geometric discriminant $\Delta \equiv {g_2}^3 - 27{g_3}^2 \neq 0$. We call $\Delta(z)$ the geometric discriminant to avoid confusion if we consider the discriminant of $\Delta(z)$, with respect to z. Again we can apply this to every regular fiber of an elliptic fibration. This yields the equation $x_0{x_2}^2 - 4{x_1}^3 + g_2(c){x_0}^2x_1 + g_3(c){x_0}^3 = 0$, in $\mathbb{P}({L_c}^0 \times {L_c}^2 \times {L_c}^3)$, where $c \in C^{\text{reg}}$ and $L = \mathfrak{g}^*$. The $\mathbb{P}({L_c}^0 \times {L_c}^2 \times {L_c}^3)$ extends to a holomorphic bundle $\mathbb{P}(L^0 \times L^2 \times L^3)$ over C. The projective Weierstrass equation within this bundle also extends to the entire C in particular to the singular points of the elliptic fibration where where the geometric discriminant $\Delta(c)$ is zero. The order with which the geometric discriminant $\Delta(c)$ and $g_2(c)$ and $g_3(c)$ disappear at a singular point is indicative of the Kodaira type of the singular fiber. The order order of the zero of Δ is in fact equal to the Euler number of the singular fiber. In table 1 we have listed the orders of the zeros of g_2, g_3 and Δ for each Kodaira type. The converse is also true, if we start with some g_2, g_3 and $\Delta = {g_2}^3 - 27{g_3}^2$ of which the orders of the zeros are as listed in table 1, we can reconstruct the relatively minimal elliptic surface without multiple singular fibers which gives rise to the g_2, g_3 and Δ. This reconstruction goes as follows we take

$$W = \{(c, [x]) \in \mathbb{P}(L^0 \oplus L^2 \oplus L^3) | x_0{x_2}^2 - 4{x_1}^3 + g_2(c){x_0}^2x_1 + g_3(c){x_0}^3 = 0\},$$

denote the projection of W to C by p and apply a minimal resolution of singularities f on W, the surface S found after the resolution is a relatively minimal elliptic surface without multiple singular fibers and moreover $\varphi = p \circ f : S \to C$. For a full discussion and all proves of the above statements see section ????? of [2].

We now concentrate on giving the definition of a confluence of singular fibers and discussing some properties of these confluences. This discussion partially relies on remark 7.3.4 of [2] and the article of Naruki [4], although our definition of confluence differs from that of Naruki.

Let us now consider a deformation of elliptic surfaces. Furthermore let the perturbation depend on the parameter ϵ. One will often observe several singular fibers $S_{c_1}(\epsilon), \ldots, S_{c_N}(\epsilon)$ with $c_1(\epsilon), \ldots, c_N(\epsilon)$ singular points of surfaces nearby some special surface $S(0)$ of the deformation flowing together into one singular fiber $S_c(0)$.[1] Such a phenomenon is called a confluence of singular fibers. In the next chapter we shall often adhere to the opposite view, where we start with a singular fiber $S_c(0)$ which is perturbed into several singular fibers $S_{c_1}(\epsilon), \ldots, S_{c_N}(\epsilon)$. This is of course simply a matter of point of view. We may associate monodromy matrices (or rather conjugacy classes thereof) $M_{S_{c_i}}$ to the singular fibers S_{c_i}, by defining the monodromy associated to the singular fibers as the monodromy of curve γ_i which encircles the singular fiber S_{c_i} once and counterclockwise. For a certain perturbation σ of indices, the concatenation $\gamma_{\sigma(1)} * \ldots * \gamma_{\sigma(N)}$ is homotopic to γ_0, after deformation. For this perturbations of indices σ we have the identity $M_{S_c} = M_{S_{c_{\sigma(1)}}} \cdot \ldots \cdot M_{S_{c_{\sigma(N)}}}$.
We will now make this statement more precise. Suppose that we are given a commutative diagram of the form



where Σ, Γ and $\mathcal{E}$ are complex analytic manifolds of dimension $n+2$, $n+1$ and n respectively and φ, δ and η are proper surjections. Let us set

$$S_\epsilon = \eta^{-1}(\epsilon), \qquad C_\epsilon = \delta^{-1}(\epsilon), \qquad \epsilon \in \mathcal{E}$$

and further assume that δ and η are complex analytic submersions and the fibers S_ϵ are compact.[2] The data $(\Sigma, \Gamma, \mathcal{E}, \varphi, \delta, \eta)$ above is called a C^∞ n-parameter deformation of elliptic surfaces, if the following conditions are satisfied

[1] We shall assume that $\epsilon = 0$ is the special value of the perturbation parameter, this assumption is of course made without loss of generality.
[2] In this we differ from Naruki.

i) δ and η are locally trivial C^∞-fibrations.

ii) The restriction $\varphi|_\epsilon : S_\epsilon \to C_\epsilon$ is an elliptic fibration.

In the work presented below we will sometimes take $\mathcal{E}$ to be a closed subset of $\mathbb{C}^n$ with its origin in the boundary of $\mathcal{E}$ and allow δ and η to exhibit root-like behaviour. This means more precisely that δ and η are locally trivial C^0-fibrations. In this case we will refer to the data $(\Sigma, \Gamma, \mathcal{E}, \varphi, \delta, \eta)$ as a C^0 n-parameter deformation. We are in general only interested in local confluence of singular fibers, so we may often even take δ and η to be trivial C^∞ or C^0 n-parameter deformations.
From this point onward we will always consider a C^∞ n-parameter deformation. We denote by C_ϵ^{sing} the set of singular points on C_ϵ and by C_ϵ^{reg} its complement in C_ϵ. We further define

$$\Gamma^{\text{sing}} = \bigcup_{\epsilon\in\mathcal{E}} C_\epsilon^{\text{sing}}, \qquad \Gamma^{\text{reg}} = \bigcup_{\epsilon\in\mathcal{E}} C_\epsilon^{\text{reg}}.$$

We denote by $\gamma_{\tilde\epsilon}(t)$ the C^∞ family of loops parameterized by t, where $\gamma_{\tilde\epsilon}(t) \in C_{\tilde\epsilon}^{\text{reg}}$ for all t and $\tilde\epsilon \in \tilde{\mathcal{E}}$ a m-dimensional submanifold of $\mathcal{E}$ which contains the special point 0.[3] For every $\epsilon \in \mathcal{E}$ we may associate a monodromy to the loop $\gamma_{\tilde\epsilon}$; $M([\gamma_{\tilde\epsilon}])$. Furthermore we have that the period lattice P depends continuously on the parameter ϵ. Discreetness of the period lattice P implies that $M([\gamma_{\tilde\epsilon}])$ is constant with respect to $\tilde\epsilon$. If for some fixed $\tilde\epsilon$, $\gamma_{\tilde\epsilon}(t)$ is homotopic to the concatenation $\gamma_1 * \ldots * \gamma_k$ of loops, where naturally $\gamma_j \subset C_{\tilde\epsilon}^{\text{reg}}$, then we have that $M([\gamma_{\tilde\epsilon}]) = M([\gamma_1])\cdot\ldots\cdot M([\gamma_k])$. We now apply this notion to the confluence of singular fibers. To do so let us first give a precise definition of a confluence of singular fibers. We say that the singular fibers $S_{c_i,\tilde\epsilon}$, with $c_i(\tilde\epsilon) \in C_{\tilde\epsilon}^{\text{sing}}$, $i = 1, \ldots, N$ of S_ϵ, flow together into the same singular fiber $S_{c_0,0}$ of S_0^{sing}, if there is a curve $\beta \subset \tilde{\mathcal{E}}$, which is parameterized by τ and sends 0 to the special point 0, such that $c_i(\beta(\tau))$ are discrete for τ in a small neighbourhood of 0, but not for $\tau = 0$ itself, as well as

$$\lim_{\tau\to 0} c_i(\beta(\tau)) = c_0.$$

Notice that this definition differs from the definition given in [4]. For $\tau = 0$ we have a small neighbourhood if the origin $U_0 \subset C_0$, analytically diffeomorphic to $D = \{z \in \mathbb{C} \big| |z| < \delta\}$, where $\delta \in \mathbb{R}_{>0}$, such that δ is the only singular point in U_0. We choose $\gamma_0(t)$ to be the curve which winds around c_0 once in the counterclockwise direction. Assuming U_0 and $\mathcal{E}$ to be sufficiently small, γ_0 may be extended to some family $\gamma_{\beta(\tau)} \subset C_{\beta(\tau)}^{\text{reg}}$ of loops as mentioned above by, for example, imposing a (trivial) connection of $\delta : \Gamma \to \mathcal{E}$.[4] Note that every $\gamma_{\beta(\tau)}$ is homotopic to γ_0 in Γ^{reg}. A member $\gamma_{\beta(\tau)}(t)$ of this family with $\tau \neq 0$ runs around all the c_i's into which c_0 breaks up. Let $\gamma^i_{\beta(\tau)}(t)$ denote a counterclockwise loop winding around c_i once. Let us further fix $\tau \neq 0$ and some base point, which we choose to lie on the curve $\gamma_{\beta(\tau)}$ denoted by p. Moreover choose for each c_i a parameterized curve γ^i_v connecting p with some point on the curve $\gamma^i_{\beta(\tau)}$, such that the γ^i_v do not intersect each other nor the $\gamma^i_{\beta(\tau)}$s and $\gamma_{\beta(\tau)}$. Finally let $\tilde\delta$ be a loop starting and ending at a point on $\gamma_{\beta(\tau)}$ winding around p once, counterclockwise, such that $\tilde\delta$ intersects γ^i_v only once and $\gamma_{\beta(\tau)}$ twice. Denote by σ the permutation of indices such that $\tilde\delta$ intersects $\gamma_v^{\sigma(1)}$ first, $\gamma_v^{\sigma(2)}$ second et cetera. The concatenation $\gamma_v^{\sigma(1)} * \gamma_{\beta(\tau)}^{\sigma(1)} * (\gamma_v^{\sigma(1)})^{-1} * \ldots * \gamma_v^{\sigma(N)} * \gamma_{\beta(\tau)}^{\sigma(N)} * (\gamma_v^{\sigma(N)})^{-1}$, where $(\tilde\gamma)^{-1}$ denotes the curve $\tilde\gamma$ inversely parameterized, clearly encloses $c_1, \ldots, c_N$ and is homotopic to $\gamma_{\beta(\tau)}$. These considerations yield

$$M_{S_c} = M_{S_{c_{\sigma(1)}}} \cdot \ldots \cdot M_{S_{c_{\sigma(N)}}}. \tag{3}$$

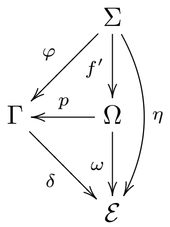
We will in in the work presented below consider deformations of Weierstrass models of elliptic surfaces, instead of the deformations of elliptic surfaces themselves, this turns out to be equivalent. As we can see as follows. If we choose to work in the local coordinate z on C, for the Weierstrass model of a relatively minimal rational elliptic surface without multiple singular fibers $\varphi : S \to C$, g_2 and g_3 can be though of simply as polynomials of fourth and sixth order in z. We now write

$$g_2(z) = \sum_{i=0}^{4} g_{2,i} z^i, \qquad g_3(z) = \sum_{i=0}^{6} g_{3,i} z^i. \tag{4}$$

[3] For our purpose it will not be necessary to endow $\tilde{\mathcal{E}}$ with a complex structure.

[4] The same holds for the set U_0.

We will now consider the $g_{2,i}$s and $g_{3,i}$s to be the parameters of deformation of a family of Weierstrass models of elliptic surfaces; corresponding to the coordinates of $\mathcal{E}$. This implies that we are faced with the following commutative diagram



where Ω denotes the family of Weierstrass models W_ϵ, p is the projection so that $p|_\epsilon$ maps W_ϵ to C_ϵ, f'_ϵ is a minimal resolution of singularities, the restriction $\varphi|_\epsilon : C_\epsilon \to S_\epsilon$ is a minimal resolution of singularities and δ, η and ω are locally trivial C^∞-fibrations. The work of Tyurina [6] guarantees that the family of Weierstrass models generates a family of elliptic surfaces in a continuous manner, in particular she has proven that the resolution of singularities $f|_\epsilon$ extends continuously to a resolution of singularities of the entire family. So investigating the deformation of Weierstrass models of elliptic surfaces is equivalent to investigating the deformation of elliptic surfaces.

We end this section with some observations mostly taken from remark 7.3.4 of [2]. The zeros of a polynomial depend continuously on the coefficients of the polynomial and the number of zeros in D, a small neighbourhood, counted with multiplicity is invariant under small perturbations of the polynomial. Moreover singular fibers correspond to zeros of the geometric discriminant Δ and the topological Euler number of a singular fiber equals the order of the zero of the geometric discriminant. Combining these two remarks we find that the Euler number is conserved in confluences, in the sense that if several singular fibers flow together into one singular fiber that then the sum of the Euler numbers of the singular fibers before the confluence is equal to the Euler number of the singular fiber which was the product of the confluence. Likewise we have that the number zeros of g_2 and g_3 in D is invariant under perturbation, but since the zeros of g_2 and g_3 do not necessarily coincide to form the zero of the geometric discriminant, we have that the sum of the orders of the zeros of g_2 (g_3) of the merging singular fibers is less or equal to the order of the zero of g_2 (g_3) of the resulting singular fiber. This implies for example that singular fibers of type I_b may only be the result of a confluence of singular fibers of type I_{b_i}, with $\sum b_i = b$. The same argument gives that two "starred" types that is two singular fibers of the set $\{\mathrm{I}_0^*, \mathrm{I}_1^*, \ldots, \mathrm{IV}^*, \mathrm{III}^*, \mathrm{II}^*\}$, cannot merge. Finally note that if g_2 and g_3 have a linear factor in common, the discriminant of the geometric discriminant $\Delta(z) = g_2(z)^3 - 27g_3(z)^2$ is zero, implying that the resultant of $g_2(z)$ and $g_3(z)$ factors discriminant of the geometric discriminant.

2 Confluence of singular fibres

In this section we consider all confluences on rational elliptic surfaces which lead to singular fibers of type I_b, II, III, IV of I_0^*. We shall provide for each confluence an exemplary family of Weierstrass models in which the confluence takes place, either explicitly or implicitly. For the confluences that do not arise we verify that the product of the equivalence classes of the monodromy matrices associated to the merging singular fibers does not lie in the same equivalence class as the monodromy matrix of the resulting singular fiber, see formula (3).
We restrict ourselves to these singular fibres because of the following considerations. Suppose that we wish to construct an example if a singular fiber itself not of type I_b which is perturbed into a great number of singular fibers of type I_{b_i} with $b_i \geq 2$. Say for example $\mathrm{II}^* \to \mathrm{I}_4 + \mathrm{I}_3 + \mathrm{I}_2 + \mathrm{I}_1$ or $\mathrm{II}^* \to 2\mathrm{I}_3 + 2\mathrm{I}_2$, then we must impose that the zeros of the geometric discriminant have the right multiplicity and the resultant of g_2 and g_3 is not equal to zero before the confluence and restricted to the singular fibers involved in the confluence. Using the methods of subsection 1.1 this demand translates into not only into imposing a great number of polynomials in the coefficients of g_2 and g_3 to be zero, as well as a significant number of algebraic inequalities. This great multitude, the fact that not all polynomials in the coefficients of g_2 and g_3 are independent and that the polynomials by themselves are not very simple makes that we have not been able to confront this problem. Furthermore in the number of singular fibres

involved in a confluence obstructed by monodromy increases the number of matrices we must conjugate in equation (3) to verify the obstruction increases accordingly. This increase also complicates our efforts if the confluences under consideration involve a large number of singular fibers.

There has been previous work on the monodromy restrictions of the confluences of singular fibers, namely by Naruki [4], which focusses on the confluence of three singular fibers of type I_b. In [4] no arguments are included to prove the existence of those confluences which are allowed by monodromy considerations. Moreover in this article we provide an explicit argument for the existence of the confluence $3\mathrm{I}_2 \to \mathrm{I}_0^*$, which according to section 5 of [4] would be disallowed.

Our object; finding confluences to a given singular fiber is a local phenomenon, but fits into a larger global picture. Namely the relation between the global configuration of singular fibers and the values of the coefficients of g_2 and g_3. We have that the discriminant of the geometric discriminant is non-trivial as a polynomial in the coefficients of g_2 and g_3. This implies that the generic configuration of singular fibers is $12\mathrm{I}_1$. We now focus on the algebraic set N_g in the space of coefficients of g_2 and g_3 defined by setting the discriminant of $\Delta(z)$ equal to zero. We clearly work in the category of algebraic varieties. It is also clear that the set of N_g is stratified. We would like to know the exact structure of N_g and find the correspondence between the strata of N_g and the global configuration of singular fibers, which is not necessarily one-to-one. Should we understand the structure of N_g completely, we would know if certain singular fibers could flow together simply because (the parts of) the strata associated to the different configurations are adjacent. Or, even stronger, we would know which (global) configurations of singular fibers can arise from the perturbation of a given configuration of singular fibers. This imposes a hierarchial structure on the list of all allowed configurations as found by Persson [5]. We are not able to determined the structure of N_g due to the same reasons we are unable to construct confluence like $\mathrm{II}^* \to \mathrm{I}_4 + \mathrm{I}_3 + \mathrm{I}_2 + \mathrm{I}_1$ or $\mathrm{II}^* \to 2\mathrm{I}_3 + 2\mathrm{I}_2$.

2.1 Confluence to singular fibers of Kodaira type I_b.

As remarked in subsection 1.2 only singular fibers of type I_{b_i} may flow together to form a singular fiber of type I_b, with $b = \sum b_i$. We note that the restriction to rational elliptic surfaces implies that $b \leq 9$, as we can verify easily in the list of Persson [5]. In this subsection we will prove the following theorem

Theorem 2.2 *Every type of confluence of singular elliptical fibers on a rational elliptic surface of type I_{b_i} into a singular fiber of type I_b with $b = \sum b_i$ occurs.*

The proof consists of three parts:

- For a perturbation of a singular fiber of Kodaira type I_b into a singular fiber of type I_{b-e} and e singular fibers of type I_1 we will give an explicit example by giving formulae for g_2 and g_3 and verify the fact that $e\mathrm{I}_1$ singular fibers are created by using the discriminant, see subsection 1.1.
- For all singular fibers of Kodaira type I_b, with $b \leq 6$, not of the above type, we are able to give explicit examples, again by giving g_2 and g_3, moreover we will give the roots of the geometric discriminant Δ.
- For every confluence not mentioned before we will use the combination of the Weierstrass preparation theorem and the implicit function theorem to prove existence of the confluence, this method is implicit when it concerns the coefficients of g_2 and g_3. [5]

$\mathrm{I}_b \to \mathrm{I}_{b-e} + \mathrm{I}_1 + \ldots + \mathrm{I}_1 = \mathrm{I}_{b-e} + e\,\mathrm{I}_1$

We will give the argument for existence of all confluences of the form $\mathrm{I}_b \to \mathrm{I}_{b-e} + \mathrm{I}_1 + \ldots + \mathrm{I}_1 = \mathrm{I}_{b-e} + e\,\mathrm{I}_1$. We start out with a singular fiber of type I_b in the origin. The g_2 and g_3 yielding the singular fiber I_b are found by setting the first b coefficients of the geometric discriminant Δ to zero consecutively.[6] Allowing

[5] Method suggested by Hans Duistermaat.

[6] One will often find it practical to solve with respect to $g_{3,0}$, $g_{2,1}$, $g_{3,2}$, $g_{2,2}$, $g_{2,4}$, $g_{3,4}$, $g_{3,5}$, $g_{3,6}$ and $g_{2,3}$, in this order.

the final e coefficients of the b coefficients set to zero to be perturbed into nonzero values generally yields a singular fiber of type I_{b-e} and e singular fibers of type I_1. The fact that only singular fibers of type I_1 arise is verified by calculating the discriminant of the geometric discriminant divided by z^{b-e}. If this discriminant is nontrivial for nonzero values of the perturbation parameter, the geometric discriminant has but zeros of order one outside the origin corresponding to singular fibers of type I_1. The explicit model and the verification of the behaviour of the geometric discriminant is given in table 2.

Table 2: In this table the geometric discriminant in denoted by Δ and the discriminant with D.

Confluence	$g_2(z)$	$g_3(z)$	Behaviour of $\Delta(z)$
$\mathrm{I}_2 \to 2\mathrm{I}_1$	3	$-1+\epsilon z+z^2$	$D(\Delta(z)) = 2^4 3^{18}\epsilon^2(8+\epsilon^2)$
$\mathrm{I}_3 \to 3\mathrm{I}_1$	3	$-1+\epsilon z+z^3$	$D(\Delta(z)) = 2^{10}3^{30}\epsilon^3(27+\epsilon^3)$
$\mathrm{I}_3 \to \mathrm{I}_2+\mathrm{I}_1$	3	$-1+\epsilon z^2+z^3$	$D(\Delta(z)/z) = 2^2 3^{24}\epsilon^2(-2^4 3^3+32\epsilon^3)$
$\mathrm{I}_4 \to 4\mathrm{I}_1$	3	$-1+\epsilon z+z^4$	$D(\Delta(z)) = 2^8 3^{45}\epsilon^4(2^{11}+27\epsilon^3)$
$\mathrm{I}_4 \to \mathrm{I}_2+2\mathrm{I}_1$	3	$-1+\epsilon z^2+z^4$	$D(\Delta(z)/z) = 2^{13}3^{36}\epsilon^3(2+\epsilon^2)^2$
$\mathrm{I}_4 \to \mathrm{I}_3+\mathrm{I}_1$	3	$-1+\epsilon z^3+z^4$	$D(\Delta(z)/z^2) = 2^2 3^{30}\epsilon^2(-2^{13}-2^4 3^3\epsilon^4)$
$\mathrm{I}_5 \to 5\mathrm{I}_1$	3	$-1+\epsilon z+z^5$	$D(\Delta(z)) = 2^{22}3^{54}\epsilon^5(5^5+16\epsilon^5)$
$\mathrm{I}_5 \to \mathrm{I}_3+2\mathrm{I}_1$	3	$-1+\epsilon z^3+z^5$	$D(\Delta(z)/z^2) = 2^{12}3^{42}\epsilon^3(5^5+27\epsilon^5)$
$\mathrm{I}_5 \to \mathrm{I}_4+\mathrm{I}_1$	3	$-1+\epsilon z^4+z^5$	$D(\Delta(z)/z^3) = 2^8 3^{36}\epsilon^2(5^5+2^7\epsilon^5)$
$\mathrm{I}_6 \to 6\mathrm{I}_1$	3	$-1+\epsilon z+z^6$	$D(\Delta(z)) = 2^{12}3^{66}5^5\epsilon^6(2^{11}3^6+5^5\epsilon^6)$
$\mathrm{I}_6 \to \mathrm{I}_2+4\mathrm{I}_1$	3	$-1+\epsilon z^2+z^6$	$D(\Delta(z)/z) = 2^{29}3^{60}\epsilon^5(27+\epsilon^3)^2$
$\mathrm{I}_6 \to \mathrm{I}_3+3\mathrm{I}_1$	3	$-1+\epsilon z^3+z^6$	$D(\Delta(z)/z^2) = 2^{10}3^{63}\epsilon^5(8+\epsilon^2)^3$
$\mathrm{I}_6 \to \mathrm{I}_4+2\mathrm{I}_1$	3	$-1+\epsilon z^4+z^6$	$D(\Delta(z)/z^3) = 2^{19}3^{48}\epsilon^3(-27+\epsilon^3)^2$
$\mathrm{I}_6 \to \mathrm{I}_5+\mathrm{I}_1$	3	$-1+\epsilon z^5+z^6$	$D(\Delta(z)/z^4) = 2^8 3^{42}\epsilon^2(2^7 3^6+5^5\epsilon^6)$
$\mathrm{I}_7 \to 7\mathrm{I}_1$	$3+z$	$-1+\epsilon z-\frac{z}{2}-\frac{z^2}{2^3 3}+\frac{z^3}{2^4 3^3}$ $-\frac{z^4}{2^7 3^3}+\frac{z^5}{2^8 3^4_{}}-\frac{7z^6}{2^{10}3^6}$	$D(\Delta(z)) = \frac{7^{13}}{2^{179}3^{31}}\epsilon^7+\mathcal{O}(\epsilon^8)$
$\mathrm{I}_7 \to \mathrm{I}_2+5\mathrm{I}_1$	$3+z$	$-1-\frac{z}{2}-\frac{z^2}{2^3 3}+\epsilon z^2+\frac{z^3}{2^4 3^3}$ $-\frac{z^4}{2^7 3^3}+\frac{z^5}{2^8 3^4}-\frac{7z^6}{2^{10}3^6}$	$D(\frac{\Delta(z)}{z}) = \frac{7^{13}5^5}{2^{174}3^{37}}\epsilon^6+\mathcal{O}(\epsilon^7)$
$\mathrm{I}_7 \to \mathrm{I}_3+4\mathrm{I}_1$	$3+z$	$-1-\frac{z}{2}-\frac{z^2}{2^3 3}+\frac{z^3}{2^4 3^3}+\epsilon z^3$ $-\frac{z^4}{2^7 3^3}+\frac{z^5}{2^8 3^4}-\frac{7z^6}{2^{10}3^6}$	$D(\frac{\Delta(z)}{z^2}) = -\frac{7^{13}}{2^{157}3^{38}}\epsilon^5+\mathcal{O}(\epsilon^6)$
$\mathrm{I}_7 \to \mathrm{I}_4+3\mathrm{I}_1$	$3+z$	$-1-\frac{z}{2}-\frac{z^2}{2^3 3}+\frac{z^3}{2^4 3^3}$ $-\frac{z^4}{2^7 3^3}+\epsilon z^4+\frac{z^5}{2^8 3^4}-\frac{7z^6}{2^{10}3^6}$	$D(\frac{\Delta(z)}{z^3}) = -\frac{7^{13}}{2^{156}3^{36}}\epsilon^4+\mathcal{O}(\epsilon^5)$
$\mathrm{I}_7 \to \mathrm{I}_5+2\mathrm{I}_1$	$3+z$	$-1-\frac{z}{2}-\frac{z^2}{2^3 3}+\frac{z^3}{2^4 3^3}$ $-\frac{z^4}{2^7 3^3}+\frac{z^5}{2^8 3^4}+\epsilon z^5-\frac{7z^6}{2^{10}3^6}$	$D(\frac{\Delta(z)}{z^4}) = \frac{7^{13}}{2^{145}3^{40}}\epsilon^3+\mathcal{O}(\epsilon^4)$
$\mathrm{I}_7 \to \mathrm{I}_6+\mathrm{I}_1$	$3+z$	$-1-\frac{z}{2}-\frac{z^2}{2^3 3}+\frac{z^3}{2^4 3^3}$ $-\frac{z^4}{2^7 3^3}+\frac{z^5}{2^8 3^4}-\frac{7z^6}{2^{10}3^6}+\epsilon z^6$	$D(\frac{\Delta(z)}{z^5}) = \frac{7^{13}}{2^{138}3^{41}}\epsilon^2+\mathcal{O}(\epsilon^3)$
$\mathrm{I}_8 \to 8\mathrm{I}_1$	$3+z+\frac{73z^2}{12}$ $+z^3+z^4$	$-1-\frac{z}{2}+\epsilon z-\frac{37z^2}{2^2 3}$ $-\frac{7\cdot 31z^3}{2^3 3^3}-\frac{5^2 z^4}{2^2 3}-\frac{z^5}{3}$	$D(\Delta(z)) = -2^{13}3^{44}7^7 2017\ \epsilon^8$ $+\mathcal{O}(\epsilon^9)$
$\mathrm{I}_8 \to \mathrm{I}_2+6\mathrm{I}_1$	$3+z+\frac{73z^2}{12}$ $+z^3+z^4$	$-1-\frac{z}{2}-\frac{37z^2}{2^2 3}+\epsilon z^2$ $-\frac{7\cdot 31z^3}{2^3 3^3}-\frac{5^2 z^4}{2^2 3}-\frac{z^5}{3}$	$D(\frac{\Delta(z)}{z}) = -2^{19}3^{45}2017\ \epsilon^7$ $+\mathcal{O}(\epsilon^8)$
$\mathrm{I}_8 \to \mathrm{I}_3+5\mathrm{I}_1$	$3+z+\frac{73z^2}{12}$ $+z^3+z^4$	$-1-\frac{z}{2}-\frac{37z^2}{2^2 3}$ $-\frac{7\cdot 31z^3}{2^3 3^3}+\epsilon z^3-\frac{5^2 z^4}{2^2 3}-\frac{z^5}{3}$	$D(\frac{\Delta(z)}{z^2}) = 2^{12}3^{34}5^5 2017\ \epsilon^6$ $+\mathcal{O}(\epsilon^7)$
$\mathrm{I}_8 \to \mathrm{I}_4+4\mathrm{I}_1$	$3+z+\frac{73z^2}{12}$ $+z^3+z^4$	$-1-\frac{z}{2}-\frac{37z^2}{2^2 3}$ $-\frac{7\cdot 31z^3}{2^3 3^3}-\frac{5^2 z^4}{2^2 3}+\epsilon z^4-\frac{z^5}{3}$	$D(\frac{\Delta(z)}{z^3}) = 2^{19}3^{29}2017\ \epsilon^5$ $+\mathcal{O}(\epsilon^6)$
$\mathrm{I}_8 \to \mathrm{I}_5+3\mathrm{I}_1$	$3+z+\frac{73z^2}{12}$ $+z^3+z^4$	$-1-\frac{z}{2}-\frac{37z^2}{2^2 3}$ $-\frac{7\cdot 31z^3}{2^3 3^3}-\frac{5^2 z^4}{2^2 3}-\frac{z^5}{3}+\epsilon z^5$	$D(\frac{\Delta(z)}{z^4}) = -2^{10}3^{27}2017\ \epsilon^4$ $+\mathcal{O}(\epsilon^5)$
$\mathrm{I}_8 \to \mathrm{I}_6+2\mathrm{I}_1$	$3+z+\frac{73z^2}{12}$ $+z^3+z^4$	$-1-\frac{z}{2}-\frac{37z^2}{2^2 3}$ $-\frac{7\cdot 31z^3}{2^3 3^3}-\frac{5^2 z^4}{2^2 3}-\frac{z^5}{3}+\epsilon z^6$	$D(\frac{\Delta(z)}{z^5}) = -2^{11}3^{19}2017\ \epsilon^3$ $+\mathcal{O}(\epsilon^4)$
$\mathrm{I}_8 \to \mathrm{I}_7+\mathrm{I}_1$	$3+z+\frac{73z^2}{12}$ $+\epsilon z^2+z^3+z^4$	$-1-\frac{z}{2}-\frac{37+2\cdot 3\epsilon}{2^2 3}z^2$ $-\frac{7\cdot 31+2\cdot 3^2\epsilon}{2^3 3^3}z^3$ $-\frac{2\cdot 19+2^2 3\epsilon+\epsilon^2}{2^3 3}z^4$ $-\frac{2^2 3^2-\epsilon^2}{2^4 3^2}z^5$ $+\frac{2^4 3^3+2^3 3^3\epsilon+5\cdot 7\epsilon^2+2\epsilon^3}{2^5 3^3}z^6$	$D(\frac{\Delta(z)}{z^6}) = \frac{3^{30}23\cdot 313}{2^{16}}\epsilon^2$ $+\mathcal{O}(\epsilon^3)$

Table 2 – continued from previous page

Confluence	$g_2(z)$	$g_3(z)$	Behaviour of $\Delta(z)$
$I_9 \to 9I_1$	$\frac{1}{12}+\frac{z}{3} +\epsilon z+\frac{z^2}{2} -\frac{5\cdot 23z^3}{3^2}-\frac{7\cdot 67z^4}{2^2 3^2}$	$-\frac{1}{2^3 3^3}-\frac{z}{2^2 3^2}-\frac{5z^2}{2^3 3^2}+z^3 +\frac{7\cdot 11z^4}{2^3 3}+\frac{13z^5}{2^2}-\frac{27143z^6}{2^3 3^4}$	$D(\frac{\Delta(z)}{z})=\frac{59^{41}}{2^4 3^{80}}\epsilon^7+\mathcal{O}(\epsilon^8)$
$I_9 \to I_2+7I_1$	$\frac{1}{12}+\frac{z}{3} +\frac{z^2}{2} -\frac{5\cdot 23z^3}{3^2}-\frac{7\cdot 67z^4}{2^2 3^2}$	$-\frac{1}{2^3 3^3}-\frac{z}{2^2 3^2}-\frac{5z^2}{2^3 3^2} +\epsilon z^2+z^3+\frac{7\cdot 11z^4}{2^3 3} +\frac{13z^5}{2^2}-\frac{27143z^6}{2^3 3^4}$	$D(\frac{\Delta(z)}{z^2})=\frac{7^7 59^{38}}{2^{12} 3^{67}}\epsilon^6+\mathcal{O}(\epsilon^7)$
$I_9 \to I_3+6I_1$	$\frac{1}{12}+\frac{z}{3} +\frac{z^2}{2}+\epsilon z^2 -\frac{5\cdot 23z^3}{3^2}-\frac{7\cdot 67z^4}{2^2 3^2}$	$-\frac{1}{2^3 3^3}-\frac{z}{2^2 3^2}-\frac{5z^2}{2^3 3^2} -\frac{\epsilon z^2}{2^2 3}+z^3+\frac{7\cdot 11z^4}{2^3 3} +\frac{13z^5}{2^2}-\frac{27143z^6}{2^3 3^4}$	$D(\frac{\Delta(z)}{z^3})=\frac{59^{35}}{2^9 3^{60}}\epsilon^6+\mathcal{O}(\epsilon^7)$
$I_9 \to I_4+5I_1$	$\frac{1}{12}+\frac{z}{3} +\frac{z^2}{2}-\frac{5\cdot 23z^3}{3^2} -\frac{7\cdot 67z^4}{2^2 3^2}+\epsilon z^4$	$-\frac{1}{2^3 3^3}-\frac{z}{2^2 3^2}-\frac{5z^2}{2^3 3^2}+z^3 +\frac{7\cdot 11z^4}{2^3 3}+\frac{13z^5}{2^2}-\frac{27143z^6}{2^3 3^4}$	$D(\frac{\Delta(z)}{z^4})=\frac{5^5 59^{32}}{2^{16} 3^{59}}\epsilon^4+\mathcal{O}(\epsilon^5)$
$I_9 \to I_5+4I_1$	$\frac{1}{12}+\frac{z}{3} +\frac{z^2}{2}-\frac{5\cdot 23z^3}{3^2} -\frac{7\cdot 67z^4}{2^2 3^2}-12\epsilon z^4$	$-\frac{1}{2^3 3^3}-\frac{z}{2^2 3^2}-\frac{5z^2}{2^3 3^2}+z^3 +\frac{7\cdot 11z^4}{2^3 3}+\epsilon z^4+\frac{13z^5}{2^2} -\frac{27143z^6}{2^3 3^4}$	$D(\frac{\Delta(z)}{z^5})=\frac{2^5 59^{29}}{3^{49}}\epsilon^3+\mathcal{O}(\epsilon^4)$
$I_9 \to I_6+3I_1$	$\frac{1}{12}+\frac{z}{3} +\frac{z^2}{2}-\frac{5\cdot 23z^3}{3^2} -\frac{7\cdot 67z^4}{2^2 3^2}-6\epsilon z^4$	$-\frac{1}{2^3 3^3}-\frac{z}{2^2 3^2}-\frac{5z^2}{2^3 3^2}+z^3 +\frac{7\cdot 11z^4}{2^3 3}+\frac{\epsilon z^4}{2}+\frac{13z^5}{2^2} +\epsilon z^5-\frac{27143z^6}{2^3 3^4}$	$D(\frac{\Delta(z)}{z^6})=\frac{59^{26}}{2^6 3^{40}}\epsilon^2+\mathcal{O}(\epsilon^3)$
$I_9 \to I_7+2I_1$	$\frac{1}{12}+\frac{z}{3} +\frac{z^2}{2}-\frac{5\cdot 23z^3}{3^2} -\frac{7\cdot 67z^4}{2^2 3^2}-12\epsilon z^4$	$-\frac{1}{2^3 3^3}-\frac{z}{2^2 3^2}-\frac{5z^2}{2^3 3^2}+z^3 +\frac{7\cdot 11z^4}{2^3 3}+\epsilon z^4+\frac{13z^5}{2^2} +2\epsilon z^5-\frac{27143z^6}{2^3 3^4}+\epsilon z^6$	$D(\frac{\Delta(z)}{z^7})=\frac{2^2 59^{24}}{3^{38}}\epsilon+\mathcal{O}(\epsilon^2)$
$I_9 \to I_8+I_1$	$\frac{1}{12}+\frac{z}{3} +\frac{(1-\epsilon)z^2}{2} -\frac{5\cdot 23z^3}{3^2}+\epsilon z^3 -\frac{7\cdot 67z^4}{2^2 3^2}+\frac{3\epsilon(2+\epsilon)z^4}{2^2}$	$-\frac{1}{2^3 3^3}-\frac{z}{2^2 3^2}-\frac{5z^2}{2^3 3^2}+\frac{\epsilon z^2}{2^3 3}+z^3 +\frac{7\cdot 11z^4}{2^3 3}+\epsilon z^4-\frac{\epsilon(2+\epsilon)z^4}{8}+\frac{13z^5}{2^2} -\frac{\epsilon(2\cdot 5\cdot 13-3^2\epsilon)z^5}{2^2 3^2}-\frac{27143z^6}{2^3 3^4} +\frac{\epsilon(233-3\cdot 5\epsilon+3\epsilon^2)z^6}{2^3 3}$	$D(\frac{\Delta(z)}{z^8})=\frac{59^{20}}{3^{31}}+\mathcal{O}(\epsilon)$

Confluences found by explicit calculation

In some very rare cases we are able to find a very elegant geometric discriminant which factors. In these cases the zeros of the geometric discriminant are easily found by explicit computation. From the fact that only I_{b_i} may merge into a I_b, as mentioned above, we derive that zeros of order b_i correspond to singular fibers of type I_{b_i}. These examples are listed in table 3.

Confluences and the Weierstrass preparation theorem

All confluences other then the ones mentioned above will not be given explicitly, but existence will be proven by making use of the Weierstrass preparation theorem and the implicit function theorem. We shall start by giving (a part of) the Weierstrass preparation theorem.

Let f be a complex analytic function in one variable, not identically equal to zero, on a convex neighbourhood U of zero. Now let $\gamma : [0,1] \to U\backslash\{0\}$ be a closed curve around the origin. For simplicity we will assume that $\gamma([0,1])$ is homotopic to a circle in $U\backslash\{0\}$. Denote the two real dimensional surface in U enclosed by γ by D. Furthermore we shall assume that $f\mid_{\gamma([0,1])} \neq 0$.

Theorem 2.2.1 *Let f be be as above and furthermore assume number of zeros inside D counted with multiplicity equals M. Then there exists a unique Weierstrass polynomial $W(z)$ of degree M*

$$W(z) = z^M + c_1 z^{M-1} + c_2 z^{M-2} + \ldots + c_M,$$

where $W(z)$ has the same zeros as f in D or alternatively $f(z) = W(z)u(z)$ with $u(z)$ a unit in D.

Table 3: In this table the geometric discriminant in denoted by Δ

Confluence	$g_2(z)$	$g_3(z)$	$\Delta(z)$	Roots near $z=0$
$I_4 \to 2I_2$	3	$-1+\frac{\epsilon^2 z^2}{4}$ $+\epsilon z^3+z^4$	$-\frac{3^3}{2^4}z^2(2z+\epsilon)^2\times$ $(-8+z^2\epsilon^2+4\epsilon z^3+4z^4)$	0, 0, $\epsilon/2$, $\epsilon/2$
$I_5 \to I_3+I_2$	3	$-1+\frac{\epsilon^2 z^3}{4}$ $+\epsilon z^4+z^5$	$-\frac{3^3}{2^4}z^3(2z+\epsilon)^2\times$ $(-8+z^3\epsilon^2+4\epsilon z^4+4z^5)$	0, 0, 0 $\epsilon/2$, $\epsilon/2$
$I_5 \to 2I_2+I_1$	3	$-1+\epsilon z^2$ $-\frac{3\epsilon^{2/3}}{2^{2/3}}z^3+z^5$	$-\frac{3^3}{2^2}z^2\times$ $(2\epsilon-3\cdot 2^{1/3}\epsilon^{2/3}z+2z^2)\times$ $(-4+2\epsilon z^2-3\cdot 2^{1/3}\epsilon^{2/3}z^3+2z^5)$	0, 0 $(\epsilon/2)^{1/3}$, $(\epsilon/2)^{1/3}$, $-2^{2/3}\epsilon^{1/3}$
$I_6 \to I_4+I_2$	3	$-1+\frac{\epsilon^2 z^4}{4}$ $+\epsilon z^5+z^6$	$-\frac{3^3}{2^4}z^4(2z+\epsilon)^2\times$ $(-8+\epsilon^2 z^4+4\epsilon z^5+4z^6)$	0, 0, 0, 0 $\epsilon/2$, $\epsilon/2$
$I_6 \to I_3+I_2$ $+I_1$	3	$-1+\epsilon z^3$ $-\frac{3\epsilon^{2/3}}{2^{2/3}}z^4+z^6$	$-\frac{3^3}{2^2}z^3\times$ $(2\epsilon-3\cdot 2^{1/3}\epsilon^{2/3}z+2z^3)\times$ $(-4+2\epsilon z^3-3\cdot 2^{1/3}\epsilon^{2/3}z^4+2z^6)$	0, 0, 0 $(\epsilon/2)^{1/3}$, $(\epsilon/2)^{1/3}$, $-2^{2/3}\epsilon^{1/3}$
$I_6 \to 2I_2+2I_1$	3	$-1+\frac{3\epsilon^{4/3}}{2^{2+2/3}}z^2$ $+\epsilon z^3+z^6$	$-\frac{3^3}{2^6}z^2\times$ $(3\cdot 3^{1/3}\epsilon^{4/3}-2^3\epsilon z+2^3z^4)\times$ $(-2^4+2^{1/3}3\epsilon^{4/3}z^2$ $+2^3\epsilon z^3+2^3z^6)$	0, 0, $\frac{1}{2}(2^{1/3}\epsilon^{1/3}+i2^{5/6}\epsilon^{1/3})$, $\frac{1}{2}(2^{1/3}\epsilon^{1/3}-i2^{5/6}\epsilon^{1/3})$, $-\frac{\epsilon^{1/3}}{2^{2/3}}$, $-\frac{\epsilon^{1/3}}{2^{2/3}}$
$I_6 \to 3I_2$	3	$-1+\frac{\epsilon^2}{2^2}z^2$ $-\epsilon z^4+z^6$	$-\frac{3^3}{2^4}z^2(2z^2-\epsilon)^2\times$ $(-2^3+\epsilon^2z^2-2^2\epsilon z^4+2^2z^6)$	0, 0, $\sqrt{\frac{\epsilon}{2}}$, $\sqrt{\frac{\epsilon}{2}}$, $-\sqrt{\frac{\epsilon}{2}}$, $-\sqrt{\frac{\epsilon}{2}}$
$I_6 \to 2I_3$	3	$-1+\epsilon^3z^3$ $+3\epsilon^2z^4$ $+3\epsilon z^5+z^6$	$-27z^3(z+\epsilon)^3\times$ $(-2+z^6+3z^5\eta+3z^4\eta^2+z^3\eta^3)$	0, 0,0, $-\epsilon$, $-\epsilon$, $-\epsilon$

In the following $\Delta(z)$ will play the role of $f(z)$. We shall start with an example is which we illustrate how we use the Weierstrass preparation theorem and the implicit function theorem to prove the existence of a confluence.

$I_7 \to I_4 + I_3$

We start with a Weierstrass model defined by

$$g_2(z) = 3 + z^3 + z^4$$
$$g_3(z) = -1 - \frac{z^3}{2} - \frac{z^4}{2} + \epsilon z^4 + \eta z^5 - \frac{z^6}{2^3 3} + \delta z^6,$$

so that

$$\begin{aligned}\Delta(z) =& 2\cdot 3^3\epsilon z^4 + 2\cdot 3^3\eta z^5 + 2\cdot 3^3\delta z^6 + \left(\frac{3^2}{2} + 3^3\epsilon\right)z^7 + \left(\frac{3^2}{2^2} + 3^3\epsilon - 3^3\epsilon^2 + 3^3\eta\right)z^8 \\ &+ \left(-\frac{1}{8} + 3^3\delta + (3^3 - 2\cdot 3^3\epsilon)\eta\right)z^9 + 3\left(\frac{5}{8} + 3^2(1-2\epsilon)\delta + \frac{3\epsilon}{2^2} - 3^2\eta^2\right)z^{10} \\ &+ \left(3 + \left(\frac{3^2}{2^2} - 2\cdot 3^3\delta\right)\right)z^{11} + \left(1 - 3^3\left(\frac{1}{2^3 3} + \delta\right)^2\right)z^{12}.\end{aligned}$$

It is clear that for $\epsilon, \eta, \delta = 0$ we find a zero of order seven in the origin, corresponding to a singular fiber of type I_7, but for $\epsilon \neq 0$ we find a zero of order four, corresponding to a singular fiber of type I_4. From the Weierstrass preparation theorem we deduce that the geometric discriminant must be of the form

$$\Delta(z) = z^4\big(z^3 + c_{1,\epsilon,\eta,\delta}z^2 + c_{2,\epsilon,\eta,\delta}z + c_{3,\epsilon,\eta,\delta}\big)u_{\epsilon,\eta,\delta}(z)$$

where we made the dependence on the perturbation parameters obvious by the under-indices and where u is again a unit in a neighbourhood of the origin. We note that the product of z^4 and the third order polynomial has been called the Weierstrass polynomial. and we therefore use the notation

$$W_{\epsilon,\eta,\delta}(z) = z^3 + c_{1,\epsilon,\eta,\delta}z^2 + c_{2,\epsilon,\eta,\delta}z + c_{3,\epsilon,\eta,\delta}$$

and refer to $W_{\epsilon,\eta,\delta}$ as he reduced Weierstrass polynomial. Our aim is the prove that there exists a curve δ' in the ϵ, η, δ-space so that

$$W_{\epsilon,\eta,\delta}(z) = (z - z_{0,\epsilon,\eta,\delta})^3, \tag{5}$$

where $z_{0,\epsilon,\eta,\delta}|_{\delta'}$ is equal to zero if and only if δ' is at the origin. Equation (5) is equivalent to

$$c_{1,\epsilon,\eta,\delta} = -3z_{0,\epsilon,\eta,\delta} \qquad c_{2,\epsilon,\eta,\delta} = 3(z_{0,\epsilon,\eta,\delta})^2 \qquad c_{3,\epsilon,\eta,\delta} = -(z_{0,\epsilon,\eta,\delta})^3,$$

which in turn yields

$$c_{2,\epsilon,\eta,\delta} - \frac{1}{3}(c_{1,\epsilon,\eta,\delta})^2 = 0 \qquad c_{3,\epsilon,\eta,\delta} - \frac{1}{3^3}(c_{1,\epsilon,\eta,\delta})^3 = 0. \tag{6}$$

We shall now view this equation as an equation in the variables ϵ, η, δ. To prove the existence of a solution curve δ' in a neighbourhood of the origin, it is sufficient to prove that

$$D(c_{2,\epsilon,\eta,\delta} - \frac{1}{3}(c_{1,\epsilon,\eta,\delta})^2, c_{3,\epsilon,\eta,\delta} - \frac{1}{3^3}(c_{1,\epsilon,\eta,\delta})^3)|_0 \in \mathrm{Aut}(\mathbb{C}^2),$$

where D indicates the total derivative with respect to two variables, any combination of ϵ, η and δ will do. In the following we will not indicated these variables explicitly. Since the value of $c_{1,\epsilon,\eta,\delta}$ is zero at $\epsilon = \eta = \delta = 0$ we need to establish that

$$D(c_{2,\epsilon,\eta,\delta}, c_{3,\epsilon,\eta,\delta})|_0 \in \mathrm{Aut}(\mathbb{C}^2).$$

Alternatively we may also prove that the rank of

$$D(c_{2,\epsilon,\eta,\delta}, c_{3,\epsilon,\eta,\delta})$$

is maximal, in this case D indicates the total derivative with respect to ϵ, η and δ. We shall always assume we take the derivative at zero, we shall make this explicit no longer.

We now determine $c_{1,\epsilon,\eta,\delta}, c_{2,\epsilon,\eta,\delta}$ and $c_{3,\epsilon,\eta,\delta}$ perturbatively by making use of the Weierstrass preparation theorem. We have that

$$\Delta_{\epsilon,\eta,\delta}(z) = z^4 W_{\epsilon,\eta,\delta} u_{\epsilon,\eta,\delta}(z)$$

so that

$$\begin{aligned}
\frac{\partial}{\partial \epsilon}\Delta_{\epsilon,\eta,\delta}(z) = \frac{\partial}{\partial \epsilon}\big(z^4 W_{\epsilon,\eta,\delta}(z) u_{\epsilon,\eta,\delta}(z)\big) \qquad &= z^4\left(\frac{\partial}{\partial \epsilon}W_{\epsilon,\eta,\delta}(z)\right)u_{\epsilon,\eta,\delta}(z) + z^4 W_{\epsilon,\eta,\delta}(z)\frac{\partial}{\partial \epsilon}u_{\epsilon,\eta,\delta}(z)\\
\frac{\partial}{\partial \eta}\Delta_{\epsilon,\eta,\delta}(z) = \frac{\partial}{\partial \eta}\big(z^4 W_{\epsilon,\eta,\delta}(z) u_{\epsilon,\eta,\delta}(z)\big) \qquad &= z^4\left(\frac{\partial}{\partial \eta}W_{\epsilon,\eta,\delta}(z)\right)u_{\epsilon,\eta,\delta}(z) + z^4 W_{\epsilon,\eta,\delta}(z)\frac{\partial}{\partial \eta}u_{\epsilon,\eta,\delta}(z)\\
\frac{\partial}{\partial \delta}\Delta_{\epsilon,\eta,\delta}(z) = \frac{\partial}{\partial \delta}\big(z^4 W_{\epsilon,\eta,\delta}(z) u_{\epsilon,\eta,\delta}(z)\big) \qquad &= z^4\left(\frac{\partial}{\partial \delta}W_{\epsilon,\eta,\delta}(z)\right)u_{\epsilon,\eta,\delta}(z) + z^4 W_{\epsilon,\eta,\delta}(z)\frac{\partial}{\partial \delta}u_{\epsilon,\eta,\delta}(z).
\end{aligned}$$

We are only interested in a neighbourhood of the origin in ϵ, η, δ-space, so we consider

$$\begin{aligned}
\frac{\partial}{\partial \epsilon}\Delta_{\epsilon,\eta,\delta}(z)\big|_0 &= z^4\left(\frac{\partial}{\partial \epsilon}W_{\epsilon,\eta,\delta}(z)\right)u_{\epsilon,\eta,\delta}(z)\big|_0 + z^4 W_{\epsilon,\eta,\delta}(z)\frac{\partial}{\partial \epsilon}u_{\epsilon,\eta,\delta}(z)\big|_0\\
\frac{\partial}{\partial \eta}\Delta_{\epsilon,\eta,\delta}(z)\big|_0 &= z^4\left(\frac{\partial}{\partial \eta}W_{\epsilon,\eta,\delta}(z)\right)u_{\epsilon,\eta,\delta}(z)\big|_0 + z^4 W_{\epsilon,\eta,\delta}(z)\frac{\partial}{\partial \eta}u_{\epsilon,\eta,\delta}(z)\big|_0\\
\frac{\partial}{\partial \delta}\Delta_{\epsilon,\eta,\delta}(z)\big|_0 &= z^4\left(\frac{\partial}{\partial \delta}W_{\epsilon,\eta,\delta}(z)\right)u_{\epsilon,\eta,\delta}(z)\big|_0 + z^4 W_{\epsilon,\eta,\delta}(z)\frac{\partial}{\partial \delta}u_{\epsilon,\eta,\delta}(z)\big|_0.
\end{aligned}$$

If we now also note that $W_{\epsilon,\eta,\delta}(z)|_0 = z^3$, this equation reduces to

$$\begin{aligned}
\frac{\partial}{\partial \epsilon}\Delta_{\epsilon,\eta,\delta}(z)\big|_0 &= z^4\left(\frac{\partial}{\partial \epsilon}W_{\epsilon,\eta,\delta}(z)\right)u_{\epsilon,\eta,\delta}(z)\big|_0 + z^7\frac{\partial}{\partial \epsilon}u_{\epsilon,\eta,\delta}(z)\big|_0\\
\frac{\partial}{\partial \eta}\Delta_{\epsilon,\eta,\delta}(z)\big|_0 &= z^4\left(\frac{\partial}{\partial \eta}W_{\epsilon,\eta,\delta}(z)\right)u_{\epsilon,\eta,\delta}(z)\big|_0 + z^7\frac{\partial}{\partial \eta}u_{\epsilon,\eta,\delta}(z)\big|_0\\
\frac{\partial}{\partial \delta}\Delta_{\epsilon,\eta,\delta}(z)\big|_0 &= z^4\left(\frac{\partial}{\partial \delta}W_{\epsilon,\eta,\delta}(z)\right)u_{\epsilon,\eta,\delta}(z)\big|_0 + z^7\frac{\partial}{\partial \delta}u_{\epsilon,\eta,\delta}(z)\big|_0.
\end{aligned}$$

The unit around zero is determined by

$$\left.\frac{\Delta_{\epsilon,\eta,\delta}(z)}{z^7}\right|_0 = u_{\epsilon,\eta,\delta}(z)|_0,$$

which together with the previous formulae implies that the first derivatives of $W_{\epsilon,\eta,\delta}(z)$ at the origin can be determined from the fourth through seventh coefficient of the polynomial $\Delta(z)$. From this we may deduce that

$$W(z) = z^3 + \left(\frac{10\epsilon}{3} - 6\eta + 12\delta\right)z^2 + (-6\epsilon + 12\eta)z + 12\epsilon + \mathcal{O}(|(\epsilon,\eta,\delta)|^2).$$

This implies that the rank of

$$D(c_{2,\epsilon,\eta,\delta}, c_{3,\epsilon,\eta,\delta})$$

is maximal, which means that we have proven the existence of a curve δ' such that

$$\Delta_{\delta'}(z) = z^4(z - z_{0,\delta'})^3 u_{\delta'}(z)$$

and thus the existence of a confluence of type $\mathrm{I}_7 \to \mathrm{I}_4 + \mathrm{I}_3$.

Let us now reflect on this proof. We did focus on a confluence of type $\mathrm{I}_7 \to \mathrm{I}_4 + \mathrm{I}_3$ and thus imposed the equation (5). If we would have been interested in a confluence of type $\mathrm{I}_7 \to \mathrm{I}_4 + \mathrm{I}_2 + \mathrm{I}_1$ we would have imposed for example[7]

$$W_{\epsilon,\eta,\delta}(z) = (z - z_{0,\epsilon,\eta,\delta})^2(z + z_{0,\epsilon,\eta,\delta})$$

This would have altered the coefficients in (6), but it still would have been sufficient to prove that the rank of

$$D(c_{2,\epsilon,\eta,\delta}, c_{3,\epsilon,\eta,\delta})$$

is maximal. So we have also found a proof of the existence of confluences of type $\mathrm{I}_7 \to \mathrm{I}_4 + \mathrm{I}_2 + \mathrm{I}_1$ and $\mathrm{I}_7 \to \mathrm{I}_4 + \mathrm{I}_1 + \mathrm{I}_1 + \mathrm{I}_1$. This statement may be generalized to the following.

Lemma 2.2.2 *Let g_2, g_3 and Δ, depending of perturbation parameters $\delta_1, \ldots, \delta_m$ be such that for $\delta_1 = \ldots = \delta_m = 0$ we have the Weierstrass model of a rational elliptic surface with a singular fiber of type I_b in the origin. Further assume that for $\delta_1, \ldots, \delta_m \neq 0$ the geometric discriminant is of the form*

$$\Delta(z) = z^2 W(z) u(z),$$

where $u(z)$ is a unit, $W(z)$ a polynomial, which we shall refer to as the reduced Weierstrass polynomial, and both $W(z)$ and $u(z)$ depend of some perturbation parameters $\delta_1, \ldots, \delta_m$. Furthermore if we write

$$W(z) = z^{b-2} + c_1 z^{b-3} + \ldots + c_b,$$

the maximality of the rank of the jacobian

$$D(c_1, c_2, \ldots, c_b)|_0$$

implies that every confluence of type $I_b \to I_{b-e} + I_{e_1} + \ldots + I_{e_j}$, with $e_1 + \ldots + e_j = e \leq b - 2$ exists.

Proof To prove the existence of any confluence of type $\mathrm{I}_b \to \mathrm{I}_{b-e} + \mathrm{I}_{e_1} + \ldots + \mathrm{I}_{e_j}$, with $e_1 + \ldots + e_j = e \leq b-2$ it is sufficient to prove that there is a curve δ' in $\delta_1, \delta_2, \ldots, \delta_m$-space such that $W(z)$ is of the form

$$W(z) = z^{b-e-2}(z - e^{i\theta_0} z_{0,\,\delta_1,\delta_2,\ldots,\delta_m})(z - e^{i\theta_1} z_{0,\,\delta_1,\delta_2,\ldots,\delta_m}) \ldots (z - e^{i\theta_e} z_{0,\,\delta_1,\delta_2,\ldots,\delta_m}),$$

[7] It is practical to let the position of all roots depend on one parameter to make sure we can distinguish roots.

where we impose that the $\theta_i \in [-\pi, \pi)$ are fixed and $\theta_1 = \ldots = \theta_{e_1}$, $\theta_{e_1+1} = \ldots = \theta_{e_2}$, $\ldots$, $\theta_{e_{j-1}+1} = \ldots = \theta_{e_j}$ and no other equalities occur. Imposing this means that

$$\begin{aligned}
W(z) &= z^{b-e-2}(z - e^{i\theta_0} z_{0,\,\delta_1,\delta_2,\ldots,\delta_m})(z - e^{i\theta_1} z_{0,\,\delta_1,\delta_2,\ldots,\delta_m}) \ldots (z - e^{i\theta_e} z_{0,\,\delta_1,\delta_2,\ldots,\delta_m}) \\
&= z^{b-e-2}\big(z^e - z^{e-1}(e^{i\theta_0} + \ldots + e^{i\theta_e}) z_{0,\,\delta_1,\delta_2,\ldots,\delta_m} \\
&\quad + z^{e-2}(e^{i\theta_0}e^{i\theta_1} + \ldots + e^{i\theta_{e-1}}e^{i\theta_e}) z^2_{0,\,\delta_1,\delta_2,\ldots,\delta_m} \\
&\quad + \ldots + (-1)^e z^e_{0,\,\delta_1,\delta_2,\ldots,\delta_m}(e^{i\theta_0} \ldots e^{i\theta_e}) \\
&= z^{b-2} + c_1 z^{b-3} + \ldots + c_{b-2},
\end{aligned}$$

in particular we have that

$$\begin{aligned}
c_1 &= -(e^{i\theta_0} + \ldots + e^{i\theta_e}) z_{0,\,\delta_1,\delta_2,\ldots,\delta_m} \\
c_2 &= (e^{i\theta_0}e^{i\theta_1} + \ldots + e^{i\theta_{e-1}}e^{i\theta_e}) z^2_{0,\,\delta_1,\delta_2,\ldots,\delta_m} \\
&\vdots \\
c_e &= (-1)^e z^e_{0,\,\delta_1,\delta_2,\ldots,\delta_m}(e^{i\theta_0} \ldots e^{i\theta_e}) \\
c_{e+1} &= 0 \\
&\vdots \\
c_{b-2} &= 0.
\end{aligned}$$

If we pick[8] the θ_is such that $(e^{i\theta_0} + \ldots + e^{i\theta_e}) \neq 0$ we may use the first equation above to substitute the $z_{0,\,\delta_1,\delta_2,\ldots,\delta_m}$ if all of the above equations, where the equation does not read $c_i = 0$. This implies that it suffices to impose equations of the sort $c_j = 0$ or $c_j - \alpha_j c_1^j = 0$, where $\alpha_j = (e^{i\theta_0} \ldots e^{i\theta_{j-1}} + \ldots + e^{i\theta_{e-j+1}} \ldots e^{i\theta_e})(e^{i\theta_0} + \ldots + e^{i\theta_e})^{-j}$. Like in our extensive discussion of the confluence $\mathrm{I}_7 \to \mathrm{I}_4 + \mathrm{I}_3$, we have that the maximality of the rank of the Jacobian[9]

$$D(c_1, c_2, \ldots, c_b)|_0$$

implies that

$$D(c_2 - \alpha_2 c_1^2, \ldots, c_{b-2})|_0$$

is an automorphism of $\mathbb{C}^{b-3}$, which via the implicit function theorem yields the existence of a solution curve δ' and thus the confluence itself. □

We have shown for $7 \leq b \leq 9$ by explicit construction of the Weierstrass models such that the conditions on Δ and W of lemma 2.2.2 hold and by calculation that the rank of the Jacobians mentioned is maximal. These explicit considerations will not be presented here. This concludes our discussion of singular fibers of type I_b.

Our method does not rely in any sense on the rationality of the elliptic surface. We therefore conjecture that the result also holds for $K3$-surfaces.

2.3 Confluence to singular fibers of Kodaira type II, III and IV.

Theorem 2.3.1 *Of all confluences to Singular Fibers of Kodaira type* II, III *and* IV*, allowed by conservation of the Euler number, the following occur:*

$\mathrm{II} \to \mathrm{I}_1 + \mathrm{I}_1$ $\quad$ $\mathrm{III} \to 3\mathrm{I}_1$ $\quad$ $\mathrm{III} \to \mathrm{I}_2 + \mathrm{I}_1$ $\quad$ $\mathrm{III} \to \mathrm{II} + \mathrm{I}_1$ $\quad$ $\mathrm{IV} \to 4\mathrm{I}_1$ $\quad$ $\mathrm{IV} \to \mathrm{I}_3 + \mathrm{I}_1$

$\mathrm{IV} \to \mathrm{III} + \mathrm{I}_1$ $\quad$ $\mathrm{IV} \to \mathrm{II} + \mathrm{I}_2$ $\quad$ $\mathrm{IV} \to 2\mathrm{II}$ $\quad$ $\mathrm{IV} \to \mathrm{I}_2 + 2\mathrm{I}_1$ $\quad$ $\mathrm{IV} \to \mathrm{II} + 2\mathrm{I}_1$.

Moreover the confluence which does not occur namely $\mathrm{IV} \to 2\mathrm{I}_2$ *is obstructed by monodromy considerations.*

[8] This is not strictly necessary but this simplifies the argument somewhat.
[9] We assume we derive with respect to the right number of coordinates.

In this subsection we shall give examples of every imaginary confluence to a singular fiber of type II, III or IV, except IV $\to$ $2\mathrm{I}_2$. The properties of these examples will be verified by explicit calculation, as can be seen in table 4, with the exception of the confluences IV $\to$ $\mathrm{II}+2\mathrm{I}_1$ and IV $\to$ $\mathrm{I}_2+2\mathrm{I}_1$. The confluences of type IV $\to$ $\mathrm{II}+2\mathrm{I}_1$ and IV $\to$ $\mathrm{I}_2+2\mathrm{I}_1$ are treated separately and rely on the same discriminant argument as used before. We will prove that there exists no confluence of the type IV $\to$ $2\mathrm{I}_2$, by using monodromy considerations, not unlike the considerations seen in [4].

Table 4: In this table the zeros of the geometric discriminant are denoted by z_0.

Confluence	$g_2(z)$	$g_3(z)$	zeros z_0 of $\Delta(z)$	$g_2(z_0)$	$g_3(z_0)$
$\mathrm{II} \to 2\mathrm{I}_1$	$3\epsilon^2$	z	$\pm\epsilon^3$	$3\epsilon^2, 3\epsilon^2$	$\pm\epsilon^3$
$\mathrm{III} \to 3\mathrm{I}_1$	$3z$	ϵ^3	$\epsilon^2, \epsilon^2 e^{\pm\frac{2\pi}{3}}$	$3\epsilon^2, 3\epsilon^2 e^{\pm\frac{2\pi}{3}}$	$\epsilon^3, \epsilon^3, \epsilon^3$
$\mathrm{III} \to \mathrm{I}_2+\mathrm{I}_1$	$z+3\epsilon^2$	$\frac{\epsilon z}{2}+\epsilon^3$	$0, 0, -\frac{9\epsilon^2}{4}$	$3\epsilon^2, 3\epsilon^2, \frac{3\epsilon}{4}$	$\epsilon^3, \epsilon^3, -\frac{\epsilon^3}{8}$
$\mathrm{III} \to \mathrm{II}+\mathrm{I}_1$	z	ϵz	$0, 0, 27\epsilon^2$	$0, 0, 27\epsilon^2$	$0, 0, 27\epsilon^3$
$\mathrm{IV} \to 4\mathrm{I}_1$	$3\epsilon^4$	z^2	$\pm\epsilon^3, \pm i\epsilon^3$	$3\epsilon^4, 3\epsilon^4$	$\epsilon^6, -\epsilon^6$
$\mathrm{IV} \to \mathrm{I}_3+\mathrm{I}_1$	$24\epsilon(z+72\epsilon^3)$	$z^2+2^5 3^2 z\epsilon^3$ $+2^9 3^3\epsilon^6$	$0, 0, 0, -64\epsilon^3$	$2^6 3^3\epsilon^4, 2^6 3^3\epsilon^4,$ $2^6 3^3\epsilon^4, 2^6 3\epsilon^4$	$2^9 3^3\epsilon^6, 2^9 3^3\epsilon^6,$ $2^9 3^3\epsilon^6, -2^9\epsilon^6$
$\mathrm{IV} \to \mathrm{III}+\mathrm{I}_1$	ϵz	z^2	$0, 0, 0, \frac{\epsilon^3}{3^3}$	$0, 0, 0, \frac{\epsilon^4}{3^3}$	$0, 0, 0, \frac{\epsilon^6}{3^6}$
$\mathrm{IV} \to \mathrm{II}+\mathrm{I}_2$	ϵz	$z^2+\frac{\epsilon^3 z}{2^2 3^3}$	$0, 0, \frac{\epsilon^3}{2^2 3^3}, \frac{\epsilon^3}{2^2 3^3}$	$0, 0, \frac{\epsilon^4}{2^2 3^3}, \frac{\epsilon^4}{2^2 3^3}$	$0, 0, \frac{\epsilon^6}{2^3 3^6}, \frac{\epsilon^6}{2^3 3^6}$
$\mathrm{IV} \to 2\mathrm{II}$	0	$z(z-\epsilon)$	$0, 0, \epsilon, \epsilon$	$0, 0, 0, 0$	$0, 0, 0, 0$

Confluence	$g_2(z)$	$g_3(z)$	Behaviour of $\Delta(z)$
$\mathrm{IV} \to \mathrm{I}_2+2\mathrm{I}_1$	$\frac{z\epsilon z}{2}+3\epsilon^2$	$z^2+\frac{\epsilon^2 z}{4}+\epsilon^3$	$D\left(\frac{\Delta(z)}{z^2}\right)=\frac{1}{64}\epsilon^3(\epsilon-72)^3$
$\mathrm{IV} \to \mathrm{II}+2\mathrm{I}_1$	ϵz	$z^2+\epsilon z$	$D\left(\frac{\Delta(z)}{z^2}\right)=\epsilon^4(\epsilon^2-108)$

Table 5: We give the g_2 and g_3 defining the Weierstrass model in affine coordinates and have already rescaled the leading term of the geometric discriminant and used a Tschirnhauser transformation on Δ.

Fixed fiber	g_2 and g_3	Parameter	Configuration
II^*	$g_2(z)=a$	$a=0$	$\mathrm{II}^*+\mathrm{II}$
	$g_3(z)=z$	$a\neq 0$	$\mathrm{II}^*+2\mathrm{I}_1$
III^*	$g_2(z)=z+9c^3$	$c=d=0$	$\mathrm{III}^*+\mathrm{III}$
	$g_3(z)=cz+d$	$5c^3-d=0,\ c\neq 0$	$\mathrm{III}^*+\mathrm{II}+\mathrm{I}_1$
		$9c^3-d=0,\ c\neq 0$	$\mathrm{III}^*+\mathrm{I}_2+\mathrm{I}_1$
		otherwise	$\mathrm{III}^*+3\mathrm{I}_1$
IV^*	$g_2(z)=az+b$	$-a^4b+2\cdot 3^3b^2+2\cdot 3^3a^2d=0$	$\mathrm{IV}^*+\mathrm{IV}$, $\mathrm{IV}^*+\mathrm{III}+\mathrm{I}_1$, $\mathrm{IV}^*+\mathrm{II}+\mathrm{I}_2$,
	$g_3(z)=z^2+\frac{a^3z}{2\cdot 3^3}+d$		$\mathrm{IV}^*+2\mathrm{II}$, $\mathrm{IV}^*+\mathrm{II}+2\mathrm{I}_1$ (generic)
		$a^{12}-3^4 11a^8b+2\cdot 3^7 5\cdot 11a^4b^2$ $-2^9 3^9 b^3+2\cdot 3^6 13a^6d$ $-2^6 3^{11}a^2bd+2^9 3^{12}d^2=0,$ $a^{12}-3^4 11a^8b+2\cdot 3^7 5\cdot 11a^4b^2\neq 0$	$\mathrm{IV}^*+\mathrm{I}_3+\mathrm{I}_1$, $\mathrm{IV}^*+\mathrm{I}_2+2\mathrm{I}_1$ (generic)
		$a=b=d=0$	$\mathrm{IV}^*+\mathrm{IV}$
		$a=b=0,\ d\neq 0$	$\mathrm{IV}^*+2\mathrm{II}$
		$b=a^4/216,\ d=a^6/15552,\ a\neq 0$	$\mathrm{IV}^*+\mathrm{II}+\mathrm{I}_2$
		$b=a^4/108,\ d=a^6/11664,\ a\neq 0$	$\mathrm{IV}^*+\mathrm{III}+\mathrm{I}_1$
		$b=7a^4/1728,\ d=37a^6/746496,\ a\neq 0$	$\mathrm{IV}^*+\mathrm{I}_3+\mathrm{I}_1$
		otherwise	$\mathrm{IV}^*+4\mathrm{I}_1$

We have that the monodromy matrices, or rather the equivalence classes thereof, before and after confluence satisfy equation (3). Taking the trace on both sides of the equation yields in the case of the confluence $\mathrm{IV} \to 2\mathrm{I}_2$

$$\mathrm{Tr}(M_{\mathrm{IV}}) = \mathrm{Tr}(M_{\mathrm{I}_2} A M_{\mathrm{I}_2} A^{-1})$$

$$\mathrm{Tr}\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} = \mathrm{Tr}\left(\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\right)$$

$$-1 = 2(1 - 2c^2),$$

which contradicts that $A \in \mathrm{SL}(2, \mathbb{Z})$.

Note that in a confluence to a singular fiber of type II the complementary singular fiber II* remains fixed in infinity. In our examples of confluences to III and IV, III* and IV*, respectively, remain fixed in infinity. This means that we have provided an explicit Weierstrass normal form for all configurations in the list of Persson [5] including a singular fiber of type II*, III* or IV*.

With respect to our ultimate goal of understanding the stratification of the space N_g we are now in a position to give more insight into the intricate structure of the set of strata corresponding to a configuration containing a singular fiber of Kodaira type II*, III* or IV*, using the methods discussed in subsection 1.1.[10] The results of this investigation are given in table 5.

2.4 Confluence to singular fibers of Kodaira type I_0^*.

In this subsection we discuss the confluences of to singular fibers of type I_0^* and prove the following theorem.

Theorem 2.4.1 *Of all confluences to Singular Fibers of Kodaira type I_0^*, allowed by conservation of Euler number, the following occur:*

$\mathrm{I}_0^* \to \mathrm{I}_4 + 2\mathrm{I}_1$	$\mathrm{I}_0^* \to \mathrm{IV} + \mathrm{II}$	$\mathrm{I}_0^* \to \mathrm{IV} + 2\mathrm{I}_1$	$\mathrm{I}_0^* \to \mathrm{I}_3 + \mathrm{II} + \mathrm{I}_1$	$\mathrm{I}_0^* \to \mathrm{I}_3 + 3\mathrm{I}_1$
$\mathrm{I}_0^* \to 2\mathrm{III}$	$\mathrm{I}_0^* \to \mathrm{III} + \mathrm{I}_2 + \mathrm{I}_1$	$\mathrm{I}_0^* \to \mathrm{III} + \mathrm{II} + \mathrm{I}_1$	$\mathrm{I}_0^* \to \mathrm{III} + 3\mathrm{I}_1$	$\mathrm{I}_0^* \to 3\mathrm{I}_2$
$\mathrm{I}_0^* \to 2\mathrm{I}_2 + 2\mathrm{I}_1$	$\mathrm{I}_0^* \to \mathrm{I}_2 + 2\mathrm{II}$	$\mathrm{I}_0^* \to \mathrm{I}_2 + \mathrm{II} + 2\mathrm{I}_1$	$\mathrm{I}_0^* \to \mathrm{I}_2 + 4\mathrm{I}_1$	$\mathrm{I}_0^* \to 3\mathrm{II}$
$\mathrm{I}_0^* \to 2\mathrm{II} + 2\mathrm{I}_1$	$\mathrm{I}_0^* \to \mathrm{II} + 4\mathrm{I}_1$	$\mathrm{I}_0^* \to 6\mathrm{I}_1$.		

Moreover the confluences which do not occur

$\mathrm{I}_0^* \to \mathrm{I}_5 + \mathrm{I}_1$	$\mathrm{I}_0^* \to \mathrm{I}_4 + \mathrm{I}_2$	$\mathrm{I}_0^* \to \mathrm{I}_4 + \mathrm{II}$	$\mathrm{I}_0^* \to \mathrm{IV} + \mathrm{I}_2$	$\mathrm{I}_0^* \to 2\mathrm{I}_3$
$\mathrm{I}_0^* \to \mathrm{I}_3 + \mathrm{III}$	$\mathrm{I}_0^* \to \mathrm{I}_3 + \mathrm{I}_2 + \mathrm{I}_1$,			

are obstructed by monodromy considerations.

The proof of this statement can be found by giving a construction of a parameterized family of Weierstrass models, which for $\epsilon = 0$ describe the configuration $2\mathrm{I}_0^*$ (one in the origin and one in infinity) and for $\epsilon \neq 0$ describe I_0^* (in infinity) together with the singular fibers which are supposed to flow together if $\epsilon \to 0$. Most examples can be grouped into sets of similar confluences. We shall discuss each set and give the family of Weierstrass models for one of the confluences of each group explicitly. The remaining will be discussed individually. We also explicitly discuss the monodromy obstructions.

For the confluences $\mathrm{I}_0^* \to \mathrm{IV} + \mathrm{II}$, $\mathrm{I}_0^* \to 2\mathrm{III}$ and $\mathrm{I}_0^* \to 3\mathrm{II}$ we choose ether g_2 or g_3 identically equal to zero and fix the orders of the zeros of the other function to the appropriate number.
Example: $\mathrm{I}_0^* \to \mathrm{IV} + \mathrm{II}$
For this confluence we choose

$$g_2(z) = 0 \qquad g_3(z) = z^2(z - \epsilon) \qquad \Delta(z) = -27z^4(z - \epsilon)^2.$$

Obviously we are now faced with a zero of fourth order and a zero of second order of the geometric discriminant $\Delta(z)$. Since $g_2(z) = 0$, they correspond to singular fibers of type IV and II.

[10] Suggested by Hans Duistermaat.

For the confluences $I_0^* \to I_4 + 2I_1$, $I_0^* \to I_3 + 3I_1$ and $I_0^* \to I_2 + 4I_1$ we fix for $\epsilon \neq 0$ a singular fiber of type I_b in the origin by consecutively setting the coefficients of the geometric discriminant equal to zero, while imposing that $g_2(0) \neq 0$. the verification that only singular fibers of type I_1 remain is done by calculating the discriminant of $\Delta(z)/z^b$.
Example: $I_0^* \to I_3 + 3I_1$
For this confluence we choose

$$g_2(z) = z^2 + 3\epsilon^2 \qquad g_3(z) = z^3 + \frac{1}{2}\epsilon z^2 + \epsilon^3 \qquad \Delta(z) = -\frac{1}{4}z^3(2^3 13 z^3 + 2^2 3^3 \epsilon z^2 - 3^2\epsilon^2 z + 2^3 3^3 \epsilon^3).$$

We are clearly faced with a singular fiber of type I_3 in the origin since z^3 factors $\Delta(z)$ but $g_2(0) \neq 0$. We may further derive that

$$D\left(\frac{\Delta(z)}{z^3}\right) = -\frac{3^6 109^3 \epsilon^6}{2^4},$$

where D denotes the discriminant. This implies that there are three singular fibers of type I_1 outside the origin.

For the confluences $I_0^* \to IV + 2I_1$, $I_0^* \to III + II + I_1$, $I_0^* \to III + 3I_1$, $I_0^* \to 2II + 2I_1$ and $I_0^* \to II + 4I_1$ we take for $\epsilon \neq 0$ the most generic function with the appropriate common zero of g_2 and g_3, while keeping the correct limit for $\epsilon \to 0$. We then verify that the remaining zeros of Δ are of first order.
Example: $I_0^* \to 2\,II + 2I_1$
The least complicated choice seems to be

$$g_2(z) = (z-\epsilon)(z+\epsilon) \qquad g_3(z) = z(z-\epsilon)(z+\epsilon) \qquad \Delta(z) = -(z-\epsilon)^2(z+\epsilon)^2(26z^2+\epsilon^2).$$

The geometric discriminant has clearly two second order zeros corresponding to singular fibers of type II, the first order zeros of the geometric discriminant automatically correspond to singular fibers of type I_1.

For the confluences $I_0^* \to I_3 + II + I_1$, $I_0^* \to III + I_2 + I_1$, $I_0^* \to I_2 + 2II$ and $I_0^* \to I_2 + II + 2I_1$ we first impose the right number of common zeros of g_2 and g_3, then we set the coefficients of $\Delta(z)$ equal to zero consecutively, while imposing that $g_2(0) \neq 0$. we then verify, if necessary, that the remaining zeros are of first order.
Example: $I_0^* \to III + I_2 + I_1$
We choose

$$g_2(z) = -3(z-\epsilon)(z+\epsilon) \qquad g_3(z) = (z-\epsilon)^2(2z+\epsilon) \qquad \Delta(z) = -27z^2(z-\epsilon)^3(5z+3\epsilon).$$

The geometric discriminant has a second order zero in the origin, a third order zero in ϵ and finally a first order zero in $-3\epsilon/5$. By construction both g_2 and g_3 are not equal to zero in the origin, which implies we have a singular fiber of type I_2. Clearly $(z-\epsilon)$ factors both g_2 and g_3, which gives us that the zero in ϵ corresponds to a singular fiber of type III. The remaining zero corresponds to a singular fiber of type I_1.

$I_0^* \to 3\,I_2$
For this confluence we need the discriminant to be a square of a third order polynomial in z. We therefore write $\Delta(z) = -27f(z)^2$. Since we assume that there is a singular fiber of type I_0^* at infinity, g_3 is of degree 2 and may be written as follows $g_3(z) = 3p(z)q(z)$, where p and q are linear functions. Moreover neither p nor q may divide g_2, since this would yield a singular fiber of type II. From the definition of the geometric discriminant we deduce that

$$p(z)^3q(z)^3 = (g_3(z) - f(z))(g_3(z) + f(z)).$$

Combining this with the fact that neither p nor q divides g_2 yields

$$p(z)^3 = C_1(g_3(z) - f(z)) \qquad q(z)^3 = C_2(g_3(z) + f(z)),$$

with C_1 and C_2 constants, which in turn may be absorbed in $p(z)$ and $q(z)$. We now have that

$$g_2(z) = 3p(z)q(z) \qquad g_3(z) = \frac{p(z)^3 + q(z)^3}{2} \qquad \Delta(z) = -\frac{27}{4}(p(z)^3 - q(z)^3).$$

By rescaling, a Tschirnhauser transformation and taking into account that g_2 and g_3 have no common factor we may set

$$p(z) = az + b \qquad\qquad q(z) = z + a^2 b,$$

where $a^3 \neq 1$.
We therefore choose

$$g_2(z) = 3(2z+\epsilon)(z+4\epsilon), \quad g_3(z) = \frac{(2z+\epsilon)^3 + (z+4\epsilon)^3}{2}, \quad \Delta(z) = -\frac{3^3 7^2}{4}(z-3\epsilon)^2(z^2+3\epsilon z+3\epsilon^2)^2.$$

The zeros of $\Delta(z)$ are therefore 3ϵ, $-\frac{1}{2}i(\sqrt{3}-3i)\epsilon$ and $-\frac{1}{2}i(\sqrt{3}+3i)\epsilon$, at which the value of g_2 is $3\cdot 7^2\epsilon^2$, $\frac{3}{2}(-13-3\sqrt{3}i)\epsilon^2$ and $\frac{3}{2}(-13+3\sqrt{3}i)\epsilon^2$ respectively. This implies that we indeed have $3\,\mathrm{I}_2$ for $\epsilon \neq 0$.

$\mathrm{I}_0^* \to 2\mathrm{I}_2 + 2\mathrm{I}_1$
Here we choose

$$\begin{aligned}
g_2(z) =& \epsilon(z+3\epsilon)\\
g_3(z) =& \frac{1}{2}(2z^3 + (3 2^{2/3} - 2^{1/3})\epsilon z^2 - \epsilon^2 z - 2\epsilon^3)\\
\Delta(z) =& -\frac{1}{4}z^2(-2^2 3^3 z^4 + 2^2 3^3 (2^{1/3} - 3 2^{2/3})\epsilon z^3 - 27(-16 + 18\, 2^{1/3} + 2^{2/3})\epsilon^2 z^2\\
&+ 2(110 - 27\, 2^{1/3} + 81\, 2^{2/3})\epsilon^3 z + 9(1 - 6\, 2^{1/3})^2\epsilon^4).
\end{aligned}$$

The geometric discriminant $\Delta(z)$ has a two zeros of order two, one in the origin and another in $z = \frac{1}{6}(2^{1/3} - 6\, 2^{2/3})\epsilon$ and two zeros of first order in $z = \frac{1}{6}(2\, 2^{1/3}\epsilon - 3\, 2^{2/3}\epsilon - 2\sqrt{-6\epsilon^2 + 18\, 2^{1/3}\epsilon^2 + 2^{2/3}\epsilon^2})$ and $z = \frac{1}{6}(2\, 2^{1/3}\epsilon - 3\, 2^{2/3}\epsilon + 2\sqrt{-6\epsilon^2 + 18\, 2^{1/3}\epsilon^2 + 2^{2/3}\epsilon^2})$. The value of g_2 in the two zeros of second order is $3\epsilon^2$ and $\frac{1}{6}(18 + 2^{1/3} - 6\, 2^{2/3})\epsilon^2$ respectively, which implies we have indeed constructed a confluence of $\mathrm{I}_0^* \to 2\mathrm{I}_2 + 2\mathrm{I}_1$.

$\mathrm{I}_0^* \to 6\,\mathrm{I}_1$
A generic perturbation of a singular fiber other then a singular fiber of type I_1 yields χ singular fibers of type I_1, where χ is the Euler number of the singular fiber before perturbation. This implies that almost any perturbation will do, however the following model will be convenient

$$g_2(z) = z^2 + \epsilon, \qquad g_3(z) = z^3, \qquad \Delta(z) = -(2z^2 - \epsilon)(13z^4 + 5\epsilon z^2 + \epsilon^2).$$

Again we use the discriminant to verify that the singular fibers are of type I_1

$$D(\Delta(z)) = 2^7 3^{18} 13\, \epsilon^{15}.$$

This completes our discussion of examples of confluences to singular fibers of type I_0^*

We shall now discuss the obstructions. Verification of the obstructions for a confluence of two singular fibers to a singular fiber of type I_0^*, is done in the same manner as for the confluence $\mathrm{IV} \to 2\,\mathrm{I}_2$ and is explicitly given in table 6. We use the same type of argument for $\mathrm{I}_0^* \to 2\mathrm{I}_2 + \mathrm{II}$. For the confluence $\mathrm{I}_0^* \to \mathrm{I}_3 + \mathrm{I}_2 + \mathrm{I}_1$ we use equation (3) fully.

Table 6

Confluence	Trace equation (one permutation given)	coefficient equation (one permutation given)
$\mathrm{I}_0^* \to \mathrm{I}_5 + \mathrm{I}_1$	$\mathrm{Tr}(M_{\mathrm{I}_1} A M_{\mathrm{I}_5} A^{-1}) = \mathrm{Tr}(M_{\mathrm{I}_0^*})$	$-2 = 2 - 5c^2$
$\mathrm{I}_0^* \to \mathrm{I}_4 + \mathrm{I}_2$	$\mathrm{Tr}(M_{\mathrm{I}_2} A M_{\mathrm{I}_4} A^{-1}) = \mathrm{Tr}(M_{\mathrm{I}_0^*})$	$-2 = -2(-1 + 4c^2)$
$\mathrm{I}_0^* \to \mathrm{I}_4 + \mathrm{II}$	$\mathrm{Tr}(M_{\mathrm{II}} A M_{\mathrm{I}_4} A^{-1})) = \mathrm{Tr}(M_{\mathrm{I}_0^*})$	$-2 = 1 - 4a^2 - 4ac - 4c^2$
$\mathrm{I}_0^* \to \mathrm{IV} + \mathrm{I}_2$	$\mathrm{Tr}(M_{\mathrm{IV}} A M_{\mathrm{I}_2} A^{-1}) = \mathrm{Tr}(M_{\mathrm{I}_0^*})$	$-2 = -1 - 2a^2 - 2ac - 2c^2$
$\mathrm{I}_0^* \to 2\mathrm{I}_3$	$\mathrm{Tr}(M_{\mathrm{I}_3} A M_{\mathrm{I}_3} A^{-1}) = \mathrm{Tr}(M_{\mathrm{I}_0^*})$	$-2 = 2 - 9c^2$
$\mathrm{I}_0^* \to \mathrm{I}_3 + \mathrm{III}$	$\mathrm{Tr}(M_{\mathrm{III}} A M_{\mathrm{I}_3} A^{-1}) = \mathrm{Tr}(M_{\mathrm{I}_0^*})$	$-2 = -3(a^2 + c^2)$

$\mathrm{I}_0^* \to \mathrm{I}_3 + \mathrm{I}_2 + \mathrm{I}_1$
For this confluence we shall consider not the eigenvalues but the full matrix product, so we verify that there are no $A_1,\, A_2 \in SL(2,\, \mathbb{Z})$ such that $M_{\mathrm{I}_3} A_1 M_{\mathrm{I}_2} A_1^{-1} A_2 M_{\mathrm{I}_1} A_2^{-1} = M_{\mathrm{I}_0^*}$, nor for any permutation of the monodromy matrices M_{I_1}, M_{I_2} and M_{I_3}. Taking $a_1,\, b_1,\, c_1,\, d_1$ and $a_2,\, b_2,\, c_2,\, d_2$ to be the coefficients of A_1 and A_2 we see that solving this equation with respect to a_1, c_1 and c_2 yields among others $c_1 = \pm\sqrt{2/3}$, which contradicts the assumption that $A_1,\, A_2 \in SL(2,\, \mathbb{Z})$. Solving the equations for a permutation of the monodromy matrices M_{I_1}, M_{I_2} and M_{I_3} yields $c = \pm 2/\sqrt{3}$, $c = \pm\sqrt{2/3}$ or $c = \pm\sqrt{2}$, again contradiction $A_1,\, A_2 \in SL(2,\, \mathbb{Z})$. This is sufficient to prove that this confluence can not be realized.

$\mathrm{I}_0^* \to 2\mathrm{I}_2 + \mathrm{II}$
Taking the trace of both sides of the equation $M_{\mathrm{II}} A_1 M_{\mathrm{I}_2} A_1^{-1} A_2 M_{\mathrm{I}_2} A_2^{-1} = M_{\mathrm{I}_0^*}$ and its permutations, where A_1 and A_2 are as before yields

$$\begin{aligned} -2 =& 1 - 2a_1^2 - 2a_2^2 - 2a_1c_1 + 4a_1a_2^2c_1 - 2c_1^2 - 2a_2c_2 - 4a_1^2a_2c_2 + 4a_1a_2c_1c_2 \\ &+ 4a_2c_1^2c_2 - 2c_2^2 - 4a_1^2c_2^2 - 4a_1c_1c_2^2 \end{aligned}$$

and likewise equalities for the permutations. This all these equalities imply that 3 is even, a contradiction.

This concludes the discussion of perturbations of a singular fiber of Kodaira type I_0^*.

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