Intern project: polynomial geodesic approximation

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The project concerns the fast computation of geodesics. Speeding up the calculations has applications in among other; transport problems, (Delaunay) triangulations, anisotropic meshing and curve approximation.

Problem: Splines approximating geodesics

Part 1

Suppose we have a hypersurface M that is the graph of a function $f : \mathbb{R}^n \to \mathbb{R}$, and assume that one has bounds (Λ) on the sectional curvatures (K) of the surface or equivalent bounds on the Hessian of the function defining the graph. We shall (at first) think of splines as parametrized (one real dimensional) curves whose parametrization is polynomial is of a given order.

We shall denote a spline of order k by s_k and the set of all splines of order k or lower, connecting $x, y \in \mathbb{R}^n$ by S_k. The curves $(s_k(t), f(s_k(t)))$ will be denoted by γ_k and the corresponding space by Γ_k. Denote by $L(\gamma_k)$ the length of a the curve γ_k with endpoint $(x, f(x))$ and $(y, f(y))$ and define

$$L_{\min}(\Gamma_k) = \min_{\gamma_i \in \Gamma_k} L(\gamma_i),$$

moreover let us call the curve where the minimum is attained $\gamma_k^{L(\min)}$.

Let us denote de geodesic curvature of a curve by $k_g(\gamma(t)) = |D(d\gamma(t)/dt)/dt|$, the maximum geodesic curvature along γ by $k_g^{L_\infty}(\gamma)$ and the square integral of the geodesic curvature by $k_g^{L_2}(\gamma)$. We now define

$$k_g^{L_\infty(\min)} = \min_{\gamma_i \in \Gamma_k} k_g^{L_\infty}(\gamma_i)$$

and denote the curve for which the minimum is attained by $\gamma^{k_g(L_\infty(\min))}$. Let us also define

$$k_g^{L_2(\min)} = \min_{\gamma_i \in \Gamma_k} k_g^{L_2}(\gamma_i)$$

and denote the curve for which the minimum is attained by $\gamma^{k_g(L_2(\min))}$.

Given a absolute bound on the distance (D) between $(x, f(x))$ and $(y, f(y))$ or x and y in parameter space do the following:

1. Prove that $\gamma_k^{L(\min)}$, $\gamma^{k_g(L_\infty(\min))}$ and $\gamma^{k_g(L_2(\min))}$ exist and are unique for sufficiently small D.

2. Give bounds on:

   - $|L_{\min}(\Gamma_k) - d_M((x, f(x)), (y, f(y)))|$
   - $|L(\gamma^{k_g(L_\infty(\min))}) - d_M((x, f(x)), (y, f(y)))|$
   - $|L(\gamma^{k_g(L_2(\min))}) - d_M((x, f(x)), (y, f(y)))|$
   - $k_g^{L_\infty}(\gamma_k^{L(\min)})$
   - $k_g^{L_\infty}(\gamma^{k_g(L_\infty(\min))})$
   - $k_g^{L_\infty}(\gamma^{k_g(L_2(\min))})$
   - $k_g^{L_2}(\gamma_k^{L(\min)})$
   - $k_g^{L_2}(\gamma^{k_g(L_\infty(\min))})$
   - $k_g^{L_2}(\gamma^{k_g(L_2(\min))})$
   - The angle between the tangent vectors of the geodesic and $\gamma_k^{L(\min)}$, $\gamma^{k_g(L_\infty(\min))}$ and $\gamma^{k_g(L_2(\min))}$, respectively, at the end points.

   These bounds should be in terms of the order k, the distance bound D and Λ the bound on the sectional curvature.

3. Hopefully the length $L(\gamma_k)$, curvatures $k_g^{L_\infty}(\gamma_k)$ and $k_g^{L_2}(\gamma_k)$ are convex functions in a part (near the minimum) of the space of parameters of Γ_k. Investigate this and if this is the case, try different descent methods to approximate the geodesic.

Part 2

Let M be a Riemannian manifold and ϕ_i an atlas of M with sufficient overlap between the domains of the charts. Suppose that for each of the coordinates domains we have bounds on the Christoffel symbols. Define the splines in the coordinate space. Investigates the three points mentioned above in this setting.

Part 3

Check in how far the results hold in the Pseudo-Riemannian context.

References

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