Canonical lossless state-space systems: staircase forms and the Schur algorithm.

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Lossless functions: transfer functions of stable allpass systems.

Lossless function (discrete-time): $G(z)$ square rational stable such that

$$G(z)^*G(z) \leq I, |z| > 1$$

with equality on the unit circle.

A lossless function $G(z)$ has a minimal realization s.t.

$$(D, C, B, A) \text{ balanced} \Leftrightarrow R = \begin{bmatrix} D & C \\ B & A \end{bmatrix} \text{ unitary.}$$

R determined up to an arbitrary unitary change of basis of the state-space

Balanced canonical forms with good truncation properties?

Continuous-time: B. Hanzon, R. Ober (1998)

The SISO case : a balanced canonical form.

A lossless function of degree n has a (unique) realization such that

(i) (A, b, c, d) balanced

$[b, Ab, A^2b, \ldots, A^{n-1}b]$ positive upper triangular

or equivalently

(ii) $R = \begin{bmatrix} d & c \\ b & A \end{bmatrix}$ is unitary and positive upper-Hessenberg.

B. Hanzon, R. Peeters (2000)

Parametrization.

$R = \begin{bmatrix} d & c \\ b & A \end{bmatrix}$ is unitary and positive upper-Hessenberg.

Then

$$\begin{bmatrix} \bar{\gamma}_n & \kappa_n & 0 \\ \kappa_n & -\gamma_n & 0 \\ 0 & 0 & I_{n-1} \end{bmatrix} \underbrace{\left[\begin{array}{c|c} d & c \\ \hline * & \\ & A \\ 0 & \end{array}\right]}_{R} = \begin{bmatrix} 1 & 0 \\ 0 & R_{n-1} \end{bmatrix},$$

$$|\gamma_n| < 1,\ \kappa_n = \sqrt{1 - |\gamma_n|}^2, \gamma_n = d.$$

By induction: $(R_k)_{k=n,\dots,0}$ unitary upper-Hessenberg order k and $(\gamma_k)_{k=n,\dots,1},\ \ \gamma_0 = R_0,\ |\gamma_0| = 1$.

Hessenberg form.

$$\begin{bmatrix} \gamma_n & \kappa_n\gamma_{n-1} & \kappa_n\kappa_{n-1}\gamma_{n-2} & \dots & & & \kappa_n\kappa_{n-1}\dots\kappa_1\gamma_0 \\ \kappa_n & -\bar{\gamma}_n\gamma_{n-1} & -\bar{\gamma}_n\kappa_{n-1}\gamma_{n-2} & \dots & & & -\bar{\gamma}_n\kappa_{n-1}\dots\kappa_1\gamma_0 \\ 0 & \kappa_{n-1} & -\bar{\gamma}_{n-1}\gamma_{n-2} & \dots & & & -\bar{\gamma}_{n-1}\kappa_{n-2}\dots\kappa_1\gamma_0 \\ 0 & 0 & \kappa_{n-2} & & & & -\bar{\gamma}_{n-2}\kappa_{n-3}\dots\kappa_1\gamma_0 \\ \vdots & \vdots & & \ddots & & & \vdots \\ 0 & 0 & \ddots & \kappa_2 & -\bar{\gamma}_2\gamma_1 & & -\bar{\gamma}_2\kappa_1\gamma_0 \\ 0 & 0 & \dots & 0 & \kappa_1 & & -\bar{\gamma}_1\gamma_0 \end{bmatrix} \tag{1}$$

Connection with the Schur algorithm.

Let $g_k(z) = \gamma_k + c_k(zI_k - A_k)^{-1}b_k$ be the lossless function whose realization is R_k.

Then g_k satisfies the interpolation condition

$$g_k(\infty) = \gamma_k, \quad |\gamma_k| < 1.$$

Moreover,

$$g_k(z) = \frac{\gamma_k\, z + g_{k-1}(z)}{z + \bar{\gamma}_k g_{k-1}(z)} \quad \Leftrightarrow \quad g_{k-1}(z) = \frac{(g_k(z) - \gamma_k)z}{1 - \bar{\gamma}_k\, g_k(z)}.$$

This is the Schur algorithm: lossless functions of McMillan degree n are parametrized by the interpolation values $\gamma_n, \gamma_{n-1}, \ldots, \gamma_1$ and γ_0.

Charts from a Schur algorithm.

Chart:

- $w_1, w_2, \ldots, w_n$ in the unit disk (interpolation points),
- $u_1, u_2, \ldots, u_n$ unit complex p-vectors (directions),

$G \in \mathcal{L}_n^p$, is in the chart if

$$G = G_n, \ldots, G_k \stackrel{LFT}{\longrightarrow} G_{k-1}, \ldots, G_0$$

$$G_k(1/\bar{w}_k)u_k = v_k, \|v_k\| < 1$$

G_k has degree k and G_0 is a constant unitary matrix.

Schur parameters:

$v_1, v_2, \ldots, v_n$ complex p-vectors (interpolation values) , $\|v_i\| < 1$.

$$\mathcal{W} = \{G \in \mathcal{L}_n^p; \ \ \|G_k(1/\bar{w}_k)u_k\| < 1\}, \quad \varphi : G \to (v_1, \ldots, v_n, G_0)$$

D. Alpay, L. Baratchart, A. Gombani, *Op. Th. and Appl.* (1994)

Balanced realizations from Schur parameters.

$w_1 = w_2 = \cdots = w_n = 0$

$$R = \Gamma_n \Gamma_{n-1} \cdots \Gamma_1 \Gamma_0 \Delta_1^* \Delta_2^* \cdots \Delta_n^*,$$

$$\Gamma_k = \begin{bmatrix} I_{n-k} & 0 & 0 \\ 0 & V_k & 0 \\ 0 & 0 & I_{k-1} \end{bmatrix}, \quad \Delta_k = \begin{bmatrix} I_{n-k} & 0 & 0 \\ 0 & U_k & 0 \\ 0 & 0 & I_{k-1} \end{bmatrix}$$

$$V_k = \begin{bmatrix} v_k & M_k \\ \kappa_k & -v_k^* \end{bmatrix}, \quad U_k = \begin{bmatrix} u_k & I_m - u_k u_k^* \\ 0 & u_k^* \end{bmatrix}, \quad V_0 = G_0.$$

$$\kappa_k = \sqrt{1 - \|v_k\|^2}, \quad M_k = I_m - (1 - \sqrt{1 - \|v_k\|^2}) \frac{v_k v_k^*}{\|v_k\|^2}$$

V_k and U_k unitary matrices.

B. Hanzon, M.O., R. Peeters, INRIA report (2004)

Permuted Hessenberg form.

Charts: $w_1 = w_2 = \cdots = w_n = 0$; $u_1, u_2, \ldots, u_n$ standard basis vectors.

$$\Gamma = \begin{bmatrix} v_n & M_n v_{n-1} & M_n M_{n-1} v_{n-2} & \ldots & & M_n M_{n-1} \ldots M_1 \\ \kappa_n & -v_n^* v_{n-1} & -v_n^* M_{n-1} v_{n-2} & \ldots & & -v_n^* M_{n-1} \ldots M_1 \\ 0 & \kappa_{n-1} & -v_{n-1}^* v_{n-2} & \ldots & & -v_{n-1}^* M_{n-2} \ldots M_1 \\ 0 & 0 & \kappa_{n-2} & & & -v_{n-2}^* M_{n-3} \ldots M_1 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & \ddots & \kappa_2 & -v_2^* v_1 & -v_2^* M_1 \\ 0 & 0 & \ldots & 0 & \kappa_1 & -v_1^* \end{bmatrix}$$

Δ^* is a permutation matrix.

R is a permutation of unitary positive p-upper Hessenberg matrix.

Admissible pivot structure.

$$B = \begin{bmatrix} * & * & + \\ & * & 0 \\ + & * & 0 \\ 0 & * & 0 \\ 0 & * & 0 \\ 0 & + & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} * & * & * & * & * & * & * \\ + & * & * & * & * & * & * \\ 0 & * & * & * & * & * & * \\ 0 & + & * & * & * & * & * \\ 0 & 0 & + & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & + & * & * & * \end{bmatrix}$$

B contains the pivot at position 1

A has a staircase structure

$\Leftrightarrow$ controllability matrix contains a column with pivot at position k for $k = 1, \ldots, n$.

Young diagrams of an admissible pivot structure.

$$Y = \begin{array}{|c|c|c|c|}\hline 4 & 6 & & \\ \hline 7 & & & \\ \hline 1 & 2 & 3 & 5 \\ \hline \end{array} \qquad Y_r = \begin{array}{|c|c|c|c|}\hline & & 4 & 6 \\ \hline & & & 7 \\ \hline 1 & 2 & 3 & 5 \\ \hline \end{array}$$

Y_{ij} pivot position in the ith column of $A^j B$ and 0 if no pivot.

$Y_{i,j} = 0 \Leftrightarrow Y_{i,k} = 0$, for $k > j$

d_i, ith dynamical index, number of non-zero entries in the ith row.

$$d_1 + d_2 + \cdots + d_m = n.$$

Its numbering is fully determined by a permutation of the row of Y_r

admissible pivot structure $\longrightarrow$ chart

Minimal atlas: $(d_1, d_2, \ldots, d_m) \longrightarrow$ chart

Connection with the Schur algorithm.

$$Y = \begin{array}{|c|c|c|c|}\hline 3 & 5 & & \\ \hline 6 & & & \\ \hline 1 & 2 & 4 & 7 \\ \hline \end{array} \qquad Y_r = \begin{array}{|c|c|c|c|}\hline & & 3 & 5 \\ \hline & & & 6 \\ \hline 1 & 2 & 4 & 7 \\ \hline \end{array}$$

For $k = 1, \ldots, n$, there exist $(i(k), j(k))$ such that $Y_{i(k),j(k)} = k$

Choose $u_{n+1-k} = e_{i(k)}$ the $i(k)$th standard basis vector:
$(v_1, \ldots, v_n, G_0) \rightarrow R$ whose controllability matrix has the pivot structure given by Y

$$\begin{aligned} (u_7, u_6, \ldots u_1) &= (e_{i(1)}, e_{i(2)}, e_{i(3)}, e_{i(4)}, e_{i(5)}, e_{i(6)}, e_{i(7)}) \\ &= (e_3, e_3, e_1, e_3, e_1, e_2, e_3) \end{aligned}$$

Schur parameters parametrize admissible pivot structures