SAW filters and Darlington synthesis

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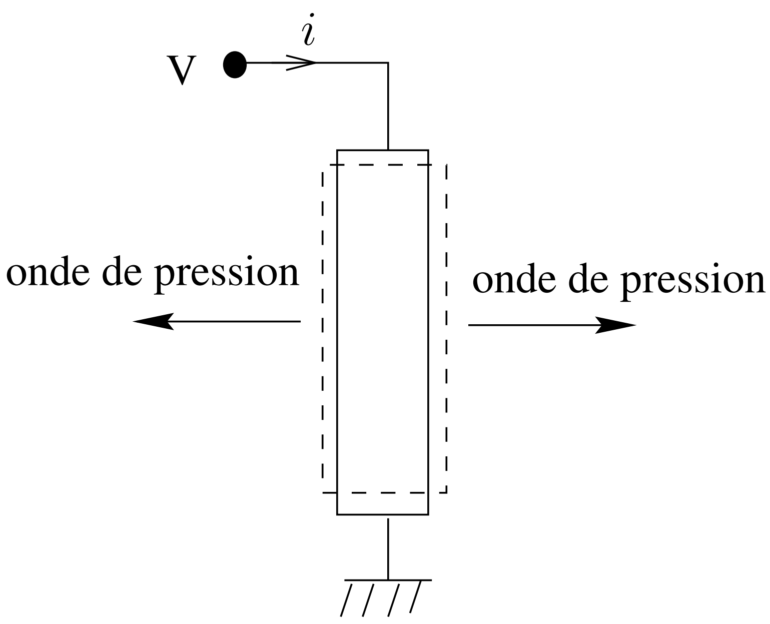
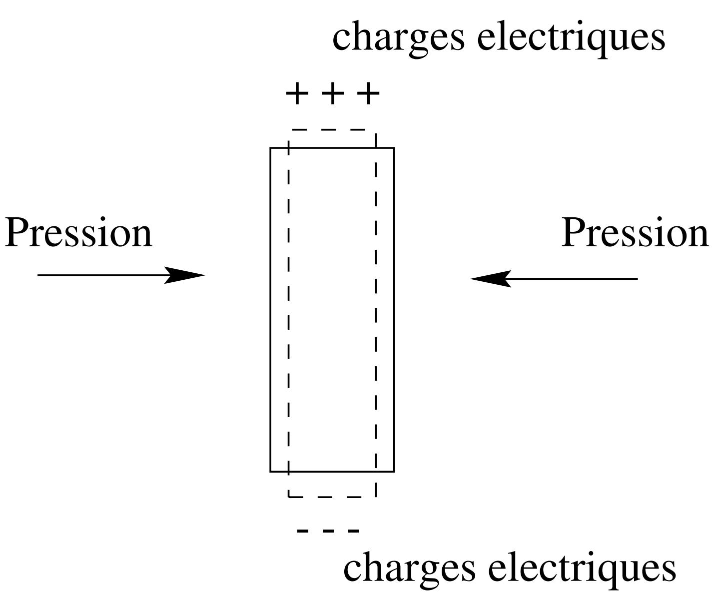
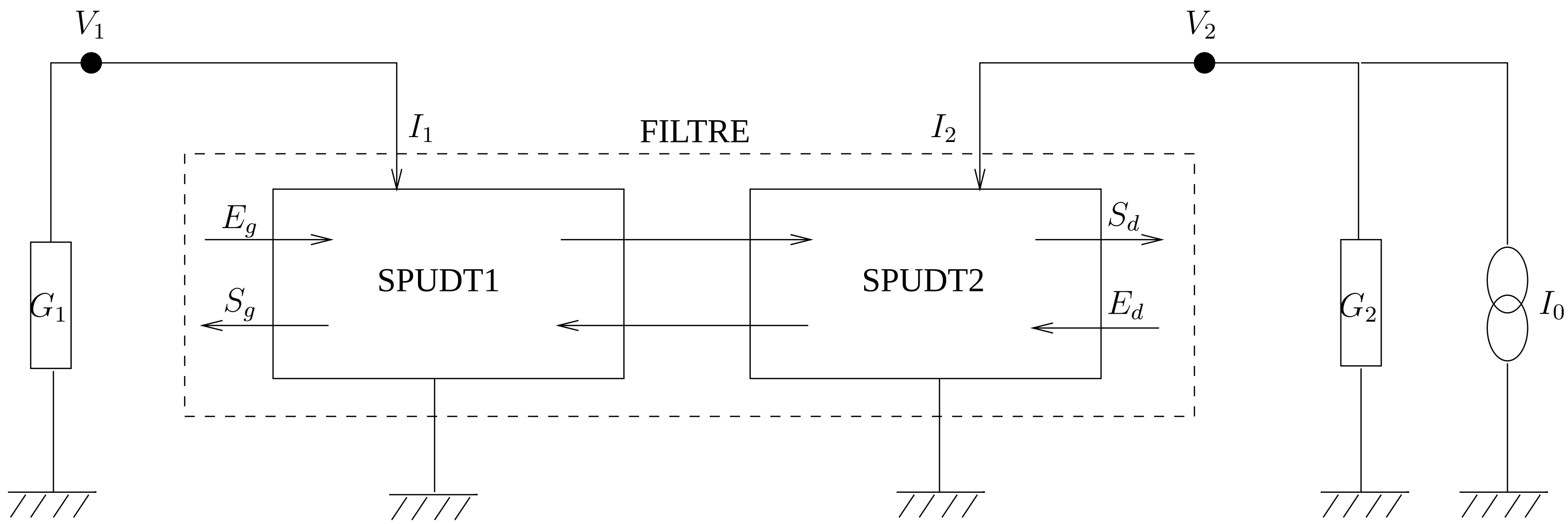
Piezoelectricity

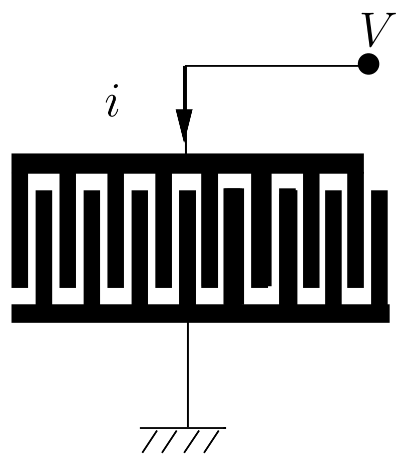
Piezoelectricity is the ability of certain crystals to generate a voltage in response to applied mechanical stress. The piezoelectric effect is reversible...





A SAW Filter

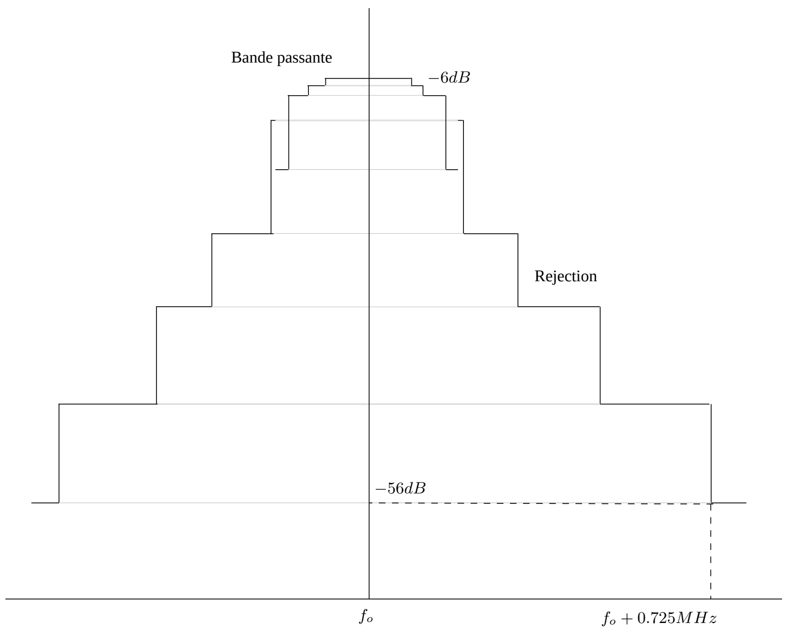




SAW Filters Design

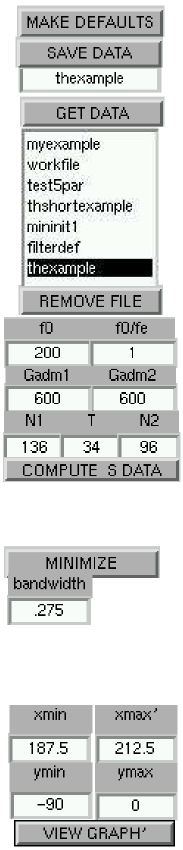
The filter is specified in the frequency domain in terms of amplitude and phase of the electrical transfer function (power transmission):

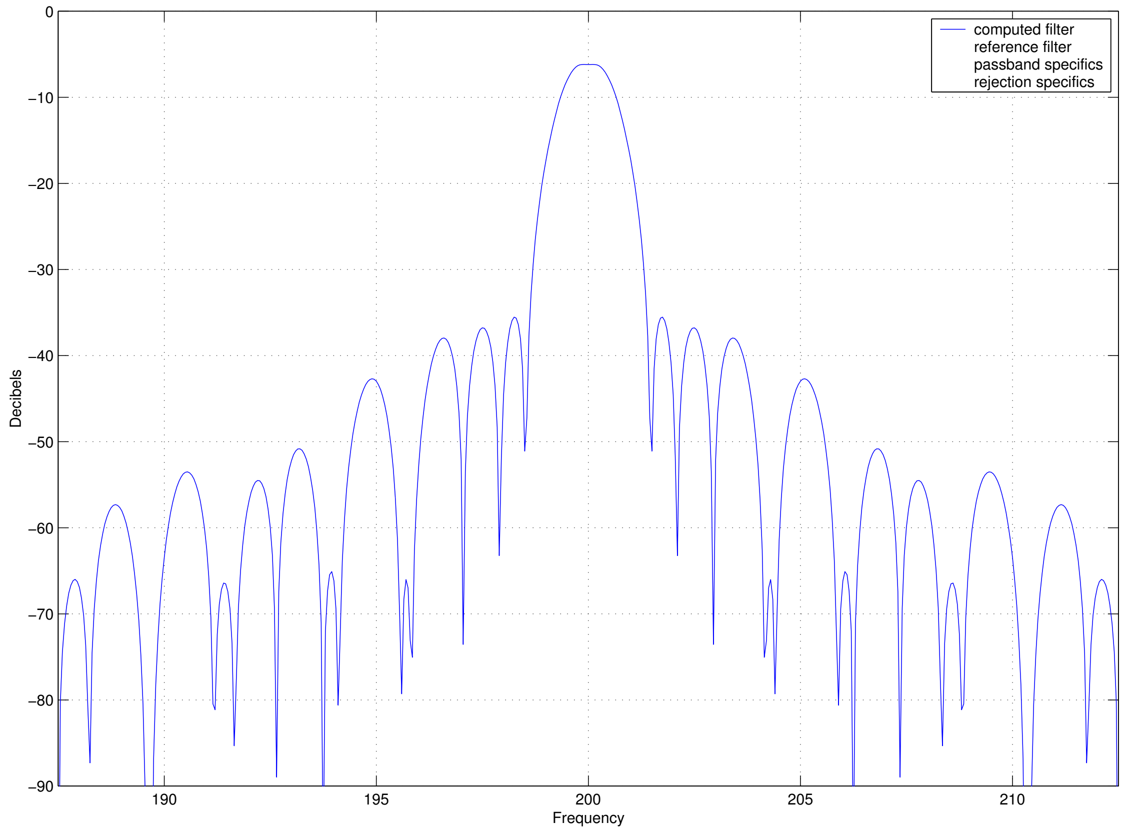
$$T = 2\sqrt{G_1 G_2}\,\frac{V_1}{I_0}.$$



J. M. Hodé, J. Desbois, P. Dufilié, M. Solal, P. Ventura (1995)

An example of filter





Mixed Matrix Representation

$$\left(\begin{array}{c}\mathcal{S}\\ \hline I\end{array}\right) = \left(\begin{array}{c|c} M & \alpha \\ \hline \beta & Y\end{array}\right)\left(\begin{array}{c}\mathcal{E}\\ \hline V\end{array}\right)$$

$\mathcal{S} = \begin{bmatrix} S_g \\ S_d \end{bmatrix}$ outgoing waves, $I = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$ currents

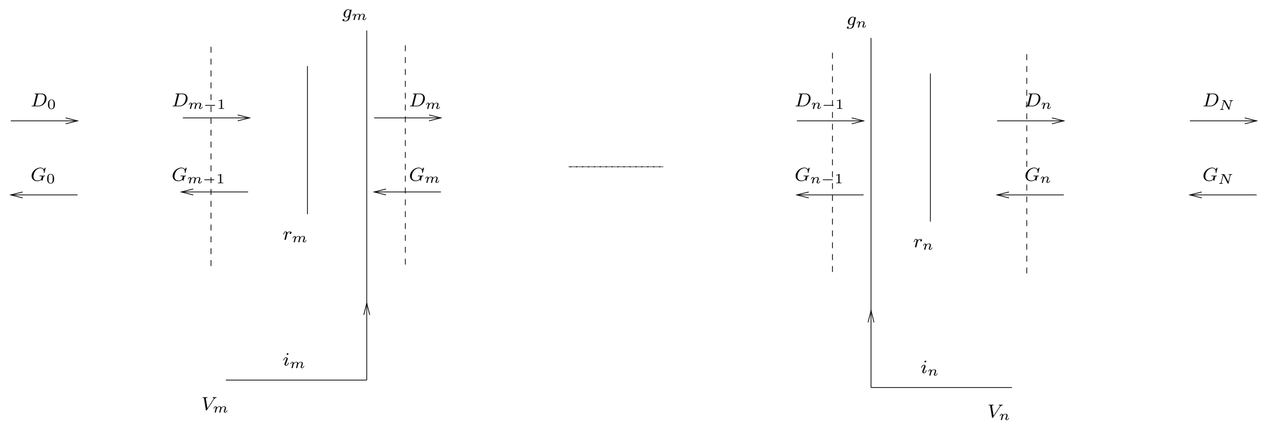
$\mathcal{E} = \begin{bmatrix} E_g \\ E_d \end{bmatrix}$ incoming waves, $V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$ voltages

M pure acoustic part, α electro-acoustic part, Y admittance, 2×2 matrices

The electrical transfer function T is the entry $(1, 2)$ of

$$S = (Y + I)^{-1}(I - Y)$$

The physical parameters



r_n reflection parameters, g_n electroacoustic parameters, $t_n = \sqrt{1 - r_n^2}$

$z = e^{2i\pi f\tau}$, τ length of the cell , f wave frequency

$\delta = e^{2i\pi f\Delta\tau}$, $\Delta\tau$ distance of the source to the boundary, $\delta \approx$ cstant

Diffraction matrix: $$\begin{bmatrix} G_{n-1} \\ D_n \end{bmatrix} = \begin{bmatrix} -i\,r_n/z & t_n/z \\ t_n/z & -i\,r_n/z \end{bmatrix} \begin{bmatrix} D_{n-1} \\ G_n \end{bmatrix}$$

Electroacoustic relations: $$\begin{cases} i_m = i\,g_m\,(D_m\delta + G_m\bar{\delta}), \\ i_n = i\,g_n\,(D_{n-1}\bar{\delta} + G_{n-1}\delta) \end{cases}$$

Mathematical properties

M, α and Y are rational functions of the complex variable z analytic outside the disk (stable).

acoustic waves $\leftrightarrow$ currents and voltages: $\left\{ \begin{array}{lcl} V & = & \frac{\mathcal{E}'+\mathcal{S}'}{\sqrt{2}} \\ I & = & \frac{\mathcal{E}'-\mathcal{S}'}{\sqrt{2}} \end{array} \right.$

Global matrix of the filter:

$$\left[\frac{\mathcal{S}}{\mathcal{S}'} \right] = \left(\begin{array}{c|c} M - \alpha(Y+I)^{-1}\beta & \sqrt{2}\,\alpha(Y+I)^{-1} \\ \hline -\sqrt{2}\,(Y+I)^{-1}\beta & \underbrace{(Y+I)^{-1}(I-Y)}_{S} \end{array} \right) \left[\frac{\mathcal{E}}{\mathcal{E}'} \right]$$

reciprocity $\Rightarrow$ symmetric matrix

non lossy system $\Rightarrow$ unitary matrix

The matrix of a SAW filter is symmetric and lossless (analytic outside the disk, takes unitary values on the circle)

Poles, zeros, McMillan degree

Smith-McMillan form: $W(s) \in \mathbb{C}^{p\times p}$

$$W(s) \;=\; U(s)\,\mathrm{diag}\left\{\frac{\phi_1}{\psi_1}(s),\ldots,\frac{\phi_r}{\psi_r}(s),0,\ldots,0\right\}\,V(s),$$

$U(s)$ and $V(s)$ unimodular polynomial matrices (inverse polynomial)

$\phi_1|\phi_2|\cdots|\phi_r$ and $\psi_r|\psi_{r-1}|\cdots|\psi_1$ polynomials

polynomial of zeros: $\phi = \prod_{1\le j\le r} \phi_j$

polynomial of poles: $\psi = \prod_{1\le j\le r} \psi_j$

McMillan degree: $\deg\psi$ number of poles

Partial multiplicities at w: $\nu_1 \le \nu_2 \le \ldots \le \nu_r$

Realizations

A rational matrix function $W(z)$, finite at infinity, admits a realization

$$W(z) \;=\; C\,(z\,I - A)^{-1}\,B \;+\; D$$

- easy to built one, not unique :
  (A, B, C, D) realization
  $\Rightarrow$ for all T invertible, $(TAT^{-1}, TB, CT^{-1}, D)$ realization
- minimal realization: size n of A minimal   McMillan degree$= n$

Realizations and pole-zero structure

$$W(z) \; = \; C\,(z\,I - A)^{-1}\,B \; + \; D$$

poles of $W(z)$ $\leftrightarrow$ eigenvalues of A

zeros of $W(z)$ $\leftrightarrow$ poles of $W(z)^{-1}$

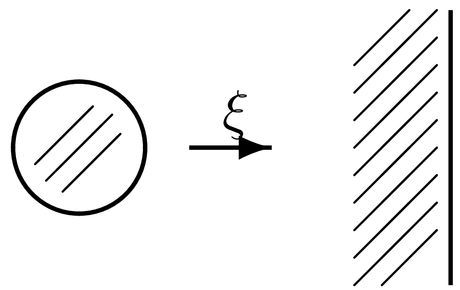
$$W(z)^{-1} \; = \; -\,D^{-1}C\,(z\,I - (A - BD^{-1}C))^{-1}\,BD^{-1} \; + \; D^{-1}$$

zeros of $W(z)$ $\leftrightarrow$ eigenvalues of $A - BD^{-1}C$

partial multiplicities $\leftrightarrow$ size of Jordan blocks

From the disk to the left half-plane

$$\xi(z) = s = \frac{1+z}{1-z}$$



$W_d(z)$ associated with

$$\left\{ \begin{array}{rcccc} x_{k+1} & = & A_d\, x_k & + & B_d\, u_k \\ y_k & = & C_d\, x_k & + & D_d\, u_k \end{array} \right.$$

W_d stable $\Leftrightarrow \sigma(A_d) \subset \mathbb{D}$

$W_c(s)$ associated with

$$\left\{ \begin{array}{rcccc} x'(t) & = & A_c\, x(t) & + & B_c\, u(t) \\ y(t) & = & C_c\, x(t) & + & D_c\, u(t) \end{array} \right.$$

W_c stable $\Leftrightarrow \sigma(A_c) \subset \mathbb{C}^-$

McMillan degree = state-space dimension

From the disk to the left half-plane

$$
\begin{aligned}
A_c &= (A_d - I)(A_d + I)^{-1} \\
B_c &= \sqrt{2}(A_d + I)^{-1} B_d \\
C_c &= \sqrt{2} C_d (A_d + I)^{-1} \\
D_c &= D_d - C_d (A_d + I)^{-1} B_d
\end{aligned}
$$

preserves the McMillan degree,

maps lossless functions onto lossless functions...

Back to our problem

N number of cells $\to$ $2N$ physical parameters $(r_k, g_k),\ k = 1, \dots N$

$$G = \begin{bmatrix} \star & \star \\ \star & S \end{bmatrix}, \quad \deg G = 2N, \quad \deg S = 2N - 2$$

G lossless symmetric

Goal: characterize the class of matrix S. Find a reference model.

Darlington synthesis

$S(s)$ $p \times p$ Schur function strictly contractive at ∞:
- $S(s)S(s)^* \leq I_p, \quad s \in \mathbb{C}^+ \quad \mathbb{C}^+ = \{s \in \mathbb{C}, \Re s > 0\}$
- $\|S(\infty)\| < \infty$

We are looking for

$$G = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S \end{bmatrix},$$

$G(s)$ lossless (analytic in $\mathbb{C}^+$ and unitary on the imaginary axis) and

$$\deg G = \deg S = n.$$

Notations: $S = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right)$ minimal realization, $S^*(s) = S(-\bar{s})^*$

The scalar case

lossless completions of $S = \frac{p}{q}$

$$\begin{bmatrix} -\frac{p^*}{q} & \frac{r^*}{q} \\ \frac{r}{q} & \frac{p}{q} \end{bmatrix}, \quad qq^* - pp^* = rr^* \text{ spectral factorization.}$$

- symmetric if $r^* = r$ i.e. $qq^* - pp^* = r^2$

- if r such that $\frac{r^*}{r}$ stable $\rightarrow \begin{bmatrix} -\frac{p^*}{q} & \frac{r^*}{q} \\ \frac{r}{q} & \frac{p}{q} \end{bmatrix} \begin{bmatrix} \frac{r^*}{r} & 0 \\ 0 & I \end{bmatrix}$ symmetric.

- if moreover $r = r_1^2 r_2 \rightarrow \begin{bmatrix} -\frac{p^*}{q} & \frac{r_1 r_1^* r_2^*}{q} \\ \frac{r_1 r_1^* r_2}{q} & \frac{p}{q} \end{bmatrix} \begin{bmatrix} \frac{r_2^*}{r_2} & 0 \\ 0 & I \end{bmatrix}$ symmetric.

The bounded real lemma

The function $S(s)$ is a Schur function strictly contractive at ∞ iff there exist P,b, d_{21} such that

$$AP + PA^* + BB^* + bb^* = 0$$

$$PC^* + BD^* + bd_{21}^* = 0$$

$$DD^* + d_{21}d_{21}^* = I$$

and P is positive definite (hermitian). Then

$$S_{21} = \left(\begin{array}{c|c} A & b \\ \hline C & d_{21} \end{array}\right)$$

is a left spectral factor of $I_p - S(s)S^*(s)$

$$I_p - S(s)S^*(s) = S_{21}(s)S_{21}^*(s)$$

Associated Riccati equation

Let $d_{21} = (I - DD^*)^{1/2}$, then $b = -(PC^* + BD^*)d_{21}^{-1}$ and P is solution to the Riccati equation

$$\mathcal{R}(P) = P\gamma P + \alpha P + P\alpha^* + \beta = 0$$

$$\begin{cases} \alpha & = & A + BD^*(I - DD^*)^{-1}C, \\ \beta & = & B(I - D^*D)^{-1}B^*, \\ \gamma & = & C^*(I - DD^*)^{-1}C, \end{cases}$$

$$\mathcal{A} = \begin{bmatrix} -\alpha^* & -\gamma \\ \beta & \alpha \end{bmatrix}, \ \ \beta^* = \beta, \ \ \gamma^* = \gamma$$

$\mathcal{A}$ Hamiltonian $\longrightarrow$ eigenvalues symmetric w.r.t. imaginary axis $(\lambda, -\bar{\lambda})$
$\mathcal{A}$ dynamic matrix of $(I - SS^*)^{-1} \longrightarrow$ zeros of $I - SS^*$

Lossless completions at degree n

There is a one to one correspondance $P \to G_P$

P hermitian solution to $\mathcal{R}(P)$

G_P lossless completion of S of degree n with prescribed value at ∞

$$G_P = \left(\begin{array}{c|c} A & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right) = \left(\begin{array}{c|cc} A & b & B \\ \hline c & d_{11} & d_{12} \\ C & d_{21} & D \end{array}\right)$$

$$d_{21} = (I - DD^*)^{1/2}, \quad d_{12} = (I - D^*D)^{1/2}, \quad d_{11} = -D^*$$

$$c = -(I - D^*D)^{-1/2}(B^*P^{-1} + D^*C)$$

$$b = -(PC^* + BD^*)(I - DD^*)^{-1/2}$$

All lossless completions

All rational lossless completions of a Schur function S, strictly contractive at ∞, can be written on the form

$$\begin{bmatrix} R & 0 \\ 0 & I \end{bmatrix} G_P \begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix}$$

where R, Q are lossless, and G_P is a minimal degree inner extension of S obtained from a solution of $\mathcal{R}(P) = 0$.

More on the Riccati equation

$$\mathcal{R}(P) = P\gamma P + \alpha P + P\alpha^* + \beta = 0, \quad \mathcal{A} = \begin{bmatrix} -\alpha^* & -\gamma \\ \beta & \alpha \end{bmatrix}$$

Important similarity relation

$$\begin{bmatrix} I & 0 \\ -P & I \end{bmatrix} \mathcal{A} \begin{bmatrix} I & 0 \\ P & I \end{bmatrix} = \begin{bmatrix} -(\alpha + P\gamma)^* & -\gamma \\ 0 & \alpha + P\gamma \end{bmatrix}$$

$$\sigma(\mathcal{A}) = \sigma(\alpha + P\gamma) \cup \sigma(-(\alpha + P\gamma)^*)$$

- S_{21}^{-1} has dynamic matrix $A - bd_{21}^{-1}C = \alpha + P\gamma$
- S_{12}^{-1} has dynamic matrix $A - Bd_{12}^{-1}c = -P(\alpha + P\gamma)^* P^{-1}$

The following statements are equivalent

(i) the Riccati equation has an hermitian solution

(ii) all pure imaginary eigenvalues of $\mathcal{A}$ have even multiplicity $(2n_0)$

The symmetry assumption

S is symmetric: $S = S^T$ $\Rightarrow$ a symmetric (complex) realization exists

$$S = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right), \quad A = A^T, \quad B = C^T, \quad D = D^T$$

Then,

$$\left\{ \begin{array}{ccc} \alpha & = & \alpha^T \\ \beta & = & \gamma^T \end{array} \right.$$

$$\mathcal{R}(P) = 0 \Leftrightarrow \mathcal{R}(P^{-T}) = 0 \text{ and } G_{P^{-T}} = G_P^T$$

Symmetric unitary completions

$$\Sigma_P = G_P \begin{bmatrix} S_{21}^{-1} S_{12}^T & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} S_{11} S_{21}^{-1} S_{12}^T & S_{12} \\ S_{12}^T & S \end{bmatrix}$$

symmetric unitary completion of S.

$Q_P = S_{21}^{-1} S_{12}^T$ is unitary and has (non minimal) realization

$$Q_P = \left(\begin{array}{c|c} \alpha + P\gamma & (P^{-T} - P) C^* d_{21}^{-1} \\ \hline -d_{21}^{-1} C & I \end{array} \right),$$

- $\deg Q_P = \mathrm{rank}(P^{-T} - P)$
- Q_P lossless iff $P^{-T} - P$ positive semi-definite. Then Σ_P is lossless and has degree $n + \deg Q_P$
- $\kappa \leq \deg Q_P$

2κ number of eigenvalues of $\mathcal{A}$ with odd multiplicity

Proof. Let $\Gamma = P^{-T} - P$ and $Z = \alpha + P\gamma$. Γ is hermitian and satisfies

$$\mathcal{R}(P^{-T}) - \mathcal{R}(P) = Z\Gamma + \Gamma Z^* + \Gamma\gamma\Gamma = 0$$

Write (SVD) $\Gamma = V \begin{bmatrix} 0 & 0 \\ 0 & \Gamma_0 \end{bmatrix} V^*$ and $V^* Z V = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}$

$V = [V_1 \ V_2]$ unitary, Γ_0 real and diagonal.

Then, $Z_{12} = 0$ and $Z_{22}^* \Gamma_0^{-1} + \Gamma_0^{-1} Z_{22} + V_2^* \gamma V_2 = 0$

$$Q_P = \left(\begin{array}{c|c} Z & V^* \Gamma C^* d_{21}^{-1} \\ \hline -d_{21}^{-1} C V & I \end{array} \right) = \left(\begin{array}{c|c} Z_{22} & \Gamma_0 V_2^* C^* d_{21}^{-1} \\ \hline -d_{21}^{-1} C V_2 & I \end{array} \right),$$

number poles of Q_P in $\mathbb{C}^+$ = number negative eigenvalues of Γ_0^{-1}

$\deg Q_P = \operatorname{rank} \Gamma$, Q_P lossless iff Γ_0 positive definite

$$\sigma(\mathcal{A}) = \sigma(Z) \cup \sigma(-PZ^*P^{-1}), \quad Z = \alpha + P\gamma$$

$$V^*ZV = \begin{bmatrix} V_1^*ZV_1 & V_1^*ZV_2 \\ V_2^*ZV_1 & V_2^*ZV_1 \end{bmatrix} = \begin{bmatrix} Z_{11} & 0 \\ Z_{21} & Z_{22} \end{bmatrix}$$

$$\widetilde{Z} = -PZ^*P^{-1} = -P(\alpha^* + \gamma P)^*P^{-1} = (\alpha P + \beta)^*P^{-1} = \alpha + \beta P^{-1}$$

We have that $V_1^*(P - P^{-T}) = 0$, or else $P^T\overline{V}_1 = P^{-1}V_1$:

$$\widetilde{Z}\overline{V}_1 = (\alpha + \beta P^T)\overline{V}_1 = Z^T\overline{V}_1$$

$$V^T\widetilde{Z}\overline{V} = \begin{bmatrix} V_1^T\widetilde{Z}\overline{V}_1 & V_1^T\widetilde{Z}\overline{V}_2 \\ V_2^T\widetilde{Z}\overline{V}_1 & V_2^T(-PZ^*P^{-1})\overline{V}_2 \end{bmatrix} = \begin{bmatrix} Z_{11}^T & \star \\ 0 & \star \end{bmatrix}$$

Then $\dim(\ker(P^{-T} - P)) \leq n - \kappa \Rightarrow \mathrm{rank}(P^{-T} - P) \geq \kappa$

A symmetric lossless completion

Partial order: $P_1 \leq P_2$ iff $P_2 - P_1$ is positive semi-definite

- there exists a maximal solution $\hat{P}$ and a minimal solution $\check{P}$

$$
\begin{aligned}
\sigma(\alpha + \hat{P}\gamma) &\subset \overline{\mathbb{C}}^{+} \\
\sigma(\alpha + \check{P}\gamma) &\subset \overline{\mathbb{C}}^{-}
\end{aligned}
$$

- $\ker(\hat{P} - \check{P}) = \sigma(\alpha + \hat{P}\gamma) \cap i\mathbb{R}$ has dimension n_0

$2n_0$ number of eigenvalues of $\mathcal{A}$ on $i\mathbb{R}$

$\check{P}$ minimal solution $\Rightarrow$ $\Sigma_{\check{P}}$ symmetric lossless completion of S.

Moreover, $\check{P}^{-T} = \hat{P} \Rightarrow \ker(\check{P}^{-T} - \check{P})$ has dimension n_0 and thus $\Sigma_{\check{P}}$ has degree $2n - n_0$

Symmetric Potapov factorization

Let $T(s)$ be a symmetric lossless function. Suppose that $T(s)$ has a zero ω (in $\mathbb{C}^+$) with algebraic multiplicity greater than 1. Then, there exists an elementary Blascke factor

$$B_{\omega,u} = I_p + (b_\omega - 1)uu^*, \quad b_\omega(s) := \frac{s-\omega}{s+\bar{\omega}},$$

and u unit vector in $\mathbb{C}^p$, such that

$$T(s) = B_{\omega,u}(s)R(s)B_{\omega,u}(s)^T$$

and $R(s)$ is lossless and symmetric.

Potapov factorization

$$T(s) = B_{\omega_1,u_1} B_{\omega_2,u_2} \dots B_{\omega_n,u_n}$$

ω zero: $\exists u, \quad Q(\omega)u = 0$

Taylor series: $Q(s) = Q(\omega) + (s - \omega)Q_\omega(s)$

SVD: $Q(\omega) = V \mathrm{diag}(0, \dots, 0, \lambda_1, \dots, \lambda_r)U^*$
V and U unitary; we can choose $U = [u \ \cdots]$

$$B(s) = U \mathrm{diag}\left(\frac{s - \omega}{s + \bar{\omega}}, 1, \dots, 1\right) U^* = B_{\omega,u}(s)$$

Then $Q_1(s) = Q(s)B(s)^{-1}$ is analytic in $\mathbb{C}^+$

$Q_1(s) = V \mathrm{diag}(0, \dots)\mathrm{diag}(\frac{s+\bar{\omega}}{s-\omega}, 1, \dots)U^* + Q_\omega(s)U \mathrm{diag}(s + \bar{\omega}, 1, \dots)U^*$

An idea of the proof:

ω has a partial multiplicity ≥ 2

$$\begin{bmatrix} b_\omega(s)^2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} b_\omega(s) & 0 \\ 0 & 1 \end{bmatrix}^2$$

all the partial multiplicities of ω are 1

$$\begin{bmatrix} b_\omega(s) & 0 \\ 0 & b_\omega(s) \end{bmatrix} = \begin{bmatrix} 1+\frac{1}{2}(b_\omega(s)-1) & -\frac{i}{2}(b_\omega(s)-1) \\ \frac{i}{2}(b_\omega(s)-1) & 1+\frac{1}{2}(b_\omega(s)-1) \end{bmatrix}$$

$$\begin{bmatrix} 1+\frac{1}{2}(b_\omega(s)-1) & \frac{i}{2}(b_\omega(s)-1) \\ -\frac{i}{2}(b_\omega(s)-1) & 1+\frac{1}{2}(b_\omega(s)-1) \end{bmatrix}$$

Minimal symmetric lossless completion

Let S be a symmetric Schur function strictly contractive at ∞. Assume that $\mathcal{A}$ has κ eighenvalues in $\mathbb{C}^+$ with odd algebraic multiplicity.

Then S has a symmetric lossless completion of degree $n+\kappa$. This completion of S has minimal degree among all the symmetric completions of S.

We start with the "maximal” lossless completion $\Sigma_{\check{P}}$

$\sigma(\alpha + \check{P}\gamma) \subset \overline{\mathbb{C}}^{-}$ $\longrightarrow$ poles of $Q_{\check{P}} = \sigma(\mathcal{A}) \cap \mathbb{C}^{-}$
eigenvalues of $\mathcal{A}$ in $\mathbb{C}^{+}$ $\leftrightarrow$ zeros of $Q_{\check{P}}$ $(n - n_0 = \kappa + 2\ell)$

Let ω be an eigenvalue of $\mathcal{A}$ in $\mathbb{C}^{+}$ of multiplicity greatest than 1. Then, we can perform a symmetric Potapov factorization

$$\Sigma_{\check{P}} = \begin{bmatrix} B_{w,x} & 0 \\ 0 & I_p \end{bmatrix} \Sigma_1 \begin{bmatrix} B_{w,x}^T & 0 \\ 0 & I_p \end{bmatrix}$$

where Σ_1 is a lossless completion of S of degree $2n - n_0 - 2$. We can make ℓ iterations, so that we finally obtain a lossless completion of degree $2n - n_0 - 2\ell = n + \kappa$.

Σ symmetric lossless completion:

$$\Sigma = \begin{bmatrix} R & 0 \\ 0 & I \end{bmatrix} G_P \begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix}$$

By symmetry: $(RS_{12})^T = S_{21}Q \Leftrightarrow Q_P = S_{21}^{-1}S_{12}^T = Q\bar{R}$

$$\kappa \leq \deg Q_P = \deg Q\bar{R} \leq \deg Q + \deg R,$$

and finally,

$$n + \kappa \leq n + \deg Q + \deg R = \deg \Sigma.$$

Conclusion

Global matrix of the SAW filter

$$G = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S \end{bmatrix}, \quad \deg G = 2N, \quad \deg S = 2N - 2$$

entries of G have either real or pure imaginary coefficients: double zero at infinity that cannot be reduced

$\longrightarrow$ real case under study

$p \times p$ symmetric lossless $\deg = 2N \rightarrow N(p+1) = 5N$ parameters

$2N$ physical parameters ... extra constraints:

- M acoustic matrix: parity conditions ($2N$ conditions)
- one source in each cell (N conditions)

New approach: find a "reference model" such that S lossless ($r_1 \rightarrow 1$ and $r_N \rightarrow 1$)

Diffraction matrices and Schur polynomials

$\phi_n,\ \psi_n$ Schur polynomials of degree n; ϕ_n stable.

$$\left\{\begin{array}{rcl} \phi_{n+1}(z) & = & z\,\phi_n(z) + r_{n+1}\,\psi_n(z) \\ \psi_{n+1}(z) & = & r_{n+1}\,z\,\phi_n(z) + \psi_n(z) \end{array}\right. \qquad \phi_1(z) = z;\ \psi_1(z) = r_1 z.$$

The matrix M

$$M = \frac{1}{\phi_N(z^2)} \begin{bmatrix} -\,i\,z^{-1}\,\widetilde{\psi}_N(z^2) & P_N\,z^N \\ P_N\,z^N & -\,i\,z\,\psi_N(z^2) \end{bmatrix}$$

$$\widetilde{\phi}_n(z) = z^n\,\phi_n(1/z),\ \widetilde{\psi}_n(z) = z^n\,\psi_n(1/z),\ P_n = t_1\ldots t_n,$$

is rational lossless of degree $2N$.

The Structure of α and Y

$$\alpha \;=\; -\,\frac{\delta}{\phi_N(z^2)}$$

$$\begin{bmatrix} i\sum_{\mathrm{T1}} g_n\, P_n\, z^n\, \left[\phi^R_{N-n}(z^2)\, -\, \psi^R_{N-n}(z^2)/z\right] & \sum_{\mathrm{T2}} g_{n+1}\, P_n\, z^n\, \left[\phi^R_{N-n}(z^2)\, +\, \psi^R_{N-n}(z^2)/z\right] \\ \sum_{\mathrm{T1}} g_n\, \frac{P_N\, z^N}{P_n\, z^n}\, \left[\phi_n(z^2)\, +\, \psi_n(z^2)/z\right] & i\sum_{\mathrm{T2}} g_{n+1}\, \frac{P_N\, z^N}{P_n\, z^n}\, \left[\phi_n(z^2)\, -\, \psi_n(z^2)/z\right] \end{bmatrix}$$

$$Y(z) = \frac{X(z)}{\Phi_N(z^2)}$$

$$X(z) = \begin{bmatrix} X_{11} & iX_{12} \\ iX_{12} & X_{22} \end{bmatrix} \text{poly matrix} \begin{cases} g_n \\ \phi_n(z^2) \pm \psi_n(z^2)/z \\ \phi^R_n(z^2) \pm \psi^R_n(z^2)/z \end{cases}$$