Parametrization of rational lossless matrices with applications to linear system theory.

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Outline

- Lossless matrices in linear system theory
- Parametrization issues
  B. Hanzon, R. Peeters
- Rational approximation and the software RARL2
  APICS, J. P. Marmorat
- Completion issues: the symmetric case
  L. Baratchart, P. Enqvist, A. Gombani

Outline

Rational approximation and model reduction

General problem

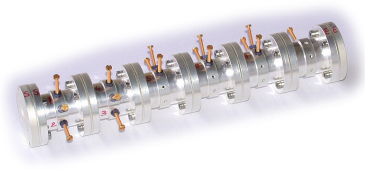
Given F rational stable transfer function, find $\hat{H}$ s.t.

$$\|F - \hat{H}\| = \min_{H} \|F - H\|$$

over the set of H rational stable of order $\leq n$

- Hankel norm
  [Adamyan, Arov, Krein, 1971]; [Kung, Lin, 1981]; [Glover, 1984]
- L^2 norm (Hilbert structure, differentiability)
  [Meier, Luenberger,1967]; [Wilson, 1970];
  [Rosencher,1978]; [Ruckebusch, 1978]

Key application

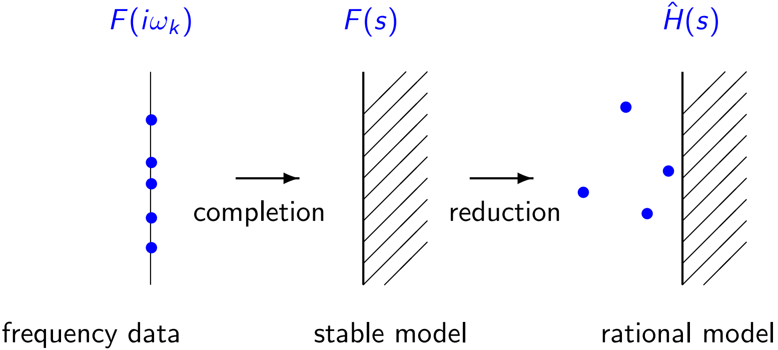


Identification of microwave filters in telecomunication systems

- Electrical model $\rightarrow \hat{H}(z)$ scattering 2×2 matrix, order ≈ 10
- Band-limited frequency data (800) $F(i\omega_k)$

Long standing collaboration (CNES, XLIM)
Software PRESTO-HF (Seyfert)

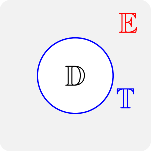
Identification vs Model Reduction



- $\longrightarrow$ Completion: analytic approximation
  Baratchart, Leblond, Seyfert ...
- $\longrightarrow$ Reduction: rational approximation
- $\longrightarrow$ Rational approximation highly depend on completion !

The discrete-time/disk framework

$$\|F\|_2^2 = \frac{1}{2\pi}\mathbf{Tr}\left\{\int_0^{2\pi} F(e^{it})F(e^{it})^* dt\right\}.$$



$$L^2(\mathbb{T}) = H_2 \oplus H_2^{\perp}$$

H_2	analytic in $\mathbb{D}$	anti-stable
$H_2^{\perp}$	analytic in $\mathbb{E}$	strictly proper stable

Discrete-time inner:

A rational matrix $Q(z)$ is inner iff it is analytic in $\mathbb{D}$ and unitary for $z \in \mathbb{T}$.

They appear in the description of LTI systems in connection with the Beurling-Lax theorem.

Fuhrmann, *Linear systems and operators in Hilbert spaces*, 1981

Douglas-Shapiro-Shield factorization

Douglas-Shapiro-Shield, 1970

Any rational matrix function $T \in H_2^\perp$ can be written as

$$T = PQ^{-1}$$

T rational stable, P anti-stable, Q inner

where Q^{-1} has same degree than T (coprime).

- T and Q^{-1} same pair (A, B) (Q brings the pole structure)
- Right generalization of a polynomial denominator

Their role in MIMO rational approximation

Given $F \in H_2^\perp$, minimize $\|F - H\|_2, \quad H$ stable, deg $H = n$

Multiplication by Q is an isometry in $L^2(\mathbb{T})$

$$\left.\begin{array}{ll} \hat{H} & \text{best approx. of } F \\ \hat{H} & = P\ Q^{-1}[DSS] \end{array}\right\} \Rightarrow P = \mathcal{P}_+(FQ) \text{ anti-stable projection}$$

- **Minimization over a bounded set:** Inner matrices degree n

$$J(Q) = \|F - Q^{-1}\mathcal{P}_+(FQ)\|_2$$

- **Classical in the SISO case**
  Gradient-based algorithm [Baratchart, Cardelli, O.,1991]

→ **Parametrization issues**

Matrix Blaschke Products

SISO inner functions: Blaschke factors and their products

$$b_w = \frac{z-w}{1-\bar{w}z}, w \in \mathbb{D},$$

Potapov factorization (1960)

Q inner of degree n

$$Q = B_{w_1,u_1} B_{w_2,u_2} \cdots B_{w_n,u_n} U_0$$

U_0 unitary

Blaschke-Potapov factors (degree one)

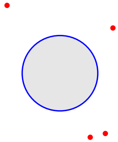
$$B_{w,u} = I + \left(\frac{z-w}{1-\bar{w}z} - 1 \right) uu^*$$

u unit vector, $\det B_{w,u} = b_w$ and $u^* B_{w,u}(w) = 0$.

Inner vs lossless

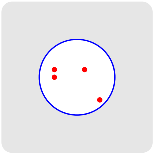
$Q \leftrightarrow Q^\sharp$

Inner:
analytic in $\mathbb{D}$
anti-stable



Interpolation theory

Lossless:
analytic in $\mathbb{E}$
stable



System theory

$$G(z) = Q^\sharp(z) = Q(1/\bar{z})^*$$

Outline

Our purpose

- Find good parametrizations for the class of degree n lossless matrices
- Establish the connections between to kind of representations:
  - Atlases of charts derived from interpolation theory
    Alpay, Baratchart, Gombani (1994)
  - Overlapping balanced canonical forms
    Ober, Hanzon (1998)

Interest of balanced realizations

- Model reduction by balanced truncation [Moore, 1981]
- Lossless case: nice state-space representation

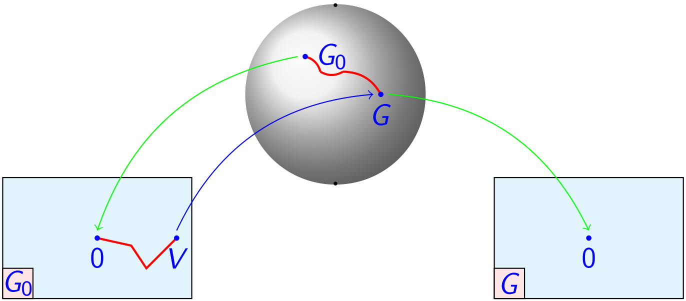
$$\left\{ \begin{array}{cl} D + C(zI - A)^{-1}B & \text{lossless} \\ (A, B, C, D) & \text{balanced} \end{array} \right. \Leftrightarrow \begin{bmatrix} D & C \\ B & A \end{bmatrix} \text{ unitary}$$

[Genin, Van Dooren, Kailath, Delosme, Morf, 1983]
Good numerical behavior

- State-space formulas for L^2-criterion $J(Q)$

$$\begin{aligned} J(A, B) &= \|F\|^2 - \mathrm{Tr}[\mathcal{C} W W^* \mathcal{C}] \\ W &= \mathcal{A} W A^* + \mathcal{B} B^* \end{aligned}$$

Atlases of charts and optimization



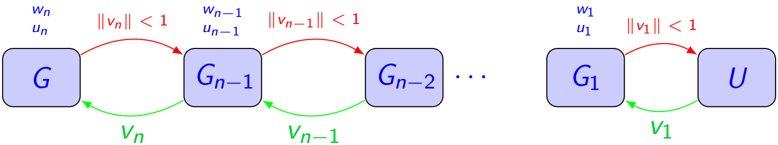
- No global smooth parametrization (lossless matrices order n)
- An atlas: family of local coordinate maps (charts)
- V (matrix of) parameters in the chart.
- Adapted chart at G: $V = 0$.

Parametrization from the TSA

Interpolation condition $G[1/\bar{w}]u = v, \quad w \in \mathbb{D}, \quad \|v\| < \|u\| = 1.$

Matrix lossless case: the tangential Schur algorithm

Always stops: either if $\|v_i\| = 1$ or after n steps if G has order n.



Interpolation values as parameters

Fixing (w_i, u_i) we get a local coordinates map (chart)

$$\phi_{(w_i,u_i)} : G \in \mathcal{D} \mapsto (v_1, v_2, \ldots, v_n, U)$$

Letting (w_i, u_i) vary, we get an atlas of charts

Alpay, Baratchart, Gombani (1994)

In the SISO case the Hessenberg form can be obtained by a product of Givens rotations parametrized by Schur parameters
Hanzon, Peeters, 2000

- Can we attach to the tangential Schur algorithm a similar construction of balanced realizations?
- Can we parametrize the balanced canonical forms of Ober and Hanzon with Schur vectors ?

The complete picture

$$\left[\begin{array}{cc|cccc} + & * & * & * & * & * \\ 0 & * & + & * & * & * \\ 0 & + & 0 & * & * & * \\ 0 & 0 & 0 & * & + & * \end{array}\right] \qquad \left[\begin{array}{cc|cccc} + & * & * & * & * & * \\ 0 & * & + & * & * & * \\ 0 & * & 0 & + & * & * \\ 0 & + & 0 & 0 & * & * \end{array}\right]$$

- Charts: $(w_i, u_i)_{i=1,\ldots,n}$
  Balanced realizations = products of unitary matrices
  Hanzon, O., Peeters (LAA, 2006)
- Charts: $w_i = 0$ and u_i standard basis vectors
  Subdiagonal pivot structure in $[B|A]$
  Hanzon, O., Peeters (Sysid, 2009)
- Admissible sequences $(u_i)_{i=1,\ldots,n}$
  Staircase structure in A - Reachability matrix full pivot structure
  We recover the atlas of [Hanzon, Ober, 1998]
  Hanzon, O., Peeters (LAA, 2007)

Outline

Rational L^2 approximation: the context

- **A huge literature**
  Meier, Luenberger, Wilson, Rosencher, Ruckebusch, Krajewski, Lepschy, Mian, Viaro, Spanos, Milman, Mingori, Hanzon, Maciejowski, Yan, Lam, Gugercin, Antoulas, Beattie, Van Dooren, Gallivan, Absil, ...
- **A difficult problem**
  - Fixed McMillan degree: constraint on the whole matrix
  - Stable rational matrices of degree n: no global smooth parametrization
  - Many local minima
- **Well-posed:** existence, normality [Baratchart, 1986], consistency [Baratchart, O., 1998]
- **Our approach:** optimize the concentrated criterion $J(Q)$ or $J(A, B)$ using an atlas of chart.

Previous implementations

- **Hyperion** (Grimm,) C++
  Atlas: Alpay, Baratchart, Gombani
  Polynomial representation (Fulcheri, O.,1998)
- **RARL2** first version (Marmorat, O.) Matlab
  Atlas: Hanzon, O., Peeters, 2006
  State-space representation

Good point:

Potapov factorization	**Adapted chart**	Realization in Schur form
$G^\sharp = \prod B_{w_i,u_i}$	(w_i, u_i) Parameters $v_i = 0$	$A = \begin{bmatrix} w_n & 0 & 0 \\ \vdots & \ddots & 0 \\ * & \cdots & w_1 \end{bmatrix}$

Bad point: does not work in the case of real functions

Lossless Mutual Encoding

A chart: specified by $\Omega = (W^*, U^*, Y^*, X^*)$ balanced realization

[Marmorat, O, 2007]

Let (A, B, C, D) balanced realization of G lossless, and S s.t.

$$S - ASW = BU$$

- G can be represented in Ω iff $S > 0$.
  Parameter: $V = DU + CSW$ (Nudelman interpolation value)
- (A, B, C, D) canonical form w.r.t. Ω iff S Hermitian $S^* = S$.
- Adapted chart at G: $\Omega_G = (A, B, C, D)$ $(S = I)$

S measure the quality of the chart at G

RARL2 new version

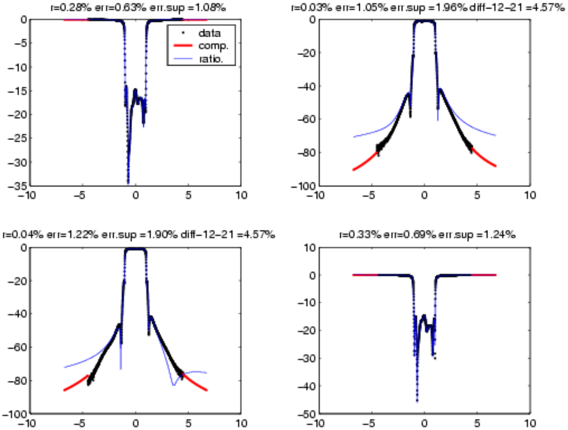
Uses and users of RARL2

- **Identification of microwave filters**
  Presto-HF [Seyfert] matlab based toolbox
  - Completion of the frequency data
  - RARL2 compute the best L^2 approximation

  Real world users: CNES, XLIM, Alcatel, Thales
- **Localization of dipolar sources in electro-encephalography**
  FindSource3d [Bassila, Leblond, Clerc, Marmorat]
  Baratchart, Partington, Yattselev, Zghal, ...:
  sources location $\leftrightarrow$ singularities of some (f_p) p index planar sections
  - data $\rightarrow f_p$
  - RARL2 find the poles of f_p

  New constraint handled by RARL2: triple poles

Identification of microwave filters



Bode diagrams of the completion and the approximant at order 8

Outline

The problem

Continuous-time framework: Π_- left half-plane, Π_+ right half-plane
S Schur: S analytic (stable) and contractive in Π_+

- **An old problem** [Darlington synthesis, 1939]
  extend S $p \times p$ Schur into a lossless matrix

$$\Sigma = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S \end{bmatrix}$$

  size $2p \times 2p$, same degree than S
- **A new requirement** Σ **symmetric** completion of S symmetric extension of size $2p + n$ [Anderson, Vongpanitlerd,1973]
  - Size $2p \times 2p$, minimal degree
  - Same degree than S, minimal size

Minimal degree symmetric extension

Baratchart, Gombani, Enqvist, O., 2007

Let S symmetric Schur function, *strictly contractive* at infinity, S_{21} *minimum phase* s.t. $I - SS^* = S_{21}S_{21}{}^*$.

- $Q = S_{21}^{-1}S_{12}^T$ lossless and

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix}$$

  symmetric lossless extension (degree $> n$)

- κ number of distinct zeros of $\det Q$ with odd multiplicity. Then S has a symmetric lossless extension of degree $n + \kappa$ (minimal degree).

To reduce the degree: Potapov symmetric factorization

Potapov symmetric factorization

Baratchart, Gombani, Enqvist, O., 2007

Let T symmetric lossless function and $\omega \in \Pi^+$ double zero of $\det T$. Then,

$$T = B_{\omega,u}^T \; R \; B_{\omega,u},$$

with R symmetric lossless and $B_{\omega,u}(s) = I + \left(\frac{s-\omega}{s+\bar{\omega}} - 1\right) uu^*$.

- Works in $\mathbb{C}$
- The case of real functions with real zeros is more involved

Minimal size symmetric extension

Start with

$$\Sigma := \begin{bmatrix} S_{11}Q & 0 & S_{12} \\ 0 & \det Q & 0 \\ S_{12}^T & 0 & S \end{bmatrix}$$

and reduce the degree using Potapov symmetric factorization

Baratchart, Gombani, Enqvist, O., 2007

S has a minimal symmetric extension of degree n and size $2p+1$

$S = \frac{p}{q}$ we get a 3×3 lossless symmetric matrix

- all the 3×3 lossless extensions are obtained in this way
- we know the number of such extensions

Avanessoff, O., Seyfert, MTNS 2010

Ongoing works and perspectives

- Boundary interpolation of lossless matrices
  SISO continuous-time O., Hanzon, Peeters, CDC08
  MIMO discrete-time Hanzon, O, Peeters, MTNS10
- Parametrization of Schur functions PHD thesis V. Lunot
- Model reduction using subdiagonal pivot structures

- Adress the passivity constraint
- Use the parametrization of pairs (A, B) for other criteria
- Find new applications

- Minimal degree realization in the real case
- Polynomial structure of 3×3 scattering matrices
- Applications to filter design

Thanks for your attention !