L^2 Rational Approximation, Model Reduction and Applications

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Plan

1. Rational approximation and system theory
2. Our approach to rational approximation
3. Applications
    - Identification of hyperfrequency filters
    - Inverse EEG source problems and approximation
    - Implementing wavelets in analog circuits

What is the use of rational approximation ?

General problem

Find a mathematical model from measured data

$\longrightarrow$ **Applications :** model reduction, identification, realization problem, simulation, control, prediction

- Which model ?
  Linear time invariant (LTI) systems represented by their transfer function a rational matrix-valued function $H(z)$.
- Which data ?
  - Time series : $u(t_k), y(t_k), k = 1, \dots N$
  - Frequency data : $H(i\omega_k), k = 1, \dots N$
- Which method ?
  - Projection
  - Optimization (criterion ?)

Rational approximation and system theory

A fertile interaction :

- Minimal partial realization/moment matching
- Nonlinear least-squares
- AAK approximation/ Hankel norm approximation
- Subspace methods
- H^2 rational approximation

A still active field.

Stability and Analyticity

Stability

A bounded input $\|u\|_\infty < \infty$ produces a bounded output $\|y\|_\infty < \infty$

- Impulse response $h(t)$ integrable
- Transfer function $H(s)$ analytic in the right half-plane $\{\mathrm{Re} s \geq 0\}$

Analytic functions have a rigid structure, they are completely determined by their values on certain subsets of their domain of definition, or of their boundary.

The H^2 norm

H^2 Hardy spaces of functions analytic in the right half-plane, square integrable on the imaginary axis

$$\|H\|^2 = \frac{1}{2\pi}\int |H(i\omega)|^2 d\omega$$

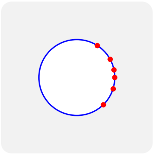
A rich structure: analyticity + Hilbert space

- $L^2 \to L^\infty$ stability
- Comes from a scalar product (orthogonality principle, differentiable)
- Stochastic interpretation : minimize the mean square output error due to a white noise
- Cauchy formula : $f(a) = \frac{1}{2\pi}\int_\partial \frac{f(z)}{z-a}\, dz, \mathrm{Re}\ a > 0$
  f completely determined by its boundary values on the imaginary axis $\partial = i\mathbb{R}$.

From continuous-time to discrete-time

s-plane

$s = \frac{z+1}{z-1}$



z-plane

$z = \frac{s+1}{s-1}$

$H(s)$ strictly proper $\quad \tilde{H}(z) = \frac{\sqrt{2}}{z-1} H\left(\frac{z+1}{z-1}\right)$

$$\frac{1}{2\pi}\int |H(i\omega)|^2 d\omega = \frac{1}{2\pi}\int |\tilde{H}(e^{i\theta})|^2 d\theta.$$

Preserves the H^2-norm and the order or McMillan degree

General inverse problem

Recover an analytic function on a domain from (pointwise band-limited) data on the boundary

Rational L^2 approximation

$H_\perp^2$ Hardy space of matrix valued functions, analytic outside $\mathbb{D}$, vanishing at ∞ (stable transfer functions)

$$\|F\|^2 = \frac{1}{2\pi}\mathbf{Tr}\left\{\int_0^{2\pi} F(e^{it})F(e^{it})^* dt\right\}$$

Rational approximation

Given $F \in H_2^\perp$, minimize

$$\|F - H\|^2$$

H rational, stable, of McMillan degree n

$F(z)$ given: a method for model reduction

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A difficult problem

- Nonlinear optimization problem
- The set of approximants (stable systems of order n) has a complex structure, specially in the MIMO case.
  It is difficult to get 'good' parameters.
- L^2 norm $\mapsto$ great number of local minima

How we cope with ?

- Eliminate the linear variables obtaining a function that involves only the nonlinear parameters (separable least square)
- Use a 'good' parametrization for the reduced optimization space
- Use an optimization strategy : start from an initial point given by another method or iterative search on the order

Implementation: RARL2 a matlab software

J. P. Marmorat, M.O. (2004)

Optimization space reduction

Given $F(z) \in H^2_\perp$ find $H(z)$ of degree n which minimizes $\|F - H\|^2$.

Elimination of linear variables: two points of view

- Lossless-unstable factorization Douglas-Shapiro-Shields (1970)
  $H(z) = \underbrace{\Phi(z)}_{\text{poles}}\underbrace{G(z)}_{\text{zeros}}$, $\Phi(z)$ lossless, $G(z) \in H^2$.
  $\mathrm{Vec}(\Phi) = \{H(z) \in H^2_\perp, H(z) = \Phi(z)G(z)\}$
- State-space framework
  $\mathrm{Vec}(C, A) = \{H(z) \in H^2_\perp, H(z) = C(zI - A)^{-1}\beta\}$

Connection: lossless embedding

(C, A) output-normal pair: $\longrightarrow$ $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ unitary matrix

$A^*A + C^*C = I$ $\qquad$ $\Phi(z) = C(zI - A)^{-1}B + D$ lossless

The concentrated criterion

The projection theorem (orthogonality principle) in a Hilbert space allows to compute

- $G(z)$ from $\Phi(z)$
- β from (C, A)

Concentrated criterion

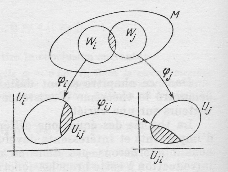
$$J(C, A) = \min_{H \in \mathrm{Vec}(C,A)} \|F - H\|^2$$

Advantages:

- The dimension of the parameter space is reduced but we also get a better-conditioned problem
- Lossless matrices enter the picture (transfer functions of conservative systems)

Which parametrizations for lossless systems?

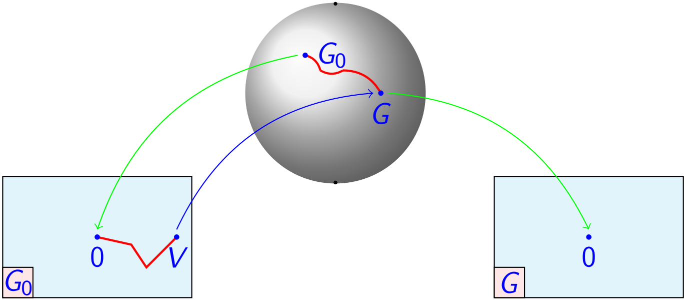
- An atlas of local coordinate maps: identifiability, differentiability



- Parametrizations with a double interpretation
  - Schur analysis and interpolation theory
    A nice way to address the constraint metric $\|F\|_\infty \leq 1$.
    Big inpact in system theory.
    Kailath (1986), Kimura, Ball, Gohberg and Rodman (1990)
  - Balanced state-space canonical forms
    Useful in computations; physical interpretation

Hanzon, M.O., Peeters, LAA (2006),
Marmorat, M.O., Automatica(2007)

Optimization process in RARL2



- $V \mapsto G(z) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ computed from V and G_0 as a product of two unitary matrices (well-conditionned)
- Optimization with respect to local coordinates V (fmincon)
  Nonlinear condition reached $\longrightarrow$ change of chart
- Nonlinear condition: eig min $Q \geq \epsilon$ where Q is the solution to $Q - A^* Q A_0 = C^* C_0$.

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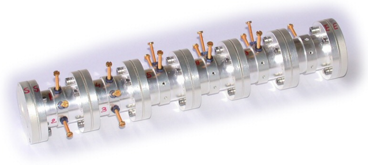
Identification of hyperfrequency filters



- Electro-magnetic waves filter made of resonant cavities, interconnected by coupling irises (orthogonal double slits). Each cavity has 3 screws.
- Works around the GHz, Passband: a few Mhz
- Used in space telecommunication (satellites transmission) for multiplexing purposes.

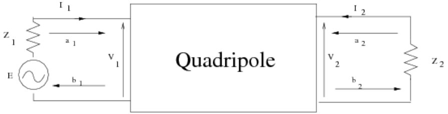
Identification problem: given measurements performed on the device find the values of parameters from the physical model
What for ? Tuning (adjusting the screws)

An equivalent electrical model (1)

- Maxwell equations
- Input and output : one spacial mode (waveguide)

  $$\begin{aligned} \text{electric field} &\approx \text{voltage} \\ \text{magnetic field} &\approx \text{current} \end{aligned}$$

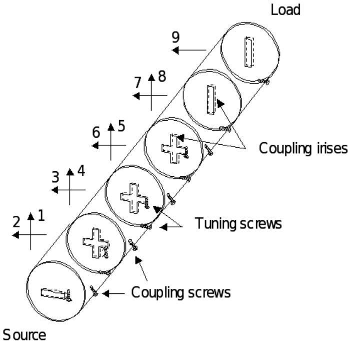
  The filter can be seen as a quadripole, with two ports.



- The electrical field in each cavity is decomposed along two orthogonal modes (approximation valid in the range of frequencies)
- For each mode, a good approximation of the Maxwell equations is given by the solution of a second order differential equation.

  One mode in one cavity $\mapsto$ a RLC circuit (order 2)

An equivalent electrical model (2)



Screws act as capacitors

Irises results in a coupling between two horizontal (or two vertical) modes of adjacent cavities

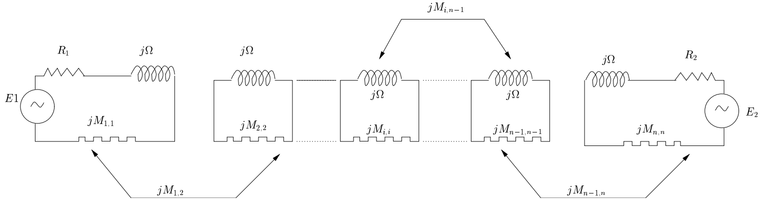
Order $= 2 \times 9$ (two times the number of modes)

Low-pass equivalent electrical model

Original filter: two (conjugate) bandwidths in the high frequencies

$\downarrow$ low-pass transformation

Low-pass equivalent : one bandwidth around 0



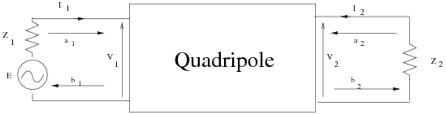
one mode = one circuit (order 1)

order = number of modes

$M_{i,j}$ coupling

State-space representation

We are interested in the power which is transmitted and reflected.



$$\begin{cases} \dot{x}(t) &= Ax(t) + Ba(t) \\ b(t) &= Cx(t) + a(t) \end{cases} \qquad a(t) = \begin{bmatrix} a_1(t) \\ a_2(t) \end{bmatrix} \qquad b(t) = \begin{bmatrix} b_1(t) \\ b_2(t) \end{bmatrix}$$

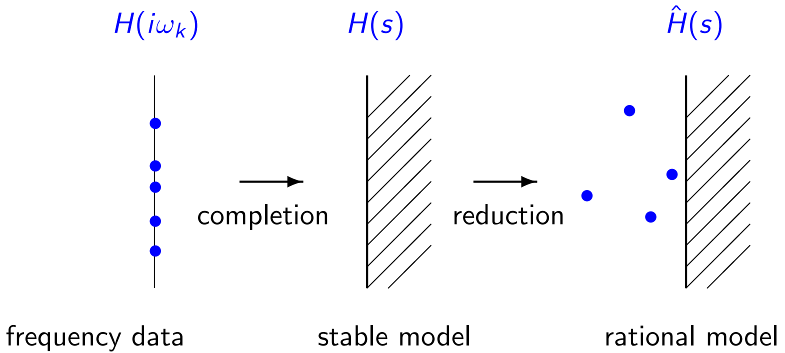
Scattering matrix: $S(z) = I + C(zI - A)^{-1}B$
lossless symmetric with complex coefficients

$$C = \begin{bmatrix} j\sqrt{2R_1} & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & j\sqrt{2R_2} \end{bmatrix} \qquad B = C^t$$

$$A = -R - jM, \quad A = A^t \qquad\qquad 2R = -C^tC$$

M **coupling matrix**

Identification vs Model Reduction



A difficult problem:

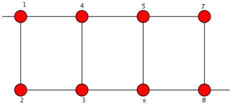
$\longrightarrow$ Interpolate and extrapolate the data (band-limited)

$\longrightarrow$ Ensure stability

$\longrightarrow$ Find a small order model

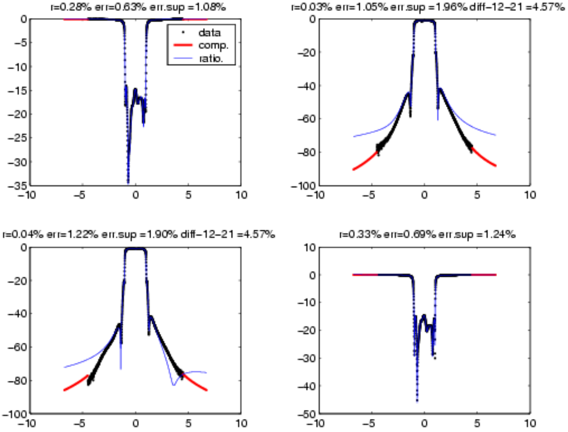
Resolution in 3 steps

- **Interpolation/extrapolation of the frequency data** dedicated method
- **Model reduction** by rational L^2-approximation RARL2
- **Computation of the physical parameters** from a realization in a particular form (coupling matrix) DEDALE (F.Seyfert)

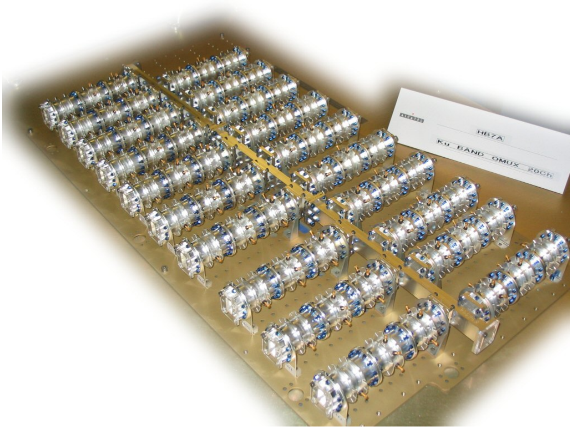


Presto-HF (F. Seyfert) is a matlab based toolbox dedicated to the identification problem of low pass coupling parameters of band pass microwave filters. Used by CNES, IRCOM, Alcatel.

Numerical results



The future



Omux identification and design

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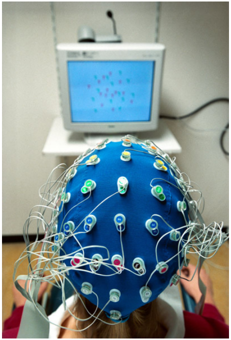
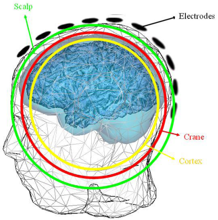
Inverse EEG (electroencephalography) problem



From measurements by electrodes of the electric potential u on the scalp, recover a distribution of m pointwise dipolar current sources C_k with moments p_k located in the brain (modeling the presence of epileptic foci).

L. Baratchart, M. Clerc, J. Leblond, J.P. Marmorat, M. Zghal

Mathematical model



The head Ω is modeled as a set of 3 spherical nested regions $\Omega_i \subset \mathbb{R}^3$, $i = 0, 1, 2$ (brain, skull, scalp), with piecewise constant conductivity σ_i, separated by interfaces S_i.

Macroscopic model + quasi-static approximation of Maxwell equations
$\longrightarrow$ Spatial behavior of u in Ω

$$(P) \begin{cases} \operatorname{div}(\sigma \nabla u) = \sum_{k=1}^{m} p_k . \nabla \delta_{C_k} & \text{in } \Omega \\ u = \text{ and } \partial_n u & \text{given on } \partial\Omega \end{cases}$$

Resolution of this inverse problem in 3 steps

- **Data propagation**

$$\boxed{\begin{array}{l} u \text{ at electrodes} \\ \partial_n u = 0 \text{ on } S_2 \end{array}} \xrightarrow{\text{Cauchy problem}} \boxed{\left.\begin{array}{l} u \\ \partial_n u \end{array}\right. \text{ on } S_1} \xrightarrow{\text{Cauchy problem}} \boxed{\left.\begin{array}{l} u \\ \partial_n u \end{array}\right. \text{ on } S_0}$$

- **Anti-harmonic projection**

$$(P) \text{ in } \Omega_0 \begin{cases} \triangle u = \frac{1}{\sigma_0} \sum_{k=1}^{m} p_k . \nabla \delta_{C_k} & \text{in } \Omega_0 \\ u \text{ and } \partial_n u & \text{given on } S_0 \end{cases}$$

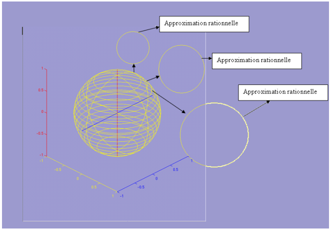
The solution u to (P) in Ω_0 assumes the form:

$$u(x) = h(x) + \sum_{k=1}^{m} \frac{< p_k,\ x - C_k >}{4\pi \|x - C_k\|^3} = \underbrace{h(x)}_{harmonic} + \underbrace{u_a(x)}_{anti-harmonic}$$

u_a obtained by expanding u on bases of spherical harmonics
u_a contains the information on the sources

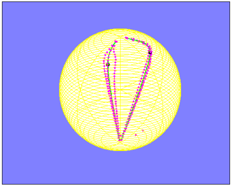
- **Best rational approximation on planar sections**
Source localization

Best rational approximation on planar sections

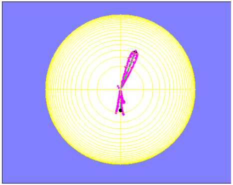


- Slice Ω_0 along a family of planes Π_p:
  $\Pi_p \cap \Omega_0 = \Gamma_p$ (disks) $\Pi_p \cap S_0 = \Gamma_p$ (circles)
- From pointwise values on Γ_p, compute the best L^2 rational approximation to $f_p = (u_a|_{\Gamma_p})^2$ on Γ_p

Why does it works ?



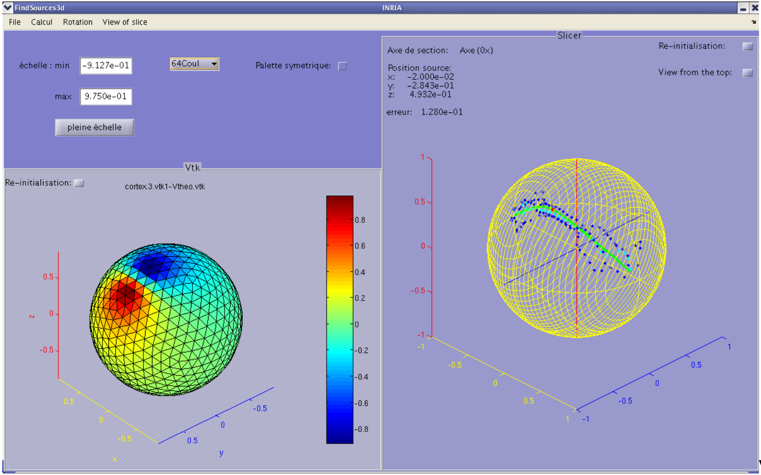
2 sources



above view

- f_p is a (meromorphic) function whose singularities $\zeta_{k,\ p}$ inside D_p are strongly and explicitely linked with the sources C_k.
  - $(\zeta_{k,\ p})$ are aligned together and also with the complex coordinates ζ_k of C_k.
  - $|\zeta_{k,\ p}|$ is maximum at $\zeta_{k,\ p} = \zeta_k$ (the k^{th} source's section)
- The poles $\widetilde{\zeta}_{k,\ p}$ of the best L^2 rational approximation to f_p on Γ_p accumulate to the singularities $\zeta_{k,p}$
  Leblond, Baratchart, Yattselev

FindSource3D



view

of u on S_2 then of C_1 into Ω_0

R. Bassila, M. Clerc, J. Leblond, J.P. Marmorat

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Wavelets in analog circuits

The continuous-time wavelet transform of a signal $f(t)$

$$W_\psi(\tau,\sigma) = \frac{1}{\sigma}\int f(t)\psi\left(\frac{t-\tau}{\sigma}\right)\, dt$$

- Widely used signal processing technique in medical applications (cardiac signal processing)
- Provides combined time and frequency localization

Can be implemented with a linear system:

$$f * h(t) = \int f(\tau)h(t-\tau)\, d\tau, \quad h(t) = \frac{1}{\sigma}\psi\left(\frac{-t}{\sigma}\right)$$

Only the implementation of strictly causal stable filters is feasible !
To ensure causality, a time-shifted (truncated) time-reversed mother wavelet $\tilde{\psi}(t) = \psi(t_0 - t)$ is considered.

L^2-approximation of wavelets functions

Motivation of an L^2-approximation criterion

- Wavelet transform involves an L^2-inner product
- Due to Parseval's theorem, L^2-approximation in the time domain can be reformulated as L^2-approximation in the frequency domain

$$\min_h \|\tilde{\psi}(t) - h(t)\| \equiv \min_H \|\tilde{\Psi}(s) - H(s)\|$$

An important admissibility property: wavelets needs to have (at least) one vanishing moment

$$\int \tilde{\psi}(t)\ dt = 0$$

Approximation method (sampling, choice of a clever initial point, use an iterative local search optimization technique)
Karel, Haddad, Westra, Serdijn, Peeters (2009)

Model reduction using RARL2

- The given function $F(z)$: the approximation obtained by Karel & al. converted into a discrete-time transfer function.
- The vanishing moment condition yields on the discrete-time approximant $H(z) = C(zI - A)^{-1}\beta$ the linear condition on β

$$C(I + A)^{-1}\beta = 0$$

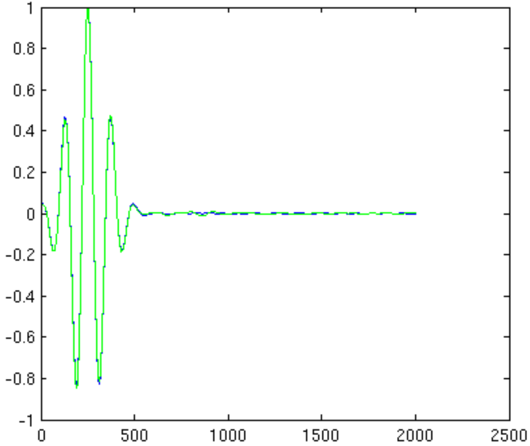
- It can be handled by a slightly different concentrated criterion

$$J(A, C) = \begin{array}{l} \min_\beta \|F(z) - H(z)\|^2 \\ \text{subject to } C(I + A)^{-1}\beta = 0 \end{array}$$

  β is easily computed from A and C using Lagrange multipliers. Same optimization space and parametrization.
- Incremental search on the order

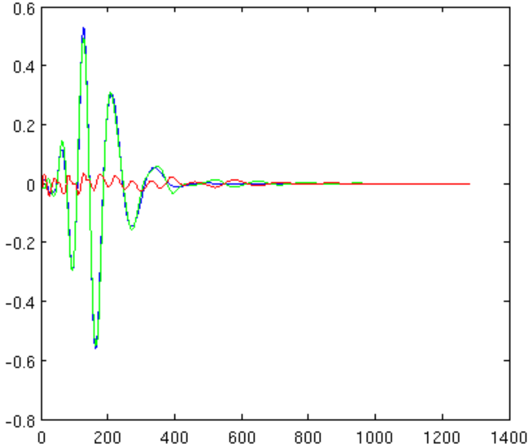
Approximation of Morlet wavelet

Initial approximation order 20 / RARL2 approximation order 8



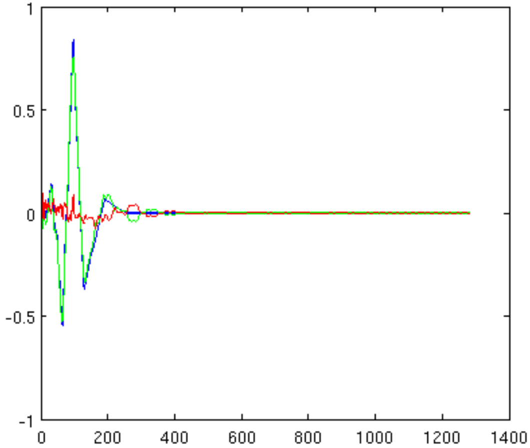
Approximation of DB7 wavelet

Initial approximation order 136 / RARL2 approximation order 8



Approximation of DB3 wavelet

Initial approximation order 225 / RARL2 approximation order 12



Perspectives

- Convex constraints on the approximant can be handled with the same optimization space (output normal pairs)
  - $\longrightarrow$ Vanishing moments (wavelets)
  - $\longrightarrow$ Positive real, bounded real (β solution to an LMI)
- The same optimization space can be used for other criterions (separable least-square)

  RARL2 subroutines divides into two independant libraries :

  arl2lib: computation of the concentrated criterion, gradient with respect to A, C

  boplib: parametrization of balanced output pairs C, A
  - $\longrightarrow$ Multi-objective control (β solution to an LMI Scherer)
  - $\longrightarrow$ (weighted) Nonlinear least-square

http://www-sop.inria.fr/apics/RARL2/rarl2-eng.html