Schur parametrizations of stable all-pass discrete-time systems and balanced realizations. Application to rational L^2 approximation.

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Outline.

- Introduction : lossless functions and rational L^2 approximation
- Schur parametrizations and balanced realizations
- The software RARL2
- New atlases

Lossless functions: transfer functions of stable allpass systems.

Lossless function (discrete-time): $G(z)$ square rational stable such that

$$G(z)^* G(z) \leq I, |z| > 1$$

with equality on the unit circle.

A lossless function $G(z)$ has a minimal realization s.t.

$$(D, C, B, A) \text{ balanced} \Leftrightarrow R = \begin{bmatrix} D & C \\ B & A \end{bmatrix} \text{ unitary.}$$

lossless transfer functions in discrete-time can be represented by *unitary realization matrices.*

Möbius transform: continuous-time $\leftrightarrow$ dicrete-time.

Lossless functions in linear systems theory.

Douglas–Shapiro–Shields factorization: S rational stable

$$S = P\,G$$

G lossless, P analytic in the unit disk (unstable).

Many problems in linear system theory can be stated in terms of lossless functions :

- stochastic identification (Separable Least Squares)
- model reduction problems via rational approximation
- control design ?

Rational L^2 approximation.

H_2 Hardy space of matrix valued functions analytic in $\mathbb{D}$,
$H_2^\perp$ Hardy space of matrix valued functions analytic outside $\mathbb{D}$,
vanishing at ∞

$$\|F\|^2 = \frac{1}{2\pi} \mathbf{Tr} \left\{ \int_0^{2\pi} F(e^{it}) F(e^{it})^* dt \right\}.$$

Given $F \in H_2^\perp$, minimize

$$\|F - H\|^2$$

H rational, stable of McMillan degree n.

Applications developped in the APICS team:

- identification from partial frequency data (hyperfrequency filters)
- sources and cracks detection (MEG, EEG)

The criterion.

- $H = P\,G$ a best approximant of F:
  P proj. of $F\,G^{-1}$ onto H_2

  For G lossless (up to a left unitary factor)

$$J(G) = \|F - P(G)\,G\|^2.$$

- $F(z) = \mathcal{C}(zI_N - \mathcal{A})^{-1}\mathcal{B} + \mathcal{D}$,
  $H(z) = \gamma(zI_n - A)^{-1}B + \mathcal{D}$.
  For (A, B) input-normal pair ($AA^* + BB^* = I$)

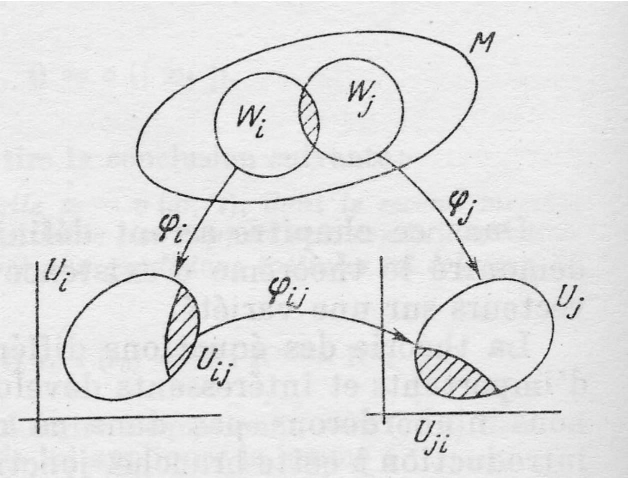
$$J(A, B) = \|F\|^2 - \mathrm{Tr}\,(\gamma\gamma^*)\,,$$

where $\gamma = \mathcal{C}W$, W solution to $\mathcal{A}WA^* + \mathcal{B}B^* = W$.

Parametrization issue in the MIMO case

In view of optimization, it is better use an atlas of charts attached to a differential manifold $\mathcal{M}$

- a collection of local coordinate maps (charts) (W_i, φ_i)
  W_i open in $\mathcal{M}$, $\varphi_i : W_i \to \mathbb{R}^d$ diffeomorphism
- their domains cover the manifold: $\cup W_i = \mathcal{M}$
- the changes of coordinates φ_{ij} are smooth



Advantages of such a parametrization

... in the representation of stable allpass systems:

- ensures identifiability
- a small perturbation of the parameters preserves the stability and the order of the system
- allows for the use of differential tools: a search algorithm can be run over the manifold changing from one local coordinate map to another when necessary.

Two approaches.

$\mathcal{L}_n^p$: $p \times p$ (complex) lossless functions of McMillan degree n is a real differentiable manifold of dimension $2np + p^2$
Subclass of real functions $G(\bar{z}) = \overline{G(z)} \longrightarrow$ dim $np + p(p-1)/2$

- charts from realizations theory (state-space)

  M. Hazewinkel, R.E. Kalman (1974)

  for proper linear systems using nice selections

  B. Hanzon, R.J. Ober (1998)

- charts from Schur type algorithms (functional)

  D. Alpay, L. Baratchart, A. Gombani (1994)

  interpolation theory, reproducing kernel Hilbert spaces

  Ball, Gohberg, Rodman, Dym, Alpay, ...

An atlas which combines these two approaches

B. Hanzon, M.O., R. Peeters (1999)

The SISO case : a balanced canonical form.

A lossless function of degree n has a (unique) realization such that

(i) (A, b, c, d) balanced

$[b, Ab, A^2b, \ldots, A^{n-1}b]$ positive upper triangular

or equivalently

(ii) $R = \begin{bmatrix} d & c \\ b & A \end{bmatrix}$ is unitary and positive upper-Hessenberg.

B. Hanzon, R. Peeters (2000)

Parametrization.

$R = \begin{bmatrix} d & c \\ b & A \end{bmatrix}$ is unitary and positive upper-Hessenberg.

Then

$$\begin{bmatrix} \bar{\gamma}_n & \kappa_n & 0 \\ \kappa_n & -\gamma_n & 0 \\ 0 & 0 & I_{n-1} \end{bmatrix} \underbrace{\left[\begin{array}{c|c} d & c \\ \hline * & \\ & A \\ 0 & \end{array}\right]}_{R} = \begin{bmatrix} 1 & 0 \\ 0 & R_{n-1} \end{bmatrix},$$

$$|\gamma_n| < 1,\ \kappa_n = \sqrt{1 - |\gamma_n|}^2, \gamma_n = d.$$

By induction: $(R_k)_{k=n,\dots,0}$ unitary upper-Hessenberg order k and $(\gamma_k)_{k=n,\dots,0}$, $\quad R_0 = \gamma_0,\ |\gamma_0| = 1$.

Hessenberg form.

$$\begin{bmatrix} \gamma_n & \kappa_n\gamma_{n-1} & \kappa_n\kappa_{n-1}\gamma_{n-2} & \cdots & & & \kappa_n\kappa_{n-1}\cdots\kappa_1\gamma_0 \\ \kappa_n & -\bar{\gamma}_n\gamma_{n-1} & -\bar{\gamma}_n\kappa_{n-1}\gamma_{n-2} & \cdots & & & -\bar{\gamma}_n\kappa_{n-1}\cdots\kappa_1\gamma_0 \\ 0 & \kappa_{n-1} & -\bar{\gamma}_{n-1}\gamma_{n-2} & \cdots & & & -\bar{\gamma}_{n-1}\kappa_{n-2}\cdots\kappa_1\gamma_0 \\ 0 & 0 & \kappa_{n-2} & & & & -\bar{\gamma}_{n-2}\kappa_{n-3}\cdots\kappa_1\gamma_0 \\ \vdots & \vdots & & \ddots & & & \vdots \\ 0 & 0 & \ddots & \kappa_2 & -\bar{\gamma}_2\gamma_1 & & -\bar{\gamma}_2\kappa_1\gamma_0 \\ 0 & 0 & \cdots & 0 & \kappa_1 & & -\bar{\gamma}_1\gamma_0 \end{bmatrix} \tag{1}$$

Connection with the Schur algorithm.

Let $g_k(z) = \gamma_k + c_k(zI_k - A_k)^{-1}b_k$ be the lossless function whose realization is R_k.

Then g_k satisfies the interpolation condition

$$g_k(\infty) = \gamma_k, \quad |\gamma_k| < 1.$$

Moreover,

$$g_k(z) = \frac{\gamma_k\, z + g_{k-1}(z)}{z + \bar{\gamma}_k g_{k-1}(z)} \quad \Leftrightarrow \quad g_{k-1}(z) = \frac{(g_k(z) - \gamma_k)z}{1 - \bar{\gamma}_k\, g_k(z)}.$$

This is the Schur algorithm: lossless functions of McMillan degree n are parametrized by the interpolation values $\gamma_n, \gamma_{n-1}, \ldots, \gamma_1$ and γ_0.

An interpolation problem.

Nevanlinna-Pick: find all the lossless functions $G(z)$

$$G(1/\bar{w})u = v, \quad |w| < 1, \quad \|u\| = 1.$$

A solution exists $\Leftrightarrow \|v\| < 1$. All the solutions are of the form

$$G = (\Theta_4 K + \Theta_3)(\Theta_2 K + \Theta_1)^{-1}.$$

for some lossless function K, where

$$\Theta(z) \;=\; \left(I_{2p} + \left(\frac{z - w}{1 - \bar{w}z} - 1 \right) C\,C^* J \right) \; H \tag{2}$$

$$\Theta = \begin{bmatrix} \Theta_1 & \Theta_2 \\ \Theta_3 & \Theta_4 \end{bmatrix}, \; J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \; C = \begin{bmatrix} u \\ v \end{bmatrix}, \quad H \text{ cstarbit. } H^* J H = J$$

ref. on interpolation pbs: Ball, Gohberg, Rodman, 1990

Charts from a Schur algorithm.

Schur algorithm: from $G \in \mathcal{L}_n^p$,

$$G = G_n, \ldots, G_k \stackrel{LFT}{\longrightarrow} G_{k-1}, \ldots, G_0$$

$$G_k(1/\bar{w}_k)u_k = v_k, \|v_k\| < 1$$

G_k has degree k and G_0 is a constant unitary matrix.

Chart and Schur parameters:

- $w_1, w_2, \ldots, w_n$ points of the unit circle,
- $u_1, u_2, \ldots, u_n$ unit complex p-vectors,
- $v_1, v_2, \ldots, v_n$ complex p-vectors, $\|v_i\| < 1$.

$$\mathcal{W} = \{G \in \mathcal{L}_n^p; \ \ \|G_k(1/\bar{w}_k)u_k\| < 1\}, \quad \varphi : G \to (v_1, \ldots, v_n, G_0)$$

Balanced realizations from Schur parameters.

For a particular choice of H in (2) depending on the interpolation data, a lossless function G in this chart has a unitary realization matrix computed by induction

$$\begin{bmatrix} D_k & C_k \\ B_k & A_k \end{bmatrix} = \begin{bmatrix} V_k & 0 \\ 0 & I_{k-1} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & D_{k-1} & C_{k-1} \\ 0 & B_{k-1} & A_{k-1} \end{bmatrix} \begin{bmatrix} U_k^* & 0 \\ 0 & I_{k-1} \end{bmatrix},$$

where A_k is $k \times k$, D_k is $p \times p$, and $D_0 = G_0$.

$\longrightarrow$ very nice numerical behavior

B. Hanzon, M.O., R. Peeters, INRIA report (2004)

The matrices U_k and V_k.

U_k and V_k are the $(p+1) \times (p+1)$ unitary matrices:

$$U_k = \begin{bmatrix} \xi_k u_k & I_p - (1 + \eta_k w_k) u_k u_k^* \\ \eta_k \overline{w_k} & \xi_k u_k^* \end{bmatrix}$$

$$V_k = \begin{bmatrix} \xi_k v_k & I_p - (1 - \eta_k) \frac{v_k v_k^*}{\|v_k\|^2} \\ \eta_k & -\xi_k v_k^* \end{bmatrix}$$

$$\xi_k = \frac{\sqrt{1 - |w_k|^2}}{\sqrt{1 - |w_k|^2 \|v_k\|^2}}, \quad \eta_k = \frac{\sqrt{1 - \|v_k\|^2}}{\sqrt{1 - |w_k|^2 \|v_k\|^2}}$$

Adapted chart

From a realization in Schur form (A lower triangular)

$$A = \begin{bmatrix} w_n & 0 & 0 \\ \vdots & \ddots & 0 \\ * & \cdots & w_1 \end{bmatrix}, \quad B = \begin{bmatrix} \sqrt{1 - |w_n|^2}\, u_n^* \\ \vdots \end{bmatrix}$$

by induction $\longrightarrow$ a chart in which $v_1 = v_2 = \ldots v_n = 0$ (Potapov factorization)

Doesn't work for real functions: $G(\bar{z}) = \overline{G(z)}$

Hanzon, Olivi, Peeters, ECC99

The RARL2 software

This software computes a stable rational L^2-approximation of specified order n to a multivariable transfer function given in one of the following forms:

- a realization
- a finite number of Fourier coefficients
- some pointwise values on the unit circle.

It has been implemented using standard MATLAB subroutines. The optimizer of the toolkit OPTIM is used to find a local minimum, given by a realization, of the nonlinear L^2-criterion.

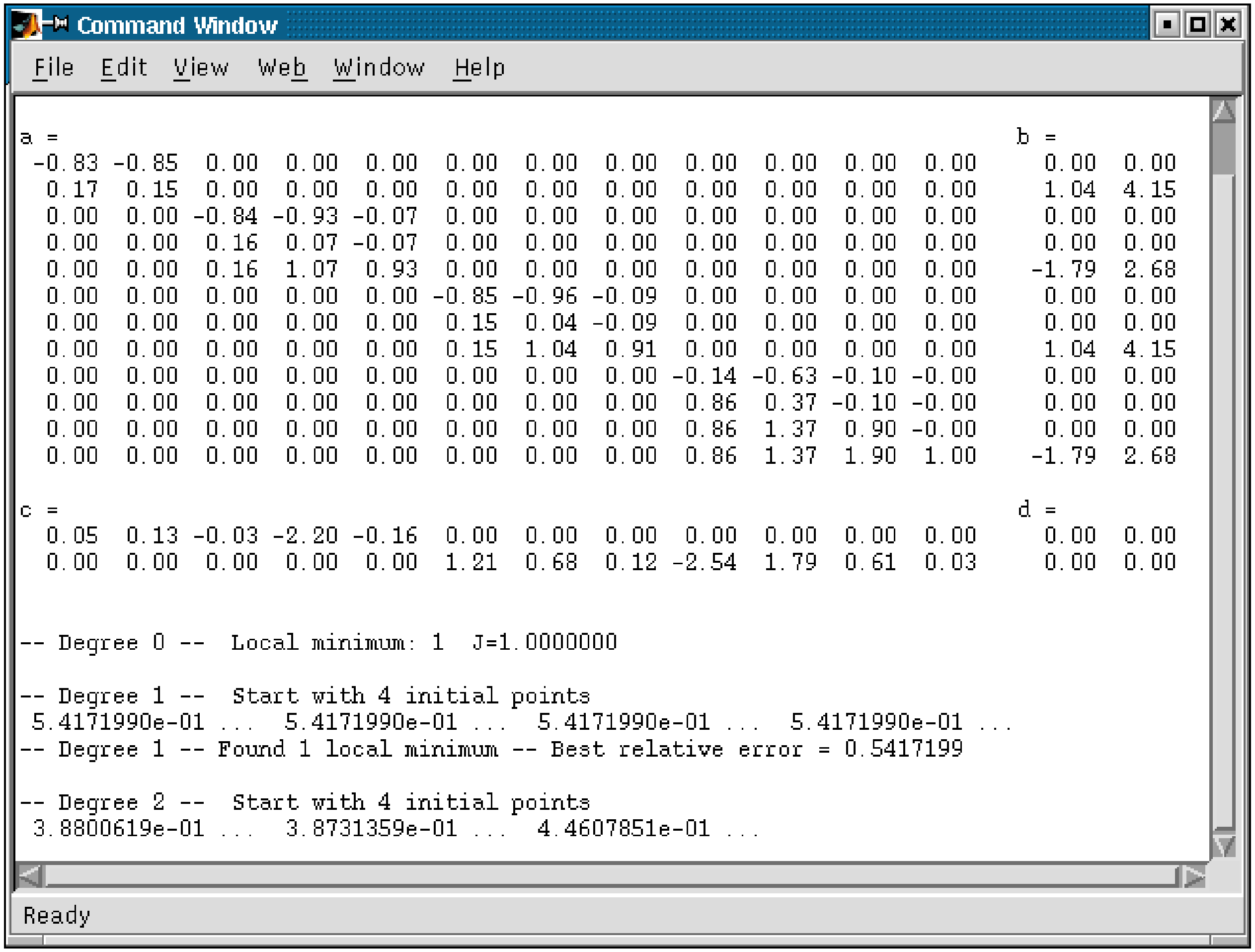
Implementation : J.P. Marmorat

`http://www-sop.inria.fr/apics/software.html`

Main features

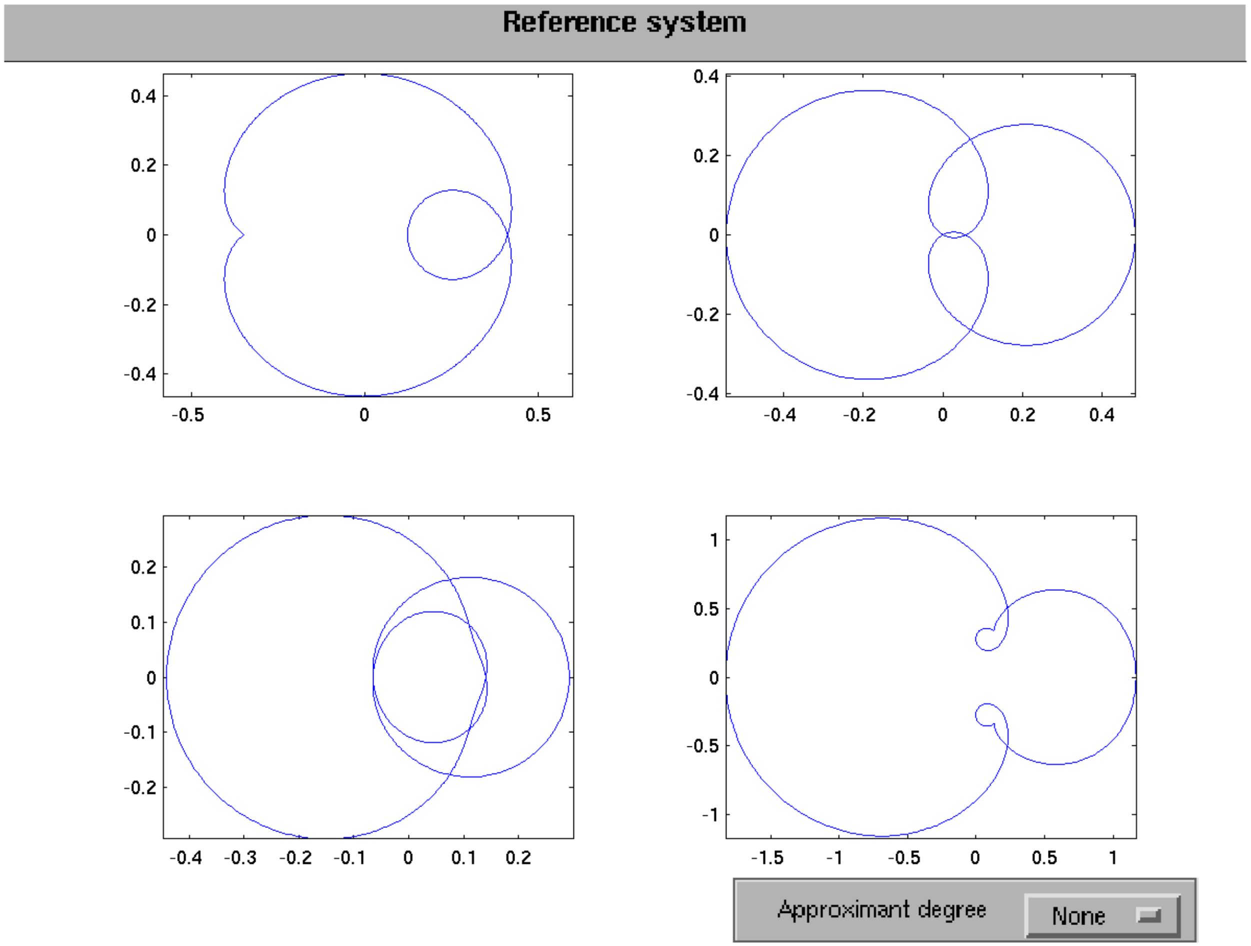
- state-space representation parametrized by Schur parameters
- optimization over a manifold
- adapted chart obtained from a realization in Schur form
- recursive search on the degree (optional):
  minimum of degree k $\longrightarrow$ initial point of degree $k+1$ (error-norm preserved)

Automobile gas turbine



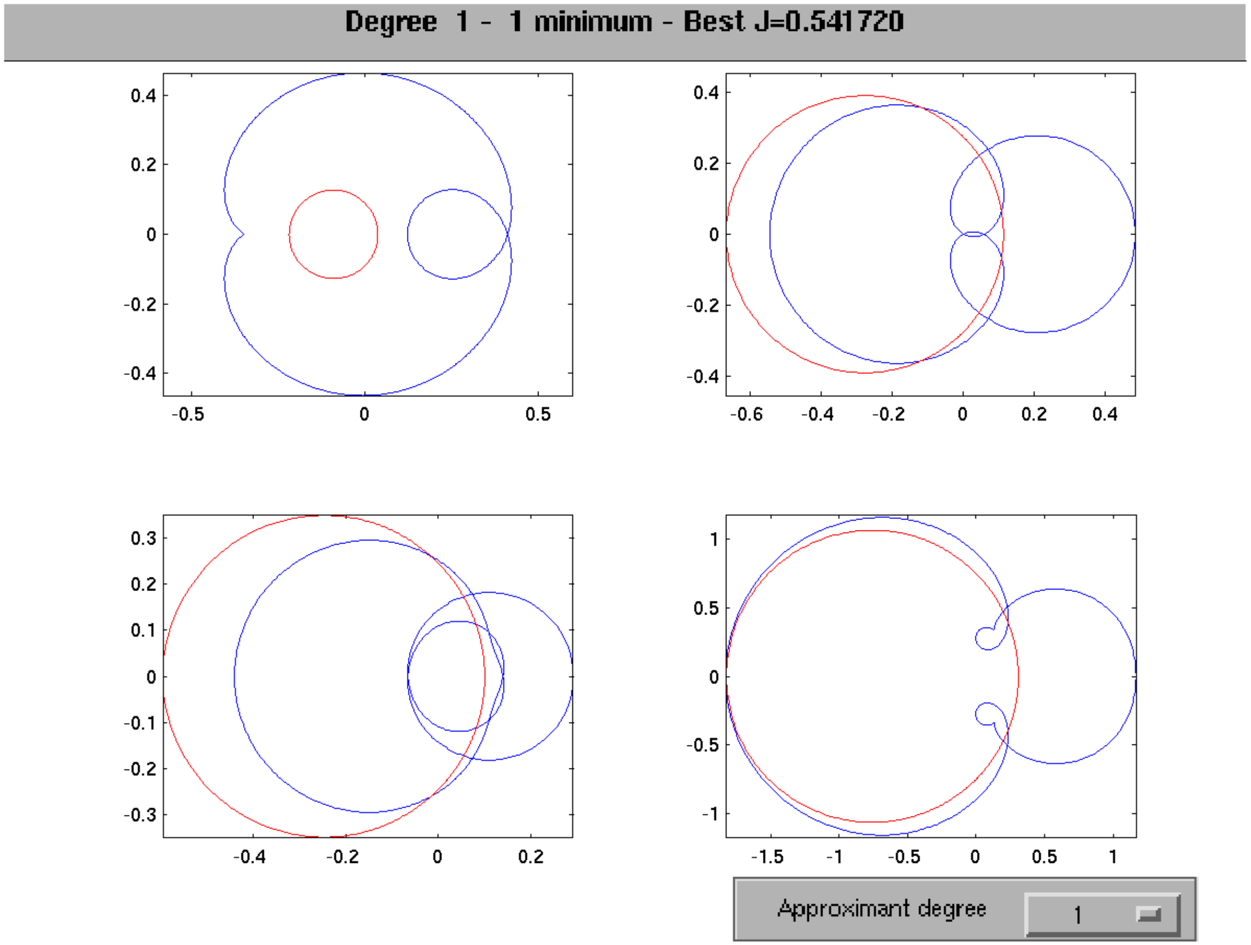
Hung, MacFarlane (1982) ; Glover (1984) ; Yan, Lam (1999)

Nyquist diagrams

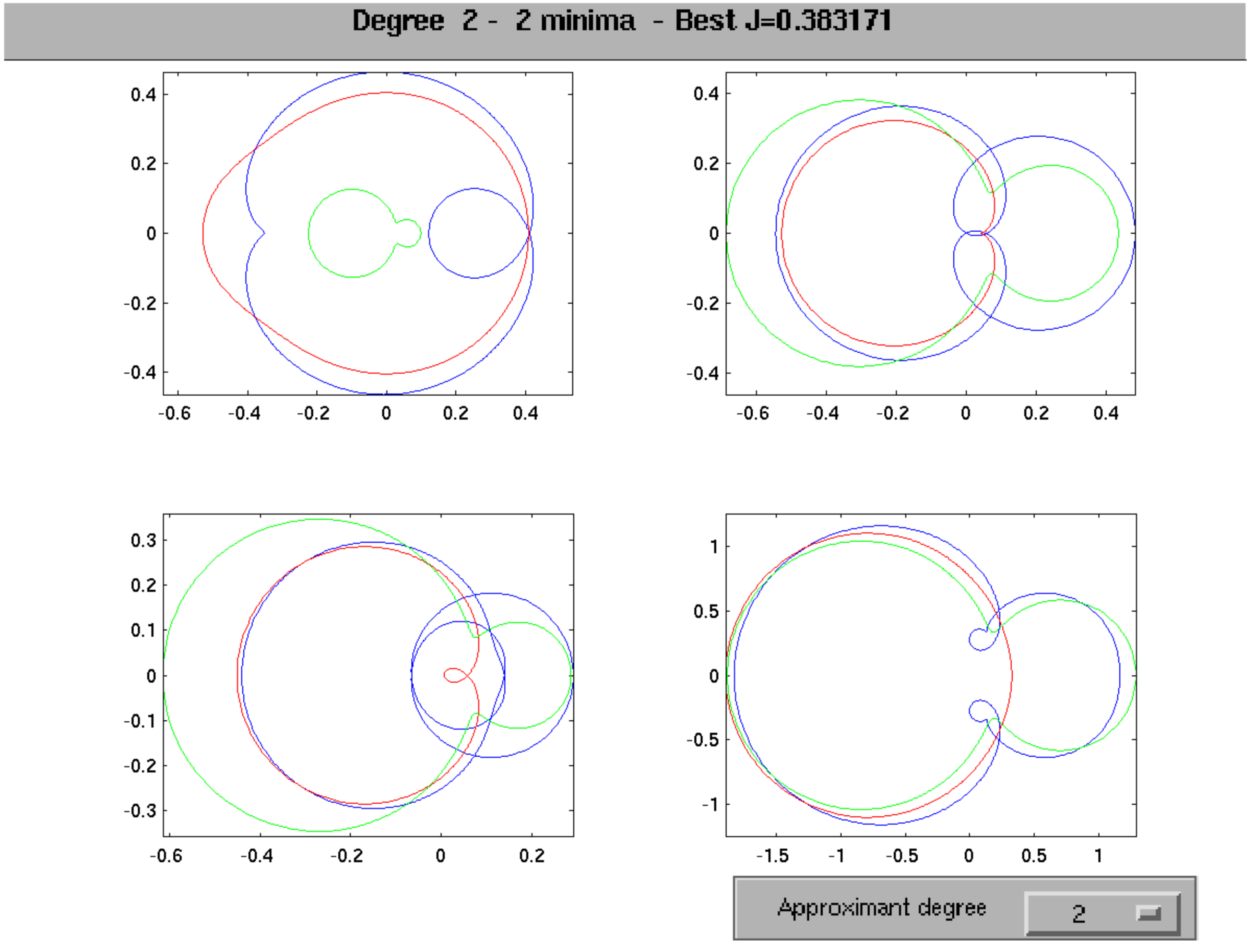


2×2; order 12.

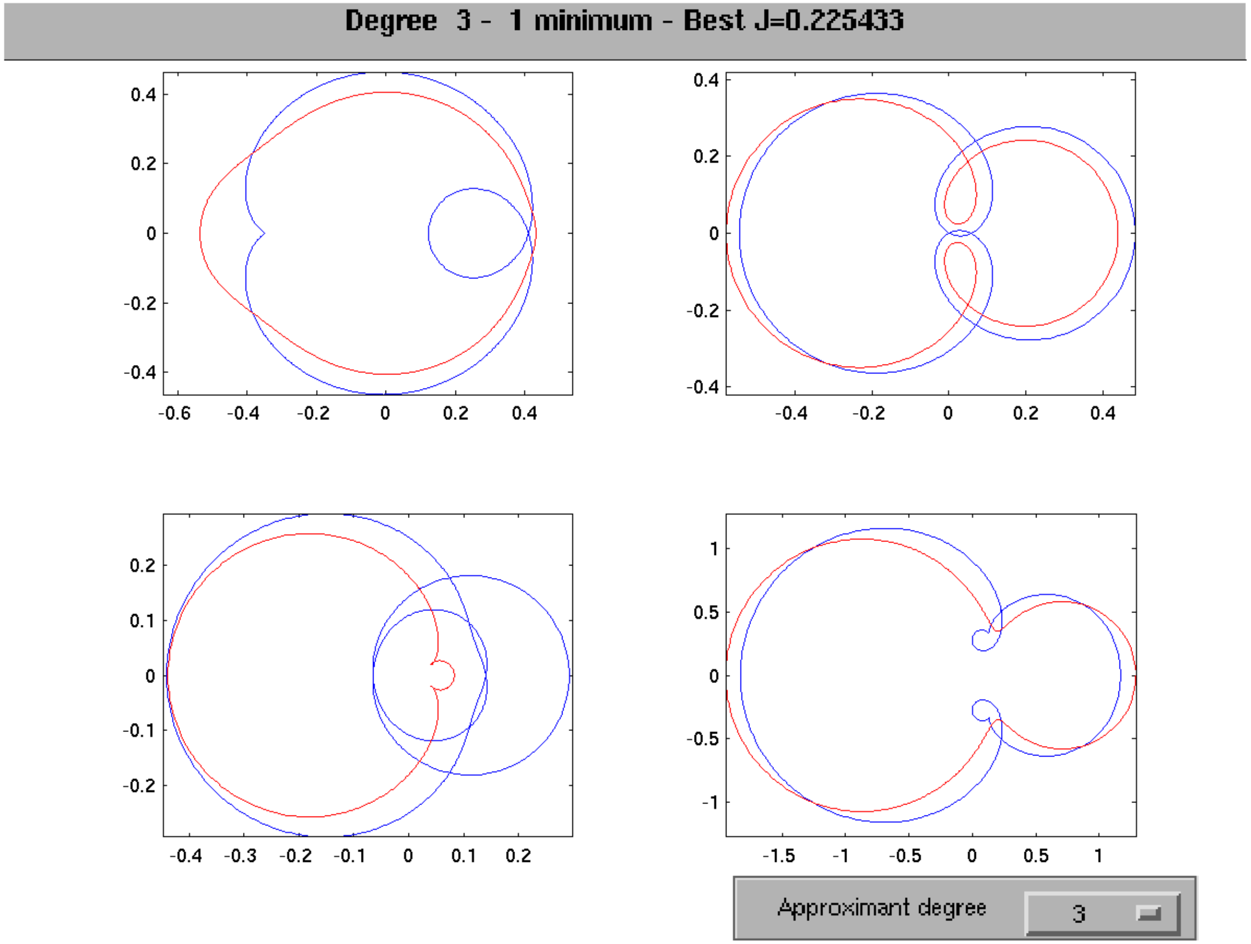
Approximants: order 1



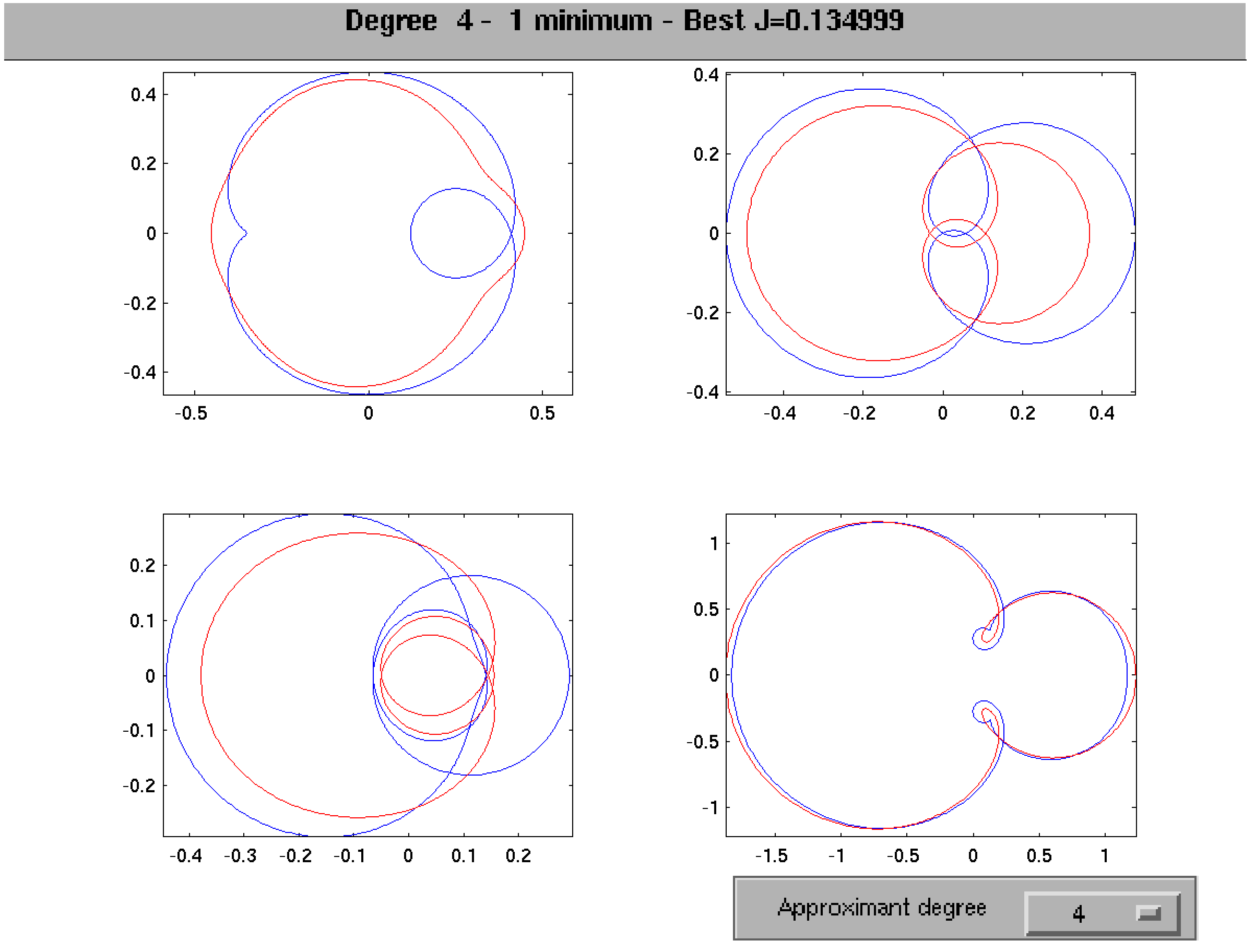
Approximants: order 2



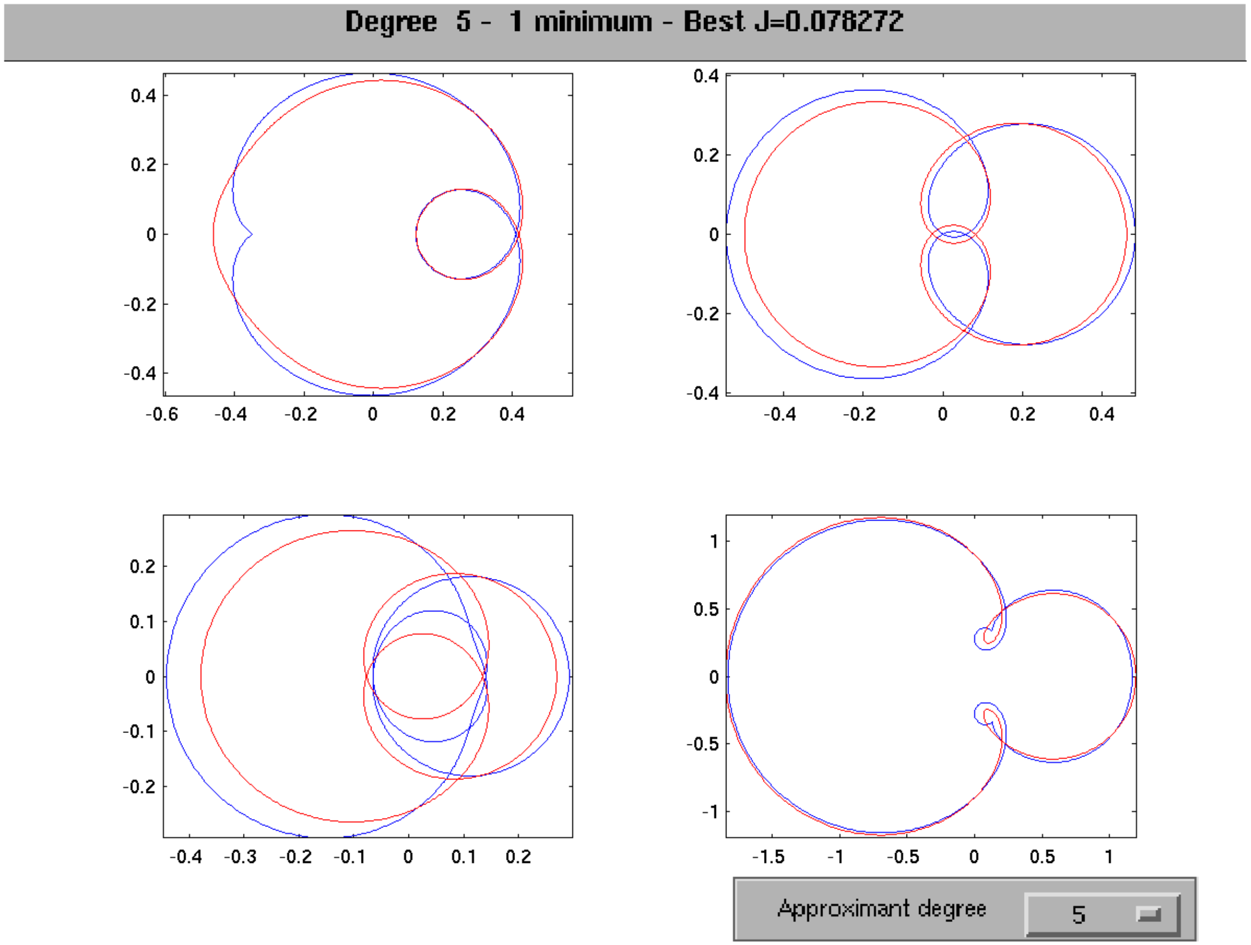
Approximants: order 3



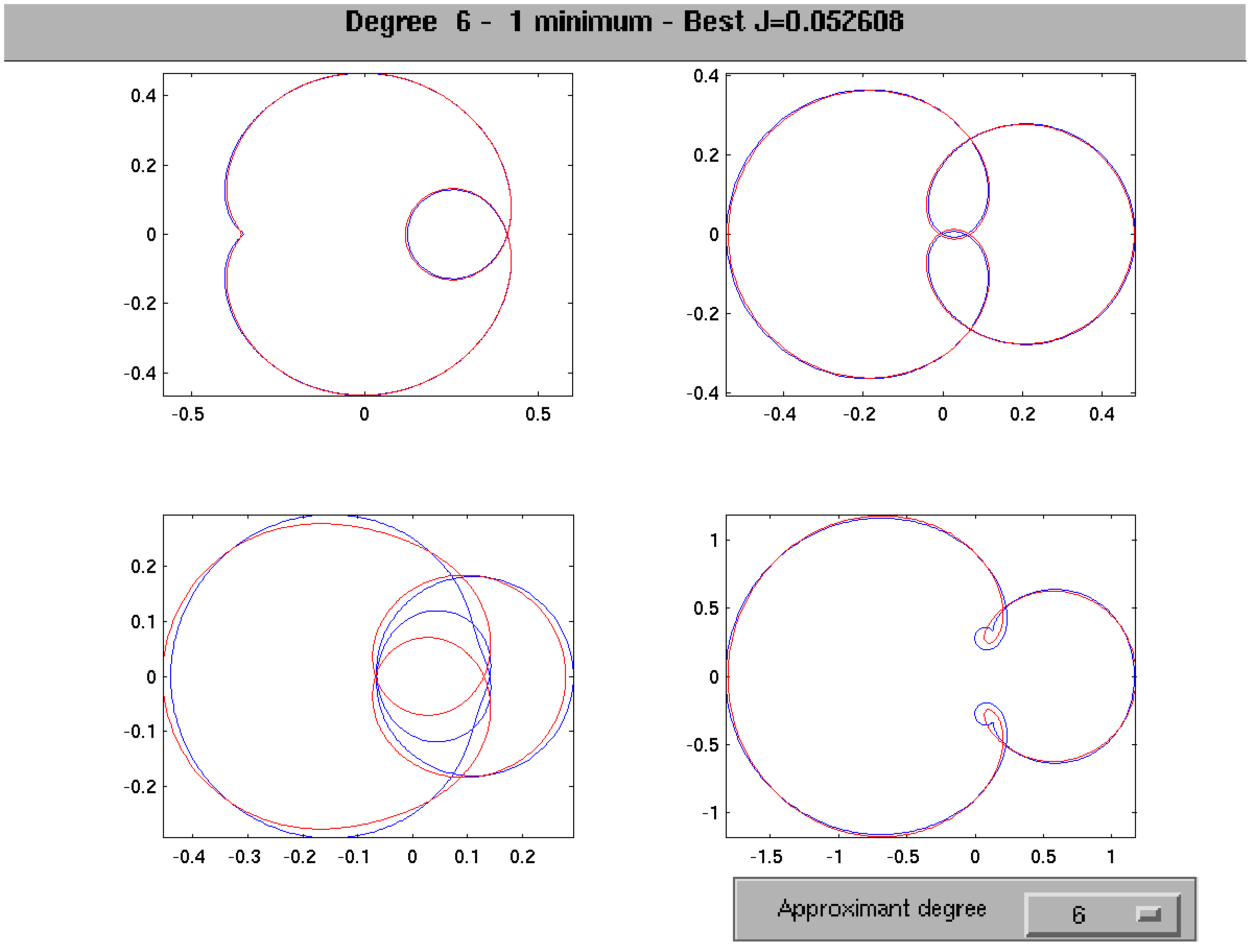
Approximants: order 4



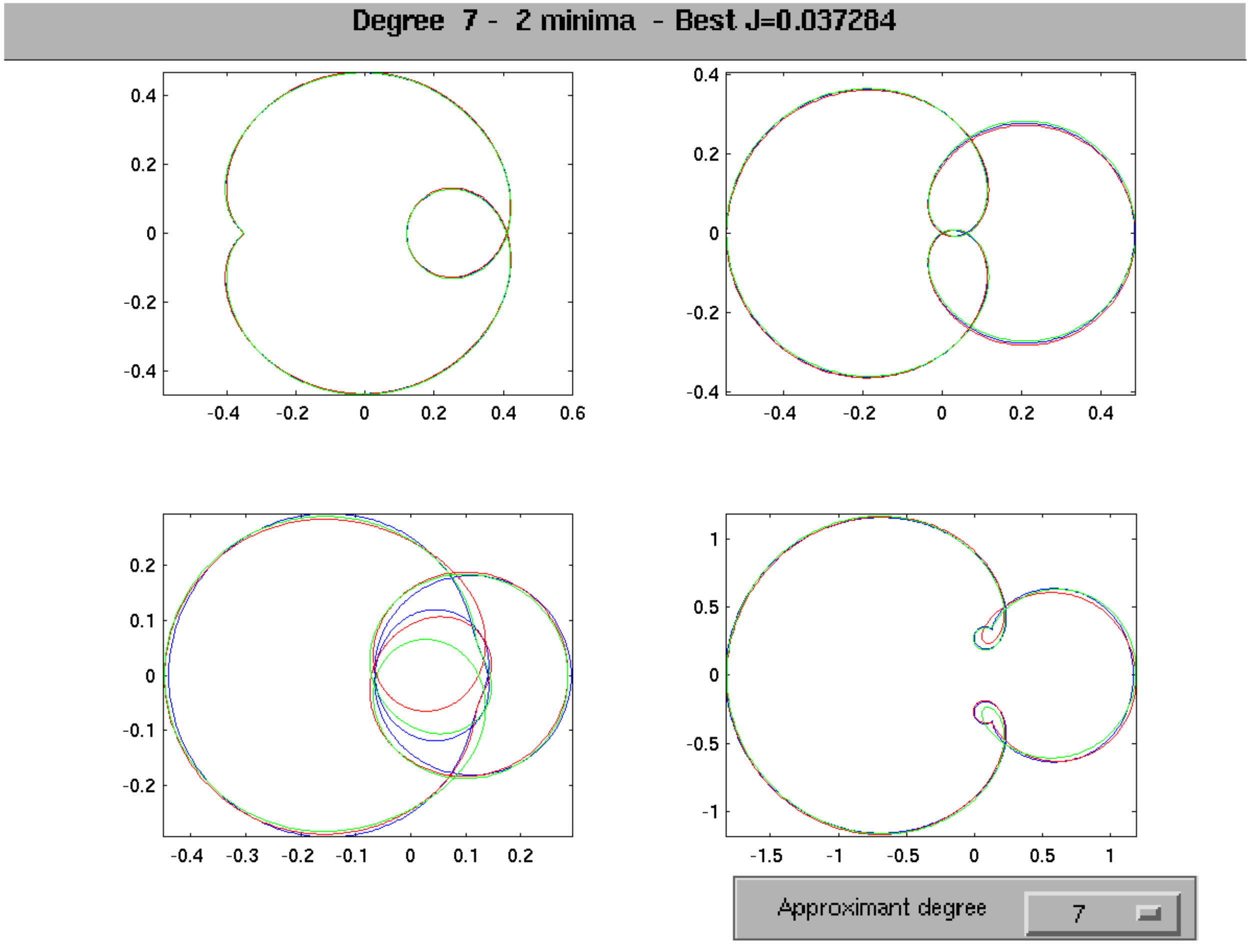
Approximants: order 5



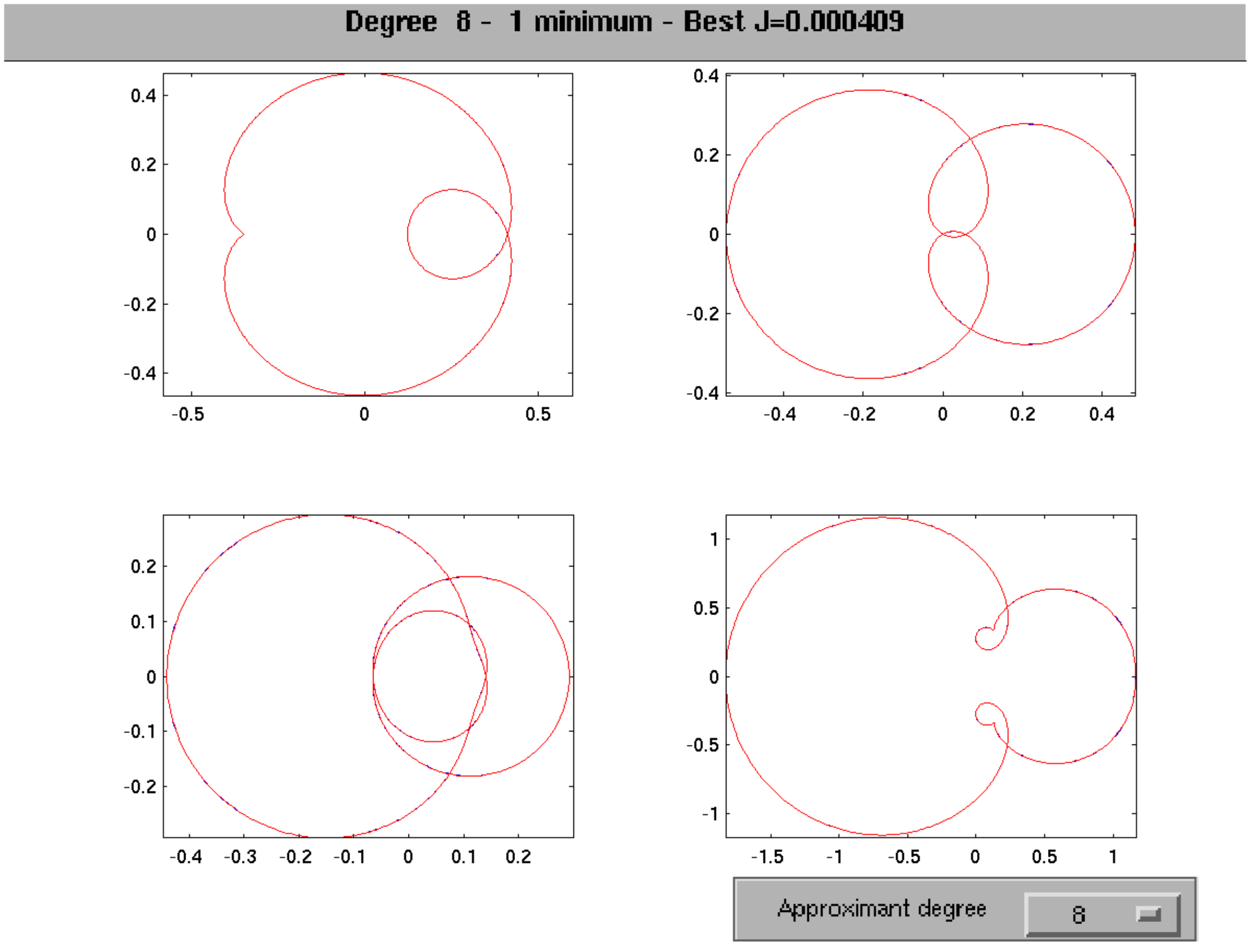
Approximants: order 6



Approximants: order 7



Approximants: order 8



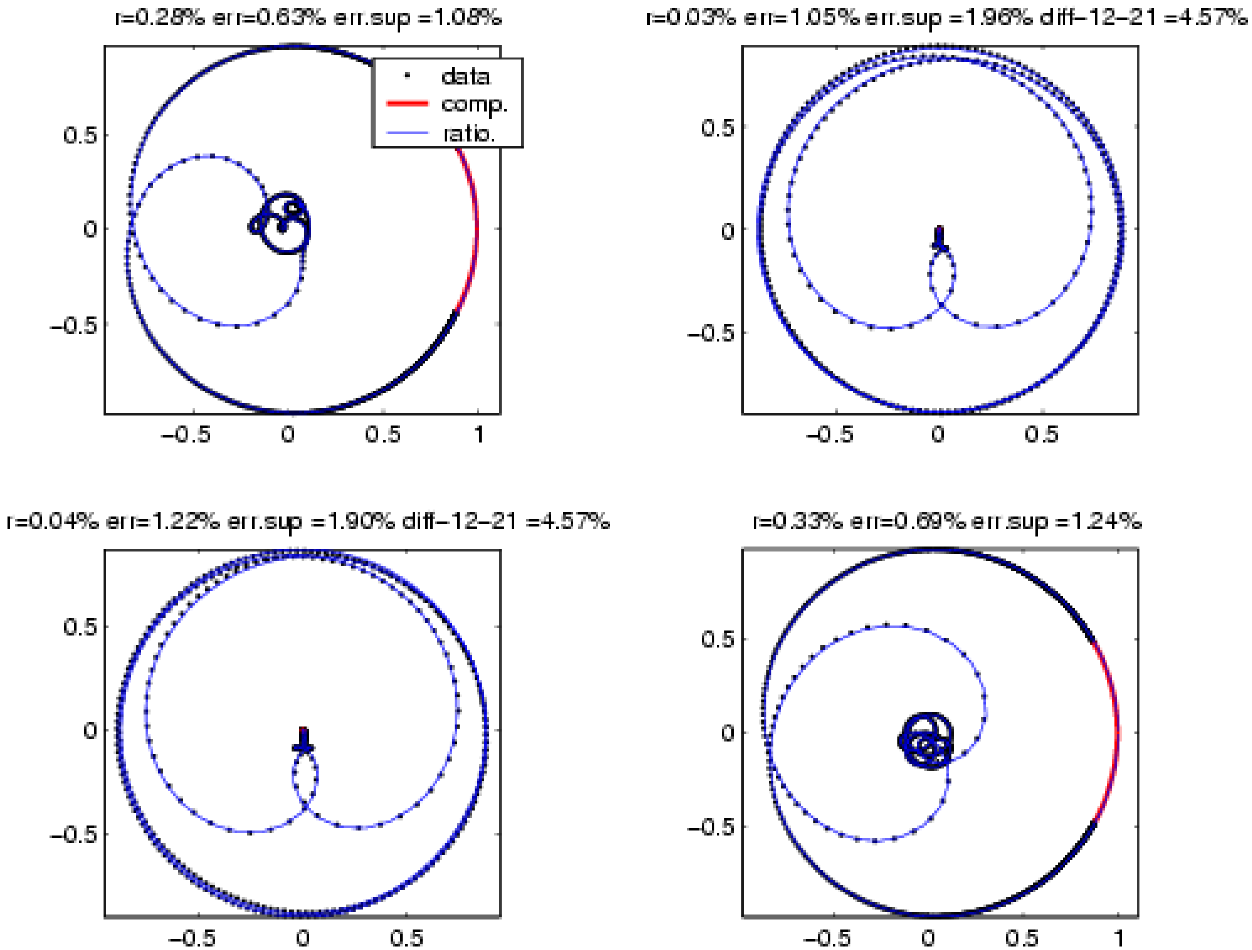
Hyperfrequency Filter

The problem: find a 8th order model of a MIMO (2×2) hyperfrequency filter, from experimental pointwise values in some range of frequencies provided by the CNES (French space agency).

Data pre-processing (interpolation/completion): compute a stable matrix transfer function of high order which approximates these data, given by a great number (800) of Fourier coefficients.

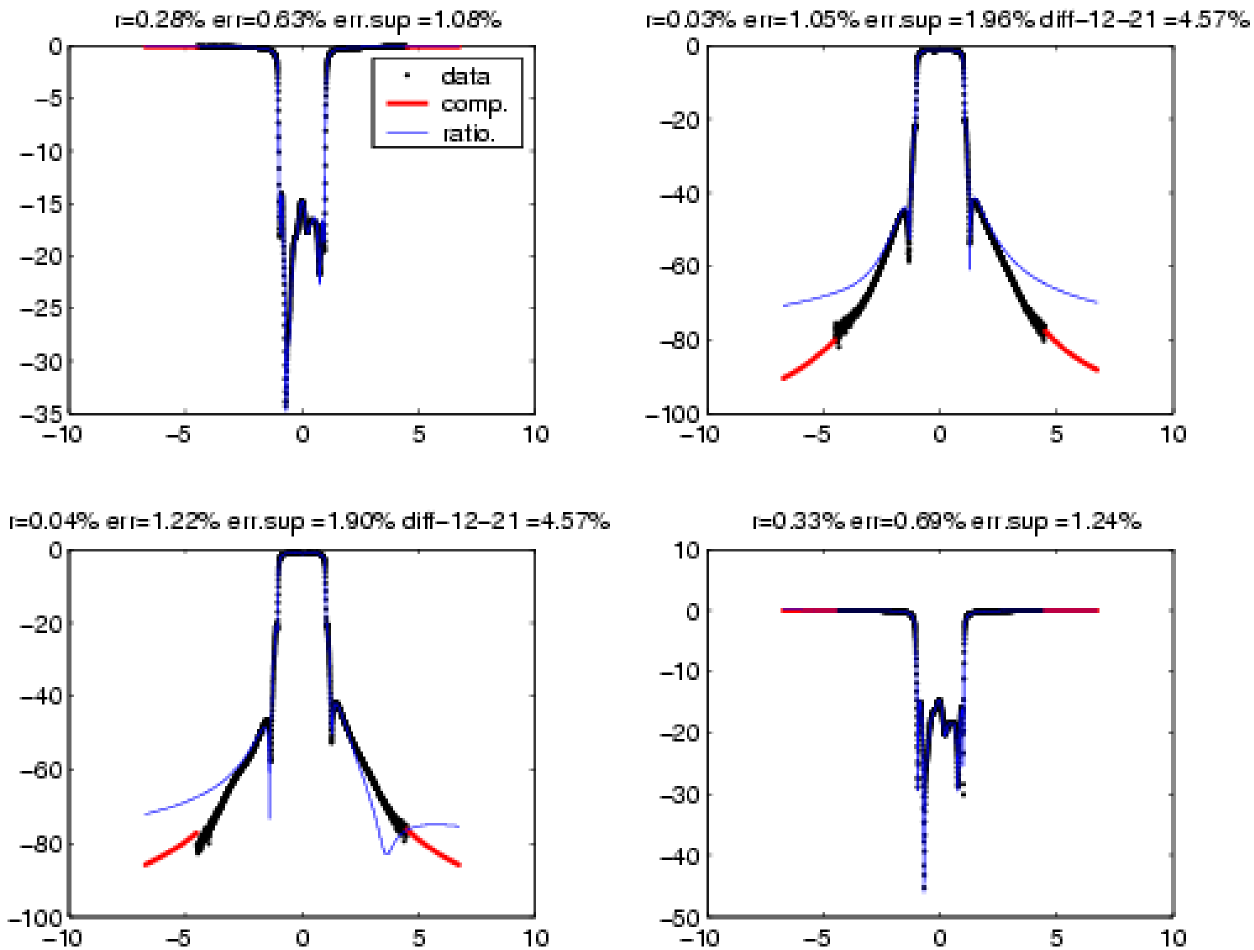
PRESTO-HF: software by F. Seyfert;
HYPERION: software by J. Grimm.

Data and approximant at order 8



Nyquist diagrams

Data and approximant at order 8



Bode diagrams

Nudelman interpolation.

$$\frac{1}{2i\pi}\int_{\mathbb{T}} G(1/z)U\left(z\,I - W\right)^{-1} dz = V,$$

- (U, W) output normal pair $W^*W + U^*U = I$ (normalization)
- condition for a solution : positive definite solution to Lyapunov eq.

$$P - W^*PW = U^*U - V^*V, \quad P > 0.$$

Then, G is the LFT of some K lossless

$$G = (\Theta_4 K + \Theta_3)(\Theta_2 K + \Theta_1)^{-1}.$$

$$\Theta(z) = \left[I_{2p} + (z-1)C(I - zW)^{-1}P^{-1}(I - W)^{-*}C^*J\right]H.$$

$$\Theta = \begin{bmatrix} \Theta_1 & \Theta_2 \\ \Theta_3 & \Theta_4 \end{bmatrix}, \quad J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \quad C = \begin{bmatrix} U \\ V \end{bmatrix}$$

Nudelman, 1977; Ball, Gohberg, Rodman, 1990

Atlases of charts

A chart: $(W_l, U_l), \ldots, (W_1, U_1)$ output normal pairs,

W_k: $n_k \times n_k$, $\quad U_k$: $p \times n_k$, $\quad \sum_k n_k = n$

$G \in \mathcal{L}_n^p$ is in the chart if we can construct :

$$G = G_l, \ldots, G_k \xrightarrow{LFT} G_{k-1}, \ldots, G_0.$$

where G_k satisfies Nudelman for (W_k, U_k, V_k) and $P_k > 0$.
The coordinate map: $G \to (V_1, \ldots, V_l)$

- $W_k = w_k, w_k \in \mathbb{D}$,
  Nudelman $\to G_k(1/\bar{w}_k)u_k = v_k, P_k > 0 : \|v_k\| < \|u_k\|$

For a suitable choice of the LFT's, balanced realizations computed recursively from interpolation data (W_k, U_k, V_k).

Charts for real lossless functions

W_k, U_k real, W_k: 1×1 or 2×2 with two conjugate eighenvalues.

Adapted chart : realization in real Schur form $\rightarrow V_1 = V_2 = \ldots V_l = 0$.

$$A = \begin{bmatrix} W_l & 0 & 0 \\ \vdots & \ddots & 0 \\ * & \cdots & W_1 \end{bmatrix}, \quad B = \begin{bmatrix} U_l^* \\ \vdots \end{bmatrix}$$

Allpass mutual encoding.

$W: \ n \times n \rightarrow$ 1-step Schur algorithm

chart $\mathcal{C}_\Omega$: $(W, U) \rightarrow \Omega(z) = Y + U(zI_n - W)^{-1}X \ \in \ \mathcal{L}_n^p$

$$P - W^*PW = U^*U - V^*V$$

- $G(z) \in \mathcal{C}_\Omega \Leftrightarrow P > 0$ (positive definite)
- adapted chart : $\Omega(z) = G(z)$, then $P = I \Leftrightarrow V = 0$

Implementation : J.P. Marmorat

A finite atlas.

A sub-atlas of Nevanlinna-Pick atlas (real case):

Charts:

$w_1 = w_2 = \cdots = w_n = 0; \quad u_1, u_2, \ldots, u_n$ standard basis vectors.

Balanced realization matrix= orthogonal positive p-upper Hessenberg matrix, up to a column permutation.

Connection with the canonical form of (Hanzon, Ober, 1998)

Peeters, Hanzon, M.O.

Hessenberg form.

$$
\begin{bmatrix}
v_n & M_n v_{n-1} & M_n M_{n-1} v_{n-2} & \dots & & M_n M_{n-1} \dots M_1 \\
\kappa_n & -v_n^* v_{n-1} & -v_n^* M_{n-1} v_{n-2} & \dots & & -v_n^* M_{n-1} \dots M_1 \\
0 & \kappa_{n-1} & -v_{n-1}^* v_{n-2} & \dots & & -v_{n-1}^* M_{n-2} \dots M_1 \\
0 & 0 & \kappa_{n-2} & & & -v_{n-2}^* M_{n-3} \dots M_1 \\
\vdots & \vdots & & \ddots & & \vdots \\
0 & 0 & \ddots & \kappa_2 & -v_2^* v_1 & -v_2^* M_1 \\
0 & 0 & \dots & 0 & \kappa_1 & -v_1^*
\end{bmatrix},
$$

$$
\kappa_k = \sqrt{1 - \|v_k\|^2}, \quad M_k = I_p + (\kappa_k - 1)\frac{v_k v_k^*}{\|v_k\|^2}.
$$

Conclusion.

- parametrizations which present a nice behavior with respect to optimization problems
- an interesting connection between these parametrizations and realizations with particular structure
- a rich framework $\rightarrow$ subclasses of functions or realizations with particular structure

optimization parameters $\neq$ physical parameters

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