Lossless scalar functions: boundary interpolation, Schur algorithm and Ober canonical form

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1. Introduction
2. Ober canonical form
3. Boundary interpolation
4. Connection with Ober form
5. Conclusion

Introduction

In this work some links are stressed between two types of parametrizations of lossless functions [1]:

interpolation data	structured realizations
$g(w) = v$	$D + C\,(z\,I_n - A)^{-1}B$
parameters: v	Hessenberg Schwarz/Ober
optimization parameters	physical parameters

Why lossless functions ?

[1] A lossless function is the transfer function of a conservative system: $|g(z)| < 1$ for z in the analyticity domain and $|g(z)| = 1$ on its boundary

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- Usefull in many applications (H^2 approximation, SLS identification, representations of orthogonal filter banks)
- Generalization to other classes of functions from the observable pair $[B\ \ A]$

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Discrete-Time: Hessenberg canonical form.

A lossless function $G(z)$ of degree n has a (unique) realization such that $R = \begin{bmatrix} d & c \\ b & A \end{bmatrix}$ is orthogonal and $\begin{bmatrix} b & A \end{bmatrix}$ positive upper-triangular

It can be parametrize as follows:

$$\begin{bmatrix} \gamma_n & \kappa_n & 0 \\ \kappa_n & -\gamma_n & 0 \\ 0 & 0 & I_{n-1} \end{bmatrix} \underbrace{\left[\begin{array}{c|c} d & c \\ \hline * & \\ 0 & A \end{array}\right]}_{R} = \begin{bmatrix} 1 & 0 \\ 0 & R_{n-1} \end{bmatrix},$$

$|\gamma_n| < 1,\ \kappa_n = \sqrt{1 - |\gamma_n|}^2, \gamma_n = d.$

By induction we get: $(R_k)_{k=n,\dots,0}$ realization (order k) in Hessenberg form and $(\gamma_k)_{k=n,\dots,0} \quad |R_0| = 1$.

B. Hanzon, R. Peeters (2000)

Connection with the Schur algorithm.

Let $g_k(z) = \gamma_k + c_k(zI_k - A_k)^{-1}b_k$ be the lossless function whose realization is R_k.
Then g_k satisfies the interpolation condition

$$g_k(\infty) = \gamma_k, \quad |\gamma_k| < 1.$$

Moreover,

$$g_k(z) = \frac{\gamma_k\, z + g_{k-1}(z)}{z + \bar{\gamma}_k g_{k-1}(z)} \quad \Leftrightarrow \quad g_{k-1}(z) = \frac{(g_k(z) - \gamma_k)z}{1 - \bar{\gamma}_k\, g_k(z)}.$$

This is the Schur algorithm: lossless functions of McMillan degree n are parametrized by the interpolation values $\gamma_n, \gamma_{n-1}, \dots, \gamma_1$ and $\gamma_0 = R_0$.

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Continuous-time: Ober canonical form

Any lossless function $G(s)$ of McMillan degree n has a unique balanced realization $G(s) = D + C(sI_n - A)^{-1}B$ parametrized by $\epsilon = \pm 1$, and $\beta, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ positive real numbers:

$$
\begin{aligned}
B^T &= \begin{bmatrix} \beta & 0 & \cdots & 0 \end{bmatrix} = \epsilon C, \\
A &= \begin{bmatrix} -\frac{\beta^2}{2} & \alpha_1 & & & & \\ -\alpha_1 & 0 & \alpha_2 & & 0 & \\ & -\alpha_2 & 0 & & & \\ & & & \ddots & & \\ & 0 & & & 0 & \alpha_{n-1} \\ & & & & -\alpha_{n-1} & 0 \end{bmatrix} \qquad (1) \\
D &= -\epsilon.
\end{aligned}
$$

Ober 1987

Parametrization (Ober 1987)

$$
\begin{aligned}
A_{n-k} &= \begin{bmatrix} 0 & \alpha_{k+1} & & 0 & \\ -\alpha_{k+1} & 0 & \alpha_{k+2} & & \\ & -\alpha_{k+2} & & & \\ & & \ddots & \ddots & \\ & 0 & & 0 & \alpha_{n-1} \\ & & & -\alpha_{n-1} & 0 \end{bmatrix} \\
\Delta_{n-k}(s) &= \det(sI_{n-k} - A_{n-k})
\end{aligned}
$$

Δ_j monic polynomial, even (odd) for j even (odd) satisfies

$$\Delta_{n-k}(s) = s\Delta_{n-k-1}(s) + \alpha_{k+1}^2 \Delta_{n-k-2}(s), \tag{2}$$

Are the α_k interpolation values ?

$$G(s) = -\epsilon \frac{\Delta_n(s) - \frac{\beta^2}{2}\Delta_{n-1}(s)}{\Delta_n(s) + \frac{\beta^2}{2}\Delta_{n-1}(s)}, \quad G(\infty) = -\epsilon, \quad G'(\infty) = \epsilon\beta^2$$

Interpolation conditions on the stability border !

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Boundary Interpolation

Let $f(s)$ be a lossless function which satisfies the interpolation condition:

$$f(\sigma) = \xi, \quad \mathrm{Re}(\sigma) = 0, \quad |\xi| = 1$$

Then $\rho = -\overline{f(\sigma)} f'(\sigma)$ is a strictly positive number called the *angular derivative*. A well-posed interpolation problem is the ADI problem: find all the lossless functions $f(s)$ such that

$$\left\{ \begin{array}{rcl} f(\sigma) & = & \xi \\ f'(\sigma) & = & -\xi\rho \end{array} \right. \tag{3}$$

Ball, Gohberg and Rodman, *Interpolation of rational matrix functions*, 1990, ch. 21

Linear fractional transformations

For a rational 2×2 rational matrix function

$$\Theta(s) = \begin{bmatrix} \Theta_{11}(s) & \Theta_{12}(s) \\ \Theta_{21}(s) & \Theta_{22}(s) \end{bmatrix}$$

we define

$$T_\Theta(f) = [\Theta_{11} f + \Theta_{12}][\Theta_{21} f + \Theta_{22}]^{-1}.$$

Used to describe the solutions to interpolation problems:

- $\Theta(s)$ satisfies itself an interpolation condition
- $\Theta(s)$ J-lossless: $\left\{ \begin{array}{lcll} \Theta(s)^* J \Theta(s) & \leq & J, & \mathrm{Re}\ s > 0 \\ \Theta(s)^* J \Theta(s) & = & J, & \mathrm{Re}\ s = 0 \end{array} \right.$,
  $J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$
  Then T_Θ maps a lossless function to a lossless function.

Solutions to the ADI problem

The ADI problem (3) always has a solution (since $\rho > 0$). All the solutions are given by

$$f = T_{\theta_{\sigma,\rho,\xi}}(g)$$

where

- $\Theta_{\sigma,\rho,\xi}$ is the J-lossless function

$$\Theta_{\sigma,\rho,\xi}(s) = J - \frac{1}{(s-\sigma)\rho} \begin{bmatrix} \xi \\ 1 \end{bmatrix} \begin{bmatrix} \xi \\ 1 \end{bmatrix}^* \tag{4}$$

- $g(s)$ is a lossless function such that $g(\sigma) \neq -\xi$

We have that $\deg f = \deg g + 1$.

Schur algorithm and parametrizations

Schur algorithm: let $\sigma = 0$ and f lossless: for $j = n, n-1, \ldots, 1$ put

$$\begin{cases} \xi_j & = & f_j(0) \\ \rho_j & = & -\xi_j f_j'(0) \end{cases}$$

and let

$$\xi_j f_{j-1} = T^{-1}_{\Theta_{\sigma_j, \rho_j, \xi_j}}(f_j).$$

Using the reverse Schur algorithm, we define a one-to-one map

$$(\xi_0, \xi_1, \ldots, \xi_n, \rho_1, \rho_2, \rho_n) \to f \quad \text{lossless}$$

on the domain: $\rho_j > 0$ and $\xi_j \neq -\xi_0$ unit complex number, $j = 1, \ldots, n$

Real functions (conjugate symmetry): $\xi_0 = \xi_1 = \cdots \xi_n = \pm 1$

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Connection with Ober form

To move the interpolation point at infinity, let $F_j(s) = f_j(1/s)$ and compute it using the reverse Schur algorithm with $\xi_0 = 1$. Let $a_j = \rho_j/2$.

$$\begin{aligned}
F_0(s) &= 1 \\
F_1(s) &= \frac{s - a_1}{s + a_1} \\
F_2(s) &= \frac{s^2 - a_2 s + a_1 a_2}{s^2 + a_2 s + a_1 a_2} \\
F_3(s) &= \frac{s^3 - a_3 s^2 + (a_1 a_2 + a_2 a_3)s - a_1 a_2 a_3}{s^3 + a_3 s^2 + (a_1 a_2 + a_2 a_3)s + a_1 a_2 a_3}
\end{aligned}$$

Connection with Ober form

Separating the odd and even part of the numerator and the denominator:

$$
\begin{aligned}
F_0(s) &= 1 \\
F_1(s) &= \frac{s - a_1}{s + a_1} \\
F_2(s) &= \frac{s^2 + a_1 a_2 - a_2 s}{s^2 + a_1 a_2 + a_2 s} \\
F_3(s) &= \frac{s^3 + (a_1 a_2 + a_2 a_3)s - a_3(s^2 + a_1 a_2)}{s^3 + (a_1 a_2 + a_2 a_3)s + a_3(s^2 + a_1 a_2)}
\end{aligned}
$$

Connection with Ober form

We get the following structure :

$$F_j(s) = \frac{P_j - a_j P_{j-1}}{P_j + a_j P_{j-1}}$$

- P_j monic polynomial, even (odd) for j even (odd)
- $P_{j+1}(s) = sP_j(s) + a_j a_{j+1} P_{j-1}(s)$
  Compare with (2) !

$$\beta^2 = \rho_n \quad \alpha_{n-j}^2 = a_j a_{j+1}$$

In the description of a ladder filter (Johns and all, 1989) the a_i's are the inverse of capacitor and inductor values.

LFTs and balanced realizations

In (Peeters, Hanzon, M.O., SSSC 2001) it was proved that if

$$\tilde{G}(s) = T_{\Theta(1/s)}(G(s))$$

then a realization of $\tilde{G}(s)$ can be computed from an *extended* realization of $G(s)$ by an LFT too !

$$\begin{bmatrix} \tilde{D} & \tilde{C} \\ \tilde{B} & \tilde{A} \end{bmatrix} = T_{\Phi}\left(\begin{bmatrix} D & 0 & C \\ 0 & 1 & 0 \\ B & 0 & A \end{bmatrix}\right)$$

Where Φ is a block matrix builts from a realization of $\Theta(s)$.

Recursive construction of Ober canonical form

A realization of $F_j(s) = D_j + C_j(sI_j - A_j)^{-1}B_j$ is obtained by the following recursion:

$$\begin{aligned}
A_j &= \begin{bmatrix} -\rho_j/2 & \frac{\sqrt{\rho_j}}{2}C_{j-1} \\ \frac{\sqrt{\rho_j}}{2}B_{j-1} & A_{j-1} - \frac{B_{j-1}C_{j-1}}{2} \end{bmatrix}, \\
B_j &= \begin{bmatrix} \sqrt{\rho_j} \\ 0 \end{bmatrix}, \\
C_j &= \begin{bmatrix} -\sqrt{\rho_j} & 0 \end{bmatrix}, \\
D_j &= 1,
\end{aligned}$$

which gives the balanced canonical form of Ober, with $\epsilon = -1$.

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- Can be view as the "limit" of a classical multipoint Schur algorithm and used to parametrize Schur functions (passive systems)
- Boundary interpolation an useful tools for the parametrization of certain classes of stable systems ... never used in that way to our knowledge.
- Generalization to the MIMO case and connection with pivot structures under study