A Schur algorithm for symmetric inner functions

M. Olivi, B. Hanzon, R. Peeters

Sophia–Antipolis, France
December 2005

Symmetric inner functions

Schur function (discrete-time): $S(z)$ square rational, analytic and contractive in the open unit disk,

$$S(z)^* S(z) \leq I, |z| < 1$$

Inner function: $Q(z)$ is Schur and

$$Q(z)^* Q(z) = I, |z| = 1$$

Symmetric: $Q(z)^T = Q(z)$

Symmetric inner rational functions arise in the description of physical systems which satisfy the conservation and reciprocity laws.

Nevanlinna-Pick interpolation problem

Find a $p \times p$ Schur function S such that

$$S(w)u = v, \quad u,\ v \in \mathbb{C}^p,\ |w| < 1,\ \|u\| = 1,\ \|v\| < 1$$

The solutions can be parametrized via a linear fractional transformation

$$S = T_\Theta(R) = (\Theta_1\, R + \Theta_2)(\Theta_3\, R + \Theta_4)^{-1} \tag{1}$$

in which R is a $p \times p$ Schur function.

$\Theta = \begin{bmatrix} \Theta_1 & \Theta_2 \\ \Theta_3 & \Theta_4 \end{bmatrix}$ is built from the interpolation data (w, u, v).

It is J-inner: $\begin{array}{l} \Theta(z)^* J \Theta(z) \leq J,\ \ |z| < 1, \\ \Theta(z)^* J \Theta(z) = J,\ \ |z| = 1 \end{array}, \quad J = \begin{bmatrix} I_p & 0 \\ 0 & -I_p \end{bmatrix}$

Parametrizations via a Schur algorithm

$\mathcal{I}_n^p$ manifold of $p \times p$ inner functions of McMillan degree n

Lemma: $w \in \mathbb{C},\ |w| < 1,\ \ \exists\, u$ such that $\|Q(w)u\| < 1$.

Schur algorithm: $Q = Q_n,\ Q_{n-1}, \ldots, Q_k \stackrel{LFT}{\longrightarrow} Q_{k-1}, \ldots, Q_0$
Q_k has McMillan degree k and Q_0 is constant unitary. The LFT is built from an interpolation condition

$$Q_k(w_k)u_k = v_k,\ |w_k| < 1,\ \|v_k\| < \|u_k\| = 1$$

Atlas of charts (local parametrizations):
The w_i's and the u_i's define the chart while the v_i's are the Schur parameters of the function Q in the chart.

Choice of the LFT

$Q(w)u = v \Rightarrow Q = T_{\Theta H}(R) = T_{\Theta}(T_H(R))$

$$\Theta(z) = I_{2p} + (z-1)\frac{1-|w|^2}{1-\|v\|^2}\frac{\begin{bmatrix} v \\ u \end{bmatrix}\begin{bmatrix} v \\ u \end{bmatrix}^*}{(z-w)(1-\bar{w})}J$$

H arbitrary constant J-unitary, $H^*JH = J$

Kimura, IEEE trans. autom. control, 1987

$H = H(vu^*)$ circuit theoretical interpretation

D. Alpay, L. Baratchart, A. Gombani, *Op. Th. and Appl.*, 1994

$H = I_{2p}$ function space approach $Q_0 = Q(1)$.

Hanzon, Olivi, Peeters, ECC99

$H = H(w, u, v) \rightarrow$ nice construction of balanced realizations

LFT's which preserve symmetry

Let $\Theta(z)$ be a J-inner function such that

$$\bar{\Theta}(1/z) = K\Theta(z)K, \quad K = \begin{bmatrix} 0 & I_p \\ I_p & 0 \end{bmatrix}$$

Then, the linear fractional transformation T_Θ preserves symmetry

Every H constant J-unitary is of the form

$$H = H(E)\begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \quad H(E) = \begin{bmatrix} (I_p - EE^*)^{-1/2} & E(I_p - E^*E)^{-1/2} \\ E^*(I_p - EE^*)^{-1/2} & (I_p - E^*E)^{-1/2} \end{bmatrix}$$

P, Q unitary matrices, $H(E)$ Halmos extension (E strictly contractive)

$$\overline{H(E)} = KH(E)K \Leftrightarrow E \text{ symmetric}$$

Interpolation data for symmetric inner functions.

Q: $p \times p$ symmetric inner function, McMillan degree $n \geq 2$

To take into account symmetry:

$$\left\{ \begin{array}{rcl} Q(w)u & = & v \\ u^T Q(w) & = & v^T \end{array} \right.$$

$\longrightarrow$ two-sided Nevanlinna-Pick problem ... same interpolation point.

To be well-posed:

$$u^T Q'(w) u = \rho$$

Interpolation data: $\delta = (w, u, v, \rho) \longrightarrow$ interpolation conditions $\mathcal{C}(\delta)$

$\longrightarrow$ two-sided Nudelman problem Ball, Gohberg, Rodman, 1990

The solutions

Pick matrix: $\Lambda_\delta = \begin{bmatrix} \sigma & \bar{\rho} \\ \rho & \sigma \end{bmatrix}$

J-inner function:

$$\Theta_\delta(z) =$$
$$I_{2p} + (z-1)C \begin{bmatrix} (z-w)^{-1} & 0 \\ 0 & (1-z\bar{w})^{-1} \end{bmatrix} \Lambda_\delta^{-1} \begin{bmatrix} (1-\bar{w})^{-1} & 0 \\ 0 & (1-w)^{-1} \end{bmatrix} C^* J$$
$$C = \begin{bmatrix} v & -\bar{u} \\ u & -\bar{v} \end{bmatrix}$$

There exists an inner function satisfying $\mathcal{C}(\delta)$ if and only if Λ_δ is positive definite. Then

$$Q = T_{\Theta_\delta}(R),$$

for some inner function R of McMillan degree $n-2$

Sketch of the proof (1)

$\delta = (\mathbf{w} = \mathbf{0}, \mathbf{u}, \mathbf{v} = \mathbf{0}, \rho), \quad |\rho| < \mathbf{1}$

Taylor series: $Q(z) = Q(0) + zQ'(0) + \dots$
SVD: $Q(0) = V\text{diag}(0, \dots, 0, \lambda_1, \dots, \lambda_r)U^*$

Since $Q(0)u = 0$, choose $U = [u \ \cdots]$ and $B(z) = U\text{diag}(z, 1, \dots, 1)U^*$

$$\begin{aligned} Q_1(z) &= Q(z)B(z)^{-1} \\ &= Q(0)U\text{diag}(1/z, 1, \dots, 1)U^* + Q'(0)U\text{diag}(1, z, \dots, z)U^* \dots \\ &= Q(0) + Q'(0)U\text{diag}(1, z, \dots, z)U^* + z \times \dots \end{aligned}$$

and Q_1 satisfies the Nevanlinna-Pick interpolation condition

$$u^T Q_1(0) = u^T Q'(0)uu^* = \rho u^*$$

Sketch of the proof (2)

$\delta = (\mathbf{w}, \mathbf{u}, \mathbf{v} = \mathbf{0}, \rho)$

$$\beta_w(z) = \frac{(1-\bar{w})(z-w)}{(1-w)(1-\bar{w}z)}, \quad \beta_w(w) = 0,$$

Then $\check{Q}(\beta_w(z)) = Q(z)$ satisfies $\mathcal{C}(\check{\delta})$

$$\check{\delta} = (0, u, v = 0, \check{\rho}), \quad \check{\rho} = \rho(1 - |w|^2)\frac{1-w}{1-\bar{w}}$$

$\delta = (\mathbf{w}, \mathbf{u}, \mathbf{v}, \rho)$

Assume $E = Q(w)$ stricktly contractive (symmetric)

Then $\hat{Q} = T_{H(-E)}(Q)$ satisfies $\mathcal{C}(\hat{\delta})$

$$\check{\delta} = (w, \hat{u}, v = 0, \hat{\rho}), \quad \hat{u} = (I - EE^*)^{1/2}\frac{u}{\sqrt{1 - \|v\|^2}}, \quad \hat{\rho} = \frac{\rho}{1 - \|v\|^2}$$

Symmetric Potapov factorization

If both v and ρ are zero: $\delta = (w, u, 0, 0)$, the linear fractional representation $Q = T_{\Theta_\delta}(R)$ is a symmetric Potapov factorization

$$Q(z) = B_w(z)^T R(z) B_w(z)$$

B_w Blaschke factor:

$$B_w(z) = I_p + (\beta_w(z) - 1)uu^*, \;\; \beta_w(z) = \frac{(1-\bar{w})(z-w)}{(1-w)(1-\bar{w}z)}$$

Schur algorithm

$$Q = Q_n \xrightarrow{\delta_n} Q_{n-1} \xrightarrow{\delta_{n-1}} \cdots Q_0$$

where Q_0 is inner symmetric of degree 1 or 0.

degree 1:
$Q_0(z) = Y\left(I_p + (\beta_w(z) - 1)yy^*\right)Y^T, \; y \in \mathbb{R}^p, \|y\| = 1, Y$ unitary

degree 0:
$Q_0 = O\Lambda O^T, \; O$ real orthogonal , $\; \Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n), \; |\lambda_i| = 1.$

Choice of the LFT ? $\Theta = \Theta_\delta H(E)$, E symmetric
$\Theta_\delta(1) = I_{2p} \Rightarrow Q(1) = Q_0$
Application to SAW filters: chaining acoustic matrices corresponds to a Schur algorithm in which $\Theta = \Theta_\delta H(vu^*)$ where $H(vu^*)$ symmetric
E symmetric ? $\longrightarrow$ nice construction of realizations