Exponential Path Order $\mathrm{EPO}^\star$

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Outline

$\mathcal{B}$	polynomial time

Stephen Bellantoni and Stephen Cook
A new Recursion-Theoretic Characterization of the Polytime Functions.
CC, pages 97–110, 1992

Outline

Toshiyasu Arai and Naohi Eguchi
A new Function Algebra of EXPTIME Functions by Safe Nested Recursion.
TCL, pages 130–146, 2008

Outline

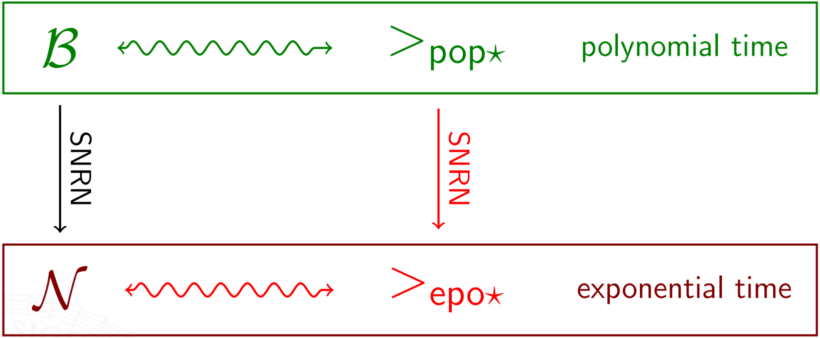
$\mathcal{B}$ $\leftsquigarrow\!\rightsquigarrow$ $>_{\mathsf{pop}\star}$ polynomial time

↓ SNRN

$\mathcal{N}$ exponential time

Martin Avanzini and Georg Moser
Complexity Analysis by Rewriting.
FLOPS '09, pages 130–146, 2008

Outline



this talk

The class $\mathcal{B}$

Recursion Theoretic Characterisation of FP

Alternative Characterisation of FP

Safe and Normal Argument Positions

Idea

break strength of recursion scheme by separation of arguments positions

$$\mathsf{f}(\underbrace{x_1, \dots, x_k}_{\text{normal}}; \underbrace{y_1, \dots, y_l}_{\text{safe}})$$

Alternative Characterisation of FP

Defining Functions by Recursion

Definition (Safe Recursion on Notation)

Suppose $g \in \mathcal{B}^{k,l}$ and $h_0, h_1 \in \mathcal{B}^{k+1,l+1}$. Then $f \in \mathcal{B}^{k+1,l}$ where

$$
\begin{aligned}
f(\epsilon, \overline{x} \,;\, \overline{y}) &= g(\overline{x} \,;\, \overline{y}) \\
f(zi, \overline{x} \,;\, \overline{y}) &= h_i(z, \overline{x} \,;\, \overline{y}, f(z, \overline{x} \,;\, \overline{y})) \qquad (i \in \{0,1\})
\end{aligned}
$$

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- . . . on Notation — recursion on binary representation

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$$\text{where } h_i(\epsilon, \overline{x}\,;\, \overline{y}) = j_i(\overline{x}\,;\, \overline{y})$$
$$h_i(zi, \overline{x}\,;\, \overline{y}) = k_{i,j}(z, \overline{x}\,;\, \overline{y}, h_i(z, \overline{x}\,;\, \overline{y}))$$

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Definition (Safe Composition)

Suppose $h \in \mathcal{B}^{k',l'}$, $\overline{r} \in \mathcal{B}^{k,0}$ and $\overline{s} \in \mathcal{B}^{k,l}$. Then $g \in \mathcal{B}^{k,l}$ where

$$g(\overline{x}\,;\,\overline{y}) = h(\overline{r(\overline{x};)}; \overline{s(\overline{x}\,;\,\overline{y})})$$

Alternative Characterisation of FP

Bellantoni and Cook, 1992

Definition

Class $\mathcal{B}$ is smallest class

1. containing certain initial functions
2. closed under safe recursion on notation
3. closed under safe composition

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Theorem (S. Bellantoni and S. Cook,1992)

$$\mathcal{B} = \mathsf{FP}$$

The class $\mathcal{N}$

Recursion Theoretic Characterisation of FEXP

Alternative Characterisation of FEXP

The class $\mathcal{N}$

$$\mathcal{N} \quad \approx \quad \mathcal{B} + \text{safe nested recursion on notation}$$

Alternative Characterisation of FEXP

Safe Nested Recursion on Notation

1. nesting of recursive function calls

$$f(\epsilon\,;\,y) = g(\,;\,y)$$
$$f(xi\,;\,y) = r_i(x\,;\,y, \qquad\qquad f(x\,;\,y)\quad)$$

Alternative Characterisation of FEXP

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$$f(\epsilon\,;\,y) = g(\,;\,y)$$
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\end{aligned}
$$

2. recursion on multiple parameters

$$
\begin{aligned}
f(\epsilon,\epsilon\,;\,\overline{z}) &= g(\,;\,\overline{z}) \\
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f(\epsilon,yi\,;\,\overline{z}) &= r_{\epsilon,j}(\epsilon,y\,;\,\overline{z}, f(x,y\,;\,s_{\epsilon,j}(\epsilon,y\,;\,f(\epsilon,y\,;\,\overline{z})))) \\
f(xi,yj\,;\,\overline{z}) &= r_{i,j}(x,y\,;\,\overline{z}, f(xi,y\,;\,s_{i,j}(x,\epsilon\,;\,f(x,yj\,;\,\overline{z}))))
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\end{aligned}
$$

3. lexicographic decreasing recursive parameters

$$
(xi,yj) >^1_{\mathsf{lex}} (xi,y) \qquad (xi,yj) >^1_{\mathsf{lex}} (x,yj)
$$

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- w is sequence of last bits (or ϵ) of $\overline{u}$

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  - $u >^1 v$ if and only if $u = vi$ for some i,

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  - $u >^1 v$ if and only if $u = vi$ for some i,
  - $(u_1, \ldots, u_n) >^1_{\text{lex}} (v_1, \ldots, v_n)$ if for some $1 \leqslant k \leqslant n$
    - $u_j = v_j$ for all $1 \leqslant j < k$, and
    - $u_k >^1 v_k$, and
    - for each $k < j \leqslant n$, $u_i \geqslant^1 v_j$ for some $1 \leqslant i \leqslant n$

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- arbitrary level of nesting allowed

Alternative Characterisation of FEXP

Eguchi, 2009

Definition

Class $\mathcal{N}$ is smallest class

1. containing the initial functions of $\mathcal{B}$
2. closed under safe nested recursion on notation
3. closed under weak safe composition

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Definition (Weak Safe Composition)

$$g(\overline{x}\,;\,\overline{y}) = h(x_{i_1},\ldots,x_{i_k}\,;\,\overline{s(\overline{x};\overline{y})})$$

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Theorem

$$\mathcal{N} = \mathsf{FEXP}$$

The order $>_{\mathsf{epo}\star}$

A Path Order based on $\mathcal{N}$

Polynomial Path Order $>_{\mathsf{pop}\star}$

- restriction of multiset path orders $>_{\mathsf{pop}\star} \subseteq >_{\mathsf{mpo}}$

Polynomial Path Order $>_{\text{pop}\star}$

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- induced by precedence *and* set of safe mapping

Polynomial Path Order $>_{pop\star}$

- restriction of multiset path orders $\qquad >_{pop\star} \subseteq >_{mpo}$
- induced by precedence *and* set of safe mapping

```
> tct -a irc -p -s "pop*" times.trs
  YES(?,POLY)

  'Polynomial Path Orders'
  ------------------------
  Answer:            YES(?,POLY)
  Input Problem:     innermost runtime-complexity with respect to
    Rules:
      {  *(x, s(y)) -> +(x, *(x, y))
       , *(x, 0()) -> 0()
       , +(s(x), y) -> s(+(x, y))
       , +(0(), y) -> y}
  Details:
    Rules in Predicative Notation:
     {  *(x, s(; y);) -> +(x; *(x, y;))
      , *(x, 0();) -> 0()
      , +(s(; x); y) -> s(; +(x; y))
      , +(0(); y) -> y}
    Precedence:
     * > +
    Safe Argument Positions:
     safe(+) = {2}, safe(*) = {}
```

Polynomial Path Order $>_{\mathsf{pop}\star}$

- restriction of multiset path orders $\qquad >_{\mathsf{pop}\star} \subseteq >_{\mathsf{mpo}}$
- induced by precedence *and* set of safe mapping

$\mathcal{R} \subseteq >_{\mathsf{pop}\star}$ $\rightsquigarrow$ "$\mathcal{R}$ obeys safe recursion and safe composition"

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$\rightsquigarrow$ innermost runtime complexity $\mathsf{rc}^{\mathsf{i}}_{\mathcal{R}}$ of $\mathcal{R}$ polynomial

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$\mathcal{R} \subseteq >_{\mathsf{pop}\star}$ $\rightsquigarrow$ "$\mathcal{R}$ obeys safe recursion and safe composition"

$\rightsquigarrow$ innermost runtime complexity $\mathsf{rc}^{\mathsf{i}}_{\mathcal{R}}$ of $\mathcal{R}$ polynomial

$$\mathsf{rc}^{\mathsf{i}}_{\mathcal{R}}(n) = \max\{\, \mathsf{dh}(t, \xrightarrow{\mathsf{i}}_{\mathcal{R}}) \mid \mathsf{size}(t) \leqslant n \text{ and arguments from } \mathcal{T}(\mathcal{C}, \mathcal{V}) \,\}$$

where $\mathsf{dh}(t, \to) = \max\{\ell \mid \exists (t_1, \ldots, t_\ell).\ t \to t_1 \to \cdots \to t_\ell\}$

Polynomial Path Order $>_{\text{pop}\star}$

- restriction of multiset path orders $>_{\text{pop}\star} \subseteq >_{\text{mpo}}$
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$\mathcal{R} \subseteq >_{\text{pop}\star}$ $\rightsquigarrow$ "$\mathcal{R}$ obeys safe recursion and safe composition"

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$\rightsquigarrow$ $\mathcal{R}$ defines only functions from $\mathcal{B} = \mathsf{FP}$

Exponential Path Order $>_{\mathsf{epo}\star}$

- restriction of lexicographic path orders $>_{\mathsf{epo}\star} \ \subseteq \ >_{\mathsf{lpo}}$
- induced by precedence *and* set of safe mapping

$\mathcal{R} \subseteq \ >_{\mathsf{epo}\star}$ $\rightsquigarrow$ "$\mathcal{R}$ obeys safe nested recursion and safe composition"

$\rightsquigarrow$ innermost runtime complexity $\mathsf{rc}^{\mathsf{i}}_{\mathcal{R}}$ of $\mathcal{R}$ exponential

$\rightsquigarrow$ $\mathcal{R}$ defines only functions from $\mathcal{N} = \mathsf{FEXP}$

Exponential Path Order $>_{\text{epo}\star}$

The Order $>_{\text{epo}\star}$

Let $s = f(s_1, \ldots, s_l \,;\, s_{l+1}, \ldots, s_m)$, $\succ$ be a precedence with $\forall c \in \mathcal{C}$ minimal

① $\dfrac{s_i \geqslant_{\text{epo}\star} t}{f(s_1, \ldots, s_l \,;\, s_{l+1}, \ldots, s_m) >_{\text{epo}\star} t}$ for some $1 \leqslant i \leqslant m$

Exponential Path Order $>_{\mathsf{epo}\star}$

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② $$\frac{s \sqsupset_{\mathsf{epo}\star} t_1 \quad \cdots \quad s \sqsupset_{\mathsf{epo}\star} t_k \qquad s >_{\mathsf{epo}\star} t_{k+1} \quad \cdots \quad s >_{\mathsf{epo}\star} t_n}{f(s_1, \ldots, s_l\ ;\ s_{l+1}, \ldots, s_m) >_{\mathsf{epo}\star} g(t_1, \ldots, t_k\ ;\ t_{k+1}, \ldots, t_n)} \quad \begin{array}{l} f \succ g, \\ f \in \mathcal{D} \end{array}$$

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$\sqsupset_{\mathsf{epo}\star} \subseteq >_{\mathsf{epo}\star}$

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Recall

Weak Safe Composition

$$g(\overline{x} \,;\, \overline{y}) \quad = \quad h(x_{i_1}, \ldots, x_{i_k} \,;\, \overline{s(\overline{x} \,;\, \overline{y})})$$

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The Order $>_{\mathsf{epo}\star}$

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② $$\frac{s \sqsupset_{\mathsf{epo}\star} t_1 \ \cdots \ s \sqsupset_{\mathsf{epo}\star} t_k \qquad s >_{\mathsf{epo}\star} t_{k+1} \ \cdots \ s >_{\mathsf{epo}\star} t_n}{f(s_1, \ldots, s_l \,;\, s_{l+1}, \ldots, s_m) >_{\mathsf{epo}\star} g(t_1, \ldots, t_k \,;\, t_{k+1}, \ldots, t_n)} \quad f \succ g,\ f \in \mathcal{D}$$

$\mathsf{e}^{\mathsf{e}}(x\,;\,) >^{?}_{\mathsf{epo}\star} \mathsf{e}(\mathsf{e}(x\,;\,)\,;\,)$ $\qquad$ $\mathsf{e}^{\mathsf{e}} \succ \mathsf{e} \quad \mathsf{e}^{\mathsf{e}}, \mathsf{e} \in \mathcal{D}$

Exponential Path Order $>_{\mathsf{epo}\star}$

The Order $>_{\mathsf{epo}\star}$

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$\mathsf{e}^{\mathsf{e}}(x\,;\,) \sqsupset^{?}_{\mathsf{epo}\star} \mathsf{e}(x\,;\,)$ $\qquad$ $\mathsf{e}^{\mathsf{e}} \succ \mathsf{e} \quad \mathsf{e}^{\mathsf{e}}, \mathsf{e} \in \mathcal{D}$

Exponential Path Order $>_{\mathsf{epo}\star}$

Auxiliary Order $\sqsupset_{\mathsf{epo}\star}$

① $$\frac{s_i \sqsupseteq_{\mathsf{epo}\star} t}{c(s_1, \ldots, s_l \,;\, s_{l+1}, \ldots, s_m) \sqsupset_{\mathsf{epo}\star} t}$$ $c \in \mathcal{C}$ and some $1 \leqslant i \leqslant m$

$\mathsf{e}^{\mathsf{e}}(x\,;\,) \sqsupset^{?}_{\mathsf{epo}\star} \quad \mathsf{e}(x\,;\,)$ $\qquad \mathsf{e}^{\mathsf{e}} \succ \mathsf{e} \quad \mathsf{e}^{\mathsf{e}}, \mathsf{e} \in \mathcal{D}$

Exponential Path Order $>_{\mathsf{epo}\star}$

Auxiliary Order $\sqsupset_{\mathsf{epo}\star}$

① $$\frac{s_i \sqsupseteq_{\mathsf{epo}\star} t}{c(s_1, \ldots, s_l \,;\, s_{l+1}, \ldots, s_m) \sqsupset_{\mathsf{epo}\star} t} \quad c \in \mathcal{C} \text{ and some } 1 \leqslant i \leqslant m$$

② $$\frac{s_i \sqsupseteq_{\mathsf{epo}\star} t}{f(s_1, \ldots, s_l \,;\, s_{l+1}, \ldots, s_m) \sqsupset_{\mathsf{epo}\star} t} \quad f \in \mathcal{D} \text{ and some } 1 \leqslant i \leqslant l$$

$$\mathsf{e}^{\mathsf{e}}(x\,;\,) \sqsupset^{?}_{\mathsf{epo}\star} \mathsf{e}(x\,;\,)$$

$$\mathsf{e}^{\mathsf{e}} \succ \mathsf{e} \qquad \mathsf{e}^{\mathsf{e}}, \mathsf{e} \in \mathcal{D}$$

Exponential Path Order $>_{\mathsf{epo}\star}$

Auxiliary Order $\sqsupset_{\mathsf{epo}\star}$

① $$\frac{s_i \sqsupseteq_{\mathsf{epo}\star} t}{c(s_1, \ldots, s_l \,;\, s_{l+1}, \ldots, s_m) \sqsupset_{\mathsf{epo}\star} t} \quad c \in \mathcal{C} \text{ and some } 1 \leqslant i \leqslant m$$

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$$x \quad \overset{?}{\sqsupset}_{\mathsf{epo}\star} \quad \mathsf{e}(x\,;\,) \qquad\qquad \mathsf{e}^{\mathsf{e}} \succ \mathsf{e} \quad \mathsf{e}^{\mathsf{e}}, \mathsf{e} \in \mathcal{D}$$

Exponential Path Order $>_{\text{epo}\star}$

Auxiliary Order $\sqsupset_{\text{epo}\star}$

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Exponential Path Order $>_{\mathsf{epo}\star}$

The Order $>_{\mathsf{epo}\star}$

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$\mathsf{e}^{\mathsf{e}}(x\,;\,) \not>_{\mathsf{epo}\star} \mathsf{e}(\mathsf{e}(x\,;\,)\,;\,)$ $\qquad$ $\mathsf{e}^{\mathsf{e}} \succ \mathsf{e} \quad \mathsf{e}^{\mathsf{e}}, \mathsf{e} \in \mathcal{D}$ $\qquad$ ☺

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Recall

Weak Safe Composition

$$g(\overline{x} \,;\, \overline{y}) \quad = \quad h(x_{i_1}, \dots, x_{i_k} \,;\, \overline{s(\overline{x} \,;\, \overline{y})})$$

Exponential Path Order $>_{\text{epo}\star}$

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Recall **Weak Safe Composition**

$$g(\overline{x} \,;\, \overline{y}) >_{\text{epo}\star} h(x_{i_1}, \dots, x_{i_k} \,;\, \overline{s(\overline{x} \,;\, \overline{y})}) \qquad \mathsf{g} \succ \mathsf{h}, \overline{\mathsf{s}}$$

☺

Exponential Path Order $>_{\text{epo}\star}$

The Order $>_{\text{epo}\star}$

Let $s = f(s_1, \ldots, s_l \,;\, s_{l+1}, \ldots, s_m)$, $\succ$ be a precedence with $\forall c \in \mathcal{C}$ minimal

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Recall — Safe Nested Recursion on Notation

$$f(xi \,;\, y) \quad = \quad r_i(x \,;\, y, f(x \,;\, s_i(x \,;\, y, f(x \,;\, y))))$$

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The Order $>_{\mathsf{epo}\star}$

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Recall

$f(xi \,;\, y) >_{\text{epo}\star}$

Safe Nested Recursion on Notation

$f(x \,;\, y)$

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The Order $>_{\text{epo}\star}$

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Exponential Path Order $>_{\mathsf{epo}\star}$

The Order $>_{\mathsf{epo}\star}$

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$f(xi \,;\, y) >_{\mathsf{epo}\star} \quad f(x \,;\, s_i(x \,;\, y, f(x \,;\, y)))$

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③ $$\frac{(s_1, \dots, s_l)\ (\sqsupset_{\mathsf{epo}\star})_{\mathsf{lex}}\ (t_1, \dots, t_l) \qquad s >_{\mathsf{epo}\star} t_{l+1} \ \cdots \ s >_{\mathsf{epo}\star} t_m}{f(s_1, \dots, s_l \,;\, s_{l+1}, \dots, s_m) >_{\mathsf{epo}\star} f(t_1, \dots, t_l \,;\, t_{l+1}, \dots, t_m)} \quad f \in \mathcal{D}$$

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Exponential Path Order $>_{\mathsf{epo}\star}$

Soundness and Completeness

Theorem

Every function from FEXP *is computed by some TRS compatible with an instance* $>_{\mathsf{epo}\star}$.

Exponential Path Order $>_{\mathsf{epo}\star}$

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Every function from FEXP *is computed by some TRS compatible with an instance* $>_{\mathsf{epo}\star}$.

Conjecture

Suppose $\mathcal{R} \subseteq >_{\mathsf{epo}\star}$ for constructor TRS $\mathcal{R}$. Then the innermost runtime complexity $\mathsf{rc}^{\mathsf{i}}_{\mathcal{R}}$ of $\mathcal{R}$ is bounded by an exponential.

Exponential Path Order $>_{\mathsf{epo}\star}$

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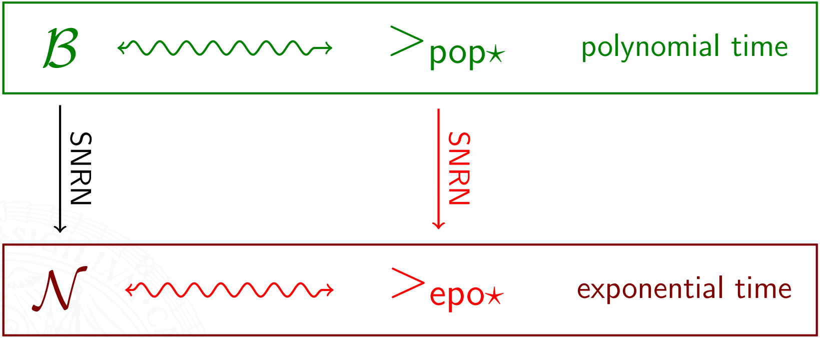
Every function from FEXP *is computed by some TRS compatible with an instance* $>_{\mathsf{epo}\star}$*.*

Theorem

Suppose $\mathcal{R} \subseteq >_{\mathsf{epo}\star}$ for constructor TRS $\mathcal{R}$. Then the innermost runtime complexity $\mathsf{rc}^{\mathsf{i}}_{\mathcal{R}}$ of $\mathcal{R}$ is bounded by an exponential.

- very sketchy, unpublished proof on paper

Conclusion


$\mathcal{B}$
$>_{\mathrm{pop}\star}$
polynomial time
SNRN
SNRN
$\mathcal{N}$
$>_{\mathrm{epo}\star}$
exponential time