Exponential Path Order EPO*

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polynomial time

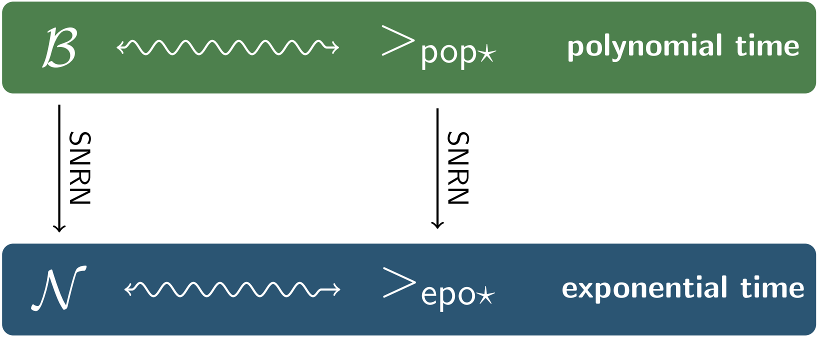
$\mathcal{B}$ polynomial time

Stephen Bellantoni and Stephen Cook
A new Recursion-Theoretic Characterization of the Polytime Functions.
CC, pages 97–110, 1992

$\mathcal{B}$ $\leftrightsquigarrow$ $>_{\mathsf{pop}\star}$ polynomial time

Martin Avanzini and Georg Moser
Complexity Analysis by Rewriting.
FLOPS '09, pages 130–146, 2008

Toshiyasu Arai and Naohi Eguchi
A new Function Algebra of EXPTIME Functions by Safe Nested Recursion.
TCL, pages 130–146, 2008



Martin Avanzini and Naohi Eguchi and Georg Moser
A Path Order for Rewrite Systems that Compute Exponential Time Functions.
RTA'11, pages 123–138, 2011

Main Result

Let FEXP denote class of functions computable in time $2^{O(n^k)}$ $(k \in \mathbb{N})$

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   Let $\mathcal{R}$ be a constructor TRS that computes a function f.
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Rewriting as Computational Model

We suppose ... in this talk

- $\mathcal{R}$ confluent and terminating
- signature $\mathcal{F}$ underlying TRS $\mathcal{R}$ partitioned into defined symbols $\mathcal{D}$ and constructors $\mathcal{C}$
- values $\mathcal{V}\mathsf{al} := \mathcal{T}(\mathcal{C}, \mathcal{V})$ are terms over constructors $\mathcal{C}$

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Definition

TRS $\mathcal{R}$ computes for each $\mathsf{f} \in \mathcal{D}$ partial function $f : \mathcal{V}\mathsf{al}^k \to \mathcal{V}\mathsf{al}_\perp$ s.t.

$$\forall \vec{s} \in \mathcal{V}\mathsf{al}^k \,.\, f(\vec{s}) = t \quad :\Longleftrightarrow \quad \mathsf{f}(\vec{s}) \to^!_{\mathcal{R}} t \text{ and } t \in \mathcal{V}\mathsf{al}$$

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Theorem

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$$\mathcal{R} \subseteq >_{\mathsf{epo}\star} \quad \Longrightarrow \quad \mathsf{rc}^{\mathsf{i}}_{\mathcal{R}} \in 2^{\mathsf{O}(n^k)}$$

$$\mathsf{rc}^{\mathsf{i}}_{\mathcal{R}}(n) = \max\{\, \mathsf{dh}(f(\vec{s}), \xrightarrow{\mathsf{i}}_{\mathcal{R}}) \mid \vec{s} \in \mathcal{V}\mathsf{al}^k \text{ and } f(\vec{s}) \text{ of size upto } n \,\}$$

$$\mathsf{dh}(t, \xrightarrow{\mathsf{i}}_{\mathcal{R}}) = \max\{\ell \mid \exists (t_1, \ldots, t_\ell).\ t \xrightarrow{\mathsf{i}}_{\mathcal{R}} t_1 \xrightarrow{\mathsf{i}}_{\mathcal{R}} \ldots \xrightarrow{\mathsf{i}}_{\mathcal{R}} t_\ell\}$$

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Ugo Dal Lago and Simone Martini
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Martin Avanzini and Georg Moser
Closing the Gap Between Runtime Complexity and Polytime Computability.
RTA'10, pages 33–48, 2010

The class $\mathcal{N}$

Syntactic, Recursion-theoretic Characterisation of FEXP

The class $\mathcal{N}$

is the smallest class . . .

1. containing certain initial function — projections, successors, . . .
2. closed under safe nested recursion on notation
3. closed under weak safe composition

Safe Recursion on Notation

- syntactical restriction of primitive recursion scheme

$$f(\underbrace{x_1, \ldots, x_k}_{\text{normal}}; \underbrace{y_1, \ldots, y_l}_{\text{safe}})$$

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- separates recursion parameters from recursively computed results

$$\begin{aligned} f(\epsilon, \vec{x}; \vec{y}) &= g(\vec{x}; \vec{y}) \\ f(\mathbf{zi}, \vec{x}; \vec{y}) &= h_i(z, \vec{x}; \vec{y}, \mathbf{f}(\mathbf{z}, \vec{\mathbf{x}}; \vec{\mathbf{y}})) \qquad (i \in \{0, 1\}) \end{aligned}$$

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$$
\begin{aligned}
\text{where } h_i(\epsilon, \vec{x}; \vec{y}, \mathbf{r}) &= r_i(\vec{x}; \vec{y}, \mathbf{r}) \\
h_i(\mathbf{zi}, \vec{x}; \vec{y}, \mathbf{r}) &= s_{i,j}(z, \vec{x}; \vec{y}, h_i(z, \vec{x}; \vec{y}, \mathbf{r}))
\end{aligned}
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no recursion on recursively computed result

Safe Nested Recursion on Notation

extends safe recursion on notation with . . .

1. nesting of recursive function calls — nested recursion

$$f(\epsilon; y) = g(; y)$$
$$f(xi; y) = r_i(x; y, f(x; s_i(x; y, f(x; \ldots))))$$

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2. simultaneous recursion on all normal arguments — multiple recursion

$$f(\epsilon, \epsilon; z) = g(; z)$$
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$$f(\epsilon, yj; z) = r_{\epsilon,j}(\epsilon, y; z, f(\epsilon, y; s_{\epsilon,j}(\epsilon, y; f(\epsilon, y; z))))$$
$$f(xi, yj; z) = r_{i,j}(x, y; z, f(xi, y; s_{i,j}(x, y; f(x, yj; z))))$$

- case analysis on least significant “bits” of recursion parameters

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- case analysis on least significant "bits" of recursion parameters
- lexicographic decreasing recursion parameters

Safe Composition

Requirements

1. composition maintains separation of safe and normal arguments

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employed in $\mathcal{B}$

$$f(\vec{x}; \vec{y}) = g(\vec{r}(\vec{x};); \vec{s}(\vec{x}; \vec{y}))$$

Safe Composition

Requirements

1. composition maintains separation of safe and normal arguments
2. reflects that FEXP is not closed under composition

Safe Composition

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$$f(\vec{x};\vec{y}) = g(\vec{r}(\vec{x};);\vec{s}(\vec{x};\vec{y}))$$

Weak Safe Composition

employed in $\mathcal{N}$

$$f(\vec{x};\vec{y}) = g(x_{i_1},\dots,x_{i_k};\vec{s}(\vec{x};\vec{y})) \qquad \{x_{i_1},\dots,x_{i_k}\} \subseteq \{\vec{x}\}$$

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Theorem

$$\mathcal{N} = \mathrm{FEXP}$$

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Exponential Path Order $>_{\mathsf{epo}\star}$

A Path Order based on $\mathcal{N}$

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- induced by precedence $>$ *and* safe mapping $\mathsf{safe} : \mathcal{F} \to 2^{\mathbb{N}}$

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```
> tct -s "epo*" fib.trs
  YES(?,EXPO)

  We consider the following Problem:
   Strict Trs:
     { fib(s(s(x)), y) -> fib(s(x), fib(x, y))
     , fib(s(0()), y)  -> s(y)
     , fib(0(), y)     -> s(y) }
   StartTerms: basic terms
   Strategy:   innermost

  The system is compatible with 'epo*' induced by
    Precedence:   fib > s,0
    Safe Mapping: safe(fib) = {2}, safe(s) = {1}
```

Exponential Path Order $>_{\mathsf{epo}\star}$

- induced by precedence $>$ *and* safe mapping $\mathsf{safe} \colon \mathcal{F} \to 2^{\mathbb{N}}$

Definition

precedence $>$ and safe mapping safe are admissible if

1. constructors are minimal $f > g \Rightarrow f \notin \mathcal{C}$
2. all argument positions of constructors are safe $f \in \mathcal{C} \Rightarrow \mathsf{safe}(f) = \{1, \ldots, \mathsf{ar}(f)\}$

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Notation

we suppose $\mathsf{safe}(f) = \{l + 1, \dots, l + m\}$, we write

$$f(s_1, \dots, s_l; s_{l+1}, \dots, s_{l+m})$$

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Preliminary Definition

Let $s = f(s_1, \dots, s_l; s_{l+1}, \dots, s_{l+m})$, let $>$ and safe be admissible.

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ST

$$\frac{s_i \geqslant_{\text{epo}\star} t}{s >_{\text{epo}\star} t}$$

WSC

$$\frac{\text{“}t_i \text{ are normal arguments of } s\text{”} \qquad s >_{\text{epo}\star} t_{k+1} \cdots s >_{\text{epo}\star} t_{k+n}}{s >_{\text{epo}\star} g(t_1, \ldots, t_k; t_{k+1}, \ldots, t_{k+n})}\ f > g$$

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Recall Weak Safe Composition ...

$$f(\vec{x}; \vec{y}) = g(x_{i_1}, \dots, x_{i_k}; \vec{s}(\vec{x}; \vec{y})) \qquad \{x_{i_1}, \dots, x_{i_k}\} \subseteq \{\vec{x}\}$$

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Recall Safe Nested Recursion on Notation ...

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Auxiliary Order $\sqsupset_{\mathsf{epo\star}}$

Order for $\mathcal{V}$al

$$\mathbf{ST_n}\ \frac{s_i \sqsupseteq_{\mathsf{epo\star}} t}{f(s_1, \dots, s_l; s_{l+1}, \dots, s_{l+m}) \sqsupset_{\mathsf{epo\star}} t}\ \text{if } f \in \mathcal{D} \text{ then } i \in \{1, \dots, l\}$$

Note

1. $\sqsupset_{\mathsf{epo\star}} = \rhd$ on $\mathcal{V}$al

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Note

1. $\sqsupset_{\mathsf{epo}\star} = \rhd$ on $\mathcal{V}$al
2. $\sqsupset_{\mathsf{epo}\star} \subsetneq \rhd$
   - if $f \in \mathcal{D}$, $\mathsf{safe}(f) = \{2\}$ then $f(x; z) \sqsupset_{\mathsf{epo}\star} x$ but $f(x; z) \not\sqsupset_{\mathsf{epo}\star} z$

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Let $s = f(s_1, \ldots, s_l; s_{l+1}, \ldots, s_{l+m})$, let $>$ and safe be admissible.

ST
$$\frac{s_i \geqslant_{\mathsf{epo}\star} t}{s >_{\mathsf{epo}\star} t}$$

WSC
$$\frac{s \sqsupset_{\mathsf{epo}\star} t_1 \quad \cdots \quad s \sqsupset_{\mathsf{epo}\star} t_k \qquad s >_{\mathsf{epo}\star} t_{k+1} \cdots s >_{\mathsf{epo}\star} t_{k+n}}{s >_{\mathsf{epo}\star} g(t_1, \ldots, t_k; t_{k+1}, \ldots, t_{k+n})} \; f > g$$

SNRN
$$\frac{(\dagger) \qquad s >_{\mathsf{epo}\star} t_{l+1} \quad \cdots \quad s >_{\mathsf{epo}\star} t_m}{s >_{\mathsf{epo}\star} f(t_1, \ldots, t_l; t_{l+1}, \ldots, t_{l+m})}$$

$(\dagger)$

1. $s_1 = t_1 \quad \cdots \quad s_{i-1} = t_{i-1}$,
2. $s_i \sqsupset_{\mathsf{epo}\star} t_i$, and
3. $s \sqsupset_{\mathsf{epo}\star} t_{i+1} \quad \cdots \quad s \sqsupset_{\mathsf{epo}\star} t_l$.

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The Good ...

the exponential path order $\mathsf{EPO}^{\star}$ is ...

- a restriction of LPO that induces exponentially bounded $\mathsf{rc}^{\mathsf{i}}_{\mathcal{R}}$
- sound *and* complete for FEXP, implemented in our tool $\mathsf{T_CT}$
  http://cl-informatik.uibk.ac.at/software/tct

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The Bad ...

- rules out some natural definitions:

$$\mathsf{d}(0) \to 0 \qquad \mathsf{e}(0) \to \mathsf{s}(0)$$

$$\mathsf{d}(\mathsf{s}(x)) \to \mathsf{s}(\mathsf{s}(\mathsf{d}(x))) \qquad \mathsf{e}(\mathsf{s}(x)) \to \mathsf{d}(\mathsf{e}(x))$$

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The Ugly ...

- $>_{\mathsf{pop}\star} \not\subseteq >_{\mathsf{epo}\star}$