Convergent Interpolation to Cauchy Integrals over Analytic Arcs of Jacobi-Type Weights

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Abstract. We consider multipoint Padé approximation to Cauchy transforms of complex measures. It is known [**?**] that if the support of a measure is an analytic Jordan arc and if the measure itself is absolutely continuous with respect to the equilibrium distribution of that arc with Dini-smooth non-vanishing density, then the diagonal multipoint Padé approximants associated with appropriate interpolation schemes converge locally uniformly to the approximated Cauchy transform in the complement of the arc. In this work we show that this convergence holds also for measures whose Radon-Nikodym derivative is a Jacobi weight modified by a Hölder continuous function. The asymptotic behavior of Padé approximants is deduced from the analysis of underlying non-Hermitian orthogonal polynomials, for which we use Riemann-Hilbert-$\bar\partial$ method.

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1. Introduction

2. Statements of Results

Let Δ be an analytic Jordan arc with endpoints -1 and 1. In other words, there exists a holomorphic univalent function Ξ, defined in some domain $D_\Xi \supset [-1,1]$, such that

$$\Delta = \Xi([-1,1]), \quad \Xi(\pm 1) = \pm 1.$$

We often shall say that Ξ is an analytic parameterization of Δ or analytically parametrizes Δ. We orient Δ from -1 to 1 and according to this orientation we distiguish the left, Δ^+, and the right, Δ^-, sides of Δ. It will be convenient for us to fix two unbounded arcs, say Δ_l and Δ_r, that connect $-\infty$ to -1 and $+\infty$ to 1, respectively, in such a manner that $\Delta_l \cup \Delta \cup \Delta_r$ is a smooth unbounded Jordan arc that coincides with the real line in some neighborhood of infinity. Put

$$w(z) = w(\alpha,\beta;z) := (1-z)^\alpha(1+z)^\beta, \quad \alpha,\beta > -1, \tag{2.1}$$

where we choose branches of $(1-z)^\alpha$ and $(1+z)^\beta$ that are holomorphic outside of Δ_l and Δ_r, respectively, and that assume value 1 at 0. In particular, w is analytic across $\Delta^\circ := \Delta \setminus \{\pm 1\}$. Further, set

$$\mathfrak{w}(z) := \sqrt{z^2-1}, \quad \mathfrak{w}(z)/z \to 1, \quad \text{as} \quad z \to \infty, \tag{2.2}$$

to be a holomorphic function outside of Δ. Then

$$\varphi(z) := z + \mathfrak{w}(z), \quad z \in D := \overline{\mathbb{C}} \setminus \Delta, \tag{2.3}$$

is holomorphic in $D \setminus \{\infty\}$, has continuous boundary values $\varphi^\pm$ on $\Delta^\pm$, respectively, and satisfies $2z/\varphi(z) \to 1$ as $z \to \infty$ and $\varphi^+\varphi^- = 1$ on Δ.

2.1. Symmetric Contours

The multipoint Padé approximants that we consider are rational interpolants in nature. As we want them to converge to the approximated function (Cauchy integral of a certain density given on Δ), we need to describe interpolation schemes compatible with Δ. We do it in terms of monic polynomials vanishing at the interpolation points.

Let $\{v_n\}$ be a sequence polynomials such that $\deg(v_n) \leq 2n$ and each v_n has no zeros on Δ. To this sequence we associate a sequence of functions, say $\{r_n\}$, given by

$$r_n(z) := \left(\frac{1}{\varphi(z)}\right)^{2n-\deg(v_n)} \prod_{v_n(e)=0} \frac{\varphi(z)-\varphi(e)}{1-\varphi(e)\varphi(z)}, \quad z \in D, \tag{2.4}$$

where the product is taken over all zeros of v_n according to their multiplicities. It is easy to see that each function r_n is holomorphic in D, has zeros at the zeros of v_n of the same order, and vanishes at infinity with order $2n - \deg(v_n)$. Hence, each r_n has exactly 2n zeros counting multiplicities. Moreover, the unrestricted boundary values $r_n^\pm$ exist everywhere on each side of Δ and satisfy $r_n^+ r_n^- \equiv 1$ by the corresponding property of φ.

Definition (Class $\mathrm{S}(\Delta)$**).** *We say that a sequence of polynomials* $\{v_n\}$ *with no zeros on* Δ *belongs to the class* $\mathrm{S}(\Delta)$ *if the following conditions hold:*

(1) *associated functions* r_n *satisfy* $|r_n^\pm| = O(1)$ *uniformly on* Δ *and* $r_n = o(1)$ *locally uniformly in* D*;*
(2) *there exists a neighborhood of* Δ *that does not contain zeros of* r_n *for all* n *large enough;*
(3) *the normalized counting measures of zeros of* r_n *form a weakly convergent sequence.*

We remark that the third requirement in the definition of $\mathrm{S}(\Delta)$ is purely technical and is placed only to simplify the forthcoming considerations.

Regarding the nature of the class $\mathrm{S}(\Delta)$, the following was shown in [**?**, Thm. 1]. For an analytic Jordan arc Δ there always exist sequences $\{v_n\}$ belonging to $\mathrm{S}(\Delta)$ and they can be constructed explicitly granted the parameterization Ξ. Conversely, let Δ be a rectifiable Jordan arc with endpoint ± 1 such that for $x = \pm 1$ and all $t \in \Delta$ sufficiently close to x it holds that $|\Delta_{t,x}| \leq \text{const.}|x-t|^\beta$, $\beta > 1/2$, where $|\Delta_{t,x}|$ is the arclenth of the subsarc of Δ joining t and x and const. is an absolute constant. If there exists a sequence of polynomials $\{v_n\}$ complying with the first and second requirements in the definition of $\mathrm{S}(\Delta)$, then Δ is necessarily an analytic arc.

The class $\mathrm{S}(\Delta)$ is intimately related to the so-called *symmetry property* of the contour Δ [**?**, **?**]. We shall not dwell on this relation here but illuminate the essence of it. If

Δ is as in the second part of the preceding paragraph and possesses the symmetry property in the field[1] $-V^\nu$, where ν is a compactly supported in D Borel measure and V^ν is the logarithmic potential of ν, then Δ is an analytic arc and subsequently there exist sequences belonging to $\mathrm{S}(\Delta)$. Conversely, if the set $\mathrm{S}(\Delta)$ is not empty then Δ possesses the symmetry property in the field $-V^\nu$ for some compactly supported Borel measure ν. In fact, this is exactly the measure coming from the third part of the definition of $\mathrm{S}(\Delta)$ if the latter has compact support, otherwise one needs to consider its *balayage* onto the boundary of any simply connected domain containing Δ.

For our investigation we need to know in greater detail the properties of just defined interpolation schemes which we gather in the following theorem. Recall that all the arcs are assumed to have ± 1 as the endpoints.

Theorem 1. *Let Δ be an analytic Jordan arc and $\{v_n\} \in \mathrm{S}(\Delta)$. Then there exists a sequence of analytic Jordan arcs $\{\Delta_n\}$ such that their analytic parameterizations Ξ_n converge to Ξ, an analytic parameterization of Δ, uniformly in some neighborhood of Δ. Moreover, the functions r_n, associated to v_n via (2.4), can be analytically deformed to be holomorphic in $D_n := \overline{\mathbb{C}} \setminus \Delta_n$ and their traces on each side of Δ_n are unimodular.*

The above theorem should be understood in the following way. Let $\mathfrak{w}_n$ and φ_n be defined relative to Δ_n as $\mathfrak{w}$ and φ were defined in (2.2) and (2.3) relative to Δ. Clearly, $\mathfrak{w}_n$ and φ_n are simply analytic deformations of $\mathfrak{w}$ and φ. Set r_n^* to be the function associated to v_n via (2.4) with φ replaced by φ_n. Trivially, r_n^* is an analytic deformation of r_n that has a jump across Δ_n rather than Δ. Theorem 1 claims that there is a choice of the contours Δ_n so that $|(r_n^*)^\pm| \equiv 1$ on Δ_n and Δ_n approach Δ in the sense made precise in the statement of the theorem. In fact, each Δ_n is the symmetric contour in the field generated by minus the logarithmic potential of the counting measure of finite zeros of r_n. Moreover, r_n^* turns out to coincide with a Blaschke product with respect to the domain D_n and that has the same zeros as r_n.

2.2. Strong Asymptotics for non-Hermitian Orthogonal Polynomials

As we already mentioned, we consider the weights on Δ that are the modifications of the Jacobi weight (2.1) by Hölder continuous non-vanishing functions. Thus, we define the following smoothness classes. Hereafter, the symbols s, m, and ς are reserved to have the following meaning: $s > 0$, $s = m + \varsigma$, $\varsigma \in (0, 1]$, and $m \in \mathbb{Z}_+ := \mathbb{N} \cup \{0\}$.

Definition (Classes $\mathrm{C}^s(K)$ and $\mathrm{C}^{s-}(K)$). *Let K be an analytic Jordan arc or curve. We say that a function θ, given on K, belongs to the class $\mathrm{C}^s(K)$ if θ is m-times continuously differentiable on K and its m-th derivative is uniformly Hölder continuous with exponent ς, i.e.*

$$|\theta^{(m)}(t_1) - \theta^{(m)}(t_2)| \leq \text{const.}|t_1 - t_2|^\varsigma, \quad t_1, t_2 \in K,$$

where const. $< \infty$ *depends only on θ. When K is a closure of a domain, we define $\mathrm{C}^s(K)$ as the space of all functions on K whose partial derivatives up to the order m are continuous and bounded in K and whose partial derivatives of order m are uniformly Hölder continuous in K with exponent ς. Finally, we say that $\theta \in \mathrm{C}^{s-}(K)$ if $\theta \in \mathrm{C}^{s-\epsilon}(K)$ for any $\epsilon \in (0, s)$.*

When $K = \Delta$, we simply write C^s instead of $\mathrm{C}^s(\Delta)$. Observe that under this definition C^{m+1}, $m \in \mathbb{Z}_+$, stands for the space of m-times continuously differentiable

[1]For information on the notions of potential theory we refer the reader to the monographs [**?**, **?**].

functions with m-th derivative being Lipschitz. Hence, C^{m+1} contains all $m+1$-times continuously differentiable functions.

In this section we investigate the asymptotic behavior of polynomials satisfying varying non-Hermitian orthogonality relation of the form

$$\int_\Delta t^j q_n(t) w_n(t) dt = 0, \quad j \in \{0,\dots,n-1\}, \tag{2.5}$$

and also the asymptotic behavior of their functions of the second kind, i.e.,

$$R_n(z) := \int_\Delta \frac{q_n(t)w_n(t)}{t-z}\frac{dt}{\pi i}, \quad z \in D, \tag{2.6}$$

where $\{w_n\}$ is a specially chosen sequence of weights that we specify later. To describe this asymptotic behavior, we need to introduce the notions of the geometric mean and the Szegő function. The *geometric mean* of $h := e^\theta$, $\theta \in \mathrm{C}^s$, is given by

$$G_h := \exp\left\{\int \theta d\omega\right\}, \quad d\omega(t) := \frac{idt}{\pi\mathfrak{w}^+(t)}_{|\Delta}. \tag{2.7}$$

As θ is Hölder continuous, G_h is well-defined and non-zero (see Section 5.3). Moreover, the *Szegő function* of h, given by

$$S_h(z) := \exp\left\{\frac{\mathfrak{w}(z)}{2}\int\frac{\theta(t)}{z-t}d\omega(t) - \frac{1}{2}\int\theta d\omega\right\}, \quad z \in D, \tag{2.8}$$

is the unique non-vanishing holomorphic function in D that has continuous unrestricted boundary values on Δ from each side and satisfies

$$h = G_h S_h^+ S_h^- \quad \text{on} \quad \Delta \quad \text{and} \quad S_h(\infty) = 1. \tag{2.9}$$

Observe also that in the case $\Delta = [-1,1]$ the measure $d\omega$ simply becomes the normalized arcsine distribution on $[-1,1]$. The following theorem takes place.

Theorem 2. *Let $\{q_n\}$ be a sequence of polynomilas satisfying orthogonality relations* (2.5) *with weights*

$$w_n := \frac{wh_nh}{v_n}, \quad h = e^\theta, \quad h_n = e^{\theta_n}, \tag{2.10}$$

where $\theta \in \mathrm{C}^s$, $\{\theta_n\}$ is a normal family in some neighborhood of Δ, $\{v_n\} \in \mathrm{S}(\Delta)$, and $w = w(\alpha,\beta;\cdot)$ is such that

$$\alpha,\beta \in (-1,\infty)\cap(-s,s). \tag{2.11}$$

Denote $s^* := \max\{|\alpha|,|\beta|\}$ and set

$$\delta_{n,\epsilon} := \begin{cases} n^{\epsilon-\frac{1}{2}(s-s^*)}, & s-s^*<1, \\ n^{\epsilon-\frac{1}{2}}, & s-s^* \geq 1. \end{cases} \tag{2.12}$$

Then, for all n large enough, polynomials q_n have exact degree n and therefore can be normalized to be monic. Under such a normalization, we have that

$$\begin{cases} q_n & = & [1+O(\delta_{n,\epsilon})]/S_n \\ R_n\mathfrak{w} & = & [1+O(\delta_{n,\epsilon})]\gamma_n S_n \end{cases} \tag{2.13}$$

locally uniformly in D for any arbitrarily small $\epsilon > 0$, where

$$S_n := (2/\varphi)^n S_{w_n\mathfrak{w}^+}, \quad \gamma_n := 2^{1-2n}G_{w_n\mathfrak{w}^+}, \tag{2.14}$$

and R_n *was defined in* (2.6)*;*

$$\begin{cases} q_n & = & [1+O(\delta_{n,\epsilon})]/S_n^+ + [1+O(\delta_{n,\epsilon})]/S_n^- \\ (R_n\mathfrak{w})^\pm & = & [1+O(\delta_{n,\epsilon})]\ \gamma_n S_n^\pm \end{cases} \tag{2.15}$$

locally uniformly in Δ°.

2.3. Multipoint Padé Approximation

Let μ be a complex Borel measure with compact support. We define the Cauchy transform of μ as

$$f_\mu(z) := \int \frac{d\mu(t)}{z-t}, \quad z \in \overline{\mathbb{C}} \setminus \operatorname{supp}(\mu). \tag{2.16}$$

Clearly, f_μ is a holomorphic function in $\overline{\mathbb{C}} \setminus \operatorname{supp}(\mu)$ that vanishes at infinity.

Classically, diagonal (multipoint) Padé approximants to f_μ are rational functions of type (n,n) that interpolate f_μ at a prescribed system of $2n+1$ points. However, when the approximated function is of the from (2.16), it is customary to place at least one interpolation point at infinity. More precisely, let $\{v_n\}$ be a sequence of monic polynomials, $\deg(v_n) \leq 2n$, with zeros in $\overline{\mathbb{C}} \setminus \operatorname{supp}(\mu)$.

Definition (Multipoint Padé Approximant). *Given* f_μ *of type* (2.16) *and a sequence* $\{v_n\}$*, the* n*-th diagonal Padé approximant to* f_μ *associated with* $\{v_n\}$ *is the unique rational function* $\Pi_n = p_n/q_n$ *satisfying:*

- $\deg p_n \leq n$, $\deg q_n \leq n$, *and* $q_n \not\equiv 0$*;*
- $(q_n(z)f_\mu(z) - p_n(z))/v_n(z)$ *is analytic in* $\overline{\mathbb{C}} \setminus \operatorname{supp}(\mu)$*;*
- $(q_n(z)f_\mu(z) - p_n(z))/v_n(z) = O\left(1/z^{n+1}\right)$ *as* $z \to \infty$.

A multipoint Padé approximant always exists since the conditions for p_n and q_n amount to solving a system of $2n+1$ homogeneous linear equations with $2n+2$ unknown coefficients, no solution of which can be such that $q_n \equiv 0$ (we may thus assume that q_n is monic); note that the required interpolation at infinity is entailed by the last condition and therefore Π_n is, in fact, of type $(n-1,n)$.

The following theorem is a standard consequence of Theorem 2 (see, for example, [**?**, Thm. 3]) and we shall not prove it here.

Theorem 3. *Let* Δ *be an analytic Jordan arc connecting* ± 1 *and* $\{v_n\}$ *be a sequence of polynomials from* $\mathrm{S}(\Delta)$*. Let also* f_μ *be given by* (2.16) *with*

$$d\mu(t) = \dot\mu(t)d\omega(t), \quad \dot\mu = w\exp\{\theta\}\mathfrak{w}^+,$$

where $\theta \in \mathrm{C}^s$*,* $s > 0$*, and* $w = w(\alpha,\beta;\cdot)$ *complies with* (2.11)*. Then the sequence of diagonal Padé approximants to* f_μ *associated with* $\{v_n\}$*,* $\{\Pi_n\}$*, is such that*

$$(f_\mu - \Pi_n)\mathfrak{w} = [2G_{\dot\mu} + O(1/n)]\, S_{\dot\mu}^2 r_n \quad \text{locally uniformly in} \quad D,$$

where functions r_n *are associated to polynomials* v_n *via* (2.4).

3. Proof of Theorem 1 and g-Functions

In this section we prove Theorem 1. Some notions that we introduce along the way, as g-functions, will be also needed for the proof of Theorem 2.

3.1. Parameterization Ξ, Functions g and $\widetilde{g}$

As required by the third property in the definition of the class $\mathrm{S}(\Delta)$, normalized counting measures of the zeros of r_n converge weak* to a Borel measure ν. Denote by V_D^ν the *Green potential* of this measure. It was shown in the course of the proof of Theorem 1 in [**?**] (see (4.34)) that the first requirement in the definition of $\mathrm{S}(\Delta)$ yields that

$$V_D^\nu(z) = -\int \log\left|\frac{\varphi(z)-\varphi(t)}{1-\varphi(z)\varphi(t)}\right| d\nu(t).$$

In other words, the usual kernel, the Green function for D with pole at any given point of D, which can be expressed through the conformal map of $\mathbb{D}$ onto D, can be replaced by the one above for this special measure ν.

Let L_ρ be a level line of V_D^ν in D_Ξ, $L_\rho := \{z : \ V_D^\nu(z) = \log\rho\}$, $\rho > 1$. Without loss of generality we may assume that L_ρ is a smooth Jordan curve. Denote by O the domain bounded by L_ρ and Δ. It is rather well-known and can be shown as in the proof of Theorem 1 in [**?**] ((4.39) and after) that

$$\Phi(z) := \exp\left\{2\int_1^z \frac{\partial V_D^\nu}{\partial z}(t)dt\right\} = \exp\left\{-\int \log\frac{\varphi(z)-\varphi(t)}{1-\varphi(z)\varphi(t)}d\nu(t)\right\} \tag{3.1}$$

is well-defined in O and maps it conformally onto the annulus $\{z : \ 1 < |z| < \rho\}$ and $\Phi(\pm1) = \pm1$. Moreover, by direct examination of the kernel in (3.1), we get that

$$\Phi^+ = \overline{\Phi^-} = 1/\Phi^- \quad \text{on} \quad \Delta. \tag{3.2}$$

This, in particular, yields that $J\circ\Phi$ is holomorphic across Δ, where $J(z) = (z+1/z)/2$ is the Joukovski transformation, and that the inverse $\Xi := (J\circ\Phi)^{-1}$ is a holomorphic univalent map in some neighborhood of $[-1,1]$ that analytically parametrizes Δ.

Based on the conformal map Φ, we define two more functions, g and $\widetilde{g}$. Set $L := \Phi^{-1}([-\rho,-1])$, $\widetilde{L} := \Phi^{-1}([1,\rho])$, and define

$$\begin{aligned} g &:= \log\Phi, & \lim_{z\to1} g(z) &= 0, & g &\in \mathrm{H}(O\setminus L), \\ \widetilde{g} &:= \log\Phi - \pi i, & \lim_{z\to-1} \widetilde{g}(z) &= 0, & \widetilde{g} &\in \mathrm{H}(O\setminus \widetilde{L}). \end{aligned} \tag{3.3}$$

It follows immediately from (3.2) that

$$g^+ = -g^- \quad \text{and} \quad \widetilde{g}^+ = -\widetilde{g}^- \quad \text{on} \quad \Delta. \tag{3.4}$$

Hence, g^2 and $\widetilde{g}^2$ are analytic in $O_g := (O\cup\Delta)\setminus L$ and $O_{\widetilde{g}} := (O\cup\Delta)\setminus\widetilde{L}$, respectively. Moreover, it holds that

$$g^2(\Delta) = \widetilde{g}^2(\Delta) = [-\pi^2, 0] \quad \text{and} \quad g^2(1) = \widetilde{g}^2(-1) = 0.$$

It is also true that g^2 and $\widetilde{g}^2$ are univalent in O_g and $O_{\widetilde{g}}$, respectively. Indeed, suppose that $g^2(z_1) = g^2(z_2)$, $z_1, z_2 \in O_g$. Then either $\Phi(z_1) = \Phi(z_2)$ and therefore $z_1 = z_2$ by conformality of Φ or $\Phi(z_1) = 1/\Phi(z_2)$, which is possible only for the boundary values, i.e. $\Phi^+(z_1) = \overline{\Phi^-(z_2)}$, but the latter holds if and only if $z_1 = z_2 \in \Delta$. The case of $\widetilde{g}^2$ is no different.

3.2. Jordan arcs Δ_n, Functions g_n and $\widetilde{g}_n$

Without loss of generality we may assume that functions r_n have no zeros in $\overline{O}$. Moreover, as r_n has $2n$ zeros outside of O, its winding number is equal to $-2n$ on any curve homologous to L_ρ. In other words, r_n has a continuous argument that decreases by $4n\pi$ as Δ is encompassed once in the positive direction. Thus, the functions

$$\Phi_n := r_n^{-1/2n} = \exp\left\{-\int \log\frac{\varphi(z)-\varphi(t)}{1-\varphi(z)\varphi(t)}d\nu_n(t)\right\},$$

are well-defined and analytic in O, where ν_n is the counting measure of zeros of r_n. Moreover, as the counting measures of zeros of r_n converge to ν, the functions Φ_n converge to Φ uniformly in $\overline{O}$.

Hence, we can define

$$\begin{aligned} g_n &:= \log\Phi_n, & \lim_{z\to 1} g_n(z) &= 0, & g_n &\in \mathrm{H}(O\setminus L),\\ \widetilde{g}_n &:= \log\Phi_n - \pi i, & \lim_{z\to -1}\widetilde{g}_n(z) &= 0, & \widetilde{g}_n &\in \mathrm{H}(O\setminus \widetilde{L}). \end{aligned} \tag{3.5}$$

It is rather straightforward to see that $\Phi_n^+\Phi_n^- \equiv 1$ on Δ and therefore

$$g_n^+ = -g_n^- \quad \text{and} \quad \widetilde{g}_n^+ = -\widetilde{g}_n^- \quad \text{on} \quad \Delta. \tag{3.6}$$

Thus, g_n^2 and $\widetilde{g}_n^2$ are analytic in O_g and $O_{\widetilde{g}}$, respectively. To simplify the discussion we choose domains $O_L\subset O_g$ and $O_{\widetilde{L}}\subset O_{\widetilde{g}}$ in such a manner that $O_L\supset\widetilde{L}$, $O_{\widetilde{L}}\supset L$, and $O_L\cup O_{\widetilde{L}} = O\cup\Delta$. Then it is an easy consequence of the convergence of Φ_n to Φ that g_n^2 and $\widetilde{g}_n^2$ converge uniformly to g^2 and $\widetilde{g}^2$ in the closures $\overline{O}_L$ and $\overline{O}_{\widetilde{L}}$, respectively.

Next, we claim that the functions g_n^2 and $\widetilde{g}_n^2$ are univalent for all n large enough in $\overline{O}_L$ and $\overline{O}_{\widetilde{L}}$, respectively. Assume to the contrary that there exist two sequences of points $\{z_{1,n}\},\{z_{2,n}\}\subset\overline{O}_L$ such that $g_n^2(z_{1,n}) = g_n^2(z_{2,n})$. As $\overline{O}_L$ is compact, we can assume that $z_{j,n}\to z_j\in\overline{O}_L$, $j=1,2$. Since g_n^2 converges to g^2 uniformly on $\overline{O}_L$, we have that $g^2(z_1)=g^2(z_2)$ and therefore $z_1=z_2$. Set $d_n(z) := (g_n^2(z)-g_n^2(z_{1,n}))/(z-z_{1,n})$. Then d_n are analytic functions on $\overline{O}_L$ that converge uniformly to $d(z) := (g^2(z)-g^2(z_1))/(z-z_1)$. Moreover, $d_n(z_{2,n}) = 0$ and converges to $d(z_1)$. Thus, $(g^2)'(z_1) = d(z_1) = 0$, which is impossible since g^2 is univalent. This finishes the proof of the claim as the case of $\widetilde{g}_n^2$ is no different.

From all the above we see that each g_n^2 maps O_L conformally into a neighborhood of zero. Set $\Delta_{n,1}$ to be the preimage of the intersection of this neighborhood with $\Sigma_2 := \{\zeta:\ \mathrm{Arg}(\zeta)=\pi\}$. Then $\Delta_{n,1}$ is an analytic arc with one endpoint being 1. Analogously, $\widetilde{g}_n^2$ maps $O_{\widetilde{L}}$ conformally into another neighborhood of zero. Thus, we can define $\Delta_{n,-1}$ to be the preimage of the intersection of this neighborhood again with Σ_2. Clearly, $\Delta_{n,-1}$ is an analytic arc with one endpoint being -1. By noticing that g_n^2 assumes negative values if and only if $\widetilde{g}_n^2$ assumes negative values, we derive that $\Delta_n := \Delta_{n,1}\cup\Delta_{n,-1}$ is an analytic arc with endpoint ± 1.

3.3. Parameterizations Ξ_n, Functions g_n^* and $\widetilde{g}_n^*$

Now, define $\mathfrak{w}_n$ and φ_n with respect to Δ_n as $\mathfrak{w}$ and φ were defined in (2.2) and (2.3) with respect to Δ. Clearly, φ_n is a holomorphic deformation of φ, i.e. $\varphi_n=\varphi$ outside of a bounded domain with the boundary $\Delta\cup\Delta_n$. Further, let r_n^* be defined by (2.4) with φ replaced by φ_n and with respect to the same zeros as r_n. Hence, r_n^* is a holomorphic deformation of r_n. Finally, define Φ_n^*, g_n^*, and $\widetilde{g}_n^*$ accordingly, so, these functions are holomorphic deformations of Φ_n, g_n, and $\widetilde{g}_n$. This immediately implies that

$$g_n^2 = (g_n^*)^2 \quad \text{and} \quad \widetilde{g}_n^2 = (\widetilde{g}_n^*)^2.$$

The latter necessarily yields that $(\Phi_n^*)^+ = \overline{(\Phi_n^*)^-}$ and $(r_n^*)^+ = \overline{(r_n^*)^-}$. Moreover, Φ_n^* maps $O_n := (O \cup \Delta) \setminus \Delta_n$ into some annular domain having the unit circle as one component of the boundary. As in the case of Φ, we deduce from the symmetries of Φ_n^* that $J \circ \Phi_n^*$ is holomorphic across Δ_n and that $\Xi_n := (J \circ \Phi_n^*)^{-1}$ is a holomorphic parameterization of Δ_n, $\Xi_n([-1,1]) = \Delta_n$. Moreover, as Φ_n converge to Φ uniformly in some annular domain encompassing Δ, we see that $J \circ \Phi_n^*$ converge locally uniformly to $J \circ \Phi$ in some neighborhood of Δ. This can be rephrased as follows. The constructed sequence of analytic parameterizations of Δ_n, $\{\Xi_n\}$, converges uniformly to analytic parameterization of Δ, Ξ, in the closure of some neighborhood of $[-1,1]$. This finishes the proof of Theorem 1.

4. Trace Theorems and Extensions

As usual in the Riemann-Hilbert approach, we will need to extend the weights of orthogonality from Δ into the complex plane or at least some part of it. As the weights are not analytic, this extension will require a special construction that we carry out in this section.

4.1. Domains with Smooth Boundaries

In this section we suppose that Ω is a bounded domain with boundary Γ which is infinitely smooth, i.e. every point of Γ possesses a neighborhood that can be parametrized by an infinitely many times differentiable function.

4.1.1. Sobolev spaces. Set $\mathrm{W}_p^0(\Omega) = \mathrm{L}^p(\Omega)$, $p \geq 1$, to be the space of all measurable functions f such that $|f|^p$ is integrable over Ω. We define $\mathrm{W}_p^k(\Omega)$, $k \in \mathbb{N}$, to be the subspace of $\mathrm{L}^p(\Omega)$ that comprises of functions with weak partial derivatives up to the order k also in $\mathrm{L}^p(\Omega)$. It is known from Sobolev's imbedding theorem [**?**, Thm. 5.4] that

$$\mathrm{W}_p^k(\Omega) \subset \begin{cases} \mathrm{W}^k_{2p/(2-p)}(\Omega), & p \in [1,2), \\ \mathrm{C}^{k-2/p}(\overline{\Omega}), & p \in (2,\infty). \end{cases} \tag{4.1}$$

To state the trace theorem for functions in $\mathrm{W}_p^k(\Omega)$, we need to introduce appropriate spaces on Γ. We define a fractional Sobolev spaces $\mathrm{W}_p^{k-1/p}(\Gamma)$ [**?**, Sec. 1.3.3] to be the subspaces of $\mathrm{L}^p(\Gamma)$ consisting of functions satisfying

$$\iint_{\Gamma\times\Gamma} \left| \frac{f^{(k-1)}(x) - f^{(k-1)}(y)}{x-y} \right|^p |dx||dy| < \infty, \tag{4.2}$$

where $|dx|$ and $|dy|$ are the arclength elements. It is a consequence of a trivial computation that

$$\mathrm{C}^s(\Gamma) \subset \mathrm{W}_p^{m+1-1/p}(\Gamma), \quad p \in \left(2, \frac{2}{1-\varsigma}\right). \tag{4.3}$$

Finally, we mention one easy property of Sobolev functions that we shall utilize on numerous occasions. Let $F \in \mathrm{W}_p^{k+1}(\Omega)$. Then $\partial F, \bar\partial F \in \mathrm{W}_p^k(\Omega)$, where ∂F and $\bar\partial F$ are analytic and anti-analytic derivatives of F, i.e.

$$\partial f := \frac{1}{2}\left(\partial_x f - i\partial_y f\right) \quad \text{and} \quad \bar\partial f := \frac{1}{2}\left(\partial_x f + i\partial_y f\right).$$

4.1.2. Trace theorem. To state the theorem, we need to introduce notions of a directional derivative and of a normal field. Let $\xi \in \mathbb{C}$. We define the derivative in the direction ξ, denoted by ∂_ξ, as

$$\partial_\xi f := \bar\xi \bar\partial f + \xi \partial f. \tag{4.4}$$

It is easy to see that usual partial derivatives can be re-expressed as $\partial_x = \partial_1$ and $\partial_y = \partial_i$. Moreover, it can be easily checked that for real-valued f definition (4.4) specializes to the usual definition $\partial_\xi f = \langle \nabla f, \vec\xi\rangle = 2\mathrm{Re}(\xi\partial u)$, where ∇f is the gradient of f, $\vec\xi$ is the vector in $\mathbb{R}^2$ corresponding to ξ, and $\langle\cdot,\cdot\rangle$ is the usual scalar product in $\mathbb{R}^2$.

Denote by $\vec n$ and $\vec\tau$ the unit normal (outer with respect to Ω) and tangential (in the positive direction) vectors on Γ, respectively. We also consider the corresponding unimodular complex numbers n and τ. Clearly, $\tau = in$. As Γ is infinitely smooth, n extends to an infinitely many times differentiable function on $\overline\Omega$, which corresponds to infinitely smooth normal vector field on $\overline\Omega$. In particular, we have for any $F \in \mathrm{W}_p^k(\Omega)$ that

$$\partial_n^{k_1}\partial_\tau^{k_2}F \in \mathrm{W}^{k-k_1-k_2}(\Omega), \quad k_1+k_2 \in \{0,\dots,k\}, \tag{4.5}$$

where

$$\partial_n F(z) := \overline{n(z)}\bar\partial F(z) + n(z)\partial F(z), \quad z \in \overline\Omega, \tag{4.6}$$

and the function $\partial_\tau F$ is defined analogously. It also will be useful for us later to observe that

$$\bar\partial F = \frac{n}{2}\left(\partial_n F + i\partial_\tau F\right) \quad \text{on} \quad \Gamma. \tag{4.7}$$

Now, we are ready to state the trace theorem for $\mathrm{W}_p^m(\Omega)$ [**?**, Thm. 1.5.1.2].

Given $\{f_k\}_{k=0}^m$, $f_k \in \mathrm{W}_p^{m+1-k-1/p}(\Gamma)$, $p \in (1,\infty)$, *there exists* $F \in \mathrm{W}_p^{m+1}(\Omega)$ *such that* $(\partial_n^k F)_{|\Gamma} = f_k$, $k \in \{0,\dots,m\}$.

One important property of F is that it depends only on $\{f_k\}$ and does not depend on p. Moreover, the following differentiability properties of F and $\{f_k\}$ take place. First, it holds that $\partial_n\partial_\tau F = \partial_\tau\partial_n F$ in $\overline\Omega$. Second, we have that $(\partial_\tau^k F)_{|\Gamma} = (F_{|\Gamma})^{(k)}$. Third, it is true that

$$\left(\partial_\tau^{k_1}\partial_n^{k_2}F\right)_{|\Gamma} = \left(\left[\partial_n^{k_2}F\right]_{|\Gamma}\right)^{(k_1)} = f_{k_2}^{(k_1)}. \tag{4.8}$$

The third property is the consequence of the first two that says that any partial derivative of F with respect to any sequence of ∂_n and ∂_τ can be written as $\partial_\tau^{k_1}\partial_n^{k_2}F$ and its boundary values can be expressed through the appropriate derivatives of f_k.

4.1.3. Smooth extensions. Let $f \in \mathrm{C}^s(\Delta)$, $f^{(k)}(\pm1) = 0$, $k \in \{0,\dots,m\}$. Let also Ω and Γ be as in the previous section and assume that $\Delta \subset \Gamma$. Then there exists $F \in \mathrm{W}_p^{m+1}(\Omega)$ for all $p \in (2, \frac{2}{1-\varsigma})$ such that

$$F_{|\Delta} = f \quad \text{and} \quad \bar\partial F \in \mathrm{C}_0^{s-1-}(\overline\Omega) \quad \text{if} \quad s > 1, \tag{4.9}$$

where $\mathrm{C}_0^s(\overline\Omega) \subset \mathrm{C}^s(\overline\Omega)$ consists of functions whose partial derivatives of all (existing) orders, including the function itself, vanish on Γ.

Indeed, by setting $f_0 = f$ on Δ and $f_0 \equiv 0$ on $\Gamma\setminus\Delta$, we see that $f_0 \in \mathrm{C}^s(\Gamma)$. Further, set $f_k := (-i)^k f_0^{(k)}$, $k \in \{1,\dots,m\}$. As $f_k \in \mathrm{C}^{s-k}(\Gamma)$, $k \in \{0,\dots,m\}$, the existence of F in $\mathrm{W}_p^{m+1}(\Omega)$ with required boundary values follows immediately from (4.3) and the just stated trace theorem. Hence, we need only to show that $\bar\partial F \in \mathrm{C}_0^{s-}(\overline{\Omega})$, or equivalently, that

$$(\partial_\tau^{k_1}\partial_n^{k_2}\bar\partial F)_{|\Gamma} \equiv 0 \tag{4.10}$$

for all $k_1 + k_2 \in \{0,\dots,m-1\}$. To verify (4.10), observe that since

$$\left(\partial_\tau^{k_1}\partial_n^{k_2+1}F\right)_{|\Gamma} = f_{k_2+1}^{(k_1)} = (-i)^{k_2+1} f_0^{(k_1+k_2+1)} = -i f_{k_2}^{(k_1+1)} = -i\left(\partial_\tau^{k_1+1}\partial_n^{k_2}F\right)_{|\Gamma},$$

by the choice of $\{f_k\}$ and (4.8), we have that $\left(\partial_\tau^{k_1}\partial_n^{k_2+1}F\right)_{|\Gamma} = -i\left(\partial_\tau^{k_1+1}\partial_n^{k_2}F\right)_{|\Gamma}$ by (4.7) and therefore (4.10) does indeed take place.

In fact, there exist trace theorems not only for Sobolev spaces but also for smoothness classes. In particular, by [**?**, Cor. 6.2.8], there exists $F \in \mathrm{C}^s(\overline{\Omega})$ (rather than in $\mathrm{C}^{s-}(\overline{\Omega})$) with the same boundary values for the directional derivatives. However, it will be important for us later to use not only Hölder continuity of the m-th partial derivatives of F but also the fact that there exist $m+1$-st integrable partial derivatives of F.

4.2. Domains with Polygonal Boundary

The previous results also hold, with some modifications, for domains with polygonal boundary. Namely, let Ω be a domain whose boundary is a curvilinear polygon (a union of smooth arcs that might form corners at the joints) consisting of two pieces, say Δ_1 and Δ_2. As we do not strive for generality, we assume that each Δ_j is an analytic arc connecting -1 and 1.

4.2.1. Trace theorem. We define fractional Sobolev spaces $\mathrm{W}_p^{k-1/p}(\Delta_j^\circ)$ as in (4.2) only with integration on $\Delta_j^\circ \times \Delta_j^\circ$, $j = 1,2$. Again, it is a consequence of a trivial computation that (4.3) holds for the corresponding spaces on Δ_j, $j = 1,2$.

As before, denote by $\vec n_j$ and $\vec\tau_j$ the unit normal (outer with respect to Ω) and tangential (in the positive direction with respect to the orientation from -1 to 1) vectors on Δ_j. As usual, n_j and τ_j stand for the corresponding unimodular complex numbers. Clearly, $\tau_j = in_j$ if Ω lies on the left side of Δ_j and $\tau_j = -in_j$ otherwise. As each arc Δ_j is analytic, each n_j extends to an infinitely many times differentiable function on $\overline{\Omega}$. In particular, (4.5), (4.6), and (4.7) holds with n, τ, and Γ replaced by n_j, τ_j, Δ_j, and plus sign replaced by minus sign in the right-hand side of (4.7) when $\tau_j = -in_j$.

With all the necessary material at hand, we can state the trace theorem for Sobolev spaces on domains with polygonal boundary [**?**, Thm. 1.5.2.8].

Given $\{f_{jk}\}_{k=0}^m$, $f_{jk} \in \mathrm{W}_p^{m+1-k-1/p}(\Delta_j^\circ)$, *satisfying* $f_{1k_1}^{(k_2)}(\pm1) = f_{2k_2}^{(k_1)}(\pm1)$, $k_1 + k_2 \in \{0,\dots,m\}$, *there exists* $F \in \mathrm{W}_p^{m+1}(\Omega)$ *such that* $(\partial_{n_j}^k F)_{|\Delta_j} = f_{jk}$, $k \in \{0,\dots,m\}$.

Again, the choice of F depends only on $\{f_{jk}\}$ and does not depend on p.

4.2.2. Smooth extensions. Let $f \in \mathrm{C}^s(\Delta)$, $f^{(k)}(\pm1) = 0$, $k \in \{0,\dots,m\}$. Let also $\Delta_\pm$ be two analytic Jordan arcs with endpoints ±1 such that the the interior domain of $\Delta\cup\Delta_+$, say Ω_+, lies to the left of Δ and the interior domain of $\Delta\cup\Delta_-$, say Ω_-, lies to the right of Δ. Then there exist functions $F_\pm \in \mathrm{W}_p^{m+1}(\Omega_\pm)$ for all $p \in (2, \frac{2}{1-\varsigma})$ such that

$$F_{\pm|\Delta} = \pm f, \quad F_{\pm|\Delta_\pm} \equiv 0, \quad \text{and} \quad \bar\partial F_\pm \in \mathrm{C}_0^{s-1-}(\overline{\Omega}_\pm) \quad \text{if} \quad s > 1. \tag{4.11}$$

First, we consider the case of Ω_+. By setting $f_{1k} := (-i)^k f^{(k)}$, $k \in \{0,\dots,m\}$, we see that $f_{1k} \in \mathrm{C}^{s-k}(\Delta)$, $k \in \{0,\dots,m\}$. Moreover, for $f_{2k} \equiv 0$, we observe that $f_{1k_1}^{(k_2)}(\pm 1) = f_{2k_2}^{(k_1)}(\pm 1)$, $k_1 + k_2 \in \{0,\dots,m\}$. Then the existence of F_+ in $\mathrm{W}_p^{m+1}(\Omega_+)$ follows from the analog of (4.3) and the above trace theorem. The fact that $\bar\partial F_+ \in \mathrm{C}_0^{s-}(\overline{\Omega}_+)$ can be shown exactly as in (4.9). In the case of Ω_- the only difference is that we need to set $f_{1k} := -i^k f^{(k)}$ since this time the normal and tangent on Δ satisfy $\tau = -in$.

5. Scalar Boundary Value Problems

In this section we dwell on smoothness properties of some integral operators.

5.1. Integral Operators

Here we introduce contour and surface integral operators and explain the solution of a certain $\bar\partial$-problem.

5.1.1. Contour integral operators. Let ϕ be an L^p, $p > 1$, function on Δ, where $\mathrm{L}^p = \mathrm{L}^p(\Delta)$ stands for the space of functions with p-summable modulus on Δ with respect to the arclength differential $|dt|$. The Cauchy integral operator on Δ is defined by

$$\mathcal{C}\phi(z) := \mathcal{C}_\Delta\phi(z) = \frac{1}{2\pi i}\int_\Delta \frac{\phi(t)}{t-z}dt, \quad z \in D. \tag{5.1}$$

It is known that $\mathcal{C}\phi$ is a holomorphic function in D with L^p traces on Δ, denoted $\mathcal{C}^\pm\phi$, that are connected by Sokhotski-Pemelj formulae [**?**, Sec. I.4.2], i.e.

$$\mathcal{C}^+\phi - \mathcal{C}^-\phi = \phi \quad \text{and} \quad \mathcal{C}^+\phi + \mathcal{C}^-\phi = \mathcal{S}\phi, \tag{5.2}$$

where $\mathcal{S}$ is the singular integral operator on Δ given by

$$\mathcal{S}\phi(\tau) := \mathcal{S}_\Delta\phi(\tau) = \frac{1}{\pi i}\int_\Delta \frac{\phi(t)}{t-\tau}dt, \quad \tau \in \Delta^\circ, \tag{5.3}$$

the integral, of course, is understood in the sense of the principal value.

Let now Ω be a simply connected bounded domain with smooth boundary Γ. We define $\mathcal{C}_\Gamma$ and $\mathcal{S}_\Gamma$ by (5.1) and (5.3) only this time with integration on Γ rather than on Δ. The Sokhotski-Plemelj formulae (5.2) still hold for $\phi \in \mathrm{L}^p(\Gamma)$, $p > 1$, with the only difference that now $\mathcal{C}_\Gamma\phi$ is a sectionally holomorphic function and therefore $\mathcal{C}_\Gamma^+\phi$ is the trace of $\mathcal{C}_\Gamma\phi$ from within Ω and $\mathcal{C}_\Gamma^-\phi$ is the trace of $\mathcal{C}_\Gamma\phi$ from within $\overline{\mathbb{C}} \setminus \overline{\Omega}$.

Concerning the smoothness of $\mathcal{C}_\Gamma\phi$ the following is known. If $\phi \in \mathrm{C}^\varsigma(\Gamma)$, $\varsigma \in (0,1]$, then $\mathcal{C}_\Gamma\phi \in \mathrm{C}^{\varsigma-}(\overline{\Omega})$ [**?**, Sec. 5.5.1]. Further, if ϕ is continuously differentiable on Γ, then $\mathcal{C}_\Gamma'\phi = \mathcal{C}_\Gamma\phi'$ [**?**, Sec. 4.4.4]. Thus, we may conclude that when $\phi \in \mathrm{C}^s(\Gamma)$, $s > 0$, then $\mathcal{C}_\Gamma\phi \in \mathrm{C}^{s-}(\overline{\Omega})$.

5.1.2. Surface integral operators. Let $\phi \in \mathrm{W}_p^0(\Omega) = \mathrm{L}^p(\Omega)$. The surface Cauchy integral on Ω is defined as

$$\mathcal{K}\phi(z) := \frac{1}{2\pi i}\iint_\Omega \frac{\phi(\zeta)}{\zeta - z}d\zeta \wedge d\bar\zeta, \quad z \in \Omega. \tag{5.4}$$

Then

$$\bar\partial\mathcal{K}\phi = \phi \quad \text{and} \quad \partial\mathcal{K}\phi = \mathcal{B}\phi, \tag{5.5}$$

where $\mathcal{B}$ is the Beurling transform, i.e.

$$\mathcal{B}\phi(z) := \frac{1}{2\pi i}\iint_\Omega \frac{\phi(\zeta)}{(\zeta-z)^2}d\zeta\wedge d\bar\zeta, \quad z\in\Omega, \tag{5.6}$$

where the integral is understood in the sense of the principal value.

The integral transformation $\mathcal{K}$ represents a bounded operator from $\mathrm{W}^0_p(\Omega)$ into $\mathrm{W}^0_{2p/(2-p)}(\Omega)$ for $p\in(1,2)$, [?, Thm. 4.8.5], and into $\mathrm{C}^{1-2/p}(\overline{\Omega})$ for $p\in(2,\infty)$, [?, Thm. 4.8.8]. Since nothing prevents us from taking z outside of $\overline{\Omega}$, $\mathcal{K}\phi$ is, in fact, defined throughout $\overline{\mathbb{C}}$ and is clearly holomorphic outside of $\overline{\Omega}$ and vanishes at infinity. Moreover, $\mathcal{K}\phi$ is continuous across Γ. The latter can be easily seen if we continue ϕ be zero to a larger domain, say $\widetilde{\Omega}$, and observe that this extension is in $\mathrm{W}^0_p(\widetilde{\Omega})$.

The Beurling transform $\mathcal{B}$ represents a bounded operator from a weighted space $\mathrm{L}^p_v(\mathbb{C})$ into itself when the non-negative function v is an A_p-weight (Muckenhoupt weight), $p\in(1,\infty)$ [?, Thm. 4.9.6]. Let $\phi\in\mathrm{W}^0_p(\Omega)$ and assume that ϕ vanishes everywhere on Γ. Thus, we can assume that $\phi\in\mathrm{L}^p(\mathbb{C})$ with $\phi\equiv0$ outside of Ω and therefore $\phi/\mathfrak{w}\in\mathrm{L}^p_{|\mathfrak{w}|^p}(\mathbb{C})$. It holds that $|\mathfrak{w}|^p$ is an A_p-weight for $p>2$ [?, Sec. 9.1.b]. Thus, $\mathcal{B}(\phi/\mathfrak{w})\in\mathrm{L}^p_{|\mathfrak{w}|^p}(\mathbb{C})$ and therefore $\mathfrak{w}\mathcal{B}(\phi/\mathfrak{w})\in\mathrm{W}^0_p(\Omega)$, $p>2$.

Finally, we point out that $\phi\in\mathrm{W}^1_p(\Omega)$ can be recovered by means $\mathcal{C}_\Gamma$ and $\mathcal{K}$ in the following fashion:

$$\phi = \mathcal{C}_\Gamma\phi + \mathcal{K}\bar\partial\phi \quad \text{in} \quad \Omega, \tag{5.7}$$

which is the Cauchy formula for non-analytic functions.

5.2. Functions of the Second Kind

Let R_n be given by (2.6) with w_n defined in (2.10). Clearly, R_n is holomorphic in D and vanishes at infinity with order at least $n+1$, i.e., $R_n=O(z^{-n-1})$ as $z\to\infty$, on account of (2.5). It is also clear, that $R_n=2\mathcal{C}(q_nw_n)$. Thus, it holds by (5.2) that

$$R_n^+ - R_n^- = 2q_nw_n$$

Further, since $q_nw_n/w = q_nh_nh/v_n\in\mathrm{C}^s$, $s>0$, it is true that

$$R_n = \begin{cases} O(|1-z|^\alpha), & \text{if} \quad \alpha<0, \\ O(\log|1-z|), & \text{if} \quad \alpha=0, \\ O(1), & \text{if} \quad \alpha>0, \end{cases}$$

and analogous asymptotics holds near -1. Indeed, the case $\alpha<0$ follows from [?, Sec. I.8.3 and I.8.4]. (Observe that we defined $(1-t)^\alpha$, $t\in\Delta^\circ$, as the values on Δ of $(1-z)^\alpha$, where the latter is holomorphic outside of the branch cut taken along Δ_r. However, $(1-t)^\alpha$ equivalently can be regared as the boundary values of $(1-z)^\alpha$ on Δ^+, where the latter is holomorphic outside of the the branch cut taken along $\Delta_l\cup\Delta$. Hence, the analysis in [?, Sec. I.8.3] indeed applies to the present situation.) The case $\alpha=0$ follows from [?, Sec. I.8.1 and I.8.4]. Finally, the case $\alpha>0$ holds since $R_n(1)$ exists for such α as $w_n(t)/(t-1)$ is integrable near 1 in this situation.

5.3. Szegő Functions

Let $\theta\in\mathrm{C}^s$ and $h:=e^\theta$. The definition of the Szegő function given in (2.8) can be rewritten as

$$S_h = \exp\left\{\mathfrak{w}\mathcal{C}\left(\frac{\theta}{\mathfrak{w}^+}\right) - \frac{1}{2}\int\theta(t)d\omega\right\}.$$

Hence, decomposition (2.9) easily follows from the Sokhotski-Plemelj formulae (5.2). Moreover, as the lemma in the next section shows, the traces $S_h^\pm$ belong to $\mathrm{C}^{s'}$, $s'<s$, and $S_h^+(\pm1)=S_h^-(\pm1)$. In particular, the functions $c_h^+:=S_h^+/S_h^-$ and $c_h^-:=S_h^-/S_h^+$ are continuous on Δ and assume value 1 at ±1. It also follows from the Sokhotski-Plemelj formulae that

$$c_h^\pm=\exp\left\{\mathfrak{w}^\pm\mathcal{S}\left(\frac{\theta}{\mathfrak{w}^+}\right)\right\}. \tag{5.8}$$

The following facts are intuitive and have been explained in detail in [**?**, Sec. 3.2 and 3.3]. If $\theta_1,\theta_2\in\mathrm{C}^s$, then $S_{h_1h_2}=S_{h_1}S_{h_2}$. If $\{\theta_n\}$ is a normal family in some neighborhood of Δ then $\{S_{h_n}\}$ is a normal family in D. If, in addition, $\{\theta_n\}$ converges then $\{S_{h_n}\}$ converges as well and convergence is *uniform* on the closure of D, in other words, including the boundary values.

It can be readily verified that the uniqueness of decomposition (2.9) implies the following formula for the Szegő function of a polynomial v_n, $\deg(v_n)\leq 2n$, with zeros in D:

$$S_{v_n}^2=\frac{1}{G_{v_n}}\frac{v_n}{r_n\varphi^{2n}}, \tag{5.9}$$

where r_n was defined in (2.4).

Next, observe that it is possible to define continuous arguments of $(z+1)/\varphi(z)$ and $(z-1)/\varphi(z)$ vanishing on the real axis in some neighborhood of infinity. Therefore it holds that

$$S_w(z)=\left(2\frac{z-1}{\varphi(z)}\right)^{\alpha/2}\left(2\frac{z+1}{\varphi(z)}\right)^{\beta/2}\quad\text{and}\quad G_w=2^{-(\alpha+\beta)}, \tag{5.10}$$

where w was defined in (2.1) and the branches of the power functions are taken such that the positive reals are mapped into the positive reals.

Finally, using (5.10) with $w=w(1/2,1/2;\cdot)$ we have that

$$S_w^+(t)S_w^-(t)=2\sqrt{1-t^2}=-2i\mathfrak{w}^+(t)\quad t\in\Delta.$$

Hence, we get that

$$S_{\mathfrak{w}^+}=\sqrt{\frac{2\mathfrak{w}}{\varphi}}\quad\text{and}\quad G_{\mathfrak{w}^+}=\frac{i}{2}, \tag{5.11}$$

where, as usual, the branch of the square root is chosen so $S_{\mathfrak{w}^+}$ is positive for positive reals large enough. It will be useful for us later to note that

$$(\varphi S_{\mathfrak{w}^+})^\pm=(S_{\mathfrak{w}^+}^\pm)^2\frac{\varphi^\pm}{S_{\mathfrak{w}^+}^\pm S_{\mathfrak{w}^+}^\mp}S_{\mathfrak{w}^+}^\mp=(S_{\mathfrak{w}^+}^\pm)^2\frac{i\varphi^\pm}{\pm2\mathfrak{w}^\pm}S_{\mathfrak{w}^+}^\mp=\pm iS_{\mathfrak{w}^+}^\mp, \tag{5.12}$$

where we used (2.9).

5.4. Smoothness of Singular Integral Operator

In this section we show that the boundary values on Δ of the Szegő function of e^θ have essentially the same smoothness as θ. It is likely that the conclusions of Lemma 4 below are known (at least in the common knowledge sense) and definitely were obtained in [**?**, Sec. 32] for $s\in(0,1/2]$. The following lemma takes place.

Lemma 4. *Let $\theta\in\mathrm{C}^s$, $s=m+\varsigma$, $m\in\mathbb{Z}_+$, $\varsigma\in(0,1]$. Then*

$$\mathfrak{w}^\pm\mathcal{S}(\theta/\mathfrak{w}^+)=\pm d+\mathfrak{w}^\pm\ell,\quad d\in\mathrm{C}^{s-},\quad d^{(k)}(\pm1)=0,$$

for all $k\in\{0,\dots,m\}$, where ℓ is a polynomial, $\deg(\ell)\leq 2m+1$.

Proof. Let T_k, $k\in\mathbb{Z}_+$, be the k-th Chebyshev polynomial. It is rather simple to verify that $\mathcal{C}(T_k/\mathfrak{w}^+)=(\varphi^n\mathfrak{w})^{-1}$. Hence, we get from Sokhotski-Plemelj formulae (5.2) that $\mathcal{S}(T_k/\mathfrak{w})=-U_{k-1}$, where U_k is the k-th Chebyshev polynomial of the second kind, $U_{-1}\equiv0$. Thus, when ℓ_0 is a polynomial, $\ell_1:=\mathcal{S}(\ell_0/\mathfrak{w}^+)$ is also a polynomial of degree one less. Therefore, if we let ℓ_0 to be the polynomial of degree at most $2m+1$ that interpolates θ at ±1 up to and including the order m, then

$$\mathfrak{w}^\pm\mathcal{S}(\theta/\mathfrak{w}^+)=\mathfrak{w}^\pm\mathcal{S}((\theta-\ell_0)/\mathfrak{w}^+)+\mathfrak{w}^\pm\ell_1. \tag{5.13}$$

Set $\theta_{e|\Delta}:=\theta-\ell_0$ and $\theta_{e|\Gamma\setminus\Delta}\equiv0$, where Γ is any infinitely smooth curve containing Δ. Then $\theta_e\in\mathrm{C}^s(\Gamma)$. Since θ_e is identically zero on $\Gamma\setminus\Delta$, it holds by (5.2) that

$$\mathfrak{w}^+\mathcal{S}((\theta-\ell_0)/\mathfrak{w}^+)=2\mathfrak{w}^+\mathcal{C}^+((\theta-\ell_0)/\mathfrak{w}^+)-(\theta-\ell_0)=2\mathfrak{w}\mathcal{C}^+_\Gamma(\theta_e/\mathfrak{w})-\theta_e, \tag{5.14}$$

where from now on we agree that $\mathfrak{w}_{|\Gamma}$ is the trace of $\mathfrak{w}$ from within Ω, i.e. it is equal to $\mathfrak{w}^+$ on Δ.

To investigate the smoothness of the product $\mathfrak{w}\mathcal{C}_\Gamma(\theta_e/\mathfrak{w})$, let us show that

$$\frac{\theta_e^{(m+1-j-k)}}{\mathfrak{w}^{2k+1}}\in\mathrm{C}^{j+\varsigma-\frac32-}(\Gamma),\quad j+\varsigma>\frac32,\quad k\in\{0,\ldots,m+1-j\}. \tag{5.15}$$

According to Section 4.1.3, for each $l\in\{0,\ldots,m\}$ there exists $\Theta_{m-l}\in\mathrm{W}^{l+1}_{p_\varsigma-}(\Omega)$ such that $\Theta_{m-l|\Gamma}=\theta_e^{(m-l)}$, where we set for brevity

$$\mathrm{W}^l_{p_\varsigma-}(\Omega):=\cap_{p\in(2,p_\varsigma)}\mathrm{W}^l_p(\Omega)\quad\text{and}\quad\mathrm{W}^l_{q_\varsigma-}(\Omega):=\cap_{q\in(4/3,q_\varsigma)}\mathrm{W}^l_q(\Omega)$$

for each $l\in\mathbb{Z}_+$, $p_\varsigma:=\frac{2}{1-\varsigma}$ and $q_\varsigma:=\frac{4}{3-2\varsigma}$. Let us show that

$$\frac{\Theta_{m+1-j-k}}{\mathfrak{w}^{2k+1}}\in\mathrm{W}^j_{q_\varsigma-}(\Omega),\quad j\in\{1,\ldots,m\},\quad k\in\{0,\ldots,m+1-j\}. \tag{5.16}$$

Clearly, by Sobolev's imbedding (4.1), (5.16) implies (5.15). Further, to show (5.16), it suffices to prove that

$$\partial^{l_1}\bar\partial^{l_2}\left(\frac{\Theta_{m+1-j-k}}{\mathfrak{w}^{2k+1}}\right)=\sum_{k=0}^{l_1}\binom{l_1}{l}v_{k,l}\frac{\Theta^{(l_1-l,l_2)}_{m+1-j-k}}{\mathfrak{w}^{2(k+l)+1}}\in\mathrm{W}^0_{q_\varsigma-}(\Omega) \tag{5.17}$$

for any $l_1+l_2=j$, where $v_{k,l}$ are some polynomials and $\Theta^{(l_1-l,l_2)}_{m+1-j-k}:=\partial^{l_1-l}\bar\partial^{l_2}\Theta_{m+1-j-k}$. As $\Theta^{(l_1-l,l_2)}_{m+1-j-k}\in\mathrm{W}^{k+l}_{p_\varsigma-}(\Omega)$ by properties of Sobolev functions, we have that $\Theta^{(l_1,l_2)}_{m+1-j}/\mathfrak{w}\in\mathrm{W}^0_{q_\varsigma-}(\Omega)$ by Hölder inequality. Thus, we may assume that $k+l>0$. Then on account of (4.1), we get that $\Theta^{(l_1-l,l_2)}_{m+1-j-k}\in\mathrm{C}^{k+l+\varsigma-1-}(\overline\Omega)$. Moreover, by the construction of $\Theta_{m+1-j-k}$ all its partial derivatives up to the total order $j+k-1$ vanish on $\Gamma\setminus\Delta$ and therefore

$$\left|\frac{\Theta^{(l_1-l,l_2)}_{m+1-j-k}(z)}{\mathfrak{w}^{2(k+l)+1}(z)}\right|\leq\text{const.}|z^2-1|^{\varsigma-\frac32-\epsilon}\quad\text{in}\quad\overline\Omega \tag{5.18}$$

for any arbitrarily small $\epsilon>0$ and this function is, in fact, continuous in $\overline\Omega\setminus\{\pm1\}$. Clearly, this shows the validity of (5.17) and respectively of (5.16) and (5.15).

As explained in Section 5.1.1, Cauhcy transform $\mathcal{C}_\Gamma$ preserves smoothness classes and therefore $\mathcal{C}_\Gamma(\theta_e/\mathfrak{w})\in\mathrm{C}^{s-\frac12-}(\overline\Omega)$ by (5.15) applied with $k=0$ and $j=m+1$. Hence, $\mathcal{C}_\Gamma(\theta_e/\mathfrak{w})$ has m (resp. $m-1$) continuous derivatives in $\overline\Omega$ when $\varsigma\in(\frac12,1]$ (resp.

$\varsigma \in (0, \frac{1}{2}]$). Let ℓ_2, $\deg(\ell_2) \leq 2m+1$, be a polynomial that interpolates $\mathcal{C}_\Gamma(\theta_e/\mathfrak{w})$ with maximal order at ± 1. Combining (5.13) with (5.14), we deduce that

$$\mathfrak{w}^+\mathcal{S}(\theta/\mathfrak{w}^+) = 2\mathfrak{w}^+(\mathcal{C}_\Gamma^+(\theta_e/\mathfrak{w}) - \ell_2) - \theta_e + \mathfrak{w}^+(\ell_1 + 2\ell_2).$$

Hence, it remains to verify that $f := \mathfrak{w}(\mathcal{C}_\Gamma(\theta_e/\mathfrak{w}) - \ell_2)$ belongs to $\mathrm{C}^{s-}(\overline{\Omega})$ and that f, as well as all its derivatives, vanishes at ± 1.

The $m+1$-st derivative of f in Ω, $f^{(m+1)}$, or equivalently $\partial^{m+1} f$, can be expressed as

$$\begin{aligned} f^{(m+1)} &= \sum_{j=0}^{m+1} \binom{m+1}{j} \frac{u_j}{\mathfrak{w}^{2j-1}} \left(\mathcal{C}_\Gamma^{(m+1-j)}\left(\frac{\theta_e}{\mathfrak{w}}\right) - \ell_2^{(m+1-j)} \right) \\ &= \sum_{j=1}^{m+1} \sum_{k=0}^{m+1-j} \binom{m+1}{j} \binom{m+1-j}{k} \frac{u_j}{\mathfrak{w}^{2j-1}} \left(\mathcal{C}_\Gamma \left(\frac{v_k \theta_e^{(m+1-j-k)}}{\mathfrak{w}^{2k+1}} \right) - \ell_{j,k} \right) \\ &\quad + \sum_{k=0}^{m} \binom{m}{k} \mathfrak{w} \left(\mathcal{C}_\Gamma' \left(\frac{v_k \theta_e^{(m-k)}}{\mathfrak{w}^{2k+1}} \right) - \ell_{0,k} \right) \\ &=: \sum_{j=1}^{m+1} \sum_{k=0}^{m+1-j} \binom{m+1}{j} \binom{m+1-j}{k} u_j \frac{I_{j,k}}{\mathfrak{w}^{2j-1}} + \sum_{k=0}^{m} \binom{m}{k} \mathfrak{w} I_{0,k}, \end{aligned}$$

where u_j and v_k are some polynomials appearing after differentiating $\mathfrak{w}$ and $1/\mathfrak{w}$ and the polynomials $\ell_{j,k}$, $\deg(\ell_{j,k}) \leq m+j$, interpolate at ± 1 the respective terms in the brackets up to the maximal order and add up to $\ell_2^{(m+1-j)}$.

By (5.15), $I_{j,k} \in \mathrm{C}^{j+\varsigma-\frac{3}{2}-}(\overline{\Omega})$, $j \in \{2,\ldots,m+1\}$, $k \in \{0,\ldots,m+1-j\}$. Thus, $I_{j,k}/\mathfrak{w}^{2j-1}$ is a continuous in $\overline{\Omega} \setminus \{\pm 1\}$ function and arguing as in (5.18) we see that it belongs to $\mathrm{W}^0_{p_\varsigma -}(\Omega)$. In fact, this also shows that $I_{1,k} \in \mathrm{W}^0_{p_\varsigma -}(\Omega)$, $k \in \{0,\ldots,m\}$, when $\varsigma \in (\frac{1}{2}, 1]$. When $j = 1$ and $\varsigma \in (0, \frac{1}{2}]$, we get by (5.7) that

$$\frac{I_{1,k}}{\mathfrak{w}} = \frac{v_k \Theta_{m-k}}{\mathfrak{w}^{2(k+1)}} - \frac{1}{\mathfrak{w}} \left(\mathcal{K} \left(\frac{v_k \bar{\partial} \Theta_{m-k}}{\mathfrak{w}^{2k+1}} \right) - \ell_{1,k} \right). \tag{5.19}$$

Using exactly the same arguments as in (5.16)–(5.18), we can show that $v_k\Theta_{m-k}/\mathfrak{w}^{2(k+1)} \in \mathrm{W}^0_{p_\varsigma -}(\Omega)$ and $v_k\bar{\partial}\Theta_{m-k}/\mathfrak{w}^{2k+1} \in \mathrm{W}^0_{q_\varsigma -}(\Omega)$. Therefore the result of applying operator $\mathcal{K}$ to the latter function belongs to $\mathrm{W}^0_r(\Omega)$ for any $r \in (4, \frac{4}{1-2\varsigma})$ (see Section 5.1.2). Hence, the second term on the right-hand side of (5.19) belongs to $\mathrm{W}^0_{p_\varsigma -}(\Omega)$ by Hölder inequality. Thus, we have shown that $I_{j,k}/\mathfrak{w}^{2j-1} \in \mathrm{W}^0_{p_\varsigma -}(\Omega)$ for any $j \in \{1,\ldots,m+1\}$ and $k \in \{0,\ldots,m+1-j\}$.

Now, we shall show that the above conclusion also holds for $\mathfrak{w}I_{0,k}$, $k \in \{0,\ldots,m\}$. By (5.7) we have that

$$\mathfrak{w}I_{0,k} = \frac{v_k \partial \Theta_{m-k}}{\mathfrak{w}^{2(k+1)}} + \frac{v_{k+1}\Theta_{m-k}}{\mathfrak{w}^{2(k+1)}} - \mathfrak{w}\mathcal{B}\left(\frac{v_k \bar{\partial}\Theta_{m-k}}{\mathfrak{w}^{2k+1}} \right) + \mathfrak{w}\ell_{0,k}. \tag{5.20}$$

Since $\partial\Theta_{m-k} \in \mathrm{W}^k_{p_\varsigma -}(\Omega)$ by the very definition of Θ_{m-k}, we have that the first summand on the right-hand side of (5.20) belongs to $\mathrm{W}^0_{p_\varsigma -}(\Omega)$ when $k = 0$. The same conclusion follows for $k > 0$ by arguing as in (5.16)–(5.18). As already explained, the second summand on the right-hand side of (5.20) also belongs to $\mathrm{W}^0_{p_\varsigma -}(\Omega)$. Finally, recall that by construction $\bar{\partial}\Theta_{m-k}$, as well as all its existing partial derivatives, vanishes on Γ. Hence, we can assume it is given through $\mathbb{C}$ (when continued by 0 outside of Ω). Then it follows that $v_k\partial\Theta_{m-k}/\mathfrak{w}^{2k} \in \mathrm{L}^p(\mathbb{C})$ for any $p \in (2, \frac{2}{1-\varsigma})$. As pointed out in Section 5.1.2, this implies that $\mathfrak{w}\mathcal{B}(v_k\bar\partial\Theta_{m-k}/\mathfrak{w}^{2k+1})$ belongs to $\mathrm{L}^p(\mathbb{C})$ and subsequently to $\mathrm{W}^0_{p_\varsigma-}(\Omega)$. So, we have shown that $\mathfrak{w}I_{0,k} \in \mathrm{W}^0_{p_\varsigma-}(\Omega)$ for all $k \in \{0,\dots,m\}$.

By what precedes, we have that $f^{(m+1)} \in \mathrm{W}^0_{p_\varsigma-}(\Omega)$. As f is holomorphic in Ω, this is equivalent to $f \in \mathrm{W}^{m+1}_{p_\varsigma-}(\Omega)$. In particular, the implies that $f \in \mathrm{C}^{s-}(\overline{\Omega})$. It is also clear that $f^{(k)}(\pm1) = 0$ for all $k \in \{0,\dots,m\}$ by the choice of ℓ_2. Thus, the claim of the lemma follows by setting $d := 2f_{|\Delta} - \theta + \ell_0$ and $\ell := \ell_1 + 2\ell_2$. $\square$

6. Riemann-Hilbert-$\bar\partial$ Problem

In what follows, σ_3 stands for the Pauli matrix $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

6.1. Initial Riemann-Hilbert Problem

Let $\mathscr{Y}$ be a 2×2 matrix function and w_n be given by (2.10). Consider the following Riemann-Hilbert problem for $\mathscr{Y}$ (RHP-$\mathscr{Y}$):

(a) $\mathscr{Y}$ is analytic in $\mathbb{C}\setminus\Delta$ and $\lim_{z\to\infty}\mathscr{Y}(z)z^{-n\sigma_3} = \mathscr{I}$, where $\mathscr{I}$ is the identity matrix;

(b) $\mathscr{Y}$ has continuous traces, $\mathscr{Y}_\pm$, on Δ° and $\mathscr{Y}_+ = \mathscr{Y}_- \begin{pmatrix} 1 & 2w_n \\ 0 & 1 \end{pmatrix}$;

(c) $\mathscr{Y}$ has the following behavior near $z=1$:

$$\mathscr{Y} = \begin{cases} O\begin{pmatrix} 1 & |1-z|^\alpha \\ 1 & |1-z|^\alpha \end{pmatrix}, & \text{if} \quad \alpha<0, \\ O\begin{pmatrix} 1 & \log|1-z| \\ 1 & \log|1-z| \end{pmatrix}, & \text{if} \quad \alpha=0, \qquad \text{as} \quad D\ni z\to1; \\ O\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & \text{if} \quad \alpha>0, \end{cases}$$

(d) $\mathscr{Y}$ has the same behavior when $D\ni z\to-1$ as in (c) only with α replaced by β and $1-z$ replaced by $1+z$.

The connection between RHP-$\mathscr{Y}$ and polynomials orthogonal with respect to w_n was realized by Fokas, Its, and Kitaev [**?**, **?**] and lies in the following.

Lemma 5. *If a solution of* RHP-$\mathscr{Y}$ *exists then it is unique. Moreover, let $n\in\mathbb{N}$, q_n be a polynomial satisfying orthogonality relations* (2.5)*, and R_n be the corresponding function of the second kind. Further, let q^*_{n-1} be a polynomial satisfying*

$$\int_\Delta t^j q^*_{n-1}(t)w_n(t)dt = 0, \quad j\in\{0,\dots,n-2\},$$

and R^*_{n-1} be its function of the second kind. If $\deg(q_n)=n$ *and there exists a constant m_n such that $m_nR^*_{n-1}(z) = z^{-n}[1+o(1)]$ near infinity, then the unique solution of* RHP-$\mathscr{Y}$ *is given by the matrix*

$$\mathscr{Y} := \begin{pmatrix} q_n & R_n \\ m_nq^*_{n-1} & m_nR^*_{n-1} \end{pmatrix}. \tag{6.1}$$

As neither the nature of the function w_n in RHP-$\mathscr{Y}$(b) nor the specific properties of $[-1,1]$ were used in [**?**, Lem. 2.3], this lemma translates without change to the present case and yields the uniqueness of the solution of RHP-$\mathscr{Y}$ whenever the latter exists. The explicit form of the solution of RHP-$\mathscr{Y}$ follows easily from Section 5.2 and the direct examination of RHP-$\mathscr{Y}$(a)–(d). The nonexistence of the solution of RHP-$\mathscr{Y}$ due to non-normality of q_n and R_{n-1}^* ($\deg(q_n) < n$ and $R_{n-1}^* = O(z^{-n-1})$ as $z \to \infty$, respectively) is rather typical, at least for small values of n, for the case of non-Hermitian orthogonality.

6.2. Renormalized Riemann-Hilbert Problem

Throughout the manuscript, unless specified otherwise, we follow the convention $\sqrt{z} = \sqrt{|z|}\exp\{i\mathrm{Arg}(z)/2\}$, $\mathrm{Arg}(z) \in (-\pi, \pi]$. Set

$$\epsilon_n := \sqrt{G_{v_n/hh_n}/2} \quad \text{and} \quad E_n := \epsilon_n \varphi^n S_{v_n/hh_n}. \tag{6.2}$$

Then E_n has continuous boundary values on each side of Δ that satisfy

$$E_n^+ E_n^- = \frac{v_n}{2hh_n} = \frac{w}{2w_n},$$

where we used (2.9) and the fact $\varphi^+\varphi^- \equiv 1$. Further, put

$$c^+ := S_h^+/S_h^-, \quad c_n^+ := S_{h_n}^+/S_{h_n}^-, \quad c^- := 1/c^+, \quad \text{and} \quad c_n^- := 1/c_n^+. \tag{6.3}$$

Then we get on account of (2.9) and (5.9) that

$$\frac{E_n^-}{E_n^+} = \frac{(S_{v_n}\varphi^n)^-}{(S_{v_n}\varphi^n)^+}\frac{S_{hh_n}^+}{S_{hh_n}^-} = \frac{v_n(c_n c)^+}{G_{v_n}\left(S_{v_n}^2\varphi^{2n}\right)^+} = (r_n c_n c)^+ \quad \text{and} \quad \frac{E_n^+}{E_n^-} = (r_n c_n c)^-. \tag{6.4}$$

Since $S_{v_n/hh_n}(\infty) = 1$ by the definition of a Szegő function and $\varphi(z)/2z \to 1$ as $z \to \infty$, it holds that $E_n(z)/[\epsilon_n(2z)^n] \to 1$ as $z \to \infty$. Then it is a quick computation to check that

$$\left(E_n^-\right)^{\sigma_3}\begin{pmatrix} 1 & 2w_n \\ 0 & 1 \end{pmatrix}\left(E_n^+\right)^{-\sigma_3} = \begin{pmatrix} (r_n c_n c)^+ & w \\ 0 & (r_n c_n c)^- \end{pmatrix}$$

and

$$\lim_{z\to\infty}(2^n\epsilon_n)^{\sigma_3}\mathscr{Y}E_n^{-\sigma_3}(z) = \mathscr{I}.$$

Let now $\mathscr{Y}$ be the solution of RHP-$\mathscr{Y}$. Define

$$\mathscr{T} := (2^n\epsilon_n)^{\sigma_3}\mathscr{Y}E_n^{-\sigma_3}. \tag{6.5}$$

Then $\mathscr{T}$ solves the following Riemann-Hilbert problem (RHP-$\mathscr{T}$):

(a) $\mathscr{T}$ is analytic in D and $\mathscr{T}(\infty) = \mathscr{I}$;

(b) $\mathscr{T}$ has continuous traces, $\mathscr{T}_\pm$, on Δ° and $\mathscr{T}_+ = \mathscr{T}_-\begin{pmatrix} (r_n c_n c)^+ & w \\ 0 & (r_n c_n c)^- \end{pmatrix}$;

(c) $\mathscr{T}$ has the following behavior near $z = 1$:

$$\mathscr{T} = \begin{cases} O\begin{pmatrix} 1 & |1-z|^\alpha \\ 1 & |1-z|^\alpha \end{pmatrix}, & \text{if} \quad \alpha < 0, \\ O\begin{pmatrix} 1 & \log|1-z| \\ 1 & \log|1-z| \end{pmatrix}, & \text{if} \quad \alpha = 0, \qquad \text{as} \quad D \ni z \to 1; \\ O\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & \text{if} \quad \alpha > 0, \end{cases}$$

(d) $\mathscr{T}$ has the same behavior when $D \ni z \to -1$ as in (c) only with α replaced by β and $1-z$ replaced by $1+z$.

Trivially, the following lemma holds.

Lemma 6. RHP-$\mathscr{T}$ *is soluble if and only if* RHP-$\mathscr{Y}$ *is soluble. When solutions of* RHP-$\mathscr{T}$ *and* RHP-$\mathscr{Y}$ *exist, they are unique and connected by* (6.5).

6.3. Opening the Lenses, Contours Σ_{ext}, Σ_n, and Σ_n^{md}

As standard in the Riemann-Hilbert approach, the second transformation of RHP-$\mathscr{Y}$ is based on the following fatoctorization of the jump matrix in RHP-$\mathscr{T}$(b):

$$\begin{pmatrix} (r_nc_nc)^+ & w \\ 0 & (r_nc_nc)^- \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ (r_nc_nc)^-/w & 1 \end{pmatrix}\begin{pmatrix} 0 & w \\ -1/w & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ (r_nc_nc)^+/w & 1 \end{pmatrix}.$$

This factorization allows to consider new Riemann-Hilbert problem with three jumps on a lens-shaped contour Σ_n (see Fig. 1 below). However, for that we need to prolongate

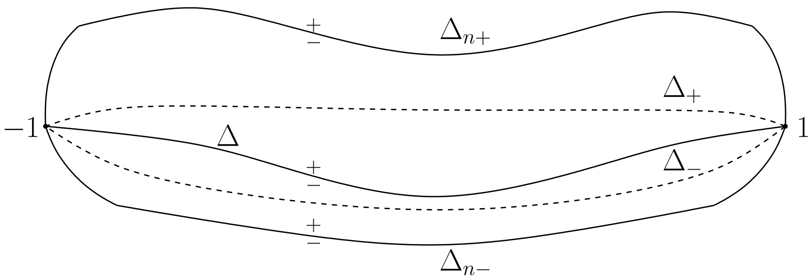

FIGURE 1. The contour $\Sigma_n := \Delta_{n+} \cup \Delta \cup \Delta_{n-} \subset D_\Xi$ (solid lines). The extension contour $\Sigma_{ext} := \Delta_+ \cup \Delta \cup \Delta_-$ (dashed lines and Δ).

$c^\pm$ and $c_n^\pm$ into the complex plane. We shall do it in such a manner that the extension functions, denoted by c and c_n, are analytic outside of a fixed lens Σ_{ext} (see Fig. 1). We postpone this task until the next section and describe here the construction of the lenses Σ_{ext} and Σ_n.

We start from Σ_{ext}. Assume first that $\Delta = [-1,1]$. Set Δ_+ to be the subarc of the circle $\{z : \ |z - ix| = |x+1|\}$ that lies in the upper half plane, where x is fixed and positive. Clearly, Δ_+ joins -1 and 1 and can be made as close to $[-1,1]$ as we want by taking x sufficiently large. We set Δ_- to be the reflection of Δ_+ across the real axis. We also denote by Ω_+ and Ω_- the upper and the lower parts of the lens Σ_{ext}, i.e. Ω_+ (resp. Ω_-) is a domain bounded by Δ_+ (resp. Δ_-) and Δ. In the general case, Σ_{ext} is the image under Ξ of the corresponding lens for $[-1,1]$ (clearly, the latter lens always can be made small enough to lie in D_Ξ).

To construct Σ_n, chose $\delta_0 > 0$ small enough that the intersection of Δ with disks of radius $\delta < \delta_0$ around 1, say U_δ, and -1, say $\widetilde{U}_\delta$, is two connected pieces of Δ. Fix $\delta < \delta_0$. As explained in Section 3.2, g_n^2, defined in (3.5), maps U_δ conformally onto some neighborhood of zero and $\widetilde{g}_n^2$ maps $\widetilde{U}_\delta$ onto another such neighborhood. Denote by K_n those points in U_δ that are mapped by g_n^2 into the rays $\Sigma_1 := \{\zeta : \ \mathrm{Arg}(\zeta) = 2\pi/3\}$ and $\Sigma_3 := \{\zeta : \ \mathrm{Arg}(\zeta) = -2\pi/3\}$. Analogously, we define $\widetilde{K}_n$. Clearly, K_n and $\widetilde{K}_n$ are Jordan arcs passing through 1 and -1, respectively. Thus, we set

$$\Sigma_n = \Delta \cup \Delta_{n+} \cup \Delta_{n-}, \quad \Delta_{n+} \cup \Delta_{n-} := K_n \cup \widetilde{K}_n \cup K_{n+} \cup K_{n-},$$

where $\Delta_{n\pm}$ are Jordan arcs connecting ±1 (Δ_{n+} contains the part of K_n that is mapped by g_n^2 onto Σ_1 and the part of $\widetilde{K}_n$ that is mapped by $\widetilde{g}_n^2$ onto Σ_3) and $K_{n\pm}$ are Jordan arcs connecting the endpoints of K_n and $\widetilde{K}_n$, having empty intersection with $U_\delta\cup\widetilde{U}_\delta$, and chosen so $\Delta_{n+}\cup\Delta_{n-}$ is a Jordan curve and the arcs $\Delta_\pm$ of the lens Σ_{ext} are contained within the interior of lens Σ_n. We point out that we have chosen the contours Σ_n in such

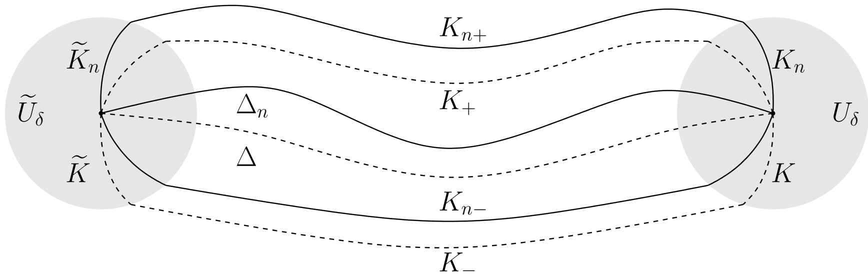

FIGURE 2. Contours Σ (dashed lines) and Σ_n^{md} (solid lines). Neighborhoods U_δ and $\widetilde{U}_\delta$ (disks around ±1).

a manner that they approach a fixed contour, say Σ, which is constructed exactly as Σ_n only with g_n^2 and $\widetilde{g}_n^2$ replaced by g^2 and $\widetilde{g}^2$, defined in (3.3) (see Fig. 2). Observe that Σ_n is well-defined since the arcs K_n (resp. $\widetilde{K}_n$) and Δ_n form angles $\pi/3$ at 1 (resp. -1) by construction. Moreover, by Theorem 1 the arcs Δ_n approach Δ in a uniform manner and therefore the tangents of Δ_n at ±1 to converge to those of Δ. Hence, it is indeed true that arcs Δ and K_n (resp. $\widetilde{K}_n$) have in common only point 1 (resp. -1).

Finally, it will be useful for us later to define one more system of contours, say Σ_n^{md}. The lens Σ_n^{md} is obtained from Σ_n simply by replacing Δ by Δ_n (see Fig. 2). By what was explained in the previous paragraph, Δ_n is contained within the lens Σ_n and therefore the lens Σ_n^{md} is well-defined.

6.4. Extension with Controlled Anti-Analytic Derivative

Without loss of generality we may assume that $\Sigma_n\subset D_\Xi$ and all the functions h_n are holomorphic in D_Ξ. By the very definition of $c_n^\pm$ we have that

$$c_n^\pm = G_{h_n}\left(S_{h_n}^\pm\right)^2 h_n^{-1}.$$

Thus, there is a natural holomorphic extension of each $c_n^\pm$ given by

$$c_n := G_{h_n}S_{h_n}^2h_n^{-1} \quad \text{in} \quad D_\Xi\setminus\Delta. \tag{6.6}$$

Concerning the extension of c, we can prove the following.

Lemma 7. *Let $\theta\in\mathrm{C}^s$. Then there exists a continuous in $\mathbb{C}\setminus\Delta$ and up to $\Delta^\pm$ function c satisfying*

$$c_{|\Delta^\pm}=c^\pm,\quad c_{|\Delta_\pm}=\exp\{\mathfrak{w}\ell\}\quad \text{and}\quad \bar\partial c = cf,$$

where ℓ is a polynomial, $\deg(\ell)\leq 2m+1$, $f\in\mathrm{L}^p(\Omega_\pm)$, $p\in(2,\frac{2}{1-s})$ *if* $s\in(0,1]$, *and* $f\in\mathrm{C}_0^{s-1-}(\overline{\Omega_\pm})$, *otherwise.*

Proof. Recall that by (5.8) we can write $c^{\pm} = \exp\{\mathfrak{w}^{\pm}\mathcal{S}(\theta/\mathfrak{w}^{+})\}$. It also was shown in Lemma 4 that $\mathfrak{w}^{\pm}\mathcal{S}(\theta/\mathfrak{w}^{+}) = \pm d + \mathfrak{w}^{\pm}\ell$, $\quad d \in \mathrm{C}^{s-}$, $d^{(k)}(\pm 1) = 0$, for all $k \in \{0,\dots,m\}$. The the conclusion of the lemma follows upon applying the considerations of Section 4.2.2 with $f = d$ and setting $c := \exp\{\mathfrak{w}\ell + F_{\pm}\}$ and $f := \bar{\partial}F_{\pm}$ in $\Omega_{\pm}$. □

6.5. Formulation of Riemann-Hilbert-$\bar{\partial}$ Problem

In this section we reformulate RHP-$\mathscr{T}$ as a Riemann-Hilbert-$\bar{\partial}$ problem. In what follows, we understand under c and c_n the extensions obtained in Section 6.4 above. Let $\mathscr{T}$ be the solution of RHP-$\mathscr{T}$. We define a matrix function $\mathscr{S}$ on $\overline{\mathbb{C}} \setminus \Sigma_n$ as follows:

$$\mathscr{S} := \begin{cases} \mathscr{T}\begin{pmatrix} 1 & 0 \\ -r_n c_n c/w & 1 \end{pmatrix}, & \text{in } \Omega_{n+}, \\ \mathscr{T}\begin{pmatrix} 1 & 0 \\ r_n c_n c/w & 1 \end{pmatrix}, & \text{in } \Omega_{n-}, \\ \mathscr{T}, & \text{outside the lens } \Sigma_n, \end{cases} \tag{6.7}$$

where the upper part, Ω_{n+}, (resp. lower part, Ω_{n-}) of the lens Σ_n is a domain bounded by Δ_{n+} (resp. Δ_{n-}) and Δ. This new matrix function is no longer analytic (in general) in the whole domain D since c is not analytic inside the extension lens Σ_{ext}. Recall that by the very construction, c coincides with a holomorphic function $\widetilde{c} = \exp\{\mathfrak{w}\ell\}$ outside the lens Σ_{ext}. To capture the non-analytic character of $\mathscr{S}$, we introduce the following matrix function that will represent the deviation from analyticity:

$$\mathscr{W}_0 := \begin{cases} \begin{pmatrix} 0 & 0 \\ \pm r_n c_n \bar{\partial} c/w & 0 \end{pmatrix}, & \text{in } \Omega_{\pm}, \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & \text{outside the lens } \Sigma_{ext}. \end{cases} \tag{6.8}$$

Then $\mathscr{S}$ solves the following Riemann-Hilbert-$\bar{\partial}$ problem (RH$\bar{\partial}$P-$\mathscr{S}$):

(a) $\mathscr{S}$ is a continuous matrix function in $\overline{\mathbb{C}} \setminus \Sigma_n$ and $\mathscr{S}(\infty) = \mathscr{I}$;

(b) $\mathscr{S}$ has continuous boundary values, $\mathscr{S}_{\pm}$, on $\Sigma_n^{\circ} := \Sigma_n \setminus \{\pm 1\}$ and

$$\begin{aligned} \mathscr{S}_+ &= \mathscr{S}_-\begin{pmatrix} 1 & 0 \\ r_n c_n c/w & 1 \end{pmatrix} \quad \text{on} \quad \Delta_{n+}^{\circ} \cup \Delta_{n-}^{\circ}, \\ \mathscr{S}_+ &= \mathscr{S}_-\begin{pmatrix} 0 & w \\ -1/w & 0 \end{pmatrix} \quad \text{on} \quad \Delta^{\circ}; \end{aligned}$$

(c) For $\alpha < 0$, $\mathscr{S}$ has the following behavior near $z = 1$:

$$\mathscr{S}(z) = O\begin{pmatrix} 1 & |1-z|^{\alpha} \\ 1 & |1-z|^{\alpha} \end{pmatrix}, \quad \text{as} \quad \mathbb{C} \setminus \Sigma_n \ni z \to 1;$$

For $\alpha = 0$, $\mathscr{S}$ has the following behavior near $z = 1$:

$$\mathscr{S}(z) = O\begin{pmatrix} \log|1-z| & \log|1-z| \\ \log|1-z| & \log|1-z| \end{pmatrix} \text{as} \quad \mathbb{C} \setminus \Sigma_n \ni z \to 1;$$

For $\alpha > 0$, $\mathscr{S}$ has the following behavior near $z = 1$:

$$\mathscr{S}(z) = \begin{cases} O\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & \text{as } z \to 1 \text{ outside the lens } \Sigma_n, \\ O\begin{pmatrix} |1-z|^{-\alpha} & 1 \\ |1-z|^{-\alpha} & 1 \end{pmatrix}, & \text{as } z \to 1 \text{ inside the lens } \Sigma_n; \end{cases}$$

(d) $\mathscr{S}$ has the same behavior when $\mathbb{C}\setminus\Sigma_n \ni z \to -1$ as in (c) only with α replaced by β and $1-z$ replaced by $1+z$;

(e) $\mathscr{S}$ deviate from an analytic matrix function according to $\bar\partial\mathscr{S} = \mathscr{S}\mathscr{W}_0$.

As for the first transformation, the following lemma holds.

Lemma 8. *RH$\bar\partial$P-$\mathscr{S}$ is soluble if and only if RHP-$\mathscr{T}$ is soluble. When solutions of RH$\bar\partial$P-$\mathscr{S}$ and RHP-$\mathscr{T}$ exist, they are unique and connected by (6.7).*

Proof. By construction, the solution of RHP-$\mathscr{T}$ yields a solution of RH$\bar\partial$P-$\mathscr{S}$. Conversely, let $\mathscr{S}^*$ be a solution of RH$\bar\partial$P-$\mathscr{S}$. It is easy to check that $\mathscr{T}^*$, obtained from $\mathscr{S}^*$ by inverting (6.5), is an analytic matrix function in $\overline{\mathbb{C}}\setminus\Sigma_n$ with continuous boundary values on Σ_n°. Moreover, it can be readily verified that $\mathscr{T}^*$ has no jumps on $\Delta_{n\pm}^\circ$ and therefore is analytic in D. It is aslo obvious that it equals to the identity matrix at infinity and has a jump on Δ described by RHP-$\mathscr{T}$(b). Thus, $\mathscr{T}^*$ complies with RHP-$\mathscr{T}$(a)–(b).

Now, if $\alpha,\beta<0$ then it follows from RH$\bar\partial$P-$\mathscr{S}$(c)–(d) and (6.5) that $\mathscr{T}^*$ has the same behavior near endpoints ±1 as $\mathscr{S}^*$. Therefore, $\mathscr{T}^*$ solves RHP-$\mathscr{T}$ in this case. It the situation when either α or β is nonnegative it is not immediate that the first column of $\mathscr{T}^*$ has the behavior near ±1 required by RHP-$\mathscr{T}$(c)–(d). The latter problem could have been resolved as in [**?**, Lem. 4.1] by considering $\mathscr{T}^*\mathscr{T}^{-1}$, where $\mathscr{T}$ is the unique solution of RHP-$\mathscr{T}$. However, in the present case it is not entirely clear that such a matrix $\mathscr{T}$ exists. Thus, we are bound to consider the first column of $\mathscr{T}^*$ by itself.

Denote by $\mathscr{T}_{11}^*$ and $\mathscr{T}_{21}^*$ the 11- and 21-entries of $\mathscr{T}^*$. Then $\mathscr{T}_{11}^*$ and $\mathscr{T}_{21}^*$ are analytic functions in D with the following behavior near 1:

$$\mathscr{T}_{j1}^*(z) = \begin{cases} O(1), & \text{if} \quad \alpha<0 \\ O(\log|1-z|), & \text{if} \quad \alpha=0, \\ O(|1-z|^{-\alpha}), & \text{if} \quad \alpha>0 \text{ and } z \text{ is inside the lens,} \\ O(1), & \text{if} \quad \alpha>0 \text{ and } z \text{ is outside the lens,} \end{cases} \tag{6.9}$$

for $j=1,2$. The behavior near -1 is completely identical only with α replaced by β and $1-z$ replaced by $1+z$. Moreover, $\mathscr{T}_{j1}^*$ are also solutions of the following scalar boundary value problem:

$$\phi^+ = \phi^-(r_nc_nc)^+ \quad \text{on} \quad \Delta, \quad \phi\in \mathrm{H}(D). \tag{6.10}$$

Now, recall that $r_n^+r_n^- \equiv 1$ on Δ and r_n has $2n$ zeros in D that lie away from the lens Σ_n. Hence, the argument of r_n^+ increases by $2\pi n$ when Δ is traversed from -1 to 1. Moreover, for c^+ and each c_n^+ the branch of the argument can be taken continuous and vanishing at ±1 (it is the imaginary part of $\mathfrak{w}^+\mathcal{S}(\theta/\mathfrak{w}^+)$, which is continuous and vanishing at ±1 by Lemma 4). Define $\varrho := \log(r_nc_nc)^+$, $\varrho(-1)=0$. This normalization is possible since $r_n^+(-1)=1$ as r_n^+ is a product of $2n$ factors each of which is equal to -1 at -1. Furthermore, this normalization necessarily yields that $\varrho(1)=2\pi ni$ and that the canonical function of the problem (6.10) is given by [**?**, Sec. 43.1]

$$\phi_c(z) := (z-1)^{-n}\exp\left\{\mathcal{C}(\varrho;z)\right\}, \quad z\in D.$$

Recall that ϕ_c is bounded in the vicinities of 1 and -1, has a zero of order n at infinty, and otherwise is non-vanishing. Hence, the functions $\phi_j := \mathscr{T}_{j1}^*/\phi_c$, $j=1,2$, are analytic in $\mathbb{C}\setminus\{\pm1\}$. Moreover, according to (6.9), the singularities of these functions at 1 and -1 cannot be essential, they are either removable or polar. In fact, since $\phi_j(z)=O(1)$ or $\phi_j(z)=O(\log|1\pm z|)$ when z approaches 1 or -1 ouside of the lens, ϕ_j can have only removable singularties at these points. Hence, $\phi_j(z)=O(1)$ and respectively $\mathscr{T}_{j1}^*=O(1)$

near 1 and -1. Thus, $\mathscr{T}^*$ satisfies RHP-$\mathscr{T}$(e)–(d) for all α and β, which means that $\mathscr{T}^*$ is the solution of RHP-$\mathscr{T}$. Therefore, indeed, the problems RHP-$\mathscr{T}$ and RH$\bar\partial$P-$\mathscr{S}$ are equivalent. □

7. Analytic Approximation of RH$\bar\partial$P-$\mathscr{S}$

Continuing on the path developed in [?], we put aside RH$\bar\partial$P-$\mathscr{S}$ and consider an analytic approximation of this problem. In other words, we seek the solution for the following Riemann-Hilbert problem (RHP-$\mathscr{A}$):

(a) $\mathscr{A}$ is a holomorphic matrix function in $\overline{\mathbb{C}}\setminus\Sigma_n$ and $\mathscr{A}(\infty)=\mathscr{I}$;
(b) $\mathscr{A}$ has continuous traces, $\mathscr{A}_\pm$, on Σ_n° that satisfy the same relations as in RH$\bar\partial$P-$\mathscr{S}$(b);
(c) the behavior of $\mathscr{A}$ near 1 is described by RH$\bar\partial$P-$\mathscr{S}$(c);
(d) the behavior of $\mathscr{A}$ near -1 is described by RH$\bar\partial$P-$\mathscr{S}$(d).

Before we proceed, observe that the function c coincides on $\Delta_{n\pm}$ with an analytic function $\widetilde c:=\exp\{\mathfrak{w}\ell\}$, where ℓ is a polynomial, by construction. Hence, we can assume that RHP-$\mathscr{A}$ is, in fact, given with $\widetilde c$.

7.1. Modified RHP-$\mathscr{A}$

The problem above almost falls into the scope of the classical approach to Riemann-Hilbert analysis. The word “almost” is used here since it is not necessarily true that the functions r_n can be written as the $2n$-th power of the same function up to a normal family. This is the reason why we constructed another lens, Σ_n^{md}, in Section 6.3. Consider the following Riemann-Hilbert problem (RHP-$\mathscr{B}$):

(a) $\mathscr{B}$ is a holomorphic matrix function in $\overline{\mathbb{C}}\setminus\Sigma_n^{md}$ and $\mathscr{B}(\infty)=\mathscr{I}$;
(b) $\mathscr{B}$ has continuous traces, $\mathscr{B}_\pm$, on $(\Sigma_n^{md})^\circ$ that satisfy

$$\begin{aligned}\mathscr{B}_+ &= \mathscr{B}_-\begin{pmatrix}1 & 0\\ r_nc_n\widetilde c/w & 1\end{pmatrix} &&\text{on}\quad \Delta_{n+}^\circ\cup\Delta_{n-}^\circ,\\ \mathscr{B}_+ &= \mathscr{B}_-\begin{pmatrix}0 & w\\ -1/w & 0\end{pmatrix} &&\text{on}\quad \Delta_n^\circ;\end{aligned}$$

(c) the behavior of $\mathscr{B}$ near 1 is described by RHP-$\mathscr{A}$(c) with respect to the lens Σ_n^{md};
(d) the behavior of $\mathscr{B}$ near -1 is described by RHP-$\mathscr{A}$(d), again, with respect to Σ_n^{md}.

In fact, this new problem is equivalent to RHP-$\mathscr{A}$.

Lemma 9. *The solutions of* RHP-$\mathscr{A}$ *and* RHP-$\mathscr{B}$ *exist and are unique simultaneously.*

Proof. Suppose that RHP-$\mathscr{B}$ is soluble and $\mathscr{B}$ is a solution. As before, let Ω_{n+} (resp. Ω_{n-}) be the upper (resp. lower) part of the lens Σ_n. Analogously define $\Omega_{n\pm}^{md}$ and set

$$\mathscr{A}^*:=\begin{cases}\mathscr{B}\begin{pmatrix}0 & w\\ -1/w & 0\end{pmatrix}, & \text{in}\quad \Omega_{n+}\cap\Omega_{n-}^{md},\\ \mathscr{B}\begin{pmatrix}0 & -w\\ 1/w & 0\end{pmatrix}, & \text{in}\quad \Omega_{n-}\cap\Omega_{n+}^{md},\\ \mathscr{B}, & \text{elsewhere.}\end{cases} \tag{7.1}$$

It is a routing exercise to verify that $\mathscr{A}^*$ complies with RHP-$\mathscr{A}$(a) and (b). Moreover, within $\Omega_{n+}\cap\Omega_{n-}^{md}$ and $\Omega_{n-}\cap\Omega_{n+}^{md}$ we have that for $\alpha<0$, $\mathscr{A}^*$ has the following behavior near $z=1$:

$$\mathscr{A}^*(z)=O\begin{pmatrix} 1 & |1-z|^\alpha \\ 1 & |1-z|^\alpha \end{pmatrix} O\begin{pmatrix} 0 & |1-z|^\alpha \\ |1-z|^{-\alpha} & 0 \end{pmatrix} = O\begin{pmatrix} 1 & |1-z|^\alpha \\ 1 & |1-z|^\alpha \end{pmatrix},$$

as $z\to1$; for $\alpha=0$, $\mathscr{A}^*$ has the same behavior near $z=1$ since we modified $\mathscr{B}$ by a bounded matrix near 1; for $\alpha>0$, $\mathscr{A}^*$ has the following behavior near $z=1$:

$$\mathscr{A}^*(z)=O\begin{pmatrix} |1-z|^{-\alpha} & 1 \\ |1-z|^{-\alpha} & 1 \end{pmatrix} O\begin{pmatrix} 0 & |1-z|^\alpha \\ |1-z|^{-\alpha} & 0 \end{pmatrix} = O\begin{pmatrix} |1-z|^{-\alpha} & 1 \\ |1-z|^{-\alpha} & 1 \end{pmatrix}.$$

Hence, $\mathscr{A}^*$ has exactly the behavior near 1 required by RHP-$\mathscr{A}$(c). In the same fashion one can check that $\mathscr{A}^*$ satisfies RHP-$\mathscr{A}$(d) and therefore it is, in fact, a solution of RHP-$\mathscr{A}$. Clearly, the arguments above could be reversed and hence each solution of RHP-$\mathscr{A}$ yields a solution of RHP-$\mathscr{B}$. □

Before we proceed, let us alleviate the notation, even though in slightly ambiguous fashion. Throughout this section, we shall understand under φ, r_n, g_n, $\widetilde{g}_n$, c_n, $\widetilde{c}$, $\mathfrak{w}$, S_{h_n}, $S_{\mathfrak{w}+}$, and S_w their holomorphic deformations that are analytic outside of Δ_n and not Δ. Note that outside the bounded set with boundary $\Delta\cup\Delta_n$ the deformed functions coincide with the original ones. Moreover, the values of the deformed functions within the bounded domain with boundary $\Delta_n\cup\Delta$ can be obtained through analytic continuation of the original functions across Δ.

7.2. Auxiliary Riemann-Hilbert Problems

In this subsection we define all the necessary objects to solve RHP-$\mathscr{B}$. All the material below is well-known and appeared in [**?**] for the case $\Delta=[-1,1]$.

7.2.1. Parametrix away from the endpoints. As the jump matrix in RHP-$\mathscr{B}$(b) is geometrically close to the identity away from Δ_n, the main term of the asymptotics for $\mathscr{B}$ in that region is determined by the following Riemann-Hilbert problem (RHP-$\mathscr{N}$):

(a) $\mathscr{N}$ is a holomorphic matrix function in D_n and $\mathscr{N}(\infty)=\mathscr{I}$;

(b) $\mathscr{N}$ has continuous traces, $\mathscr{N}_\pm$, on Δ_n° and $\mathscr{N}_+=\mathscr{N}_-\begin{pmatrix} 0 & w \\ -1/w & 0 \end{pmatrix}$;

It can be easily checked using (5.11) and (5.12) that a solution of RHP-$\mathscr{N}$ for the case $w\equiv1$ is given by

$$\mathscr{N}_*=\begin{pmatrix} S_{\mathfrak{w}+}^{-1} & i(\varphi S_{\mathfrak{w}+})^{-1} \\ -i(\varphi S_{\mathfrak{w}+})^{-1} & S_{\mathfrak{w}+}^{-1} \end{pmatrix}. \tag{7.2}$$

Then a solution of RHP-$\mathscr{N}$ for arbitrary w is given by

$$\mathscr{N}=(G_w)^{\sigma_3/2}\mathscr{N}_*(G_wS_w^2)^{-\sigma_3/2}. \tag{7.3}$$

7.2.2. Auxiliary parametrix near the endpoints. The following construction was introduced in [**?**, Thm. 6.3]. Let I_α and K_α be the modified Bessel functions and $H_\alpha^{(1)}$ and $H_\alpha^{(2)}$ be the Hankel functions [**?**, Ch. 9]. Set $\Psi=\Psi(\cdot;\alpha)$ to be the following sectionally holomorphic matrix function:

$$\Psi(\zeta):=\begin{pmatrix} I_\alpha\left(2\zeta^{1/2}\right) & \frac{i}{\pi}K_\alpha\left(2\zeta^{1/2}\right) \\ 2\pi i\zeta^{1/2}I_\alpha'\left(2\zeta^{1/2}\right) & -2\zeta^{1/2}K_\alpha'\left(2\zeta^{1/2}\right) \end{pmatrix}$$

for $|\mathrm{Arg}(\zeta)| < 2\pi/3$;

$$\Psi(\zeta) := \left(\begin{array}{cc} \frac{1}{2}H_\alpha^{(1)}\left(2(-\zeta)^{1/2}\right) & \frac{1}{2}H_\alpha^{(2)}\left(2(-\zeta)^{1/2}\right) \\ \pi\zeta^{1/2}\left(H_\alpha^{(1)}\right)'\left(2(-\zeta)^{1/2}\right) & \pi\zeta^{1/2}\left(H_\alpha^{(2)}\right)'\left(2(-\zeta)^{1/2}\right) \end{array}\right) e^{\frac{1}{2}\alpha\pi i\sigma_3}$$

for $2\pi/3 < \mathrm{Arg}(\zeta) < \pi$;

$$\Psi(\zeta) := \left(\begin{array}{cc} \frac{1}{2}H_\alpha^{(2)}\left(2(-\zeta)^{1/2}\right) & -\frac{1}{2}H_\alpha^{(1)}\left(2(-\zeta)^{1/2}\right) \\ -\pi\zeta^{1/2}\left(H_\alpha^{(2)}\right)'\left(2(-\zeta)^{1/2}\right) & \pi\zeta^{1/2}\left(H_\alpha^{(1)}\right)'\left(2(-\zeta)^{1/2}\right) \end{array}\right) e^{-\frac{1}{2}\alpha\pi i\sigma_3}$$

for $-\pi < \mathrm{Arg}(\zeta) < -2\pi/3$, where $\mathrm{Arg}(\zeta) \in (-\pi, \pi]$ is the principal part of the argument of ζ. Let further Σ_1, Σ_2, and Σ_3 be the rays $\{\zeta : \mathrm{Arg}(\zeta) = 2\pi/3\}$, $\{\zeta : \mathrm{Arg}(\zeta) = \pi\}$, and $\{\zeta : \mathrm{Arg}(\zeta) = -2\pi/3\}$, respectively, oriented from infinity to zero. Then Ψ is the solution of the following Riemann-Hilbert problem RHP-Ψ:

(a) Ψ is a holomorphic matrix function in $\mathbb{C} \setminus (\Sigma_1 \cup \Sigma_2 \cup \Sigma_3)$;

(b) Ψ has continuous traces, $\Psi_\pm$, on Σ_j°, $j \in \{1, 2, 3\}$, and

$$\begin{array}{rcll} \Psi_+ & = & \Psi_- \left(\begin{array}{cc} 1 & 0 \\ e^{\alpha\pi i} & 1 \end{array}\right) & \text{on} \quad \Sigma_1^\circ, \\ \Psi_+ & = & \Psi_- \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right) & \text{on} \quad \Sigma_2^\circ, \\ \Psi_+ & = & \Psi_- \left(\begin{array}{cc} 1 & 0 \\ e^{-\alpha\pi i} & 1 \end{array}\right) & \text{on} \quad \Sigma_3^\circ; \end{array}$$

(c) Ψ has the following behavior near ∞:

$$\Psi(\zeta) = \left(2\pi\zeta^{1/2}\right)^{-\sigma_3/2} \left(\begin{array}{cc} 1 + O\left(\zeta^{-1/2}\right) & i + O\left(\zeta^{-1/2}\right) \\ i + O\left(\zeta^{-1/2}\right) & 1 + O\left(\zeta^{-1/2}\right) \end{array}\right) e^{2\zeta^{1/2}\sigma_3}$$

uniformly in $\mathbb{C} \setminus (\Sigma_1 \cup \Sigma_2 \cup \Sigma_3)$;

(d) For $\alpha < 0$, Ψ has the following behavior near 0:

$$\Psi = O\left(\begin{array}{cc} |\zeta|^{\alpha/2} & |\zeta|^{\alpha/2} \\ |\zeta|^{\alpha/2} & |\zeta|^{\alpha/2} \end{array}\right) \quad \text{as } \zeta \to 0;$$

For $\alpha = 0$, Ψ has the following behavior near 0:

$$\Psi = O\left(\begin{array}{cc} \log|\zeta| & \log|\zeta| \\ \log|\zeta| & \log|\zeta| \end{array}\right) \quad \text{as } \zeta \to 0;$$

For $\alpha > 0$, Ψ has the following behavior near 0:

$$\Psi = \left\{\begin{array}{ll} O\left(\begin{array}{cc} |\zeta|^{\alpha/2} & |\zeta|^{-\alpha/2} \\ |\zeta|^{\alpha/2} & |\zeta|^{-\alpha/2} \end{array}\right) & \text{as } \zeta \to 0 \text{ in } |\mathrm{Arg}(\zeta)| < 2\pi/3, \\ O\left(\begin{array}{cc} |\zeta|^{-\alpha/2} & |\zeta|^{-\alpha/2} \\ |\zeta|^{-\alpha/2} & |\zeta|^{-\alpha/2} \end{array}\right) & \text{as } \zeta \to 0 \text{ in } 2\pi/3 < |\mathrm{Arg}(\zeta)| < \pi. \end{array}\right.$$

Further, if we set

$$\widetilde{\Psi} := \sigma_3\Psi(\cdot;\beta)\sigma_3,$$

then this matrix function satisfies RHP-Ψ with α replaced by β and the reversed orientation of Σ_j, $j \in \{1, 2, 3\}$.

7.2.3. Parametrix near 1. Here we describe the solution for the following Riemann-Hilbert problem (RHP-$\mathscr{P}$):

(a) $\mathscr{P}$ is a holomorphic matrix function in $U_{\delta_0} \setminus \Sigma_n^{md}$, $\delta < \delta_0$;
(b) $\mathscr{P}$ has continuous boundary values, $\mathscr{P}_\pm$, on $U_\delta \cap (\Sigma_n^{md})^\circ$ and

$$\begin{aligned} \mathscr{P}_+ &= \mathscr{P}_- \begin{pmatrix} 1 & 0 \\ r_n c_n \widetilde{c}/w & 1 \end{pmatrix} \quad \text{on} \quad U_\delta \cap (\Delta_{n+}^\circ \cup \Delta_{n-}^\circ), \\ \mathscr{P}_+ &= \mathscr{P}_- \begin{pmatrix} 0 & w \\ -1/w & 0 \end{pmatrix} \quad \text{on} \quad U_\delta \cap \Delta_n^\circ; \end{aligned}$$

(c) $\mathscr{P}\mathscr{N}^{-1} = \mathscr{I} + O(1/n)$ uniformly on ∂U_δ;
(d) For $\alpha < 0$, $\mathscr{P}$ has the following behavior near $z = 1$:

$$\mathscr{P} = O\begin{pmatrix} 1 & |1-z|^\alpha \\ 1 & |1-z|^\alpha \end{pmatrix}, \quad \text{as} \quad U_\delta \setminus \Sigma_n^{md} \ni z \to 1;$$

For $\alpha = 0$, $\mathscr{P}$ has the following behavior near $z = 1$:

$$\mathscr{P} = O\begin{pmatrix} \log|1-z| & \log|1-z| \\ \log|1-z| & \log|1-z| \end{pmatrix}, \quad \text{as} \quad U_\delta \setminus \Sigma_n^{md} \ni z \to 1;$$

For $\alpha > 0$, $\mathscr{P}$ has the following behavior near $z = 1$:

$$\mathscr{P} = \begin{cases} O\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & \text{as } z \to 1 \text{ outside the lens } \Sigma_n^{md}, \\ O\begin{pmatrix} |1-z|^{-\alpha} & 1 \\ |1-z|^{-\alpha} & 1 \end{pmatrix}, & \text{as } z \to 1 \text{ inside the lens } \Sigma_n^{md}. \end{cases}$$

To present the solution of RHP-$\mathscr{P}$, we need to introduce more notation. Denote by U_δ^+ and U_δ^- the subsets of U_δ that are mapped by g_n^2 into the upper and lower half planes, respectively. Without loss of generality we may assume that the branch cut of w in U_δ coincides with the preimage of the positive reals under g_n^2. In particular, we have that w is analytic in U_δ^+ and U_δ^- and therefore across $\Delta_{n\pm}^\circ$. Set

$$A_n(z) := \frac{\exp\left\{\frac{1}{2}\left(\theta_n(z) - \mathfrak{w}(z)\ell(z)\right)\right\}}{\sqrt{G_{h_n}} S_{h_n}(z)} (z-1)^{\alpha/2}(z+1)^{\beta/2},$$

where we choose a branch of $(z+1)^{\beta/2}$ analytic in U_δ and a branch of $(z-1)^{\alpha/2}$ analytic in $U_\delta \setminus \Delta_n$. Then

$$A_n^2 = \begin{cases} e^{\alpha\pi i} w / c_n \widetilde{c}, & \text{in} \quad U_\delta^+, \\ e^{-\alpha\pi i} w / c_n \widetilde{c}, & \text{in} \quad U_\delta^-, \end{cases} \tag{7.4}$$

by the definition of $\widetilde{c}$ and on account of (6.6). Moreover, it readily follows from (7.4) and (6.3) that

$$A_n^+ A_n^- = w \quad \text{on} \quad \Delta_n^\circ. \tag{7.5}$$

Recall that r_n has winding number $-2n$ on any Jordan curve encompassing Δ_n and therefore $r_n^{1/2}$, $\lim_{z\to1} r_n^{1/2}(z) = 1$, is a well-defined holomorphic function in $U_\delta \setminus \Delta_n$. Observe also that $g_n^{1/2}$ is a holomorphic function on $U_\delta \setminus \Delta_n$ such that

$$(g_n^{1/2})^+ = i(g_n^{1/2})^- \quad \text{on} \quad \Delta_n$$

by (3.6). Then the following lemma holds.

Lemma 10. *The solution of* RHP-$\mathscr{P}$ *is given by*

$$\mathscr{P} = \mathscr{E}\Psi\left(\frac{n^2g_n^2}{4}\right)A_n^{-\sigma_3}r_n^{\sigma_3/2}, \quad \mathscr{E} := \mathscr{N}A_n^{\sigma_3}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}(\pi n g_n)^{\sigma_3/2}.$$

Proof. Except for the technical differences, the proof is analogous to the considerations in [**?**, Sec. 6]. First, we should show that $\mathscr{E}$ is holomorphic in U_δ. This is clearly true in $U_\delta \setminus \Delta_n$. It is also clear that $\mathscr{E}$ has continuous boundary values on Δ_n°. Since

$$\begin{aligned}
\mathscr{E}_+ &= \mathscr{N}_+\left(A_n^+\right)^{\sigma_3}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}(\pi n)^{\sigma_3/2}\left(\left(g_n^{1/2}\right)^+\right)^{\sigma_3} \\
&= \mathscr{N}_-\begin{pmatrix} 0 & w \\ -1/w & 0 \end{pmatrix}\left(\frac{w}{A_n^-}\right)^{\sigma_3}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}(\pi n)^{\sigma_3/2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\left(\left(g_n^{1/2}\right)^-\right)^{\sigma_3} \\
&= \mathscr{N}_-\left(A_n^-\right)^{\sigma_3}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} i & -1 \\ 1 & -i \end{pmatrix}(\pi n)^{\sigma_3/2}\left(\left(g_n^{1/2}\right)^-\right)^{\sigma_3} = \mathscr{E}_-,
\end{aligned}$$

where we used RHP-$\mathscr{N}$(b) and (7.5), $\mathscr{E}$ is holomorphic across Δ_n°. Thus, it remains to show that $\mathscr{E}$ has no singularity at 1. For this observe that

$$\left(g_n^{1/2}(z)\right)^{\sigma_3} = O\begin{pmatrix} |1-z|^{1/4} & 0 \\ 0 & |1-z|^{-1/4} \end{pmatrix}, \quad \text{as} \quad z \to 1,$$

since g_n^2 has a simple zero at 1,

$$\mathscr{N}_* = O\begin{pmatrix} |1-z|^{-1/4} & |1-z|^{-1/4} \\ |1-z|^{-1/4} & |1-z|^{-1/4} \end{pmatrix}, \quad \text{as} \quad z \to 1,$$

by definition, and $(A_n/S_w)(z) \to 2^{-\alpha/2}$ as $z \to 1$ since $c_n\widetilde{c}$ tend to 1 in this case. Hence, the entries of $\mathscr{E}$ can have at most square-root singularity at 1, which is impossible since $\mathscr{E}$ is analytic in $U_\delta \setminus \{1\}$, and therefore $\mathscr{E}$ is analytic in the whole disk U_δ.

The analyticity of $\mathscr{E}$ implies, in particular, that the jumps of $\mathscr{P}$ are those of $\Psi\left(n^2g_n^2/4\right)A_n^{-\sigma_3}r_n^{\sigma_3/2}$. Clearly, the latter has jumps on $\Sigma_n^{md} \cap U_\delta$ by the very definition of g_n^2 and Ψ. Moreover, it is a routing exercise, using RHP-Ψ(b) and (7.4), to verify that these jumps are described exactly by RHP-$\mathscr{P}$(b). It is also clear that RHP-$\mathscr{P}$(a) is satisfied. Further, we get directly from RHP-Ψ(c) that the behavior of $\Psi\left(n^2g_n^2/4\right)$ on ∂U_δ can be described by

$$\Psi\left(\frac{n^2g_n^2}{4}\right) = (\pi n g_n)^{-\sigma_3/2}\begin{pmatrix} 1+O\left(\frac{1}{n}\right) & i+O\left(\frac{1}{n}\right) \\ i+O\left(\frac{1}{n}\right) & 1+O\left(\frac{1}{n}\right) \end{pmatrix} r_n^{-\sigma_3/2},$$

where functions $O(\cdot)$ hold uniformly on ∂U_δ. Hence, we get that

$$\begin{aligned}
\mathscr{P}\mathscr{N}^{-1} &= \mathscr{E}(\pi n g_n)^{-\sigma_3/2}\begin{pmatrix} 1+O\left(\frac{1}{n}\right) & i+O\left(\frac{1}{n}\right) \\ i+O\left(\frac{1}{n}\right) & 1+O\left(\frac{1}{n}\right) \end{pmatrix} A_n^{-\sigma_3}\mathscr{N}^{-1} \\
&= \mathscr{N}A_n^{\sigma_3}\left(\mathscr{I} + O\left(\frac{1}{n}\right)\right)A_n^{-\sigma_3}\mathscr{N}^{-1} = \mathscr{I} + O\left(\frac{1}{n}\right)
\end{aligned}$$

since the moduli of all the entries of $\mathscr{N} A_n^{\sigma_3}$ are uniformly bounded above and away from zero. Thus, RHP-$\mathscr{P}$(c) holds. Finally, RHP-$\mathscr{P}$(d) follows immediately from RHP-Ψ(d) upon recalling that $|g_n^2| = O(|1-z|)$ and $|A_n| = O(|1-z|^{\alpha/2})$ as $z \to 1$. □

7.2.4. Parametrix near -1**.** In this section we describe the solution for the following Riemann-Hilbert problem (RHP-$\widetilde{\mathscr{P}}$):

(a) $\widetilde{\mathscr{P}}$ is a holomorphic matrix function in $\widetilde{U}_{\delta_0} \setminus \Sigma_n^{md}$, $\delta < \delta_0$;
(b) $\widetilde{\mathscr{P}}$ has continuous boundary values, $\widetilde{\mathscr{P}}_\pm$, on $\widetilde{U}_\delta \cap (\Sigma_n^{md})^\circ$ and

$$\begin{aligned} \widetilde{\mathscr{P}}_+ &= \widetilde{\mathscr{P}}_- \begin{pmatrix} 1 & 0 \\ r_n c_n \widetilde{c}/w & 1 \end{pmatrix} \quad \text{on} \quad \widetilde{U}_\delta \cap (\Delta_{n+}^\circ \cup \Delta_{n-}^\circ), \\ \widetilde{\mathscr{P}}_+ &= \widetilde{\mathscr{P}}_- \begin{pmatrix} 0 & w \\ -1/w & 0 \end{pmatrix} \quad \text{on} \quad \widetilde{U}_\delta \cap \Delta_n^\circ; \end{aligned}$$

(c) $\widetilde{\mathscr{P}} \mathscr{N}^{-1} = \mathscr{I} + O(1/n)$ uniformly on $\partial \widetilde{U}_\delta$;
(d) For $\beta < 0$, $\widetilde{\mathscr{P}}$ has the following behavior near $z = -1$:

$$\widetilde{\mathscr{P}} = O \begin{pmatrix} 1 & |1+z|^\beta \\ 1 & |1+z|^\beta \end{pmatrix}, \quad \text{as} \quad \widetilde{U}_\delta \setminus \Sigma_n^{md} \ni z \to -1;$$

For $\beta = 0$, $\widetilde{\mathscr{P}}$ has the following behavior near $z = -1$:

$$\widetilde{\mathscr{P}} = O \begin{pmatrix} \log|1+z| & \log|1+z| \\ \log|1+z| & \log|1+z| \end{pmatrix}, \quad \text{as} \quad \widetilde{U}_\delta \setminus \Sigma_n^{md} \ni z \to -1;$$

For $\beta > 0$, $\widetilde{\mathscr{P}}$ has the following behavior near $z = -1$:

$$\widetilde{\mathscr{P}} = \begin{cases} O \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & \text{as } z \to -1 \text{ outside the lens } \Sigma_n^{md}, \\ O \begin{pmatrix} |1+z|^{-\beta} & 1 \\ |1+z|^{-\beta} & 1 \end{pmatrix}, & \text{as } z \to -1 \text{ inside the lens } \Sigma_n^{md}. \end{cases}$$

This problem is solved exactly in the same manner as RHP-$\mathscr{P}$. Thus, we set

$$\widetilde{A}_n(z) := \frac{\exp\left\{\frac{1}{2}\left(\theta_n(z) - \mathfrak{w}(z)\ell(z)\right)\right\}}{\sqrt{G_{h_n}} S_{h_n}(z)} (1-z)^{\alpha/2}(-1-z)^{\beta/2},$$

where $(1-z)^{\alpha/2}$ is holomorphic in $\widetilde{U}_\delta$ and $(-1-z)^{\beta/2}$ is holomorphic in $\widetilde{U}_\delta \setminus \Delta_n$. As in (7.4), we have that

$$\widetilde{A}_n^2 = \begin{cases} e^{\beta\pi i} w/c_n\widetilde{c}, & \text{in} \quad \widetilde{U}_\delta^+, \\ e^{-\beta\pi i} w/c_n\widetilde{c}, & \text{in} \quad \widetilde{U}_\delta^-, \end{cases}$$

where $\widetilde{U}_\delta^\pm$ have the same meaning as in the previous section. However, here one needs to be cautious since $\widetilde{g}_n$ reverses the orientation of Σ_2, i.e. Σ_2 is now oriented from zero to infinity, and therefore $\widetilde{U}_\delta^+$ is the "lower" part of $\widetilde{U}_\delta$ and $\widetilde{U}_\delta^-$ is the "upper" one. Again, it can be checked that

$$\widetilde{A}_n^+ \widetilde{A}_n^- = w \quad \text{on} \quad \Delta_n^\circ.$$

The following lemma can be proven exactly as Lemma 10 since $(-1)^n r_n^{1/2} = e^{n\widetilde{g}_n}$.

Lemma 11. *The solution of* RHP-$\widetilde{\mathscr{P}}$ *is given by*

$$\widetilde{\mathscr{P}} = \widetilde{\mathscr{E}} \widetilde{\Psi} \left(\frac{n^2 \widetilde{g}_n^2}{4} \right) \widetilde{A}_n^{-\sigma_3} (-1)^{n\sigma_3} r_n^{\sigma_3/2}, \; \widetilde{\mathscr{E}} := \mathscr{N} \widetilde{A}_n^{\sigma_3} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} (\pi n \widetilde{g}_n)^{\sigma_3/2}.$$

Finally, we are prepared to solve RHP-$\mathscr{A}$.

7.3. Solution of RHP-$\mathscr{A}$

Denote by Σ_n^{rd} the reduced system of contours obtained from Σ_n^{md} by removing Δ_n, K_n, and $\widetilde{K}_n$, and adding $\partial U_\delta \cup \partial\widetilde{U}_\delta$. For this new system of contours we consider the following Riemann-Hilbert problem (RHP-$\mathscr{R}$):

(a) $\mathscr{R}$ is a holomorphic matrix function in $\overline{\mathbb{C}}\setminus\Sigma_n^{rd}$ and $\mathscr{R}(\infty)=\mathscr{I}$;
(b) the traces of $\mathscr{R}$, $\mathscr{R}_\pm$, are continuous on Σ_n^{rd} except for the branching points of Σ_n^{rd}, where they have definite limits from each branch of Σ_n^{rd}. Moreover, $\mathscr{R}_\pm$ satisfy

$$\mathscr{R}_+=\mathscr{R}_-\begin{cases}\mathscr{P}\mathscr{N}^{-1} & \text{on} \quad \partial U_\delta,\\ \widetilde{\mathscr{P}}\mathscr{N}^{-1} & \text{on} \quad \partial\widetilde{U}_\delta\\ \mathscr{N}\begin{pmatrix}1 & 0\\ r_nc_n\widetilde{c}/w & 1\end{pmatrix}\mathscr{N}^{-1} & \text{on} \quad \Sigma_n^{rd}\setminus(\partial U_\delta\cup\partial\widetilde{U}_\delta).\end{cases}$$

Then the following lemma takes place.

Lemma 12. *The solution of* RHP-$\mathscr{A}$ *exists for all n large enough, is unique, and* $\det\mathscr{A}\equiv 1$. *Moreover, the solution is given by* (7.1) *with $\mathscr{B}$ defined by*

$$\mathscr{B}:=\begin{cases}\mathscr{R}\mathscr{N}, & in \quad \overline{\mathbb{C}}\setminus(\overline{U}_\delta\cup\overline{\widetilde{U}_\delta}\cup\Sigma_n),\\ \mathscr{R}\mathscr{P}, & in \quad U_\delta,\\ \mathscr{R}\widetilde{\mathscr{P}}, & in \quad \widetilde{U}_\delta,\end{cases}\tag{7.6}$$

and $\mathscr{R}$ being the solution of RHP-$\mathscr{R}$, *which exists, is unique for all n large enough, and satisfy*

$$\mathscr{R}=\mathscr{I}+O\left(\frac{1}{n}\right)\tag{7.7}$$

uniformly in $\overline{\mathbb{C}}$.

Proof. First, we show the existence of $\mathscr{R}$. By the very definition of $\mathscr{P}$ and $\widetilde{\mathscr{P}}$, we have that

$$\mathscr{R}_+=\mathscr{R}_-\left(\mathscr{I}+O\left(\frac{1}{n}\right)\right)\tag{7.8}$$

on $\partial U_\delta\cup\partial\widetilde{U}_\delta$. As $\mathscr{R}_\pm$ extends continuously to $\partial U_\delta\cup\partial\widetilde{U}_\delta\setminus\{$the branching points of $\Sigma_n^{rd}\}$, the estimate above holds uniformly on $\partial U_\delta\cup\partial\widetilde{U}_\delta$. Further, since the jump of $\mathscr{R}$ on $\Sigma_n^{rd}\setminus(\partial U_\delta\cup\partial\widetilde{U}_\delta)$ is analytic, it allows us to deform the problem RHP-$\mathscr{R}$ to a fixed contour, say Σ^{rd}, obtained from Σ as Σ_n^{rd} was obtained from Σ_n^{md} (the solutions exist, are simultaneously unique, and can be easily expressed through each other as in (7.1)). Moreover, by the properties of r_n, the jump of $\mathscr{R}$ on $\Sigma^{rd}\setminus(\partial U_\delta\cup\partial\widetilde{U}_\delta)$ is uniformly geometrically close to $\mathscr{I}$. Hence, (7.8) holds uniformly on Σ^{rd}. Thus, the existence, the uniqueness, and the asymptotic properties of the solution of RHP-$\mathscr{R}$ follow from the classical theory (see Theorem 7.8 and Corollary 7.9 in [**?**] or Theorem 7.103 and Corollary 7.108 in [**?**]). Finally, as $\mathscr{N}$, $\mathscr{P}$, and $\widetilde{\mathscr{P}}$ have determinants equal to 1 throughout $\mathbb{C}$ [**?**, Rem. 7.1], $\det\mathscr{R}$ is an analytic function in $\overline{\mathbb{C}}\setminus\Sigma^{rd}$ that is equal to 1 at infinity, has equal boundary values on Σ^{rd}, and is bounded near the branching points of Σ^{rd}. Therefore, $\det\mathscr{R}\equiv 1$. Now, it can be checked easily by the definition of $\mathscr{P}$, $\widetilde{\mathscr{P}}$, and $\mathscr{N}$, that $\mathscr{B}$, given by (7.6), is the solution of RHP-$\mathscr{B}$. Moreover, with a little bit more work around points ±1 [**?**, Sec. 7] it can be shown that a solution of RHP-$\mathscr{B}$ yields a solution of

RHP-$\mathscr{R}$. Thus, the problems RHP-$\mathscr{B}$ and RHP-$\mathscr{R}$ are equivalent. Hence, by Lemma 9 and since $\det \mathscr{B} \equiv 1$, the proof of the lemma is finished. □

8. $\bar{\partial}$-Problem

In the previous section we completed the first step in solving RH$\bar{\partial}$P-$\mathscr{S}$. In that section we solved RHP-$\mathscr{A}$, the problem with the same conditions as in RH$\bar{\partial}$P-$\mathscr{S}$ except for the deviation from analyticity, which was dropped entirely. In this section, we solve a complementary problem, namely, we show that a solution of a certain $\bar{\partial}$-problem for matrix functions exists. Set

$$\mathscr{W} = \mathscr{A}\mathscr{W}_0\mathscr{A}^{-1}, \tag{8.1}$$

where $\mathscr{W}_0$ was defined in (6.8) and $\mathscr{A}$ is the solution of RHP-$\mathscr{A}$. In what follows, we sick the solution of the following $\bar{\partial}$-problem ($\bar{\partial}$P-$\mathscr{D}$):

(a) $\mathscr{D}$ is a continuous matrix function in $\overline{\mathbb{C}}$ and $\mathscr{D}(\infty) = \mathscr{I}$;
(b) $\mathscr{D}$ deviate from an analytic matrix function according to $\bar{\partial}\mathscr{D} = \mathscr{D}\mathscr{W}$.

Then the following lemma holds.

Lemma 13. *If* (2.11) *is fulfilled, then $\mathscr{D}$, the solution of $\bar{\partial}$P-$\mathscr{D}$, uniquely exists for all n large enough and*

$$\mathscr{D} = \mathscr{I} + O(\delta_{n,\epsilon}) \tag{8.2}$$

uniformly in $\overline{\mathbb{C}}$ for any arbitrarily small ϵ, where $\delta_{n,\epsilon}$ was defined in (2.12).

Proof. We start by examining the summability and smoothness of the entries of $\mathscr{W}$. As $\mathscr{A}$ is the solution of RHP-$\mathscr{A}$, it is an analytic matrix function in $\Omega_\pm$ and its behavior near ±1 is given by RH$\bar{\partial}$P-$\mathscr{S}$(c)–(d). Since $\det\mathscr{A} = 1$ in $\overline{\mathbb{C}}$, the behavior of $\mathscr{A}^{-1}$ near ±1 is also governed by the matrices in RH$\bar{\partial}$P-$\mathscr{S}$(c)-(d) with the elements on the main diagonal interchanged. Set

$$f_{\alpha,\beta} := f_{1,\alpha}f_{-1,\beta}, \quad f_{x,\delta}(z) := \begin{cases} |z-x|^{-|\delta|}, & \delta \neq 0, \\ \log^2|z-x|, & \delta = 0. \end{cases}$$

As $|c_n c|$ are uniformly bounded in $\Omega_\pm$ with n, we get from a simple computation and Lemma 7 that

$$|\mathscr{W}_{lk}| \leq \text{const.} f_{\alpha,\beta}|r_n f|, \quad l,k = 1,2, \tag{8.3}$$

where f comes from the decomposition of $\bar{\partial}c$ in Lemma 7.

Recall the notation $s^* := \max\{|\alpha|,|\beta|\}$ and that $s^* < s$ by (2.11). First, we consider the case $s - s^* > 1$. This, in particular, means that $s > 1$ and therefore $f \in \mathrm{C}_0^{s-1-}(\overline{\Omega})$, where we set $\Omega = \Omega_+ \cup \Omega_- \cup \Delta^\circ$. (Recall that by Lemma 7, f and all the partial derivatives of f up to the order $m-1$ have well-defined vanishing boundary values on Δ and therefore f is indeed defined throughout Ω.) By choosing ϵ small enough so $s - s^* - 1 - \epsilon > 0$, we get that

$$|f_{\alpha,\beta}f| \leq \text{const.} \tag{8.4}$$

Now, we consider the case $s - s^* < 1$. Let $\delta > 0$ be such that

$$\frac{\delta}{2+\delta} < s - s^*. \tag{8.5}$$

When $s \in (0,1)$, we get for the choice of δ that $\frac{1}{2+\delta}\frac{2}{1-s} > 1$ and therefore there exists $p > 1$ such that $(2+\delta)(1-s) < \frac{2}{p} < 2-(2+\delta)s^*$. Hence, $p < \frac{1}{2+\delta}\frac{2}{1-s}$ and $(2+\delta)s^*p' < 2$, where p' is the Hölder conjugate of p. So, by applying Hölder inequality with exponents p and p', we get that

$$\|f_{\alpha,\beta}f\|_{2+\delta,\Omega} \leq \|f_{\alpha,\beta}\|_{(2+\delta)p',\Omega}\|f\|_{(2+\delta)p,\Omega} < \infty,$$

where $\|\cdot\|_{q,\Omega}$ in the usual norm in $\mathrm{L}^q(\Omega)$ and we set $f \equiv 0$ on Δ° for definitiveness. In other words, it holds that

$$f_{\alpha,\beta}f \in \mathrm{L}^{2+\delta}(\overline{\Omega}). \tag{8.6}$$

When $s = 1$, we arrive at (8.6) by the same argument as above only applied with $1-\epsilon$ for ϵ small enough. When $s \in (1,\infty)$, it holds that $f \in \mathrm{C}_0^{s-1-}(\overline{\Omega})$. This, in particular, implies that

$$|f_{\alpha,\beta}(z)f(z)| \leq \text{const.}|z^2-1|^{s-s^*-1-\epsilon}$$

for any arbitrarily small ϵ. By taking ϵ small enough that (8.5) still holds with ϵ subtracted from the right-hand side, we deduce the validity of (8.6).

Finally, when $s - s^* = 1$, we derive (8.6) as above only considering $s-\epsilon$ instead of s for ϵ small enough.

Let now $\mathscr{D}$ be a solution of $\bar\partial$P-$\mathscr{D}$ and let Γ be a smooth arc encompassing $\overline{\Omega}$. Applying Cauchy formula (5.7) to $\mathscr{D}$, we get that

$$\mathscr{D} = \mathcal{C}_\Gamma(\mathscr{D}) + \mathcal{K}_\Gamma(\mathscr{D}\mathscr{W}) = \mathscr{I} + \mathcal{K}_{\mathscr{W}}\mathscr{D}$$

since $\mathscr{W}$ has compact support $\overline{\Omega}$ and $\mathscr{D}(\infty) = \mathscr{I}$, where $\mathcal{K}_{\mathscr{W}}(\cdot) = \mathcal{K}(\cdot\mathscr{W})$. Hence, every solution of $\bar\partial$P-$\mathscr{D}$ is a solution of the following integral equation

$$\mathscr{I} = (\mathcal{I} - \mathcal{K}_{\mathscr{W}})\mathscr{D}, \tag{8.7}$$

where $\mathcal{I}$ is the identity operator. As explained in Section 5.1, $\mathcal{I} - \mathcal{K}_{\mathscr{W}}$ is a bounded operator from $\mathrm{L}^\infty(\mathbb{C})$ into itself that maps continuous functions into continuous functions preserving their value at infinity. Conversely, if $\mathscr{D}$ is a solution of (8.7) in $\mathrm{L}^\infty(\mathbb{C}^{2\times2})$ then $\mathscr{D}$ is, in fact, continuous in $\overline{\mathbb{C}}^{2\times2}$, analytic outside of $\overline{\Omega}$, $\mathscr{D}(\infty) = \mathscr{I}$, and $\bar\partial\mathscr{D} = \mathscr{D}\mathscr{W}$ by (5.5). Thus, $\bar\partial$P-$\mathscr{D}$ is equivalent to uniquely solving (8.7) in $\mathrm{L}^\infty(\mathbb{C}^{2\times2})$.

We claim that

$$\|\mathcal{K}_{\mathscr{W}}\| = O(\delta_{n,\epsilon}) \tag{8.8}$$

for any arbitrarily small ϵ, where $\|\cdot\|$ is the norm of $\mathcal{K}_{\mathscr{W}}$ as an operator from $\mathrm{L}^\infty(\mathbb{C})$ into itself. Assuming this claim to be true, we get that $(\mathcal{I} - \mathcal{K}_{\mathscr{W}})^{-1}$ exists as a Neumann series and

$$\mathscr{D} = \mathscr{I} + O\left(\frac{\|\mathcal{K}_{\mathscr{W}}\|}{1-\|\mathcal{K}_{\mathscr{W}}\|}\right),$$

which finishes the proof of the lemma granted the validity of (8.8). Thus, it only remains to prove estimate (8.8).

If $s - s^* > 1$ then (8.3) and (8.4) yield that

$$\|\mathcal{K}_{\mathscr{W}}\| \leq \text{const.}\max_{z\in\overline{\Omega}}\left\|\frac{r_n}{z-\cdot}\right\|_{1,\Omega} \leq \text{const.}'\|r_n\|_{2+\epsilon,\Omega} \tag{8.9}$$

for any arbitrarily small ϵ by Hölder inequality. If $s - s^* \leq 1$ then (8.3) and (8.6) imply that

$$\|\mathcal{K}_{\mathscr{W}}\| \leq \text{const.}\max_{z\in\overline{\Omega}}\left\|\frac{f_{\alpha,\beta}fr_n}{z-\cdot}\right\|_{1,\Omega} \leq \text{const.}'\max_{z\in\overline{\Omega}}\left\|\frac{r_n}{z-\cdot}\right\|_{\frac{2+\delta}{1+\delta},\Omega} \leq \text{const.}''\|r_n\|_{\frac{2(1+\delta)}{\delta-\epsilon},\Omega} \tag{8.10}$$

for any arbitrarily small ϵ by applying Hölder inequality twice. Using (8.5), we can restate (8.9) and (8.10) as

$$\|\mathcal{K}_{\mathscr{W}}\| \leq \text{const.}\|r_n\|_{\frac{1}{q-\epsilon},\Omega} \tag{8.11}$$

for any arbitrarily small ϵ, where $q = \frac{1}{2}$ when $s - s^* > 1$ and $q = \frac{s-s^*}{2}$ otherwise.

Let b_n be a Blaschke product with respect to the domain D that has the same zeros as r_n counting multiplicity. Then by the maximum modulus principle for analytic functions and the first requirement in the definition of the class $\mathrm{S}(\Delta)$ we have that

$$|r_n(z)| \leq \max_{t\in\Delta} |r_n^{\pm}(t)||b_n(z)| \leq \text{const.}|b_n(z)|, \quad z \in D, \tag{8.12}$$

where const. is independent of n and z. Let ψ be the conformal map of D onto $\mathbb{D}$, $\psi(\infty) = 0$, $\psi'(\infty) > 0$. Then

$$b_n(z) = \prod_{r_n(e)=0} \frac{\psi(z) - \psi(e)}{1 - \overline{\psi(e)}\psi(z)}, \quad z \in D.$$

Denote by L_ρ, $\rho \in (0, 1)$, the level line of ψ, i.e. $L_\rho := \{z \in D : \ |\psi(z)| = \rho\}$. Due to the second requirement in the definition of $\mathrm{S}(\Delta)$ there exist $1 > \rho_0 > \rho_1 > 0$ such that $\overline{\Omega}$ is contained within the bounded domain with boundary L_{ρ_0}, say Ω_{ρ_0}, and all the zeros of b_n are contained within the unbounded domain with boundary L_{ρ_1}. Then

$$\|b_n\|_{\frac{1}{q-\epsilon},\Omega} \leq \|b_n\|_{\frac{1}{q-\epsilon},\Omega_{\rho_0}} = \left\|\left(b_n \circ \psi^{-1}\right) \left((\psi^{-1})'\right)^2\right\|_{\frac{1}{q-\epsilon},\mathbb{A}_{\rho_0,1}}, \tag{8.13}$$

where $\mathbb{A}_{\rho_0,1} := \{z : \ \rho_0 < |z| < 1\}$, ψ^{-1} is the conformal map of $\mathbb{D}$ onto D that is inverse of ψ and therefore

$$|(\psi^{-1})'| \leq \text{const.} \quad \text{in} \quad \mathbb{A}_{\rho_0,1}. \tag{8.14}$$

Set

$$b_n^*(z) := (b_n \circ \psi^{-1})(z) = \prod_{b_n(\psi^{-1}(e^*))=0} \frac{z - e^*}{1 - z\overline{e^*}}, \quad z \in \mathbb{D}.$$

Then by (8.11), (8.12), (8.13), and (8.14), we get that

$$\|\mathcal{K}_{\mathscr{W}}\| \leq \text{const.}\|b_n^*\|_{\frac{1}{q-\epsilon},\mathbb{A}_{\rho_0,1}}. \tag{8.15}$$

Observe now that for $z \in \mathbb{T}_\rho$, $\rho \in (\rho_0, 1)$, it holds that

$$|b_n^*(z)| \leq \prod \frac{\rho + |e^*|}{1 + \rho|e^*|} \leq \exp\left\{-(1-\rho)\sum \frac{1 - |e^*|}{1 + \rho|e^*|}\right\} \leq \exp\left\{-2n\frac{(1-\rho)(1-\rho_1)}{1+\rho}\right\} \tag{8.16}$$

since $|e^*| < \rho_1$ by the definition of ρ_1. Clearly, (8.15) and (8.16) yield that

$$\|\mathcal{K}_{\mathscr{W}}\| \leq \text{const.}\left(\int_0^{2\pi}\int_{\rho_0}^1 \exp\left\{-\frac{2n}{q-\epsilon}\frac{(1-\rho)(1-\rho_1)}{1+\rho}\right\}\rho d\rho dt\right)^{q-\epsilon} \leq \text{const.}'n^{\epsilon-q},$$

which is exactly (8.8) by the definition of q. □

9. Solution of RH$\bar{\partial}$P-$\mathscr{S}$ (RHP-$\mathscr{Y}$) and Proof of Theorem 2

In this, last section, we gather the material from Sections 6–8 to solve RHP-$\mathscr{Y}$, which, in turn, is used to prove Theorem 2.

9.1. Solution of RH$\bar\partial$P-$\mathscr{S}$ and, respectively, RHP-$\mathscr{Y}$

It is an immediate consequence of Lemmas 12 and 13 that the following result holds.

Lemma 14. *If* (2.11) *is fulfilled, then the solution of* RH$\bar\partial$P-$\mathscr{S}$ *uniquely exists for all n large enough and is given by $\mathscr{S} = \mathscr{A}\mathscr{D}$, where $\mathscr{A}$ is the solution of* RHP-$\mathscr{A}$ *and $\mathscr{D}$ is the solution of* $\bar\partial$P-$\mathscr{D}$.

Combining the previous lemma with Lemmas 5, 6, and 8, we obtain the following.

Lemma 15. *If* (2.11) *is fulfilled, then the solution of* RHP-$\mathscr{Y}$ *uniquely exists for all n large enough and can be expressed by reversing the transformations $\mathscr{Y} \to \mathscr{T} \to \mathscr{S}$ using* (6.5) *and* (6.7) *with $\mathscr{S}$ given by Lemma* 14. *Moreover, polynomials $\{q_n\}$, satisfying orthogonality relations* (2.5) *under restrictions* (2.10) *and* (2.11), *are normal for all n large enough, i.e.* $\deg(q_n) = n$.

Proof. Clearly, we need only to explain the normality of q_n. Actually, this was already observed in [**?**, Pg. 44]. It amounts to noticing that since $\mathscr{Y}$ exists for all n large enough, $\mathscr{Y}_{11} = z^n +$ lower order terms by the normalization in RHP-$\mathscr{Y}$(a). Moreover, by RHP-$\mathscr{Y}$(b), $\mathscr{Y}_{11}$ has no jump on Δ and hence is holomorphic in the whole complex plane. Thus, $\mathscr{Y}_{11}$ is necessarily a polynomial of degree n by Liouville's theorem. Further, since $\mathscr{Y}_{12} = O(z^{-n-1})$ and satisfies RHP-$\mathscr{Y}$(b), it holds that $\mathscr{Y}_{12} = 2\mathcal{C}(\mathscr{Y}_{11}w_n)$. From the latter, we easily deduce that $\mathscr{Y}_{11}$ satisfies orthogonality relations (2.5). □

9.2. Asymptotics in the Bulk, Formula (2.13)

We claim that (2.13) holds locally uniformly in D. Clearly, for any given closed set in D, it easily can be arranged that this set lies exterior to the lens Σ_n^{rd}, and therefore to the lenses Σ_n and Σ_{ext}. Thus, the asymptotic behavior of $\mathscr{Y}$ on X is given by

$$\mathscr{Y} = (2^n\epsilon_n)^{-\sigma_3}\mathscr{R}\mathscr{N}\mathscr{D}E_n^{\sigma_3}$$

due to Lemma 15, where ϵ_n and E_n were defined in (6.2), $\mathscr{R}$ is the solution of RHP-$\mathscr{R}$ given by Lemma 12, and $\mathscr{N}$ is the solution of RHP-$\mathscr{N}$ given by (7.3). Moreover, we have that

$$\mathscr{R}\mathscr{N}\mathscr{D} = \left([1+O(\delta_{n,\epsilon})]\,\mathscr{N}_{lk}\right)_{l,k=1,2} \tag{9.1}$$

locally uniformly in $\overline{\mathbb{C}}\setminus\{\pm1\}$, including on $(\Delta^\pm)^\circ$, on account of (7.7) and (8.2). Thus, it holds that

$$\begin{cases} \mathscr{Y}_{11} & = [1+O(\delta_{n,\epsilon})]\mathscr{N}_{11}E_n/(2^n\epsilon_n) & = \varphi^n/(2^nS_{w\mathfrak{w}^+}S_{hh_n/v_n}) \\ \mathscr{Y}_{12} & = [1+O(\delta_{n,\epsilon})]\mathscr{N}_{12}/(E_n2^n\epsilon_n) & = 2iG_wG_{hh_n/v_n}S_wS_{hh_n/v_n}/(\varphi^{n+1}S_{\mathfrak{w}^+}) \end{cases}$$

locally uniformly in D by (6.2) and (7.3). Recall that the entries of $\mathscr{N}$ are, in fact, deformed Szegő functions defined with respect to Δ_n. However, we have already mentioned that they coincide with $S_{\mathfrak{w}^+}$ and S_w outside of a bounded set with boundary $\Delta_n\cup\Delta$. Thus, the equations above indeed hold true. Hence, asymptotic formulae (2.13) follow now from (6.1), (2.14), and (5.11).

9.3. Asymptotics in the Bulk, Formula (2.15)

To derive asymptotic behavior of q_n and R_n on $\Delta\setminus\{\pm1\}$, we need to consider what happens within the lens Σ_{ext} and outside the disks U_δ and $\widetilde{U}_\delta$. We shall consider the asymptotics of $\mathscr{Y}$ from within Ω_+, the upper part of the lens Σ_{ext}, the behavior of $\mathscr{Y}$ in Ω_- can be deduced in a similar fashion.

Recall that Δ_n either coincides with Δ or intersects it at finite number of points, as both arcs are images of $[-1,1]$ under holomorphic maps. Set

$$\Delta_n^* := \Delta\cap\Omega_{n+} \quad \Delta_n^{**} := \Delta\cap\Omega_{n-},$$

where Ω_{n+} and Ω_{n-} are the upper and lower parts of the lens Σ_n^{md}. Then, it holds that

$$\mathscr{A}_+ = \begin{cases} \mathscr{B}_+, & \text{on} \quad \Delta_n^*, \\ \mathscr{B}_-\begin{pmatrix} 0 & w \\ -1/w & 0 \end{pmatrix}, & \text{on} \quad \Delta_n^{**}, \end{cases} \tag{9.2}$$

by (7.1), where with a slight abuse of notation we denote by $\mathscr{B}_\pm$ the values of $\mathscr{B}$ in $\Omega_{n\pm}$ and on $\Delta_n^\pm$. Then it holds on Δ° by RHP-$\mathscr{N}$(b) and on account of Lemma 14, (9.2), and (7.6) that

$$\mathscr{S}_+ = \begin{cases} \mathscr{R}\mathscr{N}_+\mathscr{D}, & \text{on} \quad \Delta_n^* \\ \mathscr{R}\mathscr{N}_-\begin{pmatrix} 0 & w \\ -1/w & 0 \end{pmatrix}\mathscr{D}, & \text{on} \quad \Delta_n^{**} \end{cases} = \mathscr{R}\widetilde{\mathscr{N}}_+\mathscr{D},$$

where again under $\mathscr{N}_\pm$ we understand the values of $\mathscr{N}$ in $\Omega_{n\pm}$ and on $\Delta_n^\pm$ and $\widetilde{\mathscr{N}}$ is the analytic deformation of $\mathscr{N}$ that satisfies RHP-$\mathscr{N}$, only with a jump across Δ. Clearly, $\widetilde{\mathscr{N}}$ is defined by (7.2) and (7.3), where $S_{\mathfrak{w}^+}$ and S_w are the Szegő functions of $\mathfrak{w}^+$ and w with respect to Δ and not the deformed functions that actually appear in (7.2) and (7.3). Thus, we deduce from Lemma 15 and (9.1) that

$$\begin{aligned} \mathscr{Y}_+ &= (2^n\epsilon_n)^{-\sigma_3}\left([1+O(\delta_{n,\epsilon})]\,\widetilde{\mathscr{N}}_{lk}^+\right)_{l,k=1,2}\begin{pmatrix} 1 & 0 \\ (r_nc_nc)^+/w & 1 \end{pmatrix}(E_n^+)^{\sigma_3} \\ &= (2^n\epsilon_n)^{-\sigma_3}\left([1+O(\delta_{n,\epsilon})]\,\widetilde{\mathscr{N}}_{lk}^+\right)_{l,k=1,2}\begin{pmatrix} E_n^+ & 0 \\ E_n^-/w & 1/E_n^+ \end{pmatrix} \end{aligned}$$

locally uniformly on Δ°, where we used (6.4) to obtain the second equality. Therefore, it holds that

$$\begin{cases} \mathscr{Y}_{11} &= [1+O(\delta_{n,\epsilon})]\widetilde{\mathscr{N}}_{11}^+E_n^+/(2^n\epsilon_n) + [1+O(\delta_{n,\epsilon})]\widetilde{\mathscr{N}}_{12}^+E_n^-/(2^n\epsilon_nw) \\ \mathscr{Y}_{12}^+ &= [1+O(\delta_{n,\epsilon})]\widetilde{\mathscr{N}}_{12}^+/(E_n^+2^n\epsilon_n) \end{cases}$$

locally uniformly on Δ°. As in the end of the previous section, we deduce (2.15) from (6.1), the formulae

$$\frac{\widetilde{\mathscr{N}}_{11}^\pm E_n^\pm}{2^n\epsilon_n} = \frac{1}{S_n^\pm} \quad \text{and} \quad \frac{\widetilde{\mathscr{N}}_{12}^+}{E_n^+2^n\epsilon_n} = \frac{S_n^+}{\mathfrak{w}^+},$$

and by noticing that

$$\frac{1}{w}\frac{\widetilde{\mathscr{N}}_{12}^+}{\widetilde{\mathscr{N}}_{11}^-} = \frac{1}{w}\frac{G_wS_w^+S_w^-S_{\mathfrak{w}^+}^-}{\varphi^+S_{\mathfrak{w}^+}^+} = \frac{iS_{\mathfrak{w}^+}^-}{\varphi^+S_{\mathfrak{w}^+}^+} \equiv 1$$

on Δ° by (2.9) and (5.12).

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