ON THE RECOVERY OF CORE AND CRUSTAL COMPONENTS OF GEOMAGNETIC POTENTIAL FIELDS

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Abstract. In Geomagnetism it is of interest to separate the Earth's core magnetic field from the crustal magnetic field. However, measurements by satellites can only sense the sum of the two contributions. In practice, the measured magnetic field is expanded in spherical harmonics and separation into crust and core contribution is achieved empirically, by a sharp cutoff in the spectral domain. In this paper, we derive a mathematical setup in which the two contributions are modeled by harmonic potentials Φ_0 and Φ_1 generated on two different spheres $\mathbb{S}_{R_0}$ (crust) and $\mathbb{S}_{R_1}$ (core) with radii $R_1 < R_0$. Although it is not possible in general to recover Φ_0 and Φ_1 knowing their superposition $\Phi_0 + \Phi_1$ on a sphere $\mathbb{S}_{R_2}$ with radius $R_2 > R_0$, we show that it becomes possible if the magnetization $\mathbf{m}$ generating Φ_0 is localized in a strict subregion of $\mathbb{S}_{R_0}$. Beyond unique recoverability, we show in this case how to numerically reconstruct characteristic features of Φ_0 (e.g., spherical harmonic Fourier coefficients). An alternative way of phrasing the results is that knowledge of $\mathbf{m}$ on a nonempty open subset of $\mathbb{S}_{R_0}$ allows one to perform separation.

Key words. Harmonic Potentials, Hardy-Hodge Decomposition, Separation of Sources, Geomagnetic Field, Extremal Problems

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1. Introduction.

The Earth's magnetic field $\mathbf{B}$, as measured by several satellite missions, is a superposition of various contributions, e.g., of iono-/magnetospheric fields, crustal magnetic field, and of the core/main magnetic field, see [23, 24, 32] for an overview and [26, 30, 35, 40] for some recent geomagnetic field models. While iono-/magnetospheric contributions can to a certain extent be filtered out due to their temporal variations, the separation of the core/main field $\mathbf{B}_{core}$ and the crustal field $\mathbf{B}_{crust}$ is typically based on the empirical observation that the power spectra of Earth magnetic field models have a sharp knee at spherical harmonic degree 15 (see, e.g., [25, 32]). However, under this spectral separation, large-scale contributions (i.e., spherical harmonic degrees smaller than 15) are entirely neglected in crustal magnetic field models. In [22], a Bayesian approach has been proposed that addresses the separation of geomagnetic sources based on their correlation structure. The correlation of certain components, e.g., internally and externally produced magnetic fields, can (to some extent) be obtained from the underlying geophysical equations. But this approach does not explicitly address the problem that some of the involved separation problems, e.g., the separation into crustal and core magnetic field contributions, are generally not unique for the given data situation. The goal of this paper is to derive conditions under which a rigorous separation of the contributions $\mathbf{B}_{crust}$ and $\mathbf{B}_{core}$ is possible, as well as to formulate extremal problems whose solutions lead to approximations of these contributions or certain features thereof. The main assumption that we make for our approach to work is that the magnetization generating $\mathbf{B}_{crust}$ is localized in a strict subregion of the crust. By linearity, this is equivalent to assuming that this magnetization is known on a spherical cap that may, in principle, be arbitrary small. For applications, this is interesting in as much as that the crustal magnetization may be estimated in certain places of the Earth from local measurements. Thus, given

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such a local estimation, its contribution can be subtracted from global magnetic field measurements to yield a crustal contribution that stems from magnetizations localized in a strict subregion of the Earth (namely the complement of those places where a local estimate of the magnetization has been performed), thereby allowing us to apply the separation approach indicated in this paper. Similarly, if one can identify places on the Earth which are only weakly magnetized as compared to others, the separation process that we will describe may reasonably be applied by neglecting magnetizations in such places.

We assume throughout that the overall magnetic field is of the form $\mathbf{B}=\mathbf{B}_{crust}+\mathbf{B}_{core}$ in $\mathbb{R}^3\setminus\overline{\mathbb{B}_{R_0}}$, where $\mathbb{B}_{R_0}=\{x\in\mathbb{R}^3:|x|<R_0\}$ denotes the ball of radius $R_0>0$ and overline indicates closure (here R_0 can be interpreted as the radius of the Earth). Since the sources of $\mathbf{B}_{crust}$ and $\mathbf{B}_{core}$ are located inside $\mathbb{B}_{R_0}$ (hence, the corresponding magnetic fields are curl-free and divergence-free in $\mathbb{R}^3\setminus\overline{\mathbb{B}_{R_0}}$), there exist potential fields Φ, Φ_{crust}, Φ_{core} such that $\mathbf{B}=\nabla\Phi$, $\mathbf{B}_{crust}=\nabla\Phi_{crust}$, and $\mathbf{B}_{core}=\nabla\Phi_{core}$ in $\mathbb{R}^3\setminus\overline{\mathbb{B}_{R_0}}$. Therefore, from a mathematical point of view, the problem reduces to finding unique Φ_{crust}, Φ_{core} from the knowledge of Φ (but we should keep in mind that the actual measurements bear on the magnetic field $\mathbf{B}$).

It is known that $\mathbf{B}_{crust}$ is generated by a magnetization $\mathbf{M}$ confined in a thin spherical shell $\mathbb{B}_{R_0-d,R_0}=\{x\in\mathbb{R}^3:R_0-d<|x|<R_0\}$ of thickness $d>0$ (for the Earth, $d\approx 30$km is typical), therefore the corresponding magnetic potential can be expressed as (see, e.g., [8, 19])

$$\Phi_{crust}(x)=\frac{1}{4\pi}\int_{\mathbb{B}_{R_0-d,R_0}}\mathbf{M}(y)\cdot\frac{x-y}{|x-y|^3}\,\mathrm{d}\lambda(y),\quad x\in\mathbb{R}^3, \tag{1}$$

where the dot indicates the Euclidean scalar product in $\mathbb{R}^3$ and λ the Lebesgue measure. Due to the thinness of the magnetized layer relative to the Earth's radius, it is reasonable to substitute the volumetric $\mathbf{M}$ by a spherical magnetization $\mathbf{m}$ (i.e., $\mathbf{M}=\mathbf{m}\otimes\delta_{\mathbb{S}_{R_0}}$ in a distributional sense). Then, the magnetic potential (1) becomes

$$\Phi_{crust}(x)=\frac{1}{4\pi}\int_{\mathbb{S}_{R_0}}\mathbf{m}(y)\cdot\frac{x-y}{|x-y|^3}\,\mathrm{d}\omega_{R_0}(y),\quad x\in\mathbb{R}^3\setminus\mathbb{S}_{R_0}, \tag{2}$$

where $\mathbb{S}_{R_0}=\{x\in\mathbb{R}^3:|x|=R_0\}$ denotes the sphere of radius $R_0>0$ and $\mathrm{d}\omega_{R_0}$ the corresponding surface element. When interested in reconstructing the actual magnetization $\mathbf{M}$, substituting a spherical magnetization $\mathbf{m}$ is of course a significant restriction (however, one that is fairly frequent in Geomagnetism). But since our main focus is on $\mathbf{B}_{crust}$ and the corresponding potential Φ_{crust} rather than the magnetization itself, this restriction actually involves no loss of information: in Section 3 we show that, under mild summability assumptions, any potential Φ_{crust} produced by a volumetric magnetization $\mathbf{M}$ in $\mathbb{B}_{R_0-d,R_0}$ can also be generated by a spherical magnetization $\mathbf{m}$ on $\mathbb{S}_{R_0}$.

The core contribution $\mathbf{B}_{core}$ is governed by Maxwell's equations (see, e.g., [5])

$$\begin{aligned}\nabla\times\mathbf{B}_{core}&=\sigma(\mathbf{E}+\mathbf{u}\times\mathbf{B}_{core}),\\ \nabla\cdot\mathbf{B}_{core}&=0,\\ \nabla\times\mathbf{E}&=-\partial_t\mathbf{B}_{core},\\ \nabla\cdot\mathbf{E}&=\rho,\end{aligned}$$

where σ denotes the conductivity, ρ the charge density, and $\mathbf{u}$ the fluid velocity in the Earth's outer core (the constant permeability μ_0 and permittivity ε_0 have been

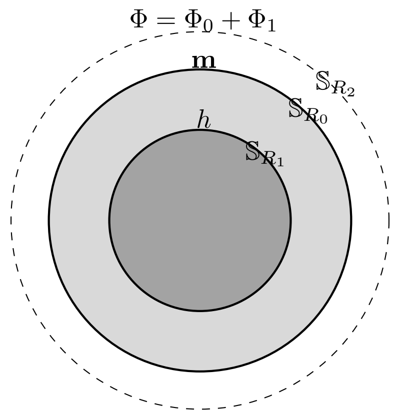

FIG. 1. *Illustration of the setup of Problem 1.1.*

set to 1). The conductivity σ is assumed to be zero outside a sphere $\mathbb{S}_{R_1}$ of radius $0 < R_1 < R_0$. The condition $R_1 < R_0$ is crucial to the forthcoming arguments and is justified by common geophysical practice and results (see, e.g., [6, 33]). In particular it implies that $\nabla \times \mathbf{B}_{core} = 0$ in $\mathbb{R}^3 \setminus \overline{\mathbb{B}_{R_1}}$, therefore, $\mathbf{B}_{core} = \nabla \Phi_{core}$ in $\mathbb{R}^3 \setminus \overline{\mathbb{B}_{R_1}}$ for some harmonic potential Φ_{core}. Although the geophysical processes in the Earth's outer core can be extremely complex, of importance to us is only that Φ_{core} can be expressed in $\mathbb{R}^3 \setminus \overline{\mathbb{B}_{R_1}}$ as a Poisson transform:

$$(3) \qquad \Phi_{core}(x) = \frac{1}{4\pi R_1} \int_{\mathbb{S}_{R_1}} h(y) \frac{|x|^2 - R_1^2}{|x-y|^3} \mathrm{d}\omega_{R_1}(y), \quad x \in \mathbb{R}^3 \setminus \overline{\mathbb{B}_{R_1}},$$

for some scalar valued auxiliary function h on $\mathbb{S}_{R_1}$; this follows from previous considerations which imply that Φ_{core} is harmonic in $\mathbb{R}^3 \setminus \overline{\mathbb{B}_{R_1}}$ and continuous in $\mathbb{R}^3 \setminus \mathbb{B}_{R_1}$. Summarizing, the problem we treat in this paper is the following (the setup is illustrated in Figure 1):

PROBLEM 1.1. *Let $\Phi \in L^2(\mathbb{S}_{R_2})$ be given on a sphere $\mathbb{S}_{R_2} \subset \mathbb{R}^3 \setminus \overline{\mathbb{B}_{R_0}}$ of radius $R_2 > R_0$. Assume Φ is decomposable into $\Phi = \Phi_0 + \Phi_1$ on $\mathbb{S}_{R_2}$, where $\Phi_0 = \Phi_0[\mathbf{m}]$ is of the form (2), with $\mathbf{m} \in L^2(\mathbb{S}_{R_0}, \mathbb{R}^3)$, and $\Phi_1 = \Phi_1[h]$ is of the form (3), with $h \in L^2(\mathbb{S}_{R_1})$ and $R_1 < R_0$. Are Φ_0 and Φ_1 uniquely determined by the knowledge of Φ on $\mathbb{S}_{R_2}$, and if yes can they be reconstructed efficiently?*

The answer to the uniqueness issue in Problem 1.1 is generally negative. But under the additional assumption that $\mathrm{supp}(\mathbf{m}) \subset \Gamma_{R_0}$ for a strict subregion $\Gamma_{R_0} \subset \mathbb{S}_{R_0}$ (i.e., $\overline{\Gamma_{R_0}} \neq \mathbb{S}_{R_0}$), uniqueness is guaranteed. This follows from results in [7, 27] and their formulation on the sphere in [17], to be reviewed in greater detail in Sections 2 and 4. In fact, we show in this case that h and the curl-free contribution of $\mathbf{m}$ can be reconstructed uniquely from the knowledge of Φ. Additionally, we provide a means of approximating $\langle \Phi_0, g \rangle_{L^2(\mathbb{S}_{R_2})}$ knowing Φ on $\mathbb{S}_{R_2}$, where g is some appropriate test function (e.g., a spherical harmonic). This allows one to separate the crustal and the core contributions to the Geomagnetic potential if, e.g., the crustal magnetization can be estimated over a small subregion on Earth by other means.

Throughout the paper, we call Φ_0 the crustal contribution and Φ_1 the core contribution. We should point out that the examples we provide at the end of the paper are not based on real Geomagnetic field data but they reflect some of the main properties of realistic scenarios (e.g., the domination of the core contribution at low spherical harmonic degrees). In Section 3, we take a closer look at harmonic potentials of the form (1) and (2) and show that the balayage onto $\mathbb{S}_{R_0}$ of a volumetric potential supported in $\mathbb{B}_{R_0-d,R_0}$ preserves divergence form. More precisely, if $\mathbf{M}$ is supported in $\mathbb{B}_{R_0-d,R_0}$ and its restriction to $\mathbb{S}_R$ is uniformly square-integrable for $R \in (R_0-d,R_0)$, then there exists a spherical magnetization $\mathbf{m}$ supported on $\mathbb{S}_{R_0}$, which is square summable and generates the same potential as $\mathbf{M}$ in $\mathbb{R}^3 \setminus \overline{\mathbb{B}_{R_0}}$. The latter property justifies the above-described modeling of the crustal magnetic field. Auxiliary material on geometry, spherical decomposition of vector fields as well as Sobolev and Hardy spaces is recapitulated in Section 2. Eventually, in Section 5 we formulate an extremal problem for the approximation of Φ_0 and $\langle \Phi_0, g\rangle_{L^2(\mathbb{S}_{R_2})}$ and provide some initial numerical examples, followed by a brief conclusion in Section 6. Some technical results on potentials of distributions and an additional numerical example are gathered in the appendix.

2. Auxiliary Notations and Results. We start with some basic definitions of function spaces and differentiation on the sphere. For $R>0$, the sphere $\mathbb{S}_R$ is a smooth, compact oriented surface embedded in $\mathbb{R}^3$. That is, $\mathbb{S}_R$ can be described by finitely many charts $\psi_j : U_j \to V_j$ (for open subsets $U_j \subset \mathbb{S}_R$ and $V_j \subset \mathbb{R}^2$, $j=1,\ldots,N$), which allows a meaningful definition of the surface area measure ω_R on the sphere $\mathbb{S}_R$ via the Lebesgue measure λ in $\mathbb{R}^2$. For $x \in U_j \subset \mathbb{S}_R$, the tangent space T_x at x is the image of the derivative $\mathrm{D}\psi_j^{-1}[\psi_j(x)] : \mathbb{R}^2 \to \mathbb{R}^3$. The tangent space may be described intrinsically as $T_x = \{y \in \mathbb{R}^3 : x \cdot y = 0\}$. A k-times differentiable or C^k-smooth function $f : \mathbb{S}_R \to \mathbb{R}$ is a function such that $f \circ \psi_j^{-1}$ is k-times differentiable or has continuous partial derivatives up to order k, respectively, for each $j=1,\ldots,N$. We simply say that f is smooth if it is C^∞-smooth. Due to the simple geometry of the sphere $\mathbb{S}_R$, this definition of differentiability is in fact equivalent to requiring that the radial extension $\bar{f}(x) = f(R\frac{x}{|x|})$ of f has the corresponding regularity in $\mathbb{R}^3 \setminus \{0\}$. This allows us to express the surface gradient $\nabla_{\mathbb{S}_R} f(x)$ of a differentiable function $f : \mathbb{S}_R \to \mathbb{R}$ at a point $x \in \mathbb{S}_R$ via the relation $\nabla_{\mathbb{S}_R} f(x) = \nabla \bar{f}(y)|_{y=x}$, where ∇ denotes the Euclidean gradient (formally, the surface gradient at x is defined as the unique vector $v \in T_x$ such that the differential $\mathrm{d}f[x] : T_x \to \mathbb{R}$ can be identified by the scalar product with v, i.e., $\mathrm{d}f[x](y) = v \cdot y$ for $y \in T_x$).

Furthermore, $L^2(\mathbb{S}_R)$ is denoted to be the space of square-integrable scalar valued functions $f : \mathbb{S}_R \to \mathbb{R}$, while $L^2(\mathbb{S}_R, \mathbb{R}^3)$ denotes the space of square integrable vector valued spherical functions $\mathbf{f} : \mathbb{S}_R \to \mathbb{R}^3$, equipped with the inner products $\langle f,h\rangle_{L^2(\mathbb{S}_R)} = \int_{\mathbb{S}_R} f(y)h(y)\mathrm{d}\omega_R(y)$ and $\langle \mathbf{f},\mathbf{h}\rangle_{L^2(\mathbb{S}_R,\mathbb{R}^3)} = \int_{\mathbb{S}_R} \mathbf{f}(y)\cdot\mathbf{h}(y)\mathrm{d}\omega_R(y)$, respectively. A vector field $\mathbf{f} : \mathbb{S}_R \to \mathbb{R}^3$ is said to be tangential if $\mathbf{f}(x) \in T_x$ for all $x \in \mathbb{S}_R$. The subspace of all tangential vector fields in $L^2(\mathbb{S}_R, \mathbb{R}^3)$ is denoted by $\mathcal{T}_R$. Note that the smooth vector fields are dense in $\mathcal{T}_R$. Clearly, if f is smooth, then $\nabla_{\mathbb{S}_R} f$ lies in $\mathcal{T}_R$. The Sobolev space $W^{1,2}(\mathbb{S}_R)$ may be defined as the completion of smooth functions with respect to the norm [20]

$$\|f\|_{W^{1,2}(\mathbb{S}_R)} = \left(\|f\|^2_{L^2(\mathbb{S}_R)} + \|\nabla_{\mathbb{S}_R} f\|^2_{L^2(\mathbb{S}_R,\mathbb{R}^3)}\right)^{1/2}.$$

Since, for an appropriate set of charts $\psi_j : U_j \to V_j$, $j=1,\ldots,N$, of the sphere, the V_j are bounded and the corresponding determinants of the metric tensors are bounded from above and below by strictly positive constants, it holds that $f \in W^{1,2}(\mathbb{S}_R)$ if and only if the functions $f \circ \psi_j^{-1}$ lie in the Euclidean Sobolev spaces $W^{1,2}(V_j)$ (see, e.g., [28]). The gradient $\nabla_{\mathbb{S}_R} f(x)$ at $x \in \mathbb{S}_R$ of a function $f \in W^{1,2}(\mathbb{S}_R)$ still satisfies the representation $\mathrm{d}f[x](y) = \nabla_{\mathbb{S}_R} f(x) \cdot y$ for $y \in T_x$, where $\mathrm{d}f$ has to be understood in the sense of distributional derivatives and $\nabla_{\mathbb{S}_R} f(x)$ needs not be a pointwise derivative in the strong sense (see [38, Ch.VIII]). Let us put

$$\mathcal{G}_R = \{\nabla_{\mathbb{S}_R} f : f \in W^{1,2}(\mathbb{S}_R)\}.$$

We claim that $\mathcal{G}_R$ is closed in $L^2(\mathbb{S}_R, \mathbb{R}^3)$. Indeed, if $\nabla_{\mathbb{S}_R} f_n$ is a Cauchy sequence in $\mathcal{G}_R$, where $f_n \in W^{1,2}(\mathbb{S}_R)$ is defined up to an additive constant, we may pick f_n so that $\int_{\mathbb{S}_R} f_n \mathrm{d}\omega_R = 0$ and then it follows from the Hölder and the Poincaré inequalities [20, Prop. 3.9] that $\|f_n - f_m\|_{L^2(\mathbb{S}_R)} \leq C \|\nabla_{\mathbb{S}_R} f_n - \nabla_{\mathbb{S}_R} f_m\|_{L^2(\mathbb{S}_R, \mathbb{R}^3)}$ for some constant C. Hence f_n is a Cauchy sequence in $W^{1,2}(\mathbb{S}_R)$, therefore it converges to some f there and consequently $\nabla_{\mathbb{S}_R} f_n$ converges to $\nabla_{\mathbb{S}_R} f$ in $L^2(\mathbb{S}_R, \mathbb{R}^3)$. Thus, $\mathcal{G}_R$ is complete and therefore it is closed in $L^2(\mathbb{S}_R, \mathbb{R}^3)$, which proves the claim.

When $\mathbf{h}$ is a smooth tangential vector field on $\mathbb{S}_R$, its surface divergence $\nabla_{\mathbb{S}_R} \cdot \mathbf{h}$ is the smooth real valued function such that

$$\int_{\mathbb{S}_R} f \, \nabla_{\mathbb{S}_R} \cdot \mathbf{h} \, \mathrm{d}\omega_R = - \int_{\mathbb{S}_R} (\nabla_{\mathbb{S}_R} f) \cdot \mathbf{h} \, \mathrm{d}\omega_R, \quad \text{for all } f \in C^\infty(\mathbb{S}_R). \tag{4}$$

When $\mathbf{h} \in \mathcal{T}_R$ is not smooth, (4) must be interpreted in a weak sense, namely $\nabla_{\mathbb{S}_R} \cdot \mathbf{h}$ is the distribution on $\mathbb{S}_R$ acting on smooth real-valued functions by $\langle f \, , \nabla_{\mathbb{S}_R} \cdot \mathbf{h} \rangle = - \int_{\mathbb{S}_R} \nabla_{\mathbb{S}_R} f \cdot \mathbf{h} \, \mathrm{d}\omega_R$, for all $f \in C^\infty(\mathbb{S}_R)$. This clearly extends by density to a linear form on $W^{1,2}(\mathbb{S}_R)$, upon letting f converge to a Sobolev function. Then it is apparent that

$$\mathcal{D}_R = \{\mathbf{h} \in \mathcal{T}_R : \nabla_{\mathbb{S}_R} \cdot \mathbf{h} = 0\}$$

is the orthogonal complement to $\mathcal{G}_R$ in $\mathcal{T}_R$. In particular,

$$\mathcal{T}_R = \mathcal{G}_R \oplus \mathcal{D}_R, \tag{5}$$

which is the so-called Helmholtz-Hodge decomposition. The particular geometry of $\mathbb{S}_R$ makes it easy to see that $\mathbf{f} \in \mathcal{D}_R$ if and only if its radial extension $\bar{\mathbf{f}}(x) = \mathbf{f}(R\frac{x}{|x|})$ is divergence free, as a $\mathbb{R}^3$-valued distribution on $\mathbb{R}^3 \setminus \{0\}$.

We now consider the operator $J_x : T_x \to T_x$ given by $J_x(y) = \frac{x}{|x|} \times y$, for $y \in T_x$, where $\times$ indicates the vector product in $\mathbb{R}^3$; that is, J_x is the rotation by $\pi/2$ in T_x. We define $J : \mathcal{T}_R \to \mathcal{T}_R$ to be the isometry acting pointwise as J_x on T_x, namely $(J\mathbf{f})(x) = J_x(\mathbf{f}(x))$ for $\mathbf{f} \in \mathcal{T}_R$. It turns out that $J(\mathcal{G}_R) = \mathcal{D}_R$. This fact holds for more general sufficiently smooth surfaces embedded in $\mathbb{R}^3$. A proof seems not easy to find in the literature and will be provided in a forthcoming publication (for the special case of continuously differentiable tangential vector fields on the sphere, the assertion essentially corresponds to [15, Thm. 2.10]). This motivates the notion of a surface curl gradient $\mathrm{L}_{\mathbb{S}_R} = x \times \nabla_{\mathbb{S}_R}$, acting at a point $x \in \mathbb{S}_R$, and justifies the representation $\mathcal{D}_R = \{\mathrm{L}_{\mathbb{S}_R} f : f \in W^{1,2}(\mathbb{S}_R)\}$. For convenience, we define the following "normalized" operators: $\nabla_{\mathbb{S}} = R\nabla_{\mathbb{S}_R}$ and $\mathrm{L}_{\mathbb{S}} = \frac{x}{|x|} \times \nabla_{\mathbb{S}}$. The Euclidean gradient then has the expression $\nabla = \frac{x}{|x|}\partial_\nu + \frac{1}{|x|}\nabla_{\mathbb{S}}$, acting at a point $x \in \mathbb{R}^3$, where $\partial_\nu = \frac{x}{|x|} \cdot \nabla$ denotes the radial derivative.

Eventually, if we let $\mathcal{N}_R$ indicate the space of radial vector fields in $L^2(\mathbb{S}_R, \mathbb{R}^3)$ (i.e., those functions whose value at x is perpendicular to T_x for each $x \in \mathbb{S}_R$), we get from (5) the orthogonal decomposition

$$L^2(\mathbb{S}_R, \mathbb{R}^3) = \mathcal{N}_R \oplus \mathcal{G}_R \oplus \mathcal{D}_R. \tag{6}$$

Related to the latter but of more relevance to our problem is the Hardy-Hodge decomposition that we now explain. For that purpose, we require the following definition.

DEFINITION 2.1. *The Hardy space $\mathcal{H}^2_{+,R}$ of harmonic gradients in $\mathbb{B}_R$ is defined by*

$$\mathcal{H}^2_{+,R} = \{\mathbf{g} = \nabla g : \text{ function } g : \mathbb{B}_R \to \mathbb{R} \text{ with } \Delta g = 0 \text{ in } \mathbb{B}_R \text{ and } \|\nabla g\|_{2,+} < \infty\},$$

where $\|\mathbf{g}\|_{2,+} = \left(\sup_{r\in[0,R)} \int_{\mathbb{S}_r} |\mathbf{g}(ry)|^2 \mathrm{d}\omega_r(y)\right)^{\frac{1}{2}}$ *and Δ is the Euclidean Laplacian in $\mathbb{R}^3$. Likewise, the Hardy space $\mathcal{H}^2_{-,R}$ of harmonic gradients in $\mathbb{R}^3 \setminus \overline{\mathbb{B}_R}$ is defined by*

$$\mathcal{H}^2_{-,R} = \left\{\mathbf{g} = \nabla g : \text{ function } g : \mathbb{R}^3 \setminus \overline{\mathbb{B}_R} \to \mathbb{R} \text{ with } \Delta g = 0 \text{ in } \mathbb{R}^3 \setminus \overline{\mathbb{B}_R} \text{ and } \|\nabla g\|_{2,-} < \infty\right\},$$

where $\|\mathbf{g}\|_{2,-} = \left(\sup_{r\in(R,\infty)} \int_{\mathbb{S}_r} |\mathbf{g}(ry)|^2 \mathrm{d}\omega_R(y)\right)^{\frac{1}{2}}$*. Note that, by Weyl's lemma [11, Theorem 24.9], it makes no difference whether the Euclidean gradient and Laplacian are understood in the distributional or in the strong sense.*

Members of $\mathcal{H}^2_{+,R}$ and $\mathcal{H}^2_{-,R}$ have non-tangential limits a.e. on $\mathbb{S}_R$, and if $\mathbf{g} \in \mathcal{H}^2_{\pm,R}$, its non-tangential limit has $L^2(\mathbb{S}_R, \mathbb{R}^3)$-norm equal to $\|\mathbf{g}\|_{2,\pm}$, see [38, VII.3.1] and [39, VI.4]. We still write $\mathbf{g}$ for this non-tangential limit and we regard it as the trace of $\mathbf{g}$ on $\mathbb{S}_R$. This way Hardy spaces can be interpreted as function spaces on $\mathbb{S}_R$ as well as on $\mathbb{B}_R$ or $\mathbb{R}^3 \setminus \overline{\mathbb{B}_R}$, but the context will make it clear if the Euclidean or the spherical interpretation is meant because the argument belongs to $\mathbb{R}^3 \setminus \mathbb{S}_R$ in the former case and to $\mathbb{S}_R$ in the latter. The Hardy-Hodge decomposition is the orthogonal sum

$$L^2(\mathbb{S}_R, \mathbb{R}^3) = \mathcal{H}^2_{+,R} \oplus \mathcal{H}^2_{-,R} \oplus \mathcal{D}_R. \tag{7}$$

Projecting (7) onto the tangent space $\mathcal{T}_R$ and grouping the first two summands into a single gradient vector field yields back the Hodge decomposition (5). The Hardy-Hodge decomposition drops out at once from [3] and (5). Its application to the study of inverse magnetization problems has been illustrated in [7, 17, 27]. Although not studied in mathematical detail, spherical versions of the Hardy-Hodge decomposition have previously been used to a various extent in Geomagnetic applications (see, e.g., [5, 16, 19, 31]).

By means of the reflection $\mathcal{R}_R(x) = \frac{R^2}{|x|^2}\, x$ across $\mathbb{S}_R$, we define the Kelvin transform $K_R[f]$ of a function f defined on an open set $\Omega \subset \mathbb{R}^3$ to be the function on $\mathcal{R}_R(\Omega)$ given by

$$K_R[f](x) = \frac{R}{|x|} f(\mathcal{R}_R(x)), \quad x \in \mathcal{R}_R(\Omega). \tag{8}$$

A function f is harmonic in Ω if and only if $K_R[f]$ is harmonic in $\mathcal{R}_R(\Omega)$ (e.g., [4, Thm. 4.7]).

Now, assume that $\mathbf{f} \in \mathcal{H}^2_{+,R}$ with $\mathbf{f} = \nabla f$ and $f(0) = 0$. Then $\nabla K_R[f] \in \mathcal{H}^2_{-,R}$. In fact, if for $\mathbf{f} \in \mathcal{H}^2_{+,R}$ (resp. $\mathbf{f} \in \mathcal{H}^2_{-,R}$) we let $\int \mathbf{f}$ indicate the harmonic function f in $\mathbb{B}_R$ (resp. in $\mathbb{R}^3 \setminus \overline{\mathbb{B}}_R$) whose gradient is $\mathbf{f}$, normalized so that $f(0) = 0$ (resp. $\lim_{|x|\to\infty} f(x) = 0$), then $\mathbf{f} \mapsto \nabla K_R \circ \int \mathbf{f}$ maps $\mathcal{H}^2_{+,R}$ continuously into $\mathcal{H}^2_{-,R}$ and back [3]. Moreover, in view of (8) we have that

$$\nabla K_R[f](x) = \frac{R^3 \nabla f(\mathcal{R}_R(x))}{|x|^3} - 2\, x \cdot \nabla f(\mathcal{R}_R(x))\, \frac{R^3 x}{|x|^5} - f(\mathcal{R}_R(x))\, \frac{Rx}{|x|^3}\,. \tag{9}$$

Clearly f and $K_R[f]$ coincide on $\mathbb{S}_R$, therefore the tangential components of ∇f and $\nabla K_R[f]$ agree on $\mathbb{S}_R$ (these are the spherical gradients $\nabla_{\mathbb{S}_R} f$ and $\nabla_{\mathbb{S}_R} K_R[f]$). The normal components $\partial_\nu f$ and $\partial_\nu K_R[f]$, though, are different. Indeed, we get from (9) that

$$\partial_\nu K_R[f](x) = -\partial_\nu f(x) - \frac{f(x)}{R}, \quad x \in \mathbb{S}_R. \tag{10}$$

We turn to some special systems of functions. First, let $\{Y_{n,k}\}_{n\in\mathbb{N}_0,\, k=1,\ldots,2n+1}$ be an $L^2(\mathbb{S})$-orthonormal system of spherical harmonics of degrees n and orders k. A possible choice is

$$Y_{n,k}(x) = \begin{cases} \sqrt{\frac{2n+1}{2\pi}\frac{(k-1)!}{(2n+1-k)!}}P_{n,n+1-k}(\sin(\theta))\cos((n+1-k)\varphi) & k=1,\ldots,n, \\ \sqrt{\frac{2n+1}{4\pi}}P_{n,0}(t), & k=n+1, \\ \sqrt{\frac{2n+1}{2\pi}\frac{(2n+1-k)!}{(k-1)!}}P_{n,k-(n+1)}(\sin(\theta))\sin((k-(n+1))\varphi) & k=n+2,\ldots,2n+1, \end{cases}$$

for $x = (\cos(\theta)\cos(\varphi), \cos(\theta)\sin(\varphi), \sin(\theta))^T \in \mathbb{S}_1$, $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}], \varphi \in [0, 2\pi)$, and $P_{n,k}$ the associated Legendre polynomials of degree n and order k (see, e.g., [15, Ch. 3] for details; another common notation is to indicate the order of the spherical harmonics by $k = -n, \ldots, n$ rather than $k = 1, \ldots, 2n+1$). Then $H^R_{n,k}(x) = \left(\frac{|x|}{R}\right)^n Y_{n,k}\left(\frac{x}{|x|}\right)$ is a homogeneous, harmonic polynomial of degree n in $\mathbb{R}^3$ (sometimes also called inner harmonic and equipped with a normalization factor $\frac{1}{R}$). In fact, every homogeneous harmonic polynomial in $\mathbb{R}^3$ can be expressed as a linear combination of inner harmonics. The Kelvin transform $H^R_{-n-1,k} = K_R[H^R_{n,k}]$ is a harmonic function in $\mathbb{R}^3 \setminus \{0\}$ with $\lim_{|x|\to\infty} H^R_{-n-1,k}(x) = 0$ (sometimes called outer harmonic). In [3, Lemma 4] the following result was shown.

LEMMA 2.2. *The vector space* $\mathrm{span}\{\nabla H^R_{-n-1,k}\}_{n\in\mathbb{N}_0,\, k=1,\ldots,2n+1}$ *is dense in* $\mathcal{H}^2_{-,R}$ *and the vector space* $\mathrm{span}\{\nabla H^R_{n,k}\}_{n\in\mathbb{N}_0,\, k=1,\ldots,2n+1}$ *is dense in* $\mathcal{H}^2_{+,R}$.

For each fixed $x \in \mathbb{R}^3 \setminus \overline{\mathbb{B}}_R$, the function $g_x(y) = \frac{1}{|x-y|}$ is harmonic in a neighborhood of $\overline{\mathbb{B}}_R$ and, therefore, its gradient

$$\mathbf{g}_x(y) = \nabla_x\ g_x(y) = -\frac{x-y}{|x-y|^3}$$

lies in $\mathcal{H}^2_{+,R}$. As a consequence of Lemma 2.2, we shall prove the following density result.

LEMMA 2.3. *The vector space* $\mathrm{span}\{\mathbf{g}_x : x \in \mathbb{R}^3 \setminus \overline{\mathbb{B}}_R\}$ *is dense in* $\mathcal{H}^2_{+,R}$ *and the vector space* $\mathrm{span}\{\mathbf{g}_x : x \in \mathbb{B}_R\}$ *is dense in* $\mathcal{H}^2_{-,R}$.

Proof. As $K_R[g_x] = \frac{1}{|x|} g_{x/|x|^2}$ and $\nabla K_R \circ \int$ is an isomorphism from $\mathcal{H}^2_{-,R}$ onto $\mathcal{H}^2_{+,R}$ (see discussion before (9)), we need only prove the second assertion. Define $g(y) = \frac{1}{|y|}$ as a function of $y \in \mathbb{R}^3 \setminus \{0\}$. For $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}_0^3$ with $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3 = n$, the derivative $\partial_\alpha g(y) = \frac{\partial^n}{\partial^{\alpha_1} y_1 \partial^{\alpha_2} y_2 \partial^{\alpha_3} y_3} g(y)$ is of the form $\frac{H_\alpha(y)}{|y|^{1+2n}}$, where H_α is a homogeneous harmonic polynomial of degree n, and actually every homogeneous harmonic polynomial H_α is a scalar multiple of $|y|^{(1+2n)}\partial_\alpha g(y)$ for some α [4, Lemma 5.15]. The discussion before Lemma 2.2 now implies that $\partial_\alpha g$ is an element of $\mathrm{span}\{H^R_{-n-1,k}\}_{n\in\mathbb{N}_0,\, k=1,\ldots,2n+1}$. Thus, by this lemma, we are done if we can show that whenever $\mathbf{f} \in \mathcal{H}^2_{-,R}$ is orthogonal in $L^2(\mathbb{S}_R, \mathbb{R}^3)$ to all $\mathbf{g}_x$, $x \in \mathbb{B}_R$, then it must be orthogonal to all $\nabla H^R_{-n-1,k}$. To this end, differentiating $\langle \mathbf{f}, \mathbf{g}_x \rangle_{L^2(\mathbb{S}_R,\mathbb{R}^3)} = 0$

with respect to x leads us to

$$0=\left\langle \mathbf{f}\,,\,\nabla\frac{H_\alpha(.-x)}{|\cdot-x|^{1+2n}}\right\rangle_{L^2(\mathbb{S}_R,\mathbb{R}^3)} \tag{11}$$

for all $\alpha\in\mathbb{N}_0^3$ and $n=|\alpha|$. Setting $x=0$ yields

$$0=\left\langle \mathbf{f}\,,\,\nabla\frac{H_\alpha}{|\cdot|^{1+2n}}\right\rangle_{L^2(\mathbb{S}_R,\mathbb{R}^3)}=R^{-2n-1}\left\langle \mathbf{f}\,,\,\nabla K_R[H_\alpha]\right\rangle_{L^2(\mathbb{S}_R,\mathbb{R}^3)}.$$

Since every inner harmonic $H_{n,k}^R$ can be expressed as a linear combination of H_α, this relation and the considerations before Lemma 2.2 imply $\langle \mathbf{f}\,,\,\nabla H_{-n-1,k}^R\rangle_{L^2(\mathbb{S}_R,\mathbb{R}^3)}=0$ for all $n\in\mathbb{N}_0$, $k=1,\ldots,2n+1$, which is the desired conclusion. □

3. Harmonic Potentials in Divergence-Form. The potential of a measure μ on $\mathbb{R}^3$ is defined by

$$p_\mu(x)=-\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{1}{|x-y|}\mathrm{d}\mu(y). \tag{12}$$

It is the solution of $\Delta\Phi=\mu$ in $\mathbb{R}^3$ which is "smallest" at infinity. If $\mu\geq 0$, the potential p_μ is a superharmonic function and therefore it is either finite quasi-everywhere or identically $-\infty$, see [2] for these properties and the definition of "quasi everywhere". Decomposing a signed measure into its positive and negative parts (the Hahn decomposition) yields that p_μ is finite quasi-everywhere if μ is finite and compactly supported (i.e., if $\mathrm{supp}(\mu)$, which is closed by definition, is also bounded). If $\mathrm{supp}(\mu)\subset\overline{\mathbb{B}}_R$, the Riesz representation theorem and the maximum principle for harmonic functions imply that there exists a unique measure $\hat\mu$ with $\mathrm{supp}(\hat\mu)\subset\mathbb{S}_R$ such that

$$\int g(y)\mathrm{d}\mu(y)=\int g(y)\mathrm{d}\hat\mu(y)$$

for every continuous function g in $\overline{\mathbb{B}}_R$ which is harmonic in $\mathbb{B}_R$. Since $y\mapsto 1/|x-y|$ is harmonic in a neighbourhood of $\overline{\mathbb{B}}_R$ when $x\notin\overline{\mathbb{B}}_R$, this entails that the potentials p_μ and $p_{\hat\mu}$ coincide in $\mathbb{R}^3\setminus\overline{\mathbb{B}}_R$, i.e.,

$$p_\mu(x)=p_{\hat\mu}(x),\quad x\in\mathbb{R}^3\setminus\overline{\mathbb{B}}_R.$$

The measure $\hat\mu$ is called the balayage of μ onto $\mathbb{S}_R$ (see, e.g., [2]). In fact, the potentials p_μ and $p_{\hat\mu}$ coincide quasi-everywhere on $\mathbb{S}_R$ as well. An expression for $\hat\mu$ easily follows from the Poisson representation of a function f which is continuous in $\overline{\mathbb{B}}_R$ and harmonic in $\mathbb{B}_R$:

$$f(x)=\frac{1}{4\pi R}\int_{\mathbb{S}_R}\frac{R^2-|x|^2}{|x-y|^3}f(y)\,\mathrm{d}\omega_R(y),\quad x\in\mathbb{B}_R. \tag{13}$$

Clearly Equation (13), Fubini's theorem, and the definition of balayage imply that

$$d\hat\mu(x)=\mathrm{d}\mu_{|\mathbb{S}_R}(x)+\left(\frac{1}{4\pi R}\int_{\mathbb{B}_R}\frac{R^2-|y|^2}{|x-y|^3}\mathrm{d}\mu(y)\right)\mathrm{d}\omega_R(x). \tag{14}$$

LEMMA 3.1. *Let the measure μ be supported in $\overline{\mathbb{B}_R}$. Furthermore, assume that μ is absolutely continuous in $\mathbb{B}_R$ with a density h (i.e., $\mathrm{d}\mu(y) = h(y)dy$) that satisfies the Hardy condition*

$$\operatorname*{ess.\,sup}_{0\le r<R} \int_{\mathbb{S}_r} |h(y)|^2 \, \mathrm{d}\omega_r(y) < \infty. \tag{15}$$

Then the balayage $\hat{\mu}$ of μ on $\mathbb{S}_R$ is absolutely continuous with respect to ω_R (i.e., $d\hat{\mu}(y) = \hat{h}(y)\mathrm{d}\omega_R(y)$) and it has a density $\hat{h} \in L^2(\mathbb{S}_R)$.

Proof. Starting from (14) and the assumption that μ is absolutely continuous, we find that the density $\hat{h}$ of $\hat{\mu}$ is

$$\hat{h}(x) = \frac{1}{4\pi R} \int_{\mathbb{B}_R} \frac{R^2 - |y|^2}{|x-y|^3} h(y) \mathrm{d}\lambda(y), \quad x \in \mathbb{S}_R.$$

Using Fubini's theorem and the identity

$$\left| \frac{x}{|x|} - |x| y \right| = \left| \frac{y}{|y|} - |y| x \right|, \qquad x, y \in \mathbb{R}^3 \setminus \{0\},$$

together with the changes of variable $\eta = \frac{\xi}{r}$, $y = \frac{rx}{R^2}$, we are led to

$$\begin{aligned}
\|\hat{h}\|^2_{L^2(\mathbb{S}_R)} &= \frac{1}{(4\pi R)^2} \int_{\mathbb{S}_R} \left(\int_{\mathbb{B}_R} \frac{R^2 - |y|^2}{|x-y|^3} h(y) \, \mathrm{d}\lambda(y) \right)^2 \mathrm{d}\omega_R(x) \\
&= \frac{1}{(4\pi R)^2} \int_{\mathbb{S}_R} \left(\int_0^R \left(\int_{\mathbb{S}_r} \frac{R^2 - |\xi|^2}{|x-\xi|^3} h(\xi) \, \mathrm{d}\omega_r(\xi) \right) dr \right)^2 \mathrm{d}\omega_R(x) \\
&\le \frac{R}{(4\pi R)^2} \int_{\mathbb{S}_R} \left(\int_0^R \left(\int_{\mathbb{S}_r} \frac{R^2 - |\xi|^2}{|x-\xi|^3} h(\xi) \, \mathrm{d}\omega_r(\xi) \right)^2 dr \right) \mathrm{d}\omega_R(x) \\
&= \frac{1}{(4\pi R)^2} \int_{\mathbb{S}_R} \left(\int_0^R \left(\int_{\mathbb{S}_r} \frac{1 - (\frac{r}{R})^2}{|\frac{x}{R} - \frac{\xi}{R}|^3} h(\xi) \, \mathrm{d}\omega_r(\xi) \right)^2 dr \right) \mathrm{d}\omega_R(x) \\
&= \frac{1}{(4\pi R)^2} \int_{\mathbb{S}_R} \left(\int_0^R \left(\int_{\mathbb{S}_r} \frac{1 - \left|\frac{rx}{R^2}\right|^2}{|\frac{\xi}{r} - \frac{rx}{R^2}|^3} h(\xi) \, \mathrm{d}\omega_r(\xi) \right)^2 dr \right) \mathrm{d}\omega_R(x) \\
&= \frac{1}{(4\pi R)^2} \int_0^R r^4 \left(\int_{\mathbb{S}_R} \left(\int_{\mathbb{S}_1} \frac{1 - \left|\frac{rx}{R^2}\right|^2}{|\eta - \frac{rx}{R^2}|^3} h(r\eta) \, \mathrm{d}\omega_1(\eta) \right)^2 \mathrm{d}\omega_R(x) \right) dr
\end{aligned}$$

$$= \int_0^R r^4 \left(\frac{1}{4\pi (\frac{r}{R})^2} \int_{\mathbb{S}_{\frac{r}{R}}} \left(\frac{1}{4\pi} \int_{\mathbb{S}_1} \frac{1 - |y|^2}{|\eta - y|^3} h(r\eta) \, \mathrm{d}\omega_1(\eta) \right)^2 \mathrm{d}\omega_{\frac{r}{R}}(y) \right) dr. \tag{16}$$

Now, the function

$$f(y) = \frac{1}{4\pi} \int_{\mathbb{S}_1} \frac{1 - |y|^2}{|\eta - y|^3} h(r\eta) \, \mathrm{d}\omega_1(\eta)$$

is the Poisson integral of $h(r\cdot)$ over the unit sphere $\mathbb{S}_1$ (and represents the middle integral on the right hand side of (16)). Thus, f is harmonic in $\mathbb{B}_1$ and its square $|f|^2$

is subharmonic there. The latter implies that the mean of $|f|^2$ over the sphere $\mathbb{S}_{\frac{r}{R}}$, $r < R$, is not greater than its mean over $\mathbb{S}_1$, i.e.,

$$\begin{aligned}\frac{1}{4\pi(\frac{r}{R})^2}\int_{\mathbb{S}_{\frac{r}{R}}}|f(y)|^2\mathrm{d}\omega_{\frac{r}{R}}(y) &\leq \lim_{\frac{s}{R}\to 1-}\frac{1}{4\pi(\frac{s}{R})^2}\int_{\mathbb{S}_{\frac{s}{R}}}|f(y)|^2\mathrm{d}\omega_{\frac{s}{R}}(y)\\ &= \frac{1}{4\pi}\int_{\mathbb{S}_1}|h(r\eta)|^2\,\mathrm{d}\omega_1(\eta)\\ &= \frac{1}{4\pi r^2}\int_{\mathbb{S}_r}|h(y)|^2\,\mathrm{d}\omega_r(y) \leq \frac{M}{4\pi r^2},\end{aligned}$$

where the constant $M > 0$ comes from the Hardy condition (15). Together with (16), we find that

$$\|\hat{h}\|^2_{L^2(\mathbb{S}_R)} \leq \frac{MR^3}{12\pi},$$

eventually showing that $\hat{h} \in L^2(\mathbb{S}_R)$ and that $\hat{\mu}$ is absolutely continuous with respect to ω_R with density $\hat{h}$. □

More generally, an arbitrary distribution D with compact support has a potential p_D given outside of supp(D) by

$$p_D(x) = D\left(-\frac{1}{4\pi}\frac{1}{|x-\cdot|}\right), \quad x \in \mathbb{R}^3 \setminus \mathrm{supp}(D). \tag{17}$$

Compactness of supp(D) easily implies that D indeed acts on $-1/(4\pi|x-\cdot|)$ when $x \notin$ supp(D) so that p_D is well-defined (cf. Section SM1 in the supplementary materials for details). If D is supported in $\overline{\mathbb{B}}_R$ (in particular, if it is supported in some shell $\overline{\mathbb{B}_{R-d,R}}$), we define the balayage of D onto $\mathbb{S}_R$ to be the distribution $\hat{D}$ on $\mathbb{S}_R$ that satisfies

$$p_{\hat{D}}(x) = p_D(x), \quad x \in \mathbb{R}^3 \setminus \overline{\mathbb{B}}_R.$$

Strictly speaking, $\hat{D}$ is a distribution on $\mathbb{S}_R$ so that $p_{\hat{D}}$ should rather be denoted by $p_{\hat{D}\otimes\delta_{\mathbb{S}_R}}$, where $\hat{D}\otimes\delta_{\mathbb{S}_R}$ is the distribution on $\mathbb{R}^3$ which is the tensor product of $\hat{D}$ with the measure $\delta_{\mathbb{S}_R}$ corresponding in spherical coordinates to a Dirac mass at $r = R$ (see [37]). Nevertheless, to alleviate notation, we do write $p_{\hat{D}}$. Thus, what is meant in (17) when $D = \hat{D}$ is that $\hat{D}$ is applied to the restriction to $\mathbb{S}_R$ of $-1/(4\pi|x-\cdot|)$.

We briefly comment on the existence and uniqueness of such a balayage in Section SM1 of the supplementary materials. If D is (associated with) a measure μ, then (17) coincides with (12) and the balayage was given in (14). The main difference between the case of a finite compactly supported measure μ and the case of a general compactly supported distribution D is that usually $p_D(x)$ cannot be assigned a meaning when $x \in$ supp(D) whereas p_μ is well-defined quasi everywhere on supp(μ). We say that D is in divergence form if

$$D = \nabla\cdot\mathbf{M}, \tag{18}$$

where $\nabla\cdot$ is to be understood as the distributional divergence and $\mathbf{M}$ is a $\mathbb{R}^3$-valued distribution. If, e.g., $\mathbf{M} \in L^2(\mathbb{B}_{R-d,R},\mathbb{R}^3)$ and supp$(\mathbf{M}) \subset \overline{\mathbb{B}_{R-d,R}}$, then the corresponding potential p_D coincides with Φ_{crust} in (1). Now we can formulate the main result of this section, namely, that balayage preserves divergence form for those $\mathbf{M}$ satisfying a Hardy condition.

LEMMA 3.2. *Let* $D = \nabla\cdot\mathbf{M}$*, where* $\mathbf{M} \in L^2(\mathbb{B}_R,\mathbb{R}^3)$ *satisfies the Hardy condition*

$$\operatorname*{ess.\,sup}_{0\leq r<R} \int_{\mathbb{S}_r} |\mathbf{M}(y)|^2 \,\mathrm{d}\omega_r(y) < \infty.$$

Then there exists $\mathbf{m} \in L^2(\mathbb{S}_R,\mathbb{R}^3)$ *such that* $\hat{D} = \nabla\cdot(\mathbf{m}\otimes\delta_{\mathbb{S}_R})$ *is the balayage of* D *onto* $\mathbb{S}_R$*.*

Proof. Let $\mathbf{M} = (M_1, M_2, M_3)^T$ denote the components of $\mathbf{M}$. The definition of p_D yields

$$\begin{aligned} p_D(x) &= \frac{1}{4\pi}\int_{\mathbb{B}_R} \mathbf{M}(y)\cdot\frac{x-y}{|x-y|^3}\,\mathrm{d}\lambda(y) \\ &= \frac{1}{4\pi}\sum_{j=1}^{3}\int_{\mathbb{B}_R} M_j(y)\frac{x_j-y_j}{|x-y|^3}\,\mathrm{d}\lambda(y), \quad x\in\mathbb{R}^3\setminus\overline{\mathbb{B}}_R. \end{aligned} \tag{19}$$

If we choose the measure μ_j such that $\mathrm{d}\mu_j(y) = M_j(y)dy$, we get from Lemma 3.1 and the Hardy condition on $\mathbf{M}$ that there exists a $m_j \in L^2(\mathbb{S}_R)$ such that balayage of μ_j onto $\mathbb{S}_R$ is given by the measure $\hat{\mu}_j$ with $d\hat{\mu}_j = m_j\mathrm{d}\omega_R$, $j = 1, 2, 3$. Setting $\mathbf{m} = (m_1, m_2, m_3)^T$ and observing that $g_{x,j}(y) = \frac{x_j-y_j}{|x-y|^3} = -\partial_{x_j}\frac{1}{|x-y|}$ is harmonic in $\mathbb{B}_R$ and continuous in $\overline{\mathbb{B}_R}$, for fixed $x\in\mathbb{R}^3\setminus\overline{\mathbb{B}}_R$, then the definition of balayage yields together with (19) that

$$\begin{aligned} p_D(x) &= \frac{1}{4\pi}\sum_{j=1}^{3}\int_{\mathbb{S}_R} m_j(t)\frac{x_j-y_j}{|x-y|^3}\,\mathrm{d}\omega_R(y) \\ &= \frac{1}{4\pi}\int_{\mathbb{S}_R}\mathbf{m}(y)\cdot\frac{x-y}{|x-y|^3}\,\mathrm{d}\omega_R(y) = p_{\hat{D}}(x), \quad x\in\mathbb{R}^3\setminus\overline{\mathbb{B}}_R. \end{aligned} \tag{20}$$

The latter implies that $\hat{D} = \nabla\cdot(\mathbf{m}\otimes\delta_{\mathbb{S}_R})$, as announced. □

REMARK 3.3. *Lemma 3.2 eventually justifies the statement made in the introduction that, to every square summable volumetric magnetization* $\mathbf{M}$ *in the Earth's crust* $\mathbb{B}_{R-d,R}$ *that satisfies the Hardy condition, there exists a spherical magnetization* $\mathbf{m}$ *on* $\mathbb{S}_R$ *that produces the same magnetic potential and therefore also the same magnetic field in the exterior of the Earth.*

4. Separation of Potentials.

We are now in a position to approach Problem 1.1. For this we study the nullspace of the potential operator Φ^{R_1,R_0,R_2} (cf. Definition 4.1), mapping a magnetization $\mathbf{m}$ on $\mathbb{S}_{R_0}$ and an auxiliary function $h\in L^2(\mathbb{S}_{R_1})$ to the sum of the potentials (2) and (3) on $\mathbb{S}_{R_2}$. First, we show in Section 4.1 that uniqueness holds in Problem 1.1 if $\operatorname{supp}\mathbf{m}\neq\mathbb{S}_{R_0}$. Similar results hold for the magnetic field operator $\mathbf{B}^{R_1,R_0,R_2} = \nabla\Phi^{R_1,R_0,R_2}$ (cf. Theorem 4.5). In Section 4.2, we discuss how the previous results can be used to approximate quantities like the Fourier coefficients $\langle\Phi_0, Y_{n,k}\rangle_{L^2(\mathbb{S}_{R_2})}$ of Φ_0. Finally, in Section 4.3, we show that $\Phi = \Phi_0 + \Phi_1$ may well vanish though $\Phi_0, \Phi_1 \neq 0$. This follows from Lemma 4.13 and answers the uniqueness issue of Problem 1.1 in the negative when $\operatorname{supp}\mathbf{m} = \mathbb{S}_{R_0}$.

4.1. Uniqueness Issues.

In accordance with the notation from Problem 1.1, we define two operators: one mapping a spherical magnetization $\mathbf{m}$ to the potential $p_{\hat{D}}$ with $\hat{D} = \nabla\cdot(\mathbf{m}\otimes\delta_{\mathbb{S}_{R_0}})$, and the other mapping an auxiliary function $h\in L^2(\mathbb{S}_{R_1})$ to its Poisson integral, both evaluated on $\mathbb{S}_{R_2}$.

Definition 4.1. *Let $0 < R_1 < R_0 < R_2$ be fixed radii and Γ_{R_0} a closed subset of $\mathbb{S}_{R_0}$. Let*

$$\Phi_0^{R_0,R_2} : L^2(\Gamma_{R_0},\mathbb{R}^3) \to L^2(\mathbb{S}_{R_2}), \quad \mathbf{m} \mapsto \frac{1}{4\pi}\int_{\Gamma_{R_0}} \mathbf{m}(y)\cdot\frac{x-y}{|x-y|^3}\,\mathrm{d}\omega_{R_0}(y), \;\; x\in\mathbb{S}_{R_2},$$

and

$$\Phi_1^{R_1,R_2} : L^2(\mathbb{S}_{R_1}) \to L^2(\mathbb{S}_{R_2}), \quad h \mapsto \frac{1}{4\pi R_1}\int_{\mathbb{S}_{R_1}} h(y)\frac{|x|^2-R_1^2}{|x-y|^3}\mathrm{d}\omega_{R_1}(y), \quad x\in\mathbb{S}_{R_2}.$$

The superposition of the two operators above is denoted by

$$\Phi^{R_1,R_0,R_2} : L^2(\Gamma_{R_0},\mathbb{R}^3)\times L^2(\mathbb{S}_{R_1}) \to L^2(\mathbb{S}_{R_2}), \quad (\mathbf{m},h)\mapsto \Phi_0^{R_0,R_2}[\mathbf{m}] + \Phi_1^{R_1,R_2}[h].$$

We start by characterizing the potentials $p_{\hat{D}}$, with $\hat{D}$ in divergence-form, which are zero in $\mathbb{R}^3\setminus\overline{\mathbb{B}_R}$.

Lemma 4.2. *Let $\mathbf{m}\in L^2(\mathbb{S}_R,\mathbb{R}^3)$ and $\hat{D} = \nabla\cdot(\mathbf{m}\otimes\delta_{\mathbb{S}_R})$ be in divergence-form. Let further $\mathbf{m} = \mathbf{m}_+ + \mathbf{m}_- + \mathbf{d}$ be the Hardy-Hodge decomposition of $\mathbf{m}$, i.e., $\mathbf{m}_+\in\mathcal{H}^2_{+,R}$, $\mathbf{m}_-\in\mathcal{H}^2_{-,R}$, and $\mathbf{d}\in\mathcal{D}_R$. Then $p_{\hat{D}}(x) = 0$, for all $x\in\mathbb{R}^3\setminus\overline{\mathbb{B}_R}$, if and only if $\mathbf{m}_+\equiv 0$. Analogously, $p_{\hat{D}}(x) = 0$, for all $x\in\mathbb{B}_R$, if and only if $\mathbf{m}_-\equiv 0$.*

Proof. We already know that $\mathbf{g}_x(y) = \frac{x-y}{|x-y|^3}$ lies in $\mathcal{H}^2_{+,R}$ for every fixed $x\in\mathbb{R}^3\setminus\overline{\mathbb{B}_R}$. The orthogonality of the Hardy-Hodge decomposition and the representation (20) of $p_{\hat{D}}$ yield that $\mathbf{m}_-$ and $\mathbf{d}$ do not change $p_{\hat{D}}$ in $\mathbb{R}^3\setminus\overline{\mathbb{B}_R}$. Conversely, if $p_{\hat{D}}(x) = 0$ for all $x\in\mathbb{R}^3\setminus\overline{\mathbb{B}_R}$, then

$$p_{\hat{D}}(x) = \langle\mathbf{g}_x,\mathbf{m}\rangle_{L^2(\mathbb{S}_R,\mathbb{R}^3)} = \langle\mathbf{g}_x,\mathbf{m}_+\rangle_{L^2(\mathbb{S}_R,\mathbb{R}^3)} = 0, \quad x\in\mathbb{R}^3\setminus\overline{\mathbb{B}_R}.$$

Since Lemma 2.3 asserts that $\mathrm{span}\{\mathbf{g}_x : x\in\mathbb{R}^3\setminus\overline{\mathbb{B}}_R\}$ is dense in $\mathcal{H}^2_{+,R}$, the above relation implies $\mathbf{m}_+\equiv 0$. The assertion for the case where $p_{\hat{D}}(x) = 0$, for all $x\in\mathbb{B}_R$ likewise follows by observing that $\mathbf{g}_x(y) = \frac{x-y}{|x-y|^3}$ lies in $\mathcal{H}^2_{-,R}$ for fixed $x\in\mathbb{B}_R$. ☐

Since $\Phi_0^{R_0,R_2}[\mathbf{m}] = p_{\hat{D}}$, we may use Lemma 4.2 to characterize the nullspace of $\Phi_0^{R_0,R_2}$ (extending the magnetization $\mathbf{m}\in L^2(\Gamma_{R_0},\mathbb{R}^3)$ by zero on $\mathbb{S}_{R_0}\setminus\Gamma_{R_0}$ if the latter is nonempty). As to $\Phi_1^{R_1,R_2}$, we know its nullspace reduces to zero because the Poisson integral (3) yields the unique harmonic extension of $h\in L^2(\mathbb{S}_{R_1})$ to $\mathbb{R}^3\setminus\overline{\mathbb{B}_{R_1}}$ which is zero at infinity (i.e., h is the nontangential limit of its Poisson extension a.e. on $\mathbb{S}_{R_1}$, see [4, Thm. 6.13]). This motivates the following statement on the nullspace $N(\Phi^{R_1,R_0,R_2})$ of Φ^{R_1,R_0,R_2}.

Theorem 4.3. *Let the setup be as in Definition 4.1 and assume that $\Gamma_{R_0}\neq\mathbb{S}_{R_0}$. Then the nullspace of Φ^{R_1,R_0,R_2} is given by*

$$N(\Phi^{R_1,R_0,R_2}) = \{(\mathbf{d},0) : \mathbf{d}\in\mathcal{D}_{R_0},\ \mathrm{supp}(\mathbf{d})\subset\Gamma_{R_0}\}.$$

Proof. Clearly $\Phi^{R_1,R_0,R_2}[(\mathbf{m},h)]$ is harmonic in $\mathbb{R}^3\setminus\{\Gamma_{R_0}\cup\mathbb{S}_{R_1}\}$ and vanishes at infinity. If $\Phi^{R_1,R_0,R_2}[(\mathbf{m},h)](x) = 0$ for $x\in\mathbb{S}_{R_2}$, then it follows from the maximum principle that $\Phi^{R_1,R_0,R_2}[(\mathbf{m},h)](x) = 0$ for all $x\in\mathbb{R}^3\setminus\mathbb{B}_{R_2}$. Subsequently, by real analyticity, $\Phi^{R_1,R_0,R_2}[(\mathbf{m},h)]$ must vanish identically in $\mathbb{R}^3\setminus\{\Gamma_{R_0}\cup\overline{\mathbb{B}_{R_1}}\}$ which is connected because $\Gamma_{R_0}\neq\mathbb{S}_{R_0}$. Thus, $\Phi^{R_1,R_0,R_2}[(\mathbf{m},h)]$ extends harmonically (by the zero function) across Γ_{R_0}:

$$\Phi^{R_1,R_0,R_2}[(\mathbf{m},h)](x) = 0, \quad x\in\mathbb{R}^3\setminus\overline{\mathbb{B}_{R_1}}. \tag{21}$$

Since $\Phi^{R_1,R_0,R_2}[(\mathbf{m},h)] = \Phi_0^{R_0,R_2}[\mathbf{m}] + \Phi_1^{R_1,R_2}[h]$, where $\Phi_1^{R_1,R_2}[h]$ is harmonic on $\mathbb{R}^3 \setminus \overline{\mathbb{B}_{R_1}}$, we find that $\Phi_0^{R_0,R_2}[\mathbf{m}]$ in turn extends harmonically across Γ_{R_0}, therefore it is harmonic in all of $\mathbb{R}^3$. Additionally $\Phi_0^{R_0,R_2}[\mathbf{m}]$ vanishes at infinity, hence $\Phi_0^{R_0,R_2}[\mathbf{m}](x) = 0$ for all $x \in \mathbb{R}^3$ by Liouville's theorem. Since $\Phi_0^{R_0,R_2}[\mathbf{m}] = p_{\hat{D}}$ for $\hat{D} = \nabla \cdot (\mathbf{m} \otimes \delta_{\mathbb{S}_{R_0}})$, Lemma 4.2 now implies that $\mathbf{m} = \mathbf{d} \in \mathcal{D}_{R_0}$ with $\operatorname{supp} \mathbf{d} \subset \Gamma_{R_0}$. Next, as $\Phi_0^{R_0,R_2}[\mathbf{m}]$ vanishes identically on $\mathbb{R}^3$, we get from (21) that $\Phi_1^{R_1,R_2}[h](x) = 0$ for all $x \in \mathbb{R}^3 \setminus \overline{\mathbb{B}_{R_1}}$. Then, injectivity of the Poisson transform entails that $h \equiv 0$, hence $N(\Phi^{R_1,R_0,R_2}) \subset \{(\mathbf{m},0) : \mathbf{m} \in \mathcal{D}_{R_0} \operatorname{supp}(\mathbf{m}) \subset \Gamma_{R_0}\}$.

The reverse inclusion $N(\Phi^{R_1,R_0,R_2}) \supset \{(\mathbf{m},0) : \mathbf{m} \in \mathcal{D}_{R_0}, \operatorname{supp}(\mathbf{m}) \subset \Gamma_{R_0}\}$ is clear because Lemma 4.2 yields that $\Phi^{R_1,R_0,R_2}[(\mathbf{m},0)](x) = \Phi_0^{R_0,R_2}[\mathbf{m}](x) = 0$, for all $x \in \mathbb{R}^3 \setminus \Gamma_{R_0}$ if $\mathbf{m} \in \mathcal{D}_{R_0}$. ☐

COROLLARY 4.4. *Notation being as in Definition 4.1 with $\Gamma_{R_0} \neq \mathbb{S}_{R_0}$, let $\Phi = \Phi^{R_1,R_0,R_2}[(\mathbf{m},h)]$ for some $\mathbf{m} \in L^2(\Gamma_{R_0},\mathbb{R}^3)$ and some $h \in L^2(\mathbb{S}_{R_1})$. Then, a pair of potentials of the form $\bar{\Phi}_0 = \Phi_0^{R_0,R_2}[\bar{\mathbf{m}}]$ and $\bar{\Phi}_1 = \Phi_1^{R_1,R_2}[\bar{h}]$, with $\bar{\mathbf{m}} \in L^2(\Gamma_{R_0},\mathbb{R}^3)$ and $\bar{h} \in L^2(\mathbb{S}_{R_1})$, is uniquely determined by the condition $\Phi(x) = \bar{\Phi}_0(x) + \bar{\Phi}_1(x)$, $x \in \mathbb{S}_{R_2}$.*

Proof. From Theorem 4.3 we get that h is uniquely determined by the values of Φ on $\mathbb{S}_{R_2}$, and also that the components $\mathbf{m}_+ \in \mathcal{H}^2_{+,R_0}$ and $\mathbf{m}_- \in \mathcal{H}^2_{-,R_0}$ of the Hardy-Hodge decomposition of $\mathbf{m}$ are uniquely determined. The former implies $\bar{h} \equiv h$ and the latter $\bar{\mathbf{m}} \equiv \mathbf{m} + \bar{\mathbf{d}}$, for some $\bar{\mathbf{d}} \in \mathcal{D}_{R_0}$. By Lemma 4.2 we have that $\Phi_0^{R_0,R_2}[\mathbf{m}](x) = \Phi_0^{R_0,R_2}[\mathbf{m}+\bar{\mathbf{d}}](x)$ for $x \in \mathbb{R}^3 \setminus \mathbb{S}_{R_0}$, so we eventually find that $\bar{\Phi}_0$ and $\bar{\Phi}_1$ are uniquely determined. ☐

Corollary 4.4 answers the uniqueness issue of Problem 1.1 in the positive provided that $\operatorname{supp}(\mathbf{m}) \neq \mathbb{S}_{R_0}$. In other words, assuming a locally supported magnetization, it is possible to separate the contribution of the Earth's crust from the contribution of the Earth's core if only the superposition of both magnetic potentials is known on some external orbit $\mathbb{S}_{R_2}$. Of course, in Geomagnetism, it is the magnetic field $\mathbf{B} = \nabla\Phi$ which is measured rather than the magnetic potential Φ. However, the result carries over at once to this setting. More in fact is true: if $\operatorname{supp}(\mathbf{m}) \neq \mathbb{S}_{R_0}$, separation is possible if only the normal component of $\mathbf{B}$ is known on $\mathbb{S}_{R_2}$. Indeed, we have the following theorem.

THEOREM 4.5. *Let the setup be as in Definition 4.1 with $\Gamma_{R_0} \neq \mathbb{S}_{R_0}$, and consider the operator*

$$\begin{aligned}\mathbf{B}^{R_1,R_0,R_2} : L^2(\Gamma_{R_0},\mathbb{R}^3) \times L^2(\mathbb{S}_{R_1},\mathbb{R}^3) &\to L^2(\mathbb{S}_{R_2},\mathbb{R}^3),\\ (\mathbf{m},h) &\mapsto \nabla\Phi_0^{R_0,R_2}[\mathbf{m}] + \nabla\Phi_1^{R_1,R_2}[h].\end{aligned}$$

Define further the normal operator:

$$\begin{aligned}\mathbf{B}_\nu^{R_1,R_0,R_2} : L^2(\Gamma_{R_0},\mathbb{R}^3) \times L^2(\mathbb{S}_{R_1},\mathbb{R}^3) &\to L^2(\mathbb{S}_{R_2}),\\ (\mathbf{m},h) &\mapsto \partial_\nu \left(\Phi_0^{R_0,R_2}[\mathbf{m}] + \Phi_1^{R_1,R_2}[h]\right).\end{aligned}$$

Then the nullspaces of $\mathbf{B}^{R_1,R_0,R_2}$ and $\mathbf{B}_\nu^{R_1,R_0,R_2}$ are all given by

$$N(\mathbf{B}^{R_1,R_0,R_2}) = N(\mathbf{B}_\nu^{R_1,R_0,R_2}) = \{(\mathbf{d},0) : \mathbf{d} \in \mathcal{D}_{R_0},\ \operatorname{supp}(\mathbf{d}) \subset \Gamma_{R_0}\}.$$

Proof. Let $\mathbf{B}_\nu^{R_1,R_0,R_2}[(\mathbf{m},h)](x) = 0$ for $x \in \mathbb{S}_{R_2}$. Then $\Phi^{R_1,R_0,R_2}[(\mathbf{m},h)]$ has vanishing normal derivative on $\mathbb{S}_{R_2}$, and is otherwise harmonic in $\mathbb{R}^3 \setminus \overline{\mathbb{B}_{R_2}}$. Note that $\Phi^{R_1,R_0,R_2}[(\mathbf{m},h)]$ is even harmonic across $\mathbb{S}_{R_2}$ onto a slightly larger open set, hence there is no issue of smoothness to define derivatives everywhere on $\mathbb{S}_{R_2}$. Since $\Phi^{R_1,R_0,R_2}[(\mathbf{m},h)]$ vanishes at infinity, its Kelvin transform $u = K_{R_2}[\Phi^{R_1,R_0,R_2}[(\mathbf{m},h)]]$ is harmonic in $\mathbb{B}_{R_2}$ with $u(0) = 0$ [4, Thm. 4.8], and by (10) it holds that $\partial_\nu u(x) + u(x)/R_2 = 0$ for $x \in \mathbb{S}_{R_2}$. Now, if u is nonconstant and x is a maximum place for u on $\mathbb{S}_{R_2}$, then $\partial_\nu u(x) > 0$ by the Hopf lemma [4, Ch. 1, Ex. 25]. Hence $u(x) < 0$, implying that $u < 0$ on $\mathbb{B}_{R_2}$, which contradicts the maximum principle because $u(0) = 0$. Therefore u vanishes identically and so does $\Phi^{R_1,R_0,R_2}[(\mathbf{m},h)]$ on $\mathbb{S}_{R_2}$. Appealing to Theorem 4.3 now achieves the proof. □

The next corollary follows in the exact same manner as Corollary 4.4. To state it, we indicate with a subscript ν the normal component of a field in $L^2(\mathbb{S}_{R_2},\mathbb{R}^3)$ while a subscript τ denotes the tangential component.

Corollary 4.6. *Let the setup be as in Definition 4.1 with $\Gamma_{R_0} \neq \mathbb{S}_{R_0}$, and let the operator $\mathbf{B}^{R_1,R_0,R_2}$, be as in Theorem 4.5. Define further the operators*

$$\mathbf{B}_0^{R_0,R_2} : L^2(\Gamma_{R_0},\mathbb{R}^3) \to L^2(\mathbb{S}_{R_2},\mathbb{R}^3), \quad \mathbf{m} \mapsto \nabla\Phi_0^{R_0,R_2}[\mathbf{m}],$$

and

$$\mathbf{B}_1^{R_1,R_2} : L^2(\mathbb{S}_{R_1}) \to L^2(\mathbb{S}_{R_2},\mathbb{R}^3), \quad h \mapsto \nabla\Phi_1^{R_1,R_2}[h].$$

Let further $\mathbf{B} = \mathbf{B}^{R_1,R_0,R_2}[(\mathbf{m},h)]$, with $\mathbf{m} \in L^2(\Gamma_{R_0},\mathbb{R}^3)$ and $h \in L^2(\mathbb{S}_{R_1})$. A pair of fields of the form $\bar{\mathbf{B}}_0 = \mathbf{B}_0^{R_0,R_2}[\bar{\mathbf{m}}]$ and $\bar{\mathbf{B}}_1 = \mathbf{B}_1^{R_1,R_2}[\bar{h}]$, with $\bar{\mathbf{m}} \in L^2(\Gamma_{R_0},\mathbb{R}^3)$ and $\bar{h} \in L^2(\mathbb{S}_{R_1})$, is uniquely determined by the condition $\mathbf{B}_\nu(x) = (\bar{\mathbf{B}}_0)_\nu(x) + (\bar{\mathbf{B}}_1)_\nu(x)$ and thus, a fortiori, by the condition $\mathbf{B}(x) = \bar{\mathbf{B}}_0(x) + \bar{\mathbf{B}}_1(x)$ for $x \in \mathbb{S}_{R_2}$.

Remark 4.7. *Opposed to the normal component, it does not suffice to know the tangential component $\mathbf{B}_\tau$ on $\mathbb{S}_{R_2}$ in order to obtain uniqueness of $\mathbf{B}_0$ and $\mathbf{B}_1$. Namely, letting $\mathbf{m} \equiv 0$ and h be any nonzero constant function on $\mathbb{S}_{R_1}$, then $\mathbf{B}_\tau(x) = (\mathbf{B}_0)_\tau(x) + (\mathbf{B}_1)_\tau(x) = \nabla_{\mathbb{S}_{R_2}}\Phi_0^{R_0,R_2}[\mathbf{m}](x) + \nabla_{\mathbb{S}_{R_2}}\Phi_1^{R_1,R_2}[h](x) = 0$ and $\mathbf{B}_0(x) = \nabla\Phi_0^{R_0,R_2}[\mathbf{m}](x) = 0$ but $\mathbf{B}_1(x) = \nabla\Phi_1^{R_1,R_2}[h](x) = -\frac{hR_1}{|x|^3}x \neq 0$ for $x \in \mathbb{S}_{R_2}$.*

4.2. Reconstruction Issues. In this section, we discuss how quantities such as the Fourier coefficients $\langle\Phi_0, Y_{n,k}\rangle_{L^2(\mathbb{S}_{R_2})}$ of Φ_0 can be approximated knowing Φ, without having to reconstruct Φ_0 itself. Such Fourier coefficients are of interest, e.g., when looking at the power spectra of Φ and Φ_0 (cf. the empirical way of separating the crustal and the core magnetic fields mentioned in the introduction). As an extra piece of notation, given $\Gamma_R \subset \mathbb{S}_R$ and $f : \mathbb{S}_R \to \mathbb{R}^k$, we let $f_{|\Gamma_R} : \Gamma_R \to \mathbb{R}^k$ designate the restriction of f to Γ_R.

Theorem 4.8. *Let the setup be as in Definition 4.1 and assume that $\Gamma_{R_0} \neq \mathbb{S}_{R_0}$. Then, for every $\varepsilon > 0$ and every function $\mathbf{g} \in \mathcal{H}^2_{+,R_0} \oplus \mathcal{H}^2_{-,R_0}$, there exists $f \in L^2(\mathbb{S}_{R_2})$ (depending on ε and $\mathbf{g}$) such that*

$$\left|\langle\Phi^{R_1,R_0,R_2}[\mathbf{m},h], f\rangle_{L^2(\mathbb{S}_{R_2})} - \langle\mathbf{m}, \mathbf{g}_{|\Gamma_{R_0}}\rangle_{L^2(\Gamma_{R_0},\mathbb{R}^3)}\right| \leq \varepsilon\|(\mathbf{m},h)\|_{L^2(\Gamma_{R_0},\mathbb{R}^3)\times L^2(\mathbb{S}_{R_1})},$$

for all $\mathbf{m} \in L^2(\Gamma_{R_0},\mathbb{R}^3)$ and $h \in L^2(\mathbb{S}_{R_1})$.

Proof. According to Theorem 4.3 and the orthogonality of the Hardy-Hodge decomposition, $(\mathbf{g}_{|\Gamma_{R_0}}, 0)$ is orthogonal to the nullspace $N(\Phi^{R_1,R_0,R_2})$ of Φ^{R_1,R_0,R_2}, for if

$\mathbf{d} \in \mathcal{D}_{R_0}$ and $\mathrm{supp}(\mathbf{d}) \subset \Gamma_{R_0}$, then $\langle \mathbf{g}_{|\Gamma_{R_0}}, \mathbf{d}\rangle_{L^2(\Gamma_{R_0},\mathbb{R}^3)} = \langle \mathbf{g}, \mathbf{d}\rangle_{L^2(\mathbb{S}_{R_0},\mathbb{R}^3)} = 0$. Therefore, $(\mathbf{g}_{|\Gamma_{R_0}}, 0)$ lies in the closure of the range of the adjoint operator $\left(\Phi^{R_1,R_0,R_2}\right)^*$, i.e., to each $\varepsilon > 0$ there is $f \in L^2(\mathbb{S}_{R_2})$ with

$$\left\| \left(\Phi^{R_1,R_0,R_2}\right)^* [f] - (\mathbf{g}_{|\Gamma_{R_0}}, 0) \right\|_{L^2(\Gamma_{R_0},\mathbb{R}^3)\times L^2(\mathbb{S}_{R_1})} \leq \varepsilon. \tag{22}$$

Taking the scalar product with $(\mathbf{m}, h)$, we get from (22) and the Cauchy-Schwarz inequality:

$$\begin{aligned}
&\left| \langle \Phi^{R_1,R_0,R_2}[\mathbf{m}, h], f\rangle_{L^2(\mathbb{S}_{R_2})} - \langle \mathbf{m}, \mathbf{g}_{|\Gamma_{R_0}}\rangle_{L^2(\Gamma_{R_0},\mathbb{R}^3)} \right| \\
&= \left| \left\langle (\mathbf{m}, h), \left(\Phi^{R_1,R_0,R_2}\right)^* [f] - (\mathbf{g}_{|\Gamma_{R_0}}, 0) \right\rangle_{L^2(\Gamma_{R_0},\mathbb{R}^3)\times L^2(\mathbb{S}_{R_2})} \right| \\
&\leq \left\| \left(\Phi^{R_1,R_0,R_2}\right)^* [f] - (\mathbf{g}_{|\Gamma_{R_0}}, 0) \right\|_{L^2(\Gamma_{R_0},\mathbb{R}^3)\times L^2(\mathbb{S}_{R_1})} \|(\mathbf{m}, h)\|_{L^2(\Gamma_{R_0},\mathbb{R}^3)\times L^2(\mathbb{S}_{R_1})} \\
&\leq \varepsilon \|(\mathbf{m}, h)\|_{L^2(\Gamma_{R_0},\mathbb{R}^3)\times L^2(\mathbb{S}_{R_1})},
\end{aligned}$$

which is the desired result. □

COROLLARY 4.9. *Let the setup be as in Definition 4.1 with $\Gamma_{R_0} \neq \mathbb{S}_{R_0}$. Then, for every $\varepsilon > 0$ and every function $g \in L^2(\mathbb{S}_{R_2})$, there exists $f \in L^2(\mathbb{S}_{R_2})$ (depending on ε and g) such that*

$$\left| \langle \Phi^{R_1,R_0,R_2}[\mathbf{m}, h], f\rangle_{L^2(\mathbb{S}_{R_2})} - \langle \Phi_0^{R_0,R_2}[\mathbf{m}], g\rangle_{L^2(\mathbb{S}_{R_2})} \right| \leq \varepsilon \|(\mathbf{m}, h)\|_{L^2(\Gamma_{R_0},\mathbb{R}^3)\times L^2(\mathbb{S}_{R_1})},$$

for all $\mathbf{m} \in L^2(\Gamma_{R_0}, \mathbb{R}^3)$ and $h \in L^2(\mathbb{S}_{R_1})$.

Proof. First observe that

$$\left\langle \Phi_0^{R_0,R_2}[\mathbf{m}], g \right\rangle_{L^2(\mathbb{S}_{R_2})} = \left\langle \mathbf{m}, \left(\Phi_0^{R_0,R_2}\right)^* [g] \right\rangle_{L^2(\Gamma_{R_0},\mathbb{R}^3)}, \tag{23}$$

where the adjoint operator of $\Phi_0^{R_0,R_2}$ is given by

$$\begin{aligned}
&\left(\Phi_0^{R_0,R_2}\right)^* : L^2(\mathbb{S}_{R_2}) \to L^2(\Gamma_{R_0}, \mathbb{R}^3), \quad g \mapsto \mathbf{H}[g]_{|\Gamma_{R_0}}, \\
&\mathbf{H}[g](x) = -\frac{1}{4\pi} \int_{\mathbb{S}_{R_2}} g(y) \frac{x-y}{|x-y|^3} \mathrm{d}\omega_{R_2}(y), \quad x \in \mathbb{S}_{R_0}.
\end{aligned} \tag{24}$$

Clearly $\mathbf{H}[g] \in \mathcal{H}^2_{+,R_0}$ whenever $g \in L^2(\mathbb{S}_{R_2})$, therefore, (23) together with Theorem 4.8 yield the desired result. □

REMARK 4.10. *The interest of Corollary 4.9 from the Geophysical viewpoint lies with the fact that $\Phi^{R_1,R_0,R_2}[\mathbf{m}, h]$ (more specifically: its gradient) corresponds to the measurements on $\mathbb{S}_{R_2}$ of the superposition of the core and crustal contributions, whereas $\Phi_0^{R_0,R_2}[\mathbf{m}]$ corresponds to the crustal contribution alone. Thus, if we can compute f knowing g, we shall in principle be able to get information on the crustal contribution up to arbitrary small error. Note also that $(\mathbf{g}, 0) \notin \mathrm{Ran}\left(\Phi^{R_1,R_0,R_2}\right)^*$ unless $\mathbf{g} \equiv 0$, due to the injectivity of the adjoint of the Poisson transform (which is again a Poisson transform). Therefore we can only hope for an approximation of $\langle \Phi_0^{R_0,R_2}[\mathbf{m}], g\rangle_{L^2(\mathbb{S}_{R_2})}$ in Corollary 4.9, up to a relative error of $\varepsilon > 0$, but not for an exact reconstruction.*

Results analogous to Theorem 4.8 and Corollary 4.9 mechanically hold in the setup of Theorem 4.5 and Corollary 4.6 (i.e., separation of the crustal and core magnetic fields $\mathbf{B}_0$ and $\mathbf{B}_1$ instead of the potentials). Below we state the corresponding results but we omit the proofs for they are similar to the previous ones.

THEOREM 4.11. *Let the setup be as in Theorem 4.5. Then, for every* $\varepsilon > 0$ *and every field* $\mathbf{g} \in \mathcal{H}^2_{+,R_0} \oplus \mathcal{H}^2_{-,R_0}$, *there exists* $\mathbf{f} \in L^2(\mathbb{S}_{R_2}, \mathbb{R}^3)$ *(depending on* ε *and* $\mathbf{g}$*) such that*

$$\left|\langle \mathbf{B}^{R_1,R_0,R_2}[\mathbf{m},h], \mathbf{f}\rangle_{L^2(\mathbb{S}_{R_2},\mathbb{R}^3)} - \langle \mathbf{m}, \mathbf{g}_{|\Gamma_{R_0}}\rangle_{L^2(\Gamma_{R_0},\mathbb{R}^3)}\right| \leq \varepsilon \|(\mathbf{m},h)\|_{L^2(\Gamma_{R_0},\mathbb{R}^3)\times L^2(\mathbb{S}_{R_1})},$$

for all $\mathbf{m} \in L^2(\Gamma_{R_0}, \mathbb{R}^3)$ *and* $h \in L^2(\mathbb{S}_{R_1})$. *The same holds if* $\mathbf{B}^{R_1,R_0,R_2}[\mathbf{m},h]$ *gets replaced by* $\mathbf{B}^{R_1,R_0,R_2}_\nu[\mathbf{m},h]$, *this time with* $\mathbf{f} \in L^2(\mathbb{S}_{R_2})$.

COROLLARY 4.12. *Let the setup be as in Theorem 4.5 and Corollary 4.6. Then, for every* $\varepsilon > 0$ *and every field* $\mathbf{g} \in L^2(\mathbb{S}_{R_2}, \mathbb{R}^3)$, *there exists* $\mathbf{f} \in L^2(\mathbb{S}_{R_2}, \mathbb{R}^3)$ *(depending on* ε *and* $\mathbf{g}$*) such that*

$$\left|\langle \mathbf{B}^{R_1,R_0,R_2}[\mathbf{m},h], \mathbf{f}\rangle_{L^2(\mathbb{S}_{R_2},\mathbb{R}^3)} - \langle \mathbf{B}_0^{R_0,R_2}[\mathbf{m}], \mathbf{g}\rangle_{L^2(\mathbb{S}_{R_2},\mathbb{R}^3)}\right| \leq \varepsilon \|(\mathbf{m},h)\|_{L^2(\Gamma_{R_0},\mathbb{R}^3)\times L^2(\mathbb{S}_{R_1})},$$

for all $\mathbf{m} \in L^2(\Gamma_{R_0}, \mathbb{R}^3)$ *and* $h \in L^2(\mathbb{S}_{R_1})$. *The same holds if* $\mathbf{B}^{R_1,R_0,R_2}[\mathbf{m},h]$ *gets replaced by* $\mathbf{B}^{R_1,R_0,R_2}_\nu[\mathbf{m},h]$, *this time with* $\mathbf{f} \in L^2(\mathbb{S}_{R_2})$.

4.3. The Case $\Gamma_{R_0} = \mathbb{S}_{R_0}$. We turn to the case where $\Gamma_{R_0} = \mathbb{S}_{R_0}$. Then, uniqueness no longer holds in Problem 1.1, but one can obtain the singular value decomposition of Φ^{R_1,R_0,R_2} fairly explicitly and thereby quantify non-uniqueness. Indeed basic computations using spherical harmonics yield:

$$\begin{aligned}
&(\Phi_0^{R_0,R_2})^*[Y_{n,k}](x) \\
&= \frac{1}{4\pi}\int_{\mathbb{S}_{R_2}} Y_{n,k}\left(\frac{y}{|y|}\right)\nabla_x \frac{1}{|x-y|}\mathrm{d}\omega_{R_2}(y) \\
&= \frac{1}{4\pi}\sum_{m=0}^{\infty}\nabla_x\int_{\mathbb{S}_{R_2}}\frac{1}{|y|}\left(\frac{|x|}{|y|}\right)^m Y_{n,k}\left(\frac{y}{|y|}\right)P_m\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right)\mathrm{d}\omega_{R_2}(y) \qquad (25)\\
&= \sum_{m=0}^{\infty}\sum_{l=1}^{2m+1}\frac{1}{2m+1}\frac{1}{R_2^{m+1}}\nabla_x\left(|x|^m Y_{m,l}\left(\frac{x}{|x|}\right)\right)\int_{\mathbb{S}_{R_2}}Y_{n,k}\left(\frac{y}{|y|}\right)Y_{m,l}\left(\frac{y}{|y|}\right)\mathrm{d}\omega_{R_2}(y) \\
&= \frac{R_2}{2n+1}\nabla H_{n,k}^{R_2}(x) = \frac{R_2}{2n+1}\left(\frac{R_0}{R_2}\right)^n\nabla H_{n,k}^{R_0}(x), \quad x\in\mathbb{S}_{R_0},
\end{aligned}$$

and

$$\begin{aligned}
&(\Phi_1^{R_1,R_2})^*[Y_{n,k}](x) \\
&= \frac{1}{4\pi R_1}\int_{\mathbb{S}_{R_2}} Y_{n,k}\left(\frac{y}{|y|}\right)\frac{|y|^2-R_1^2}{|x-y|^3}\mathrm{d}\omega_{R_2}(y) \\
&= \frac{1}{4\pi R_1}\sum_{m=0}^{\infty}(2m+1)\int_{\mathbb{S}_{R_2}}\frac{1}{|y|}\left(\frac{|x|}{|y|}\right)^m P_m\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right)Y_{n,k}\left(\frac{y}{|y|}\right)\mathrm{d}\omega_{R_2}(y) \qquad (26)\\
&= \frac{1}{R_1R_2}\sum_{m=0}^{\infty}\sum_{l=1}^{2m+1}\left(\frac{R_1}{R_2}\right)^m Y_{m,l}\left(\frac{x}{|x|}\right)\int_{\mathbb{S}_{R_2}}Y_{n,k}\left(\frac{y}{|y|}\right)Y_{m,l}\left(\frac{y}{|y|}\right)\mathrm{d}\omega_{R_2}(y) \\
&= \left(\frac{R_1}{R_2}\right)^{n-1}Y_{n,k}\left(\frac{x}{|x|}\right), \quad x\in\mathbb{S}_{R_1},
\end{aligned}$$

where $H^{R_2}_{n,k}$, $H^{R_0}_{n,k}$ are the inner harmonics from Section 2 and P_m the Legendre polynomial of degree m (see, e.g., [13, 15, Ch. 3] for details). So, we get for the adjoint operator $(\Phi^{R_1,R_0,R_2})^*$ that

$$(\Phi^{R_1,R_0,R_2})^*[Y_{n,k}] = \left(\frac{R_2}{2n+1} \left(\frac{R_0}{R_2}\right)^n \nabla H^{R_0}_{n,k}, \left(\frac{R_1}{R_2}\right)^{n-1} Y_{n,k} \right)^T. \tag{27}$$

Similar calculations also yield that

$$\Phi_0^{R_0,R_2}[\nabla H^{R_0}_{n,k}](x) = \frac{n}{R_2}\left(\frac{R_0}{R_2}\right)^n Y_{n,k}\left(\frac{x}{|x|}\right), \quad x \in \mathbb{S}_{R_2},$$

and

$$\Phi_1^{R_1,R_2}[Y_{n,k}](x) = \left(\frac{R_1}{R_2}\right)^{n+1} Y_{n,k}\left(\frac{x}{|x|}\right), \quad x \in \mathbb{S}_{R_2},$$

so we obtain for Φ^{R_1,R_0,R_2} that

$$\Phi^{R_1,R_0,R_2}[\alpha\nabla H^{R_0}_{n,k}, \beta Y_{m,l}] = \alpha\frac{n}{R_2}\left(\frac{R_0}{R_2}\right)^n Y_{n,k} + \beta\left(\frac{R_1}{R_2}\right)^{m+1} Y_{m,l}, \tag{28}$$

with $\alpha, \beta \in \mathbb{R}$. Based on the representations (27) and (28), further computation leads us to a characterization of the nullspace of Φ^{R_1,R_0,R_2} in Lemma 4.13. Note that $\Phi^{R_1,R_0,R_2} : L^2(\Gamma_{R_0}, \mathbb{R}^3) \times L^2(\mathbb{S}_{R_1}) \to L^2(\mathbb{S}_{R_2})$ is a compact operator, being the sum of two compact operators (for $\Phi_0^{R_0,R_2}$ and $\Phi_1^{R_1,R_2}$ have continuous kernels).

Lemma 4.13. *Let $\Gamma_{R_0} = \mathbb{S}_{R_0}$, then the nullspace of Φ^{R_1,R_0,R_2} is given by*

$$N(\Phi^{R_1,R_0,R_2}) = \{(\mathbf{m}_- + \mathbf{d}, 0) : \mathbf{m}_- \in \mathcal{H}^2_{-,R_0}, \mathbf{d} \in \mathcal{D}_{R_0}\} \cup \overline{\operatorname{span}\left\{ \left(\nabla H^{R_0}_{n,k}, -\frac{n}{R_1}\left(\frac{R_0}{R_1}\right)^n Y_{n,k}\right)^T : n \in \mathbb{N}, k = 1,\dots,2n+1 \right\}},$$

while the orthogonal complement reads

$$N(\Phi^{R_1,R_0,R_2})^\perp = \overline{\operatorname{span}\left\{ \left(\nabla H^{R_0}_{n,k}, \frac{2n+1}{R_1}\left(\frac{R_1}{R_0}\right)^n Y_{n,k}\right)^T : n \in \mathbb{N}, k = 1,\dots,2n+1 \right\}}.$$

All non-zero eigenvalues values of $(\Phi^{R_1,R_0,R_2})^*\Phi^{R_1,R_0,R_2}$ are of the form

$$\sigma_n = \frac{n}{2n+1}\left(\frac{R_0}{R_2}\right)^{2n} + \left(\frac{R_1}{R_2}\right)^{2n}, \quad n \in \mathbb{N},$$

and the corresponding eigenvectors in $L^2(\mathbb{S}_{R_0}, \mathbb{R}^3) \times L^2(\mathbb{S}_{R_1})$ are

$$\left(\nabla H^{R_0}_{n,k}, \frac{2n+1}{R_1}\left(\frac{R_1}{R_0}\right)^n Y_{n,k}\right)^T, \quad n \in \mathbb{N},\ k = 1,\dots,2n+1.$$

Lemma 4.13 entails that the nullspace of Φ^{R_1,R_0,R_2} contains elements of the form $(\mathbf{m}, h)$ with $h \neq 0$, hence $\Phi^{R_1,R_0,R_2}[(\mathbf{m}, h)]$ may well vanish on $\mathbb{S}_{R_2}$ even though $\Phi_1^{R_1,R_2}[h]$ is nonzero there, by injectivity of the Poisson representation. In other words, separation of the potentials $\Phi_0^{R_0,R_2}$ and $\Phi_1^{R_1,R_2}$ knowing their sum on $\mathbb{S}_{R_2}$ is no longer possible in general if $\Gamma_{R_0} = \mathbb{S}_{R_0}$.

5. Extremal Problems and Numerical Examples. In this section, we provide some first approaches on how the results from the previous sections can be used to approximate the Fourier coefficients of Φ_0. For brevity, we treat only separation of the crustal and core magnetic potentials (underlying operator Φ^{R_1,R_0,R_2}) and not the separation of the crustal and core magnetic fields (underlying operator $\mathbf{B}^{R_1,R_0,R_2}$). The procedure in such case is of course similar. In the supplementary material (Section SM2) we illustrate a similar example aiming at a reconstruction of Φ_0 as a whole rather than its single Fourier coefficients.

5.1. Reconstruction of Fourier Coefficients of Φ_0. To get a feeling of how functions f in Corollary 4.9 behave, let us derive some of their basic properties. Recall they where isentified to be those $f \in L^2(\mathbb{S}_{R_2})$ satisfying (22) with $\mathbf{g} = (\Phi_0^{R_0,R_2})^*[g]$.

Lemma 5.1. *Let $0 \neq g \in L^2(\mathbb{S}_{R_2})$ and set $\mathbf{g} = (\Phi_0^{R_0,R_2})^*[g]$. To each $\varepsilon > 0$, let $f_\varepsilon \in L^2(\mathbb{S}_{R_2})$ satisfy $\|(\Phi^{R_1,R_0,R_2})^*[f_\varepsilon] - (\mathbf{g}_{|\Gamma_{R_0}}, 0)\|_{L^2(\Gamma_{R_0},\mathbb{R}^3)\times L^2(\mathbb{S}_{R_1})} \leq \varepsilon$. Then:*
(a) $\lim_{\varepsilon\to 0} \|f_\varepsilon\|_{L^2(\mathbb{S}_{R_2})} = \infty$,
(b) $\lim_{\varepsilon\to 0} \|(\Phi_1^{R_1,R_2})^*[f_\varepsilon]\|_{L^2(\mathbb{S}_{R_1})} = 0$,
(c) $\lim_{\varepsilon\to 0} \langle f_\varepsilon, Y_{n,k}\rangle_{L^2(\mathbb{S}_{R_2})} = 0$, *for fixed $n \in \mathbb{N}_0$, $k = 1, \ldots, n$.*

Proof. From Remark 4.10 we know that $(\mathbf{g}_{|\Gamma_{R_0}}, 0) \in \overline{\mathrm{Ran}\big((\Phi^{R_1,R_0,R_2})^*\big)}$ but also $(\mathbf{g}_{|\Gamma_{R_0}}, 0) \notin \mathrm{Ran}\big((\Phi^{R_1,R_0,R_2})^*\big)$. Thus, $\|f_\varepsilon\|_{L^2(\mathbb{S}_{R_2})}$ cannot remain bounded as $\varepsilon \to 0$, otherwise a weak limit point $f_0 \in L^2(\mathbb{S}_{R_2})$ would meet $(\Phi^{R_1,R_0,R_2})^*[f_0] = (\mathbf{g}_{|\Gamma_{R_0}}, 0)$, a contradiction which proves (a). Next, the relation

$$\begin{aligned}&\|(\Phi^{R_1,R_0,R_2})^*[f_\varepsilon] - (\mathbf{g}_{|\Gamma_{R_0}}, 0)\|^2_{L^2(\Gamma_{R_0},\mathbb{R}^3)\times L^2(\mathbb{S}_{R_1})}\\ &= \|(\Phi_0^{R_0,R_2})^*[f_\varepsilon] - \mathbf{g}_{|\Gamma_{R_0}}\|^2_{L^2(\Gamma_{R_0},\mathbb{R}^3)} + \|(\Phi_1^{R_1,R_2})^*[f_\varepsilon]\|^2_{L^2(\mathbb{S}_{R_1})} \leq \varepsilon^2\end{aligned}$$

immediately implies that $\lim_{\varepsilon\to 0} \|(\Phi_1^{R_1,R_2})^*[f_\varepsilon]\|_{L^2(\mathbb{S}_{R_1})} = 0$ which is (b). Finally, expanding f_ε in spherical harmonics, one readily verifies that (26) together with (b) yields part (c). □

Next, we give a quantitative appraisal of the fact that the Fourier coefficients of $\Phi_0^{R_0,R_2}$ on $\mathbb{S}_{R_2}$, to be estimated up to relative precision ε by choosing $g = Y_{p,q}$ in Corollary 4.9, can be approximated directly by those of Φ^{R_1,R_0,R_2} (i.e., neglecting entirely the core contribution) when $\frac{R_1}{R_2}$ is small enough (i.e., the core is far from the measurement orbit) and the degree p is large enough. We also give a quantitative version of Lemma 5.1 point (c). This provides us with bounds on the validity of the separation technique consisting merely of a sharp cutoff in the frequency domain.

Lemma 5.2. *Let $\varepsilon > 0$ and choose $\mathbf{g} = (\Phi_0^{R_0,R_2})^*[Y_{p,q}]$ for some $p \in \mathbb{N}_0$ and $q \in \{1, \ldots, 2p+1\}$. Then the following assertions hold true.*
(a) If $R_1^2\big(\frac{R_1}{R_2}\big)^{p-1} \leq \varepsilon$, then $f = Y_{p,q}$ satisfies

$$\|(\Phi^{R_1,R_0,R_2})^*[f] - (\mathbf{g}_{|\Gamma_{R_0}}, 0)\|_{L^2(\Gamma_{R_0},\mathbb{R}^3)\times L^2(\mathbb{S}_{R_1})} \leq \varepsilon. \tag{29}$$

(b) If $f \in L^2(\mathbb{S}_{R_2})$ satisfies $\|(\Phi^{R_1,R_0,R_2})^*[f] - (\mathbf{g}_{|\Gamma_{R_0}}, 0)\|_{L^2(\Gamma_{R_0},\mathbb{R}^3)\times L^2(\mathbb{S}_{R_1})} \leq \varepsilon$, then, for all $n \in \mathbb{N}_0$, $k = 1, \ldots, 2n+1$,

$$|\langle f, Y_{n,k}\rangle_{L^2(\mathbb{S}_{R_2})}| \leq \varepsilon \frac{R_2^{n-1}}{R_1^{n+1}}. \tag{30}$$

Proof. To prove (a), note that $(\Phi^{R_1,R_0,R_2})^* = \left((\Phi_0^{R_0,R_2})^*, (\Phi_1^{R_1,R_2})^*\right)$ and by (26) that

$$\|(\Phi_1^{R_1,R_2})^*[f]\|_{L^2(\mathbb{S}_{R_1})} = R_1^2\left(\frac{R_1}{R_2}\right)^{p-1} \le \varepsilon,$$

while $\|(\Phi_0^{R_0,R_2})^*[f] - \mathbf{g}_{|\Gamma_{R_0}}\|_{L^2(\Gamma_{R_0},\mathbb{R}^3)} = 0$ if $f = Y_{p,q}$. Hence (29) holds.

As to (b), any $f \in L^2(\mathbb{S}_{R_2})$ with $\|(\Phi^{R_1,R_0,R_2})^*[f] - (\mathbf{g}, 0)\|_{L^2(\Gamma_{R_0},\mathbb{R}^3)\times L^2(\mathbb{S}_{R_1})} \le \varepsilon$ satisfies in particular, in view of (26):

$$\|(\Phi_1^{R_1,R_2})^*[f]\|^2_{L^2(\mathbb{S}_{R_1})} = \sum_{n=0}^{\infty}\sum_{k=1}^{2n+1} R_1^4\left(\frac{R_1}{R_2}\right)^{2(n-1)} |\langle f, Y_{n,k}\rangle_{L^2(\mathbb{S}_{R_2})}|^2 \le \varepsilon^2,$$

from which (30) follows at once. □

We turn to the computation of a function f as in Corollary 4.9, regardless of assumptions on $\frac{R_1}{R_2}$ or on the degree of a spherical harmonics $Y_{n,k}$ for which we want to estinate $\langle \Phi_0^{R_0,R_2}, Y_{n,k}\rangle_{L^2(\mathbb{S}_{R_2})}$. One way is to solve the following extremal problem. Note that finding f requires no data on the potential Φ that we eventually want to separate into $\Phi_0 + \Phi_1$.

Problem 5.3. *Let the setup be as in Definition 4.1 with $\Gamma_{R_0} \neq \mathbb{S}_{R_0}$. Fix $g \in L^2(\mathbb{S}_{R_2})$ as well as $\varepsilon > 0$, and set $\mathbf{g} = (\Phi_0^{R_0,R_2})^*[g]$. Then, find $f \in W^{1,2}(\mathbb{S}_{R_2})$ such that*

$$\|f\|_{W^{1,2}(\mathbb{S}_{R_2})} = \inf_{\substack{\bar f \in W^{1,2}(\mathbb{S}_{R_2}),\\ \|(\Phi^{R_1,R_0,R_2})^*[\bar f] - (\mathbf{g}_{|\Gamma_{R_0}}, 0)\|_{L^2(\Gamma_{R_0},\mathbb{R}^3)\times L^2(\mathbb{S}_{R_1})} \le \varepsilon}} \|\bar f\|_{W^{1,2}(\mathbb{S}_{R_2})}. \tag{31}$$

It may look strange to seek $f \in W^{1,2}(\mathbb{S}_{R_2})$ whereas Corollary 4.9 merely deals with scalar products in $L^2(\mathbb{S}_{R_2})$. This extra-smoothness requirement, though, helps regularizing the problem.

Lemma 5.4. *Let the setup be as in Problem 5.3 and additionally $g \in L^2(\mathbb{S}_{R_2})$ with $\|\mathbf{g}_{|\Gamma_{R_0}}\|_{L^2(\Gamma_{R_0},\mathbb{R}^3)} > \varepsilon$. Then, there exists a unique solution $0 \not\equiv f \in W^{1,2}(\mathbb{S}_{R_2})$ to Problem 5.3. Moreover, the constraint in (31) is saturated, i.e. $\|(\Phi^{R_1,R_0,R_2})^*[f] - (\mathbf{g}_{|\Gamma_{R_0}}, 0)\|_{L^2(\Gamma_{R_0},\mathbb{R}^3)\times L^2(\mathbb{S}_{R_1})} = \varepsilon$.*

Proof. Since $\mathbf{H}[g]$ given by (24) lies in $\mathcal{H}^2_{+,R_0}$, the same argument as in the proof of Theorem 4.8 and the density of $W^{1,2}(\mathbb{S}_{R_2})$ in $L^2(\mathbb{S}_{R_2})$ together imply the existence of $\bar f \in W^{1,2}(\mathbb{S}_{R_2})$ such that $\|(\Phi^{R_1,R_0,R_2})^*[\bar f] - (\mathbf{g}_{|\Gamma_{R_0}}, 0)\|_{L^2(\Gamma_{R_0},\mathbb{R}^3)\times L^2(\mathbb{S}_{R_1})} \le \varepsilon$ is satisfied, which ensures that the closed convex subset of $W^{1,2}(\mathbb{S}_{R_2})$ defined by

$$\mathcal{C}_\varepsilon = \left\{\bar f \in W^{1,2}(\mathbb{S}_{R_2}) : \|(\Phi^{R_1,R_0,R_2})^*[\bar f] - (\mathbf{g}_{|\Gamma_{R_0}}, 0)\|_{L^2(\Gamma_{R_0},\mathbb{R}^3)\times L^2(\mathbb{S}_{R_1})} \le \varepsilon\right\}$$

is non-empty. Existence and uniqueness of a minimizer f now follows from that of a projection of minimum norm on any nonempty convex set in a Hilbert space. From the assumption that $\|\mathbf{g}\|_{L^2(\Gamma_{R_0},\mathbb{R}^3)} > \varepsilon$, we get that $f \not\equiv 0$ because $0 \notin \mathcal{C}_\varepsilon$. If the constraint is not saturated, then there is $\delta > 0$ such that, for every $\bar f \in W^{1,2}(\mathbb{S}_{R_2})$ with $\|\bar f\|_{W^{1,2}(\mathbb{S}_{R_2})} \le 1$, also $f + t\bar f$ satisfies the constraint $\|(\Phi^{R_1,R_0,R_2})^*[f + t\bar f] - (\mathbf{g}, 0)\|_{L^2(\Gamma_{R_0},\mathbb{R}^3)\times L^2(\mathbb{S}_{R_1})} \le \varepsilon$ for $t \in (-\delta, \delta)$. Since f is a minimizer, this implies

$$0 = \partial_t \|f + t\bar f\|^2_{W^{1,2}(\mathbb{S}_{R_2})}\Big|_{t=0} = 2\left\langle f, \bar f\right\rangle_{W^{1,2}(\mathbb{S}_{R_2})},$$

for every $\bar{f} \in W^{1,2}(\mathbb{S}_{R_2})$ with $\|\bar{f}\|_{W^{1,2}(\mathbb{S}_{R_2})} \leq 1$. Thus $f \equiv 0$, contradicting what precedes. □

REMARK 5.5. *Lemma 5.1 together with the exponential decay of the eigenvalues of $(\Phi_1^{R_1,R_2})^*$ in (26) suggest that most of the relevant information of a solution $f \in W^{1,2}(\mathbb{S}_{R_2})$ of Problem 5.3 must be contained in Fourier coefficients $\langle f, Y_{n,k}\rangle_{L^2(\mathbb{S}_{R_2})}$ of increasingly high degrees n as $\varepsilon \to 0$. Lemma 5.2 provides a hint at the range of accuracies ε for which numerical solutions of Problem 5.3 with $\mathbf{g} = (\Phi_0^{R_0,R_2})^*[Y_{p,q}]$ behave differently for small and large p.*

Discretization. For the actual solution of Problem 5.3, we from now on assume that $\|\mathbf{g}_{|\Gamma_{R_0}}\|_{L^2(\Gamma_{R_0},\mathbb{R}^3)} > \varepsilon$, hence the constraint is saturated by Lemma 5.4, and we use a Lagrangian formulation and obtain from [9, Thm. 2.1] that $f \in W^{1,2}(\mathbb{S}_{R_2})$ solves for

$$\left(\mathrm{Id} + \lambda \left(\Phi^{R_1,R_0,R_2}\right)^{**} \left(\Phi^{R_1,R_0,R_2}\right)^*\right)[f] = \lambda \left(\Phi^{R_1,R_0,R_2}\right)^{**}[(\mathbf{g}_{|\Gamma_{R_0}}, 0)], \tag{32}$$

where $\lambda > 0$ is such that $\|(\Phi^{R_1,R_0,R_2})^*[f] - (\mathbf{g}_{|\Gamma_{R_0}}, 0)\|_{L^2(\Gamma_{R_0},\mathbb{R}^3)\times L^2(\mathbb{S}_{R_1})} = \varepsilon$. Here, the operator $\left(\Phi^{R_1,R_0,R_2}\right)^{**}$ stands for the adjoint of the restriction of $\left(\Phi^{R_1,R_0,R_2}\right)^*$ to the domain $W^{1,2}(\mathbb{S}_{R_2})$. In order to avoid computing $\left(\Phi^{R_1,R_0,R_2}\right)^{**}$, we rewrite (32) in variational form:

$$\begin{aligned} &\langle f, \varphi\rangle_{W^{1,2}(\mathbb{S}_{R_2})} + \lambda \left\langle \left(\Phi^{R_1,R_0,R_2}\right)^*[f], \left(\Phi^{R_1,R_0,R_2}\right)^*[\varphi]\right\rangle_{L^2(\Gamma_{R_0},\mathbb{R}^3)\times L^2(\mathbb{S}_{R_1})} \\ &= \lambda \left\langle (\mathbf{g}_{|\Gamma_{R_0}}, 0), \left(\Phi^{R_1,R_0,R_2}\right)^*[\varphi]\right\rangle_{L^2(\Gamma_{R_0},\mathbb{R}^3)\times L^2(\mathbb{S}_{R_1})}, \end{aligned} \tag{33}$$

for all $\varphi \in W^{1,2}(\mathbb{S}_{R_2})$. Remark 5.5 indicates that a discretization of f in terms of finitely many spherical harmonics is generally not advisable. As a remedy, we use a discretization in terms of the Abel-Poisson kernels

$$K_\gamma(t) = \frac{1}{4\pi} \frac{1-\gamma^2}{(1+\gamma^2-2\gamma t)^{\frac{3}{2}}}, \quad t \in [-1,1]. \tag{34}$$

More precisely, we expand f as

$$f(x) = \sum_{m=1}^{M} \alpha_m K_{\gamma,m}(x) = \sum_{m=1}^{M} \alpha_m \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} \gamma^n Y_{n,k}\left(\frac{x}{|x|}\right) Y_{n,k}(x_m), \quad x \in \mathbb{S}_{R_2}, \tag{35}$$

where $K_{\gamma,m}(x) = K_\gamma(\frac{x}{|x|} \cdot x_m)$. The parameter $\gamma \in (0,1)$ is fixed and controls the spatial localization of $K_{\gamma,m}$ (a parameter γ close to one means a strong localization) while $x_m \in \mathbb{S}_1$, $m = 1, \dots M$, denote the spatial centers of the kernels $K_{\gamma,m}$. Furthermore, one can see from (35) that γ relates to the influence of higher spherical harmonic degrees in the discretization of f. Some general properties of the Abel-Poisson kernel K_γ can be found, e.g., in [13, Ch. 5]. Computations based on the representations in

Section 4.3 yield

$$
\begin{aligned}
&(\Phi^{R_1,R_0,R_2})^*[K_{\gamma,m}] \\
(36) \quad &= \sum_{n=0}^{\infty}\sum_{k=1}^{2n+1} Y_{n,k}(x_m)\gamma^n \left(\frac{R_2}{2p+1}\left(\frac{R_0}{R_2}\right)^n \nabla H_{n,k}^{R_0}, \left(\frac{R_1}{R_2}\right)^{n-1} Y_{n,k}\right)^T \\
&= \left(\nabla \sum_{n=0}^{\infty}\sum_{k=1}^{2n+1} \gamma^n \frac{R_2}{2p+1}\left(\frac{R_0}{R_2}\right)^n \left(\frac{|\cdot|}{R_0}\right)^n Y_{n,k}(x_m) Y_{n,k}\left(\frac{\cdot}{|\cdot|}\right), \left(\frac{R_2}{R_1}\right) K_{\frac{\gamma R_1}{R_2},m}\right)^T \\
&= \left(\frac{R_2}{4\pi}\nabla F_{\frac{\gamma|\cdot|}{R_2},m}, \left(\frac{R_2}{R_1}\right) K_{\frac{\gamma R_1}{R_2},m}\right)^T,
\end{aligned}
$$

where $F_{\gamma,m}(x) = F_\gamma(\frac{x}{|x|}\cdot x_m)$, with $F_\gamma(t) = (1+\gamma^2-2\gamma t)^{-\frac{1}{2}}$ for $t\in[-1,1]$. Inserting (35) and (36) into (33), fixing $\mathbf{g} = (\Phi_0^{R_0,R_2})^*[Y_{p,q}]$ and choosing $\varphi = K_{\gamma,n}$ for $n = 1,\ldots,M$, as test functions, we are lead to the following system of linear equations

$$
\mathbf{M}\boldsymbol{\alpha} = \mathbf{d}, \tag{37}
$$

where

$$
\begin{aligned}
\mathbf{M} &= \begin{pmatrix} \frac{1}{\lambda}\left\langle K_{\gamma,m}, K_{\gamma,n}\right\rangle_{W^{1,2}(\mathbb{S}_{R_2})} + \left(\frac{R_2}{4\pi}\right)^2 \left\langle \nabla F_{\frac{\gamma|\cdot|}{R_2},m}, \nabla F_{\frac{\gamma|\cdot|}{R_2},n}\right\rangle_{L^2(\Gamma_{R_0},\mathbb{R}^3)} \\ + \left(\frac{R_2}{R_1}\right)^2 \left\langle K_{\frac{\gamma R_1}{R_2},m}, K_{\frac{\gamma R_1}{R_2},n}\right\rangle_{L^2(\mathbb{S}_{R_1})} \end{pmatrix}_{n,m=1,\ldots,M}, \\
\boldsymbol{\alpha} &= \left(\alpha_m\right)_{m=1,\ldots,M}, \\
\mathbf{d} &= \left(\frac{R_2^2}{4\pi(2p+1)}\left(\frac{R_0}{R_2}\right)^p \left\langle \nabla H_{p,q}^{R_0}, \nabla F_n\right\rangle_{L^2(\Gamma_{R_0},\mathbb{R}^3)}\right)_{n=1,\ldots,M}.
\end{aligned}
$$

A function f of the form (35), determined by coefficients α_m, $m = 1,\ldots,M$, which solves (37) will from now on be denoted as $f_{p,q}$. We use $f_{p,q}$ as an approximation of the solution to (33) for the choice $\mathbf{g} = (\Phi_0^{R_0,R_2})^*[Y_{p,q}]$.

A Numerical Example. In order to generate input data $\Phi = \Phi^{R_1,R_0,R_2}[\mathbf{m},h]$ for a test example, we choose

$$
\begin{aligned}
&\mathbf{m}(x) = b_1\frac{x}{|x|}L_{\gamma_1}\left(\frac{x}{|x|}\cdot y_1\right) + b_2\frac{x}{|x|}L_{\gamma_2}\left(\frac{x}{|x|}\cdot y_2\right), \\
&\qquad b_1 = 15,\ b_2 = 10, \\
(38) \quad &h(x) = \sum_{n=0}^{5}\sum_{k=1}^{2n+1} a_{n,k}Y_{n,k}\left(\frac{x}{|x|}\right), \\
&\qquad a_{0,1} = a_{1,1} = 2^5,\ a_{2,5} = a_{3,5} = a_{4,5} = 2^4,\ a_{5,5} = 2^3, \quad a_{n,k} = 0 \text{ else,}
\end{aligned}
$$

with $y_1 = (0,0,-1)^T$ and $y_2 = (0,\frac{1}{2},-\frac{\sqrt{3}}{2})^T$. The corresponding crustal and core contributions are then given by $\Phi_0 = \Phi_0^{R_0,R_2}[\mathbf{m}]$ and $\Phi_1 = \Phi_1^{R_1,R_2}[h]$. The functions L_{γ_i} are chosen as follows:

$$
L_{\gamma_i}(t) = \begin{cases} 0, & t\in[-1,\gamma_i), \\ \frac{(t-\gamma_i)^k}{(1-\gamma_i)^k}, & t\in[\gamma_i,1], \end{cases} \tag{39}
$$

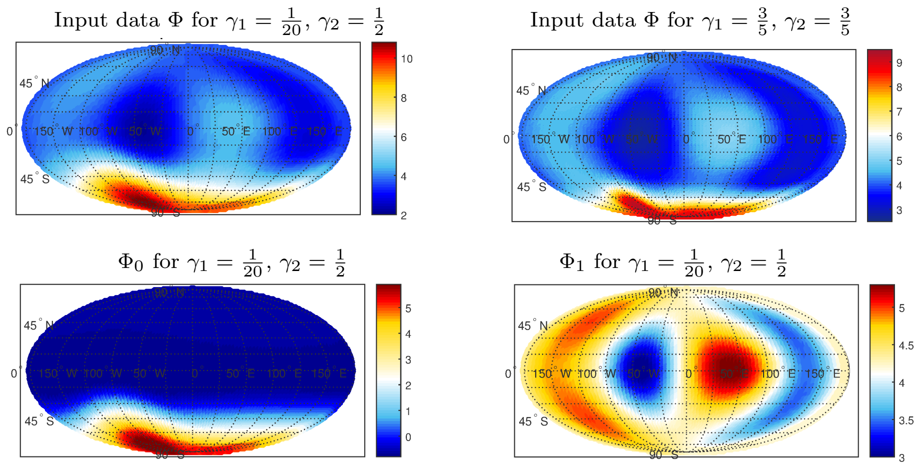

FIG. 2. *Spatial plot of the input data $\Phi = \Phi_0 + \Phi_1$ with parameters $\gamma_1 = \frac{1}{20}$, $\gamma_2 = \frac{1}{2}$ (top left) and $\gamma_1 = \frac{3}{5}$, $\gamma_2 = \frac{3}{5}$ (top right) for the magnetization $\mathbf{m}$ from (38), as well as the underlying crustal contribution Φ_0 and core contribution Φ_1, exemplarily for $\gamma_1 = \frac{1}{20}$, $\gamma_2 = \frac{1}{2}$ (bottom).*

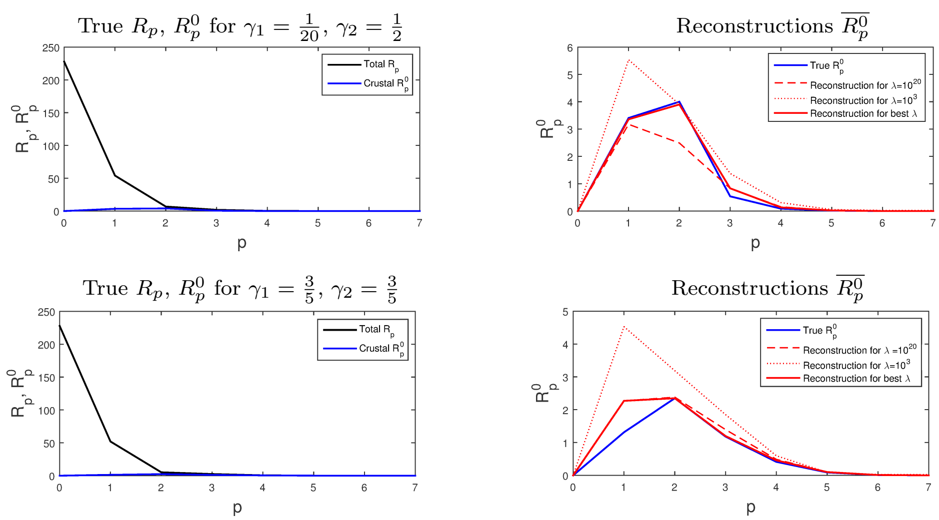

FIG. 3. *Left: Power spectrum R_p of the input data Φ and power spectrum R_p^0 of the crustal contribution Φ_0. Right: True crustal power spectrum R_p^0 (blue) and reconstructed power spectrum $\overline{R_p^0}$ (red) for different parameters λ. The top row shows the results for the parameters $\gamma_1 = \frac{1}{20}, \gamma_2 = \frac{1}{2}$ and the bottom row for $\gamma_1 = \frac{3}{5}, \gamma_2 = \frac{3}{5}$.*

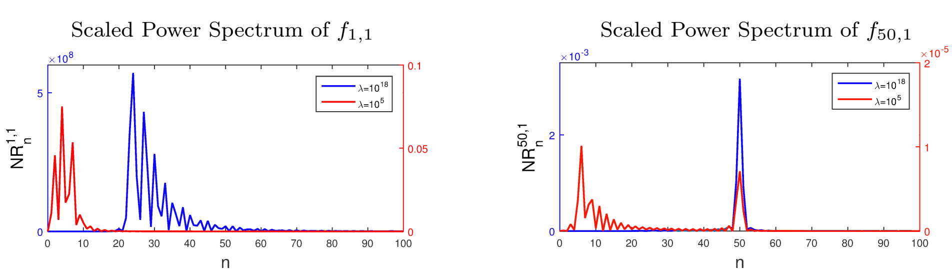

FIG. 4. *Scaled power spectrum $NR_n^{p,q}$ for $p = 1$, $q = 1$ (left) and $p = 50$, $q = 1$ (right).*

for $k = 3$. These functions have been studied in more detail in [36] and are suited for our purposes since they are compactly supported and allow a recursive computation of the Fourier coefficients of $\mathbf{m}$. The parameters $\gamma_i \in (-1, 1)$ reflect the localization of L_{γ_i} (a parameter γ_i close to one means a strong localization). In our test examples, we investigate the two setups $\gamma_1 = \frac{1}{20}, \gamma_2 = \frac{1}{2}$ and $\gamma_1 = \frac{3}{5}, \gamma_2 = \frac{3}{5}$, where latter reflects a slightly stronger localization of the underlying magnetization. For the involved radii, we choose $R_0 = 1$ and $R_2 = 1.06$ (at scales of the Earth, the latter indicates a realistic satellite altitude of about 380km above the Earth's surface) and $R_1 = 0.5$ (at scales of the Earth, this is a rough approximation of the radius of the outer core). The subregion $\Gamma_{R_0} = \{x \in \mathbb{S}_{R_0} : x \cdot (0, 0, 1)^T \leq 0\}$ is set to be the Southern hemisphere and the chosen magnetizations of the form (38) satisfy $\mathrm{supp}(\mathbf{m}) \subset \Gamma_{R_0}$. For our computations, we use the localization parameter $\gamma = 0.95$ and choose $M = 8,499$ uniformly distributed centers $x_m \in \mathbb{S}_1$, $m = 1, \ldots, M$, for the kernels $K_{\gamma,m}$. All numerical integrations necessary during the procedure are performed via the methods of [10] (when the integration region comprises the entire sphere $\mathbb{S}_{R_0}$, $\mathbb{S}_{R_1}$, or $\mathbb{S}_{R_2}$, respectively) and [21] (when the integration is only performed over the spherical cap $\mathbb{S}_{R_0} \setminus \Gamma_{R_0}$). The input data for the two different setups associated with γ_1, γ_2 are shown in the top row of Figure 2. Furthermore, the underlying potentials Φ_0 and Φ_1 are exemplarily indicated for the case $\gamma_1 = \frac{1}{20}, \gamma_2 = \frac{1}{2}$ in the bottom row. These setups are not based on real geomagnetic data but they reflect a typical geomagnetic situation in the sense that the core contribution clearly dominates the crustal contribution at low spherical harmonic degrees. The power spectra of Φ_0 in Figure 3 show that an empirical separation of Φ_0 and Φ_1 by a sharp cut-off at degree $p = 2$ or $p = 3$ would neglect relevant information in the crustal contribution Φ_0.

Corollary 4.9 states that a reasonable approximation of the Fourier coefficient $\langle \Phi_0, Y_{p,q} \rangle_{L^2(\mathbb{S}_{R_2})}$ of the crustal contribution Φ_0 is now given by $\langle \Phi, f_{p,q} \rangle_{L^2(\mathbb{S}_{R_2})}$, with $f_{p,q}$ of the form described in the previous subsection. We do this for various degrees p and orders q and we illustrate the results in terms of power spectra (which allow an easy comparison to the empirical method of a sharp cutoff in spectral domain): The crustal power spectrum is defined as

$$R_p^0 = R_p[\Phi_0] = \sum_{q=1}^{2p+1} \left| \langle \Phi_0, Y_{p,q} \rangle_{L^2(\mathbb{S}_{R_2})} \right|^2, \quad p \in \mathbb{N}_0.$$

Our approximated power spectrum is then of the form

$$\overline{R_p^0} = \sum_{q=1}^{2p+1} \left| \langle \Phi, f_{p,q} \rangle_{L^2(\mathbb{S}_{R_2})} \right|^2, \quad p \in \mathbb{N}_0.$$

The power spectrum of the input signal Φ (i.e., the superposition of the crustal and core contribution) is analogously defined by $R_p = R_p[\Phi] = \sum_{q=1}^{2p+1} |\langle \Phi, Y_{p,q} \rangle_{L^2(\mathbb{S}_{R_2})}|^2$.

Figure 3 shows the true and the reconstructed power spectra and we see that they yield good results (for a well-chosen parameter λ) in both setups under investigation. Stronger deviations mainly occur at lower spherical harmonic degrees p. The solid red spectrum in Figure 3 indicated as 'Reconstruction for best λ' does not reflect the result for a single global choice of λ but rather for λ that might vary depending on each degree p of the spectrum. The setup for magnetizations $\mathbf{m}$ with parameters $\gamma_1 = \frac{3}{5}, \gamma_2 = \frac{3}{5}$ was chosen to investigate magnetizations with a slightly stronger localization, meaning that the corresponding potential Φ_0 has slightly stronger contributions at higher spherical harmonic degrees than for the setup $\gamma_1 = \frac{1}{20}, \gamma_2 = \frac{1}{2}$ (indicated in the right hand images in Figure 3).

In Figure 4, we illustrate the effects mentioned in Remark 5.5 by observing the scaled power spectrum $NR_n^{p,q} = \frac{1}{2n+1} R_n[f_{p,q}] = \frac{1}{2n+1} \sum_{k=1}^{2n+1} |\langle f_{p,q}, Y_{n,k}\rangle_{L^2(\mathbb{S}_{R_2})}|^2$ of $f_{p,q}$ for $p = 1$, $q = 1$, and $p = 50$, $q = 1$ (we scaled by a factor $\frac{1}{2n+1}$ solely to get a better idea of the average influence of a single $|\langle f_{p,q}, Y_{n,k}\rangle_{L^2(\mathbb{S}_{R_2})}|^2$, $k = 1, \ldots, 2n+1$, rather than the total power $\sum_{k=1}^{2n+1} |\langle f_{p,q}, Y_{n,k}\rangle_{L^2(\mathbb{S}_{R_2})}|^2$ for fixed degree n). As expected from Remark 5.5, larger Lagrange parameters λ (which correspond to smaller ε) result in a general shift of the major contributions of the power spectrum of $f_{p,q}$ towards higher spherical harmonic degrees for both choices of p, q. On the other hand, a slightly different behaviour between $p = 1$, $q = 1$ and $p = 50$, $q = 1$ can be observed: the spike around $n = 50$ for the $p = 50$, $q = 1$ remains while such a thing does not happen for the smaller degree $p = 1$, $q = 1$.

6. Conclusion. In this paper, we set up a geophysically reasonable model of the core and crustal magnetic field potentials Φ_1 and Φ_0 respectively, for which we showed that each single potential can be recovered uniquely if only the superposition $\Phi = \Phi_0 + \Phi_1$ is known on an external sphere $\mathbb{S}_{R_2}$. Furthermore, we supplied first approaches to the reconstruction of Φ_0 and of its Fourier coefficients. The latter is particularly interesting as it would allow a comparison with the empirical approach to separation based on a sharp cut-off in the power spectrum of Φ. Two main directions call for further study: (1) the geophysical post-processing of real geomagnetic data in order to back up (or deny) the assumption that $\mathbf{m}$ is supported in a subregion Γ_{R_0} of the Earth's surface; (2) improving numerical schemes allowing reconstruction of Φ_0 or its Fourier coefficients when the core contribution Φ_1 is clearly dominating (as is expected at lower spherical harmonic degrees in realistic geomagnetic field models) and when $\mathbb{S}_{R_1}$ is close to $\mathbb{S}_{R_0}$. The domination of the core contribution has been simulated to some extent in the presented examples but is expected to be stronger in real scenarios.

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