Multi-Phase Averaging of Time-Optimal Low-Thrust Transfers

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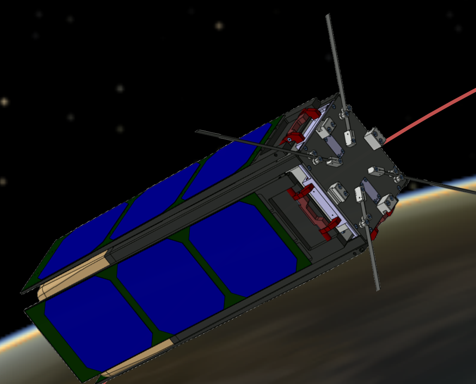
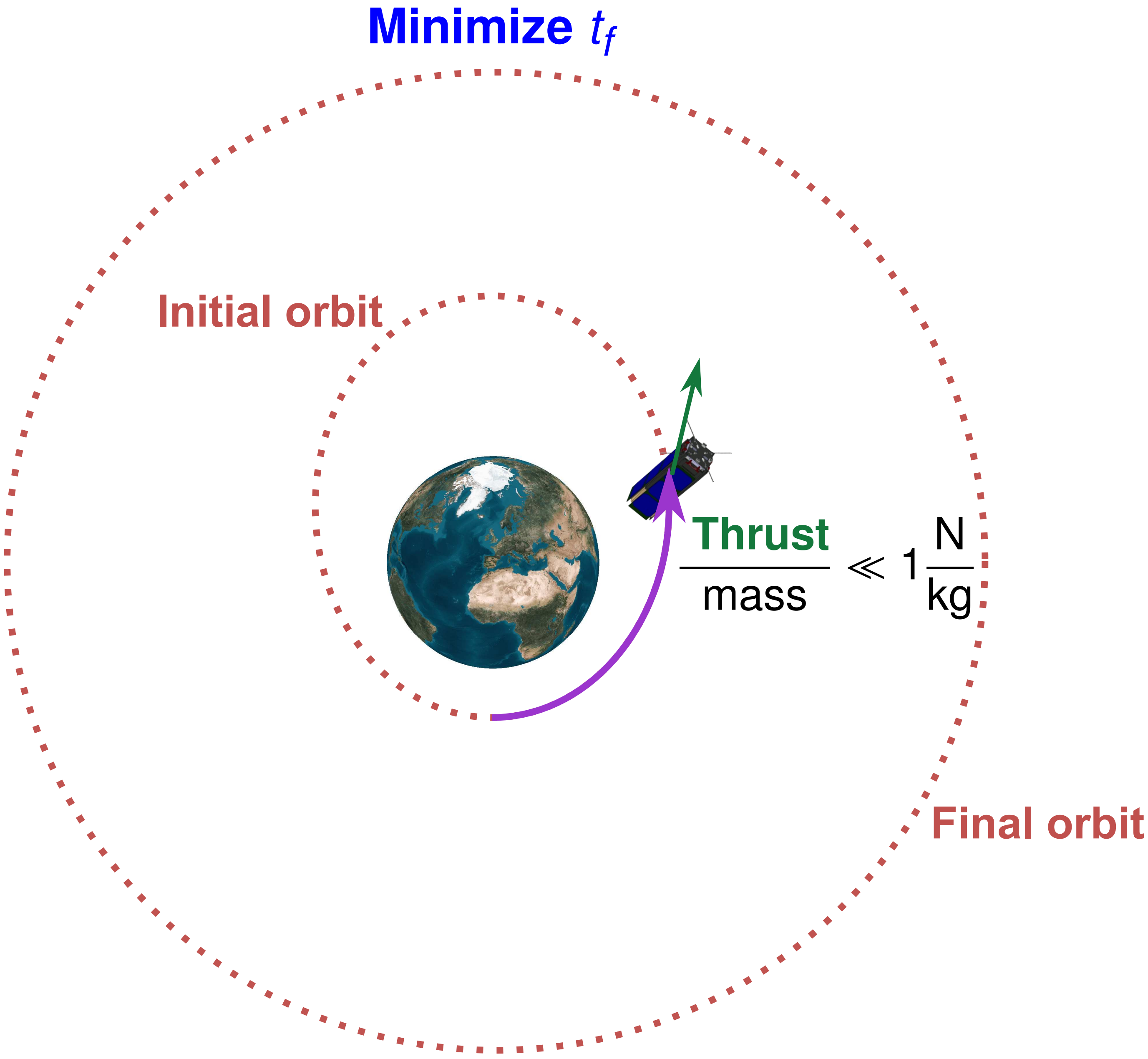
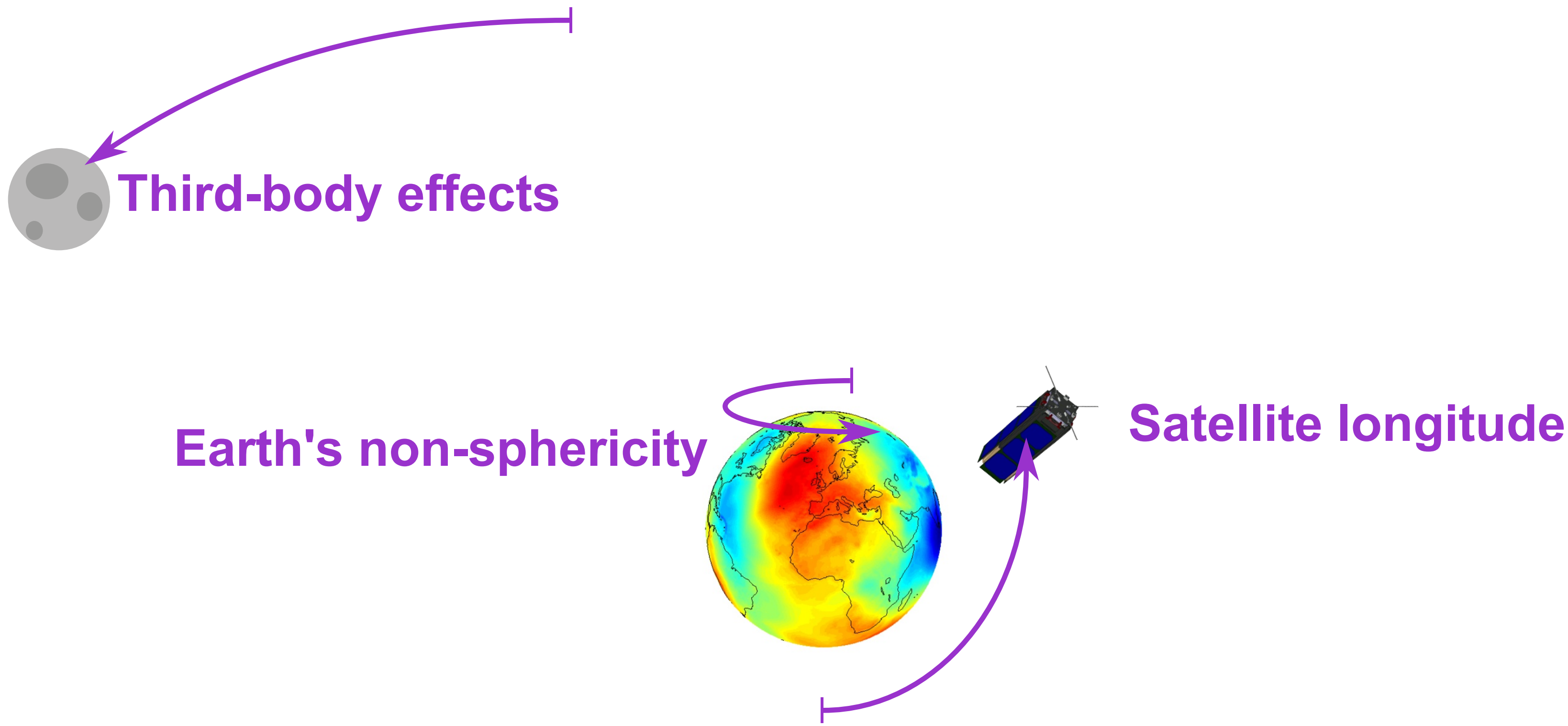


Logrono, 25/04/2019

Low-thrust transfer: a fast-oscillating control problem



Orbital perturbations may introduce new frequencies

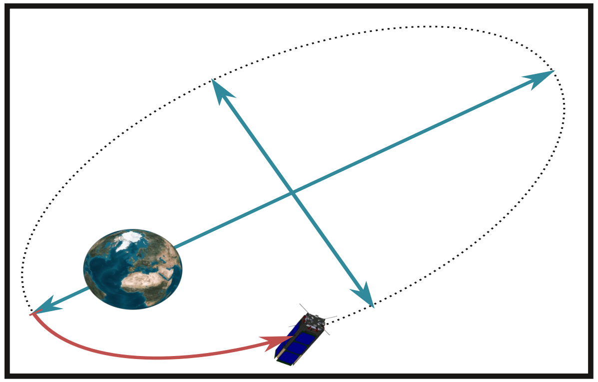


Objective: Simplify dynamics by averaging

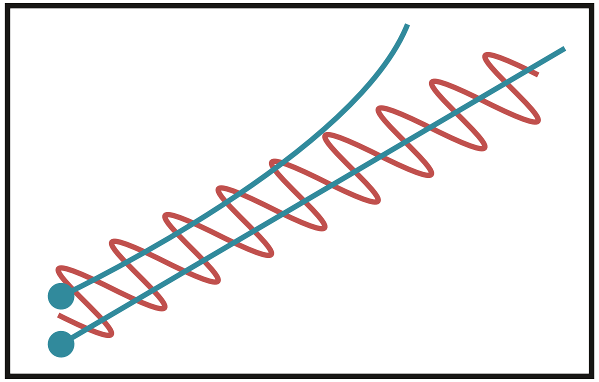
Motivation: Initial guess to shooting algorithms

Challenges: Do adjoint variables introduce additional fast dynamics?
What happens when resonances are crossed?

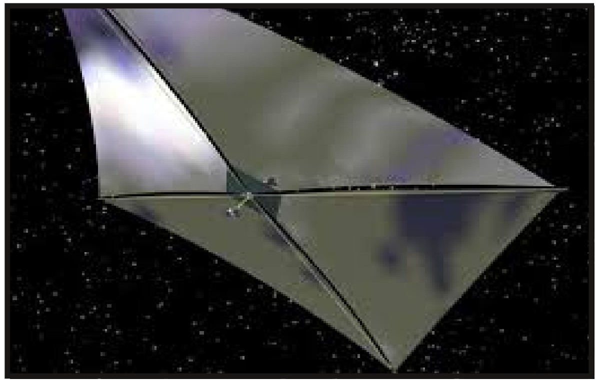
Outline



1. Minimum time control of fast oscillating systems

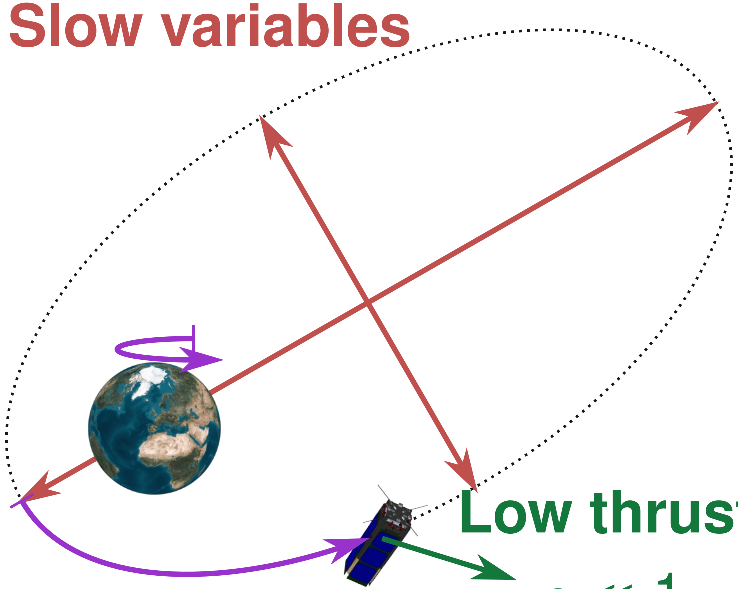


2. Averaging the optimal control Hamiltonian



3. Time optimal deorbiting of a solar sail

1. Minimum time control of fast oscillating systems



$$\min_{\|\boldsymbol{u}\| \leq 1} t_f \text{ subject to:}$$

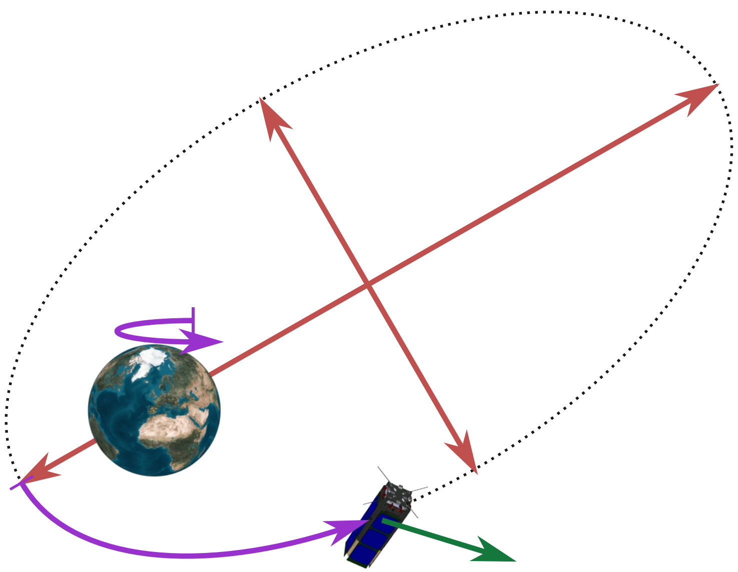
$$\frac{\mathrm{d}\boldsymbol{I}}{\mathrm{d}t} = \epsilon \left[\boldsymbol{f}_0(\boldsymbol{I}, \phi) + \sum_{i=1}^{m} \boldsymbol{f}_i(\boldsymbol{I}, \phi) u_i \right]$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \omega(\boldsymbol{I})$$

$$\boldsymbol{I}(0) = \boldsymbol{I}_0$$

$$\boldsymbol{I}(t_f) = \boldsymbol{I}_f$$

1. Hamiltonian of the extremal flow

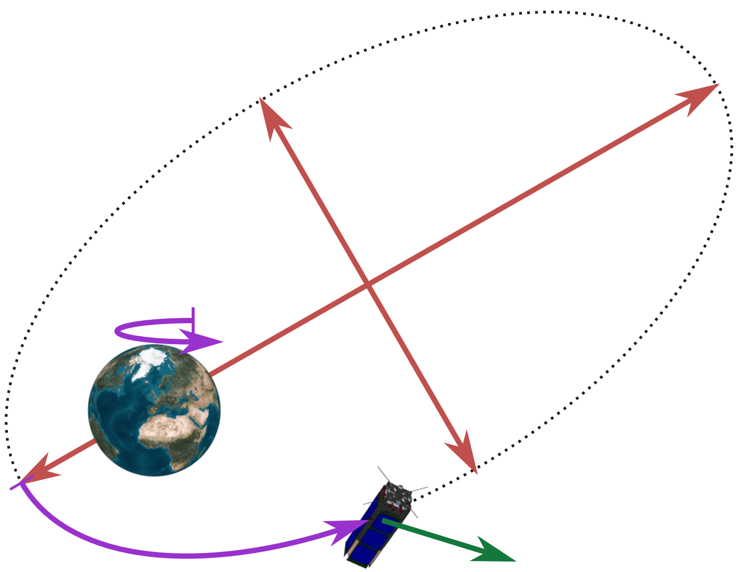


Denote by $\boldsymbol{p_I}$ and $\boldsymbol{p_\phi}$ the adjoints to $\boldsymbol{I}$ and ϕ

Define the pre-Hamiltonian

$$\mathcal{H}' = \omega(\boldsymbol{I}) \cdot \boldsymbol{p_\phi} + \epsilon \left[\boldsymbol{f}_0(\boldsymbol{I}, \phi) + \sum_{i=1}^{m} \boldsymbol{f}_i(\boldsymbol{I}, \phi) u_i \right] \cdot \boldsymbol{p_I}$$

1. Hamiltonian of the extremal flow



Denote by $\boldsymbol{p_I}$ and $\boldsymbol{p_\phi}$ the adjoints to I and ϕ

Define the pre-Hamiltonian

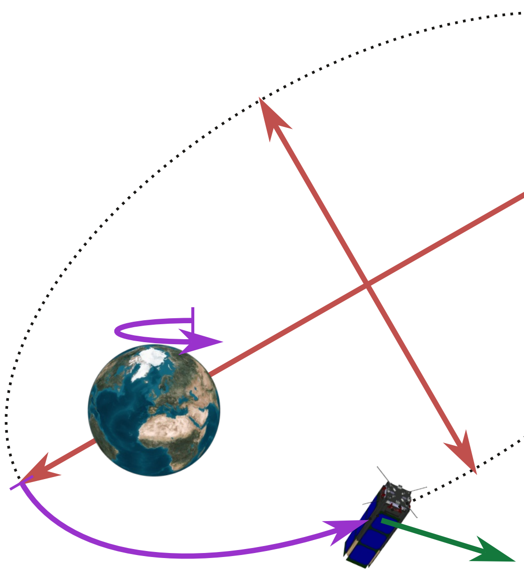
$$\mathcal{H}' = \omega(I)\cdot\boldsymbol{p_\phi} + \epsilon\left[\boldsymbol{f}_0(I,\phi) + \sum_{i=1}^{m}\boldsymbol{f}_i(I,\phi)u_i\right]\cdot\boldsymbol{p_I}$$

Apply Pontryagin maximum principle

$$\mathcal{H} = \max_{\|\boldsymbol{u}\|\leq 1}\mathcal{H}'(I,\phi,\boldsymbol{p_I},\boldsymbol{p_\phi},\boldsymbol{u})$$

$$= \omega(I)\cdot\boldsymbol{p_\phi} + \epsilon\underbrace{\left[\boldsymbol{f}_0(I,\phi)\cdot\boldsymbol{p_I} + \sqrt{\sum_{i=1}^{m}(\boldsymbol{f}_i(I,\phi)\cdot\boldsymbol{p_I})^2}\right]}_{\doteqdot K}$$

1. Necessary conditions for optimality



Boundary conditions

$$\boldsymbol{I}(0) = \boldsymbol{I_0} \qquad \boldsymbol{p_\phi}(0) = \boldsymbol{0}$$

$$\boldsymbol{I}(t_f) = \boldsymbol{I_f} \qquad \boldsymbol{p_\phi}(t_f) = \boldsymbol{0}$$

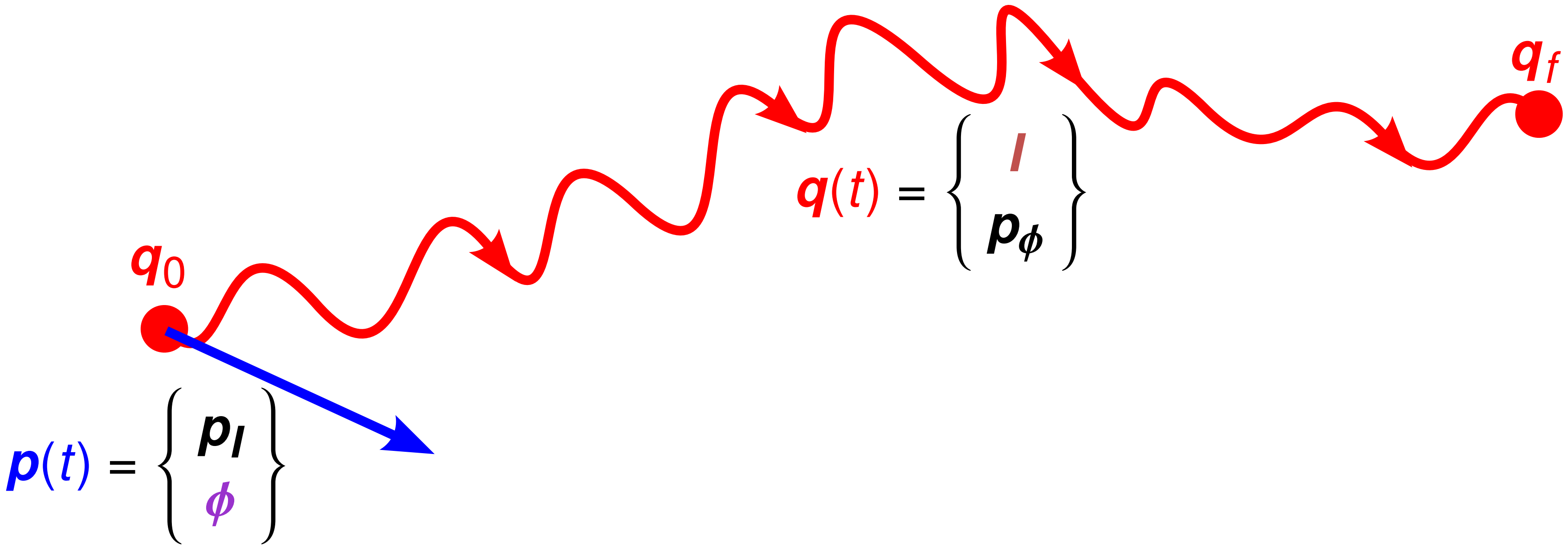
Equations of motion

$$\frac{\mathrm{d}\,\boldsymbol{I}}{\mathrm{d}\,t} = \frac{\partial\,\mathcal{H}}{\partial \boldsymbol{p_I}} \qquad \frac{\mathrm{d}\,\boldsymbol{p_I}}{\mathrm{d}\,t} = -\frac{\partial\,\mathcal{H}}{\partial \boldsymbol{I}}$$

$$\frac{\mathrm{d}\,\boldsymbol{\phi}}{\mathrm{d}\,t} = \frac{\partial\,\mathcal{H}}{\partial \boldsymbol{p_\phi}} \qquad \frac{\mathrm{d}\,\boldsymbol{p_\phi}}{\mathrm{d}\,t} = -\frac{\partial\,\mathcal{H}}{\partial \boldsymbol{\phi}}$$

1. Solution of the problem via shooting

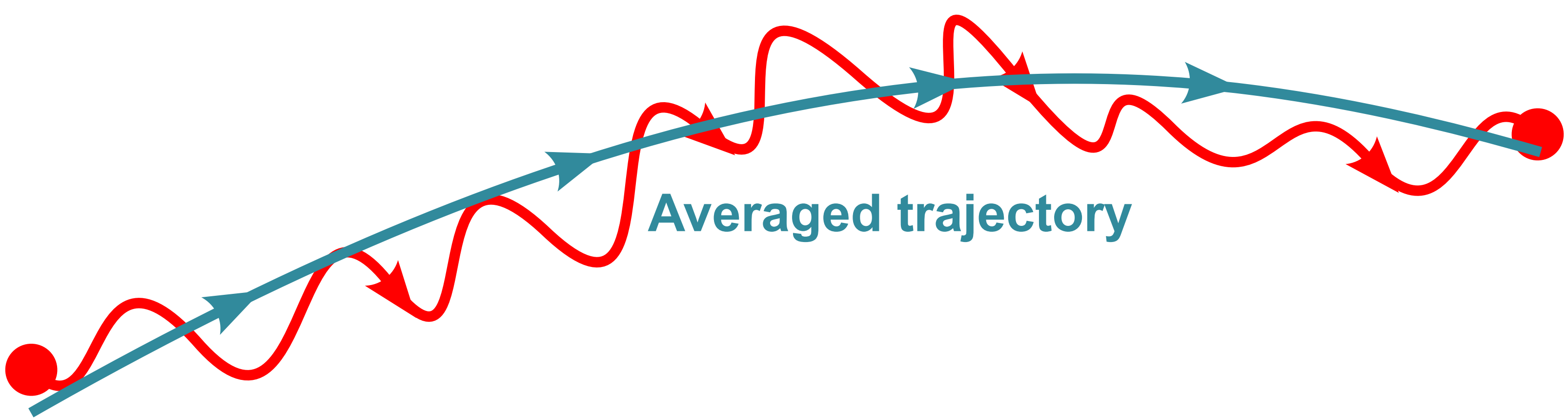
Find t_f and $\boldsymbol{p}(0)$ such that $\boldsymbol{q}(t_f) = \boldsymbol{q}_f$



1. How averaging can facilitate the solution via shooting?

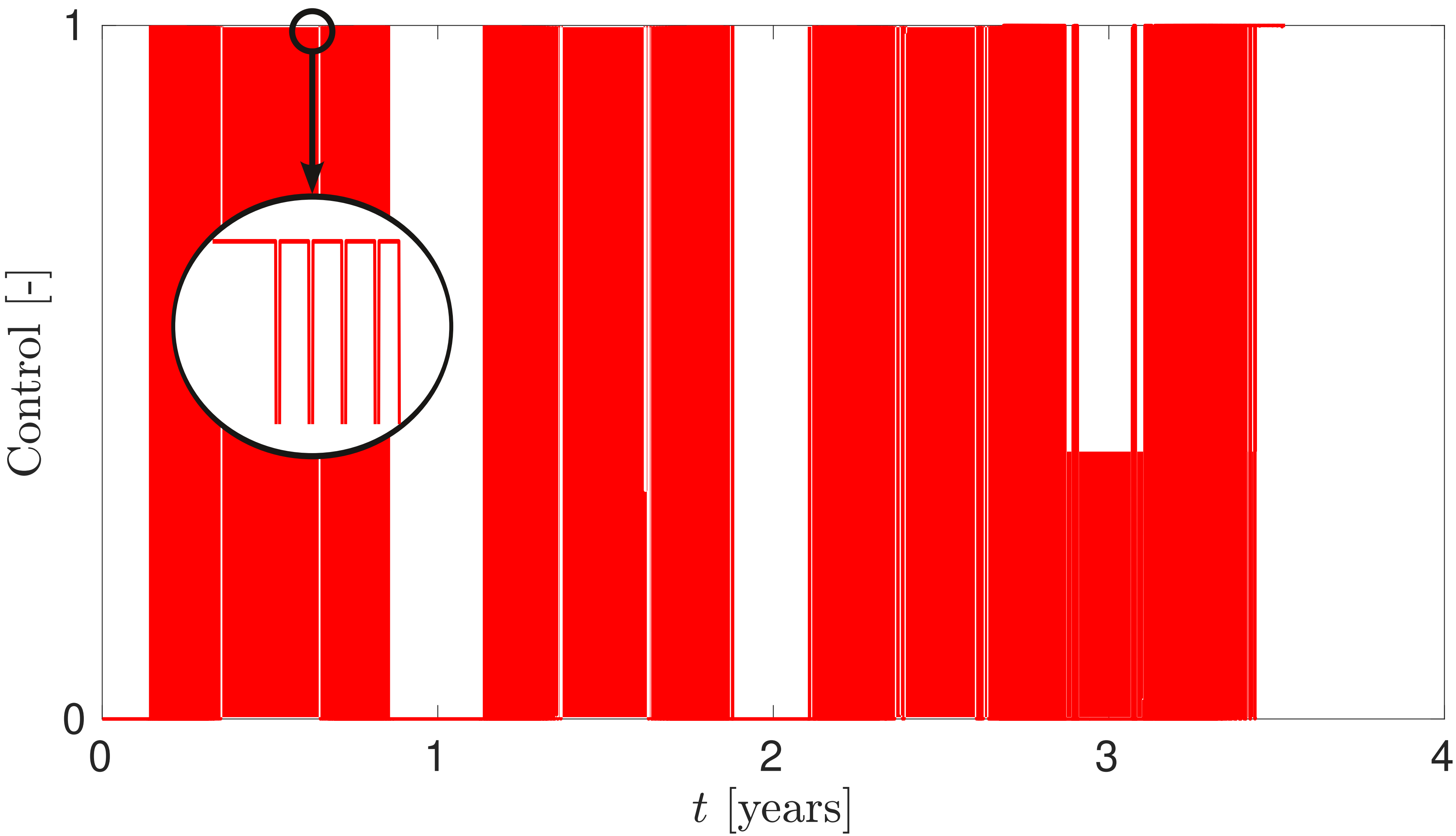
Smoothing: Less local minima, facilitates convergence

Reduced system: Independent of $\boldsymbol{\phi}$, $\boldsymbol{p_\phi}$ is constant

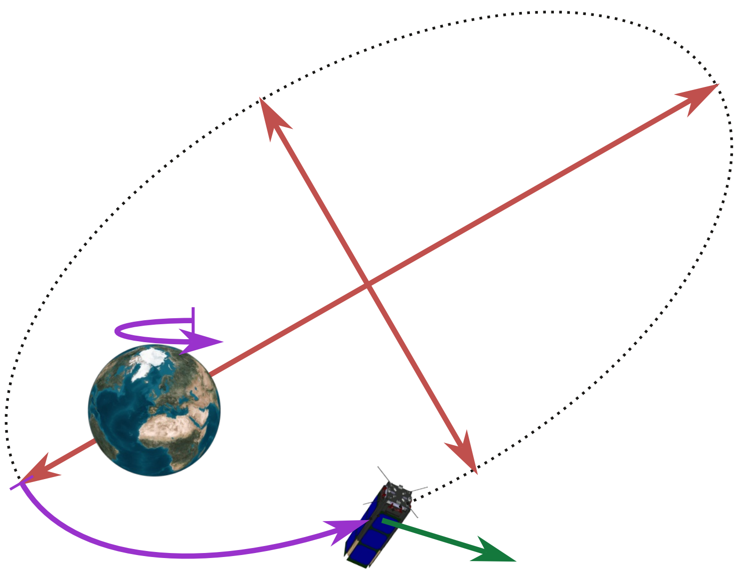


1. How averaging can facilitate the solution via shooting?

***A priori* knowledge of the control structure is not needed!**



2. Can we use averaging? Are adjoints **slow** or **fast**?



Hamiltonian:

$$\mathcal{H} = \boldsymbol{p}_{\boldsymbol{\phi}} \overset{?}{\cdot} \omega(\boldsymbol{I}) + \epsilon K\left(\boldsymbol{I}, \boldsymbol{\phi}, \boldsymbol{p}_{\boldsymbol{I}}, \boldsymbol{p}_{\boldsymbol{\phi}}\right)$$

Equations of motion of the adjoints:

$$\frac{\mathrm{d}\,\boldsymbol{p}_{\boldsymbol{I}}}{\mathrm{d}\,t} = -\epsilon\frac{\partial K}{\partial \boldsymbol{I}} - \boldsymbol{p}_{\boldsymbol{\phi}}\overset{?}{\frac{\partial \omega}{\partial \boldsymbol{I}}}$$

$$\frac{\mathrm{d}\,\boldsymbol{p}_{\boldsymbol{\phi}}}{\mathrm{d}\,t} = -\epsilon\frac{\partial K}{\partial \boldsymbol{\phi}}$$

2. The averaged control system

Assume that $\boldsymbol{I}$ is in a non-resonant zone (i.e., incommensurate frequencies $\omega(\boldsymbol{I})$)

Averaged Hamiltonian

$$\begin{aligned}\overline{\mathcal{H}} &= \int_{\mathbb{T}^r} \mathcal{H}\left(\boldsymbol{I}, \boldsymbol{\phi}, \boldsymbol{p_I}, \boldsymbol{p_\phi}\right)\, \mathrm{d}\boldsymbol{\phi} \\ &= \int_{\mathbb{T}^r} \left[\omega(\boldsymbol{I}) \cdot \boldsymbol{p_\phi} + \epsilon K\left(\boldsymbol{I}, \boldsymbol{\phi}, \boldsymbol{p_I}, \boldsymbol{p_\phi}\right)\right] \mathrm{d}\boldsymbol{\phi}\end{aligned}$$

?

For trajectories of interest: $\boldsymbol{p_\phi}(t)$ is **ϵ-slow and ϵ-small** (not proven here)

2. A "non-conventional" fast-oscillating problem

Classical fast-oscillating system

$$\frac{\mathrm{d}\,\boldsymbol{x}}{\mathrm{d}\,t} = \epsilon\, \boldsymbol{f}(\boldsymbol{x}, \phi)$$
$$\frac{\mathrm{d}\,\phi}{\mathrm{d}\,t} = \omega(\boldsymbol{x})$$

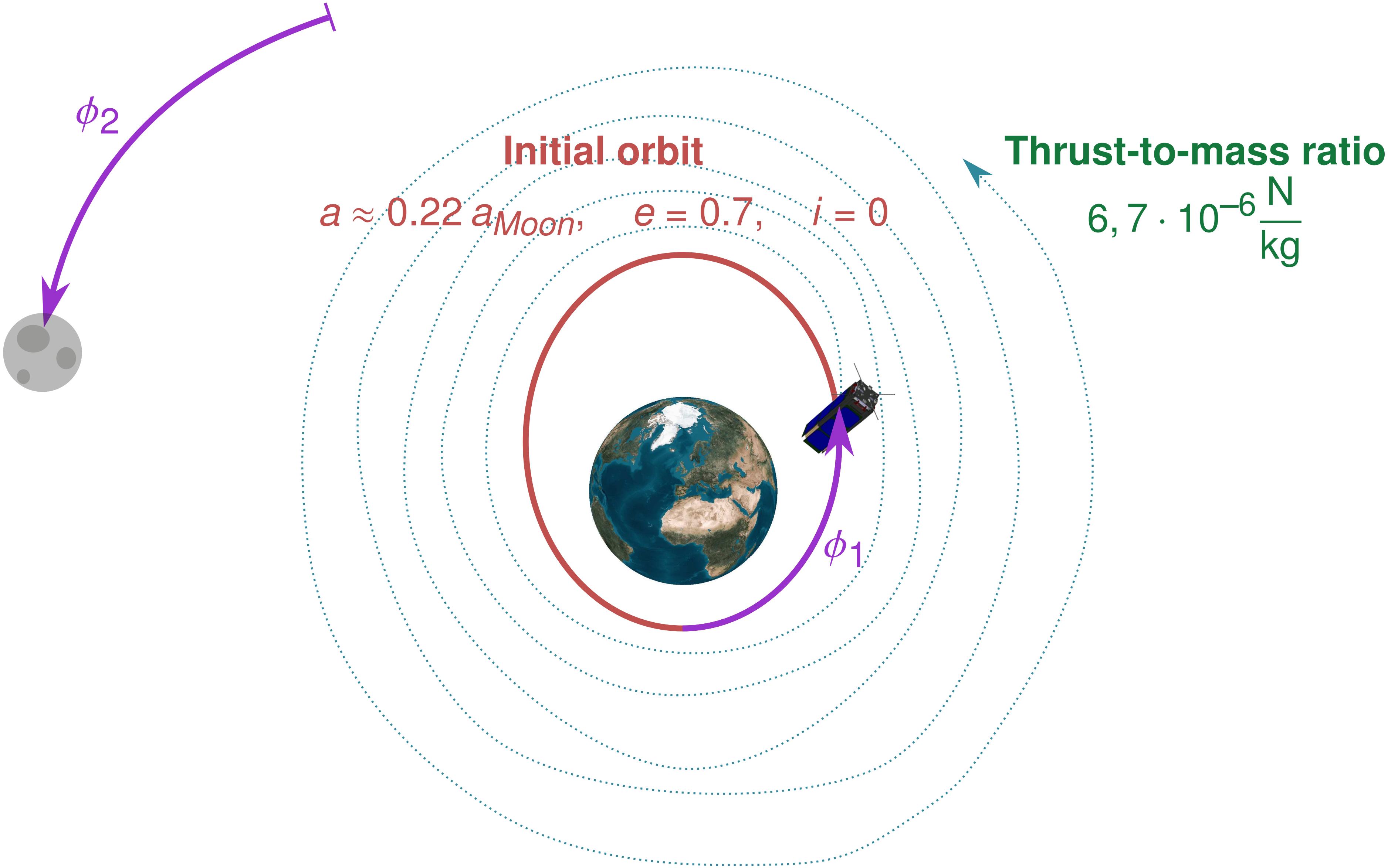
Problem studied in this talk

$$\frac{\mathrm{d}\,\boldsymbol{x}}{\mathrm{d}\,t} = \epsilon\, \boldsymbol{f}(\boldsymbol{x}, \phi, \boldsymbol{\eta}) + \boldsymbol{g}(\boldsymbol{x})\, \boldsymbol{\eta}$$
$$\frac{\mathrm{d}\,\boldsymbol{\eta}}{\mathrm{d}\,t} = \epsilon\, \boldsymbol{h}(\boldsymbol{x}, \phi, \boldsymbol{\eta})$$
$$\frac{\mathrm{d}\,\phi}{\mathrm{d}\,t} = \omega(\boldsymbol{x})$$

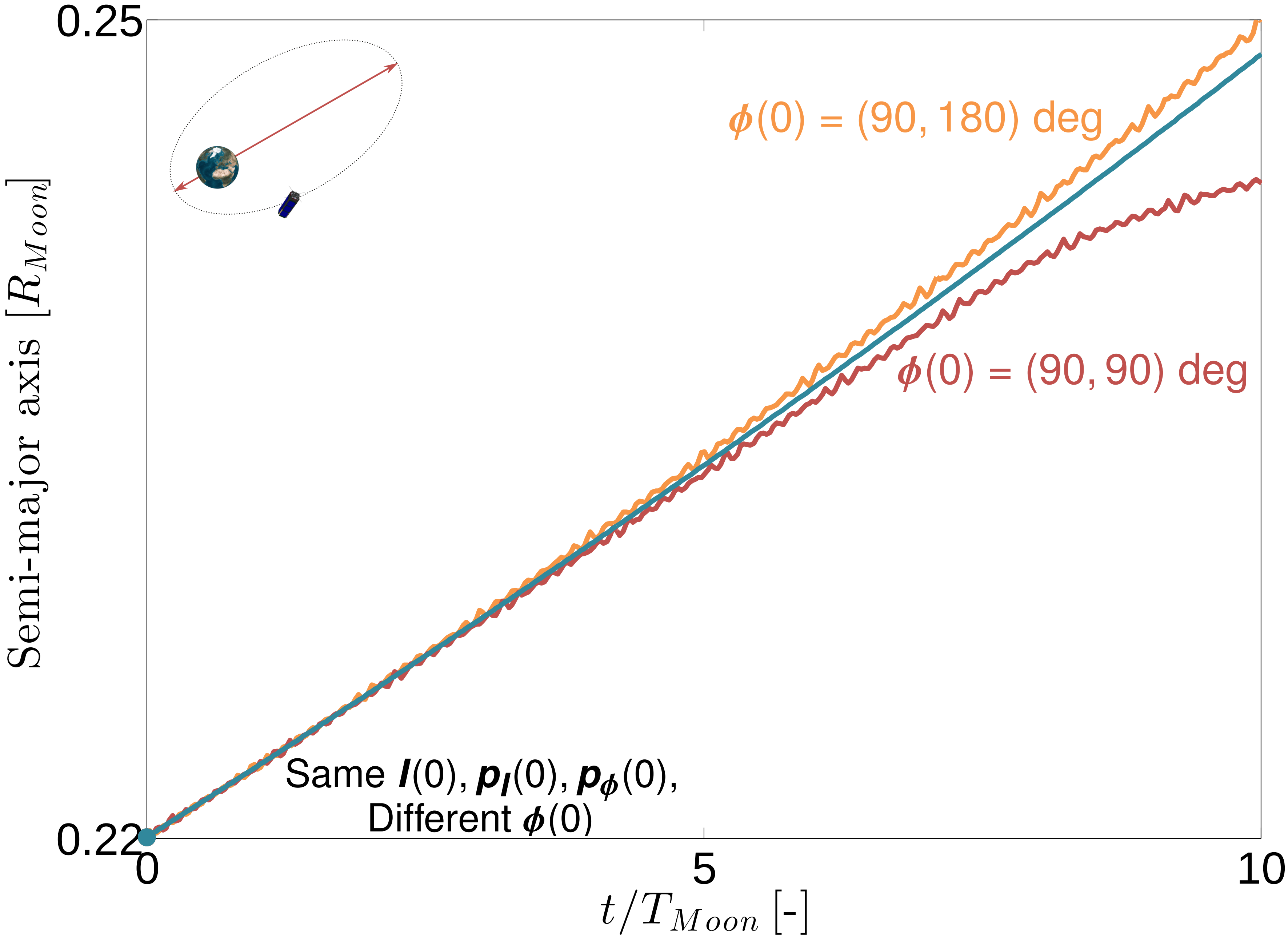
Initial conditions such that

$$\boldsymbol{\eta}(t) = \mathcal{O}(\epsilon) \quad \forall\ t \in [0, t_f]$$

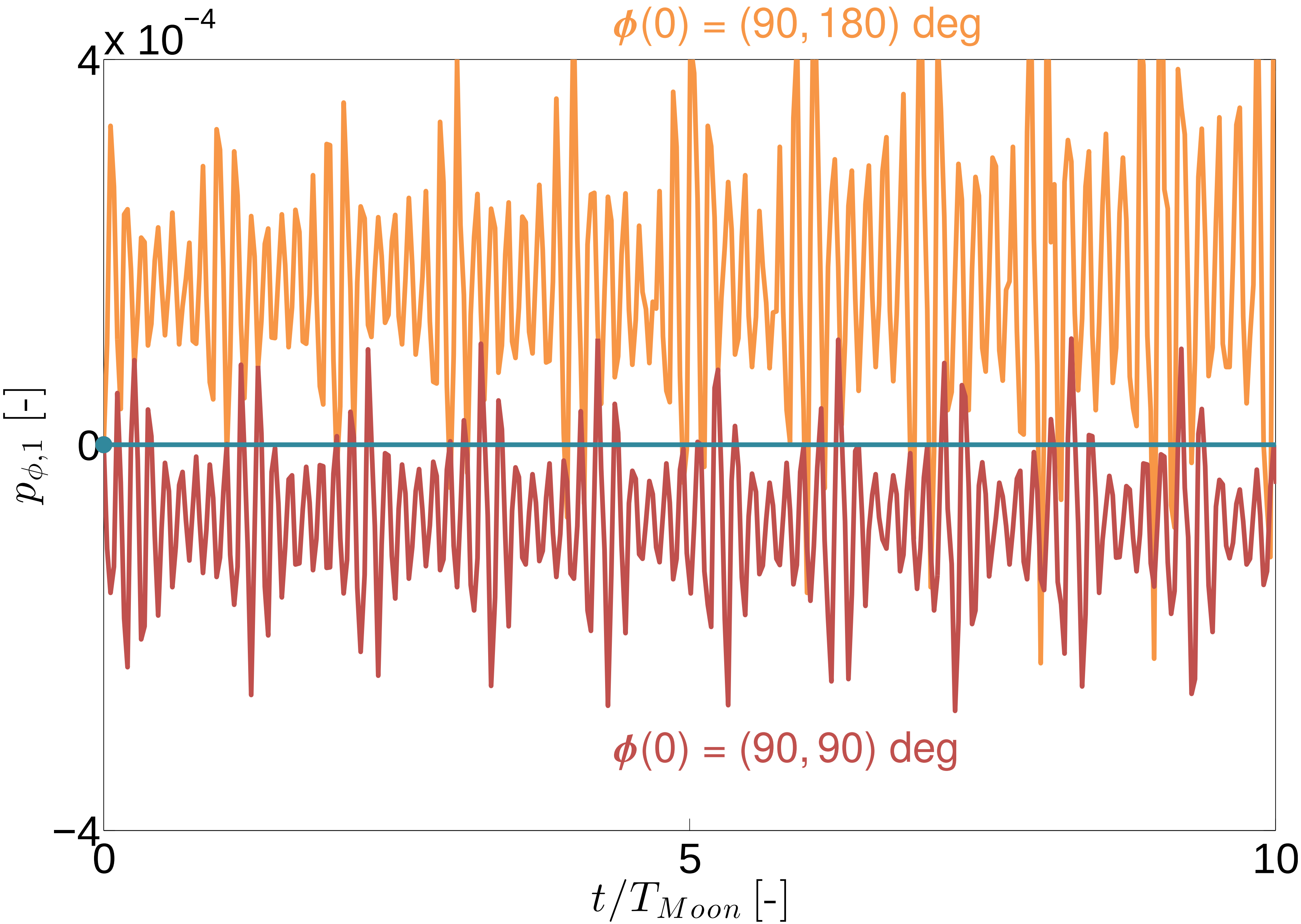
2. Case study: transfer in the Earth-Moon system



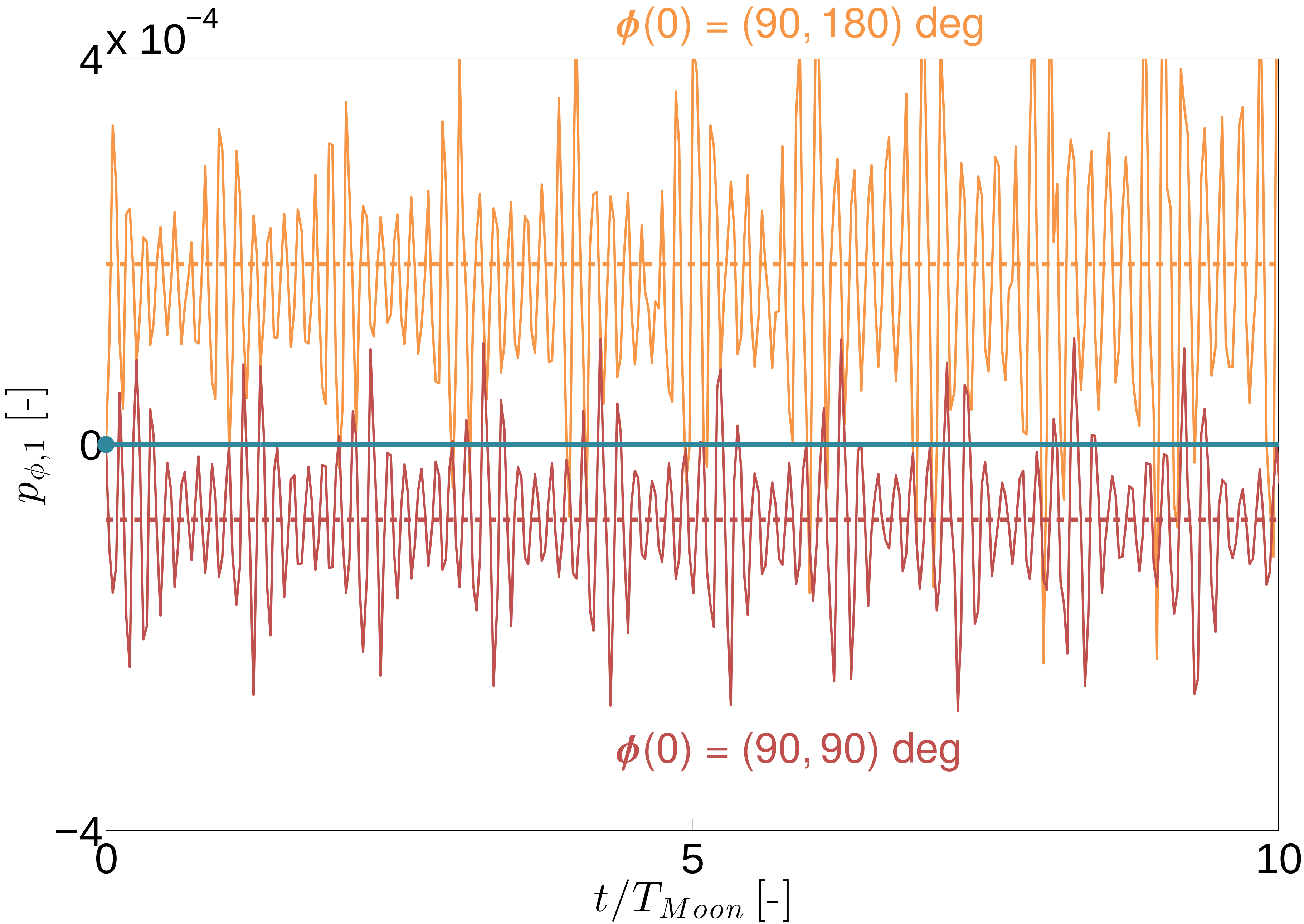
2. How to generate "reliable" averaged trajectories?



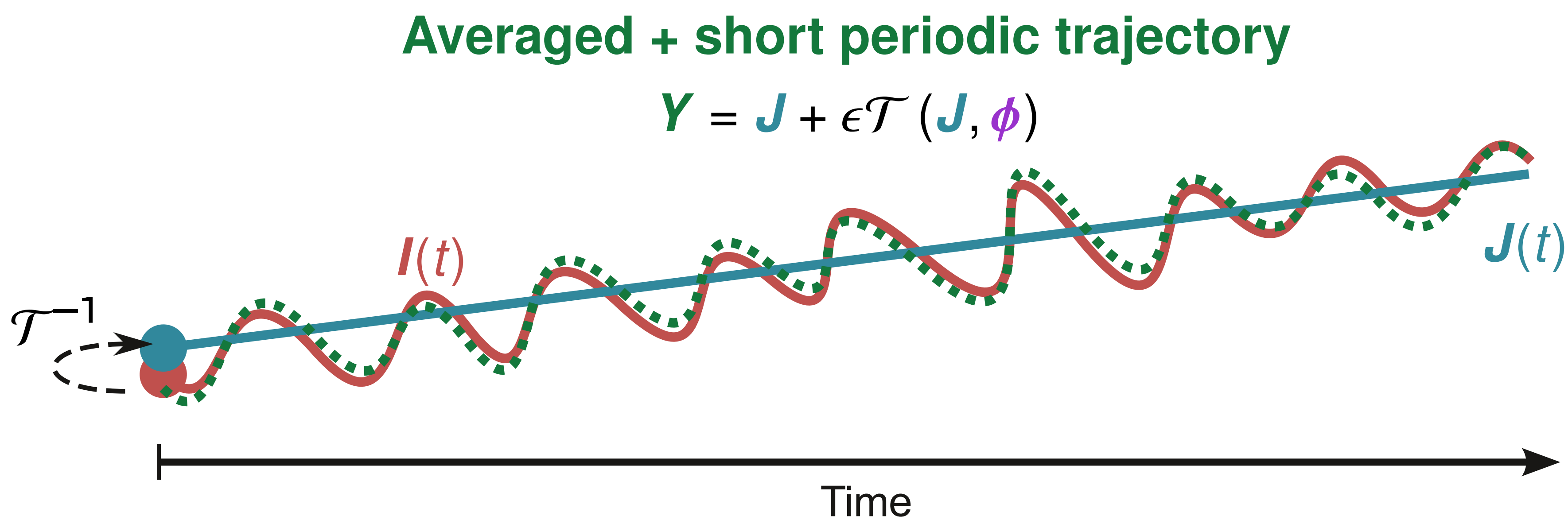
2. What makes the averaged control system different?



2. This is because $\boldsymbol{p}_{\phi}$ is constant and $\frac{\mathrm{d}\,\boldsymbol{p}_{\boldsymbol{I}}}{\mathrm{d}\,t} = -\epsilon \frac{\partial K}{\partial \boldsymbol{I}} - \boldsymbol{p}_{\phi} \frac{\partial \omega}{\partial \boldsymbol{I}}$



2. Short-periodic variations

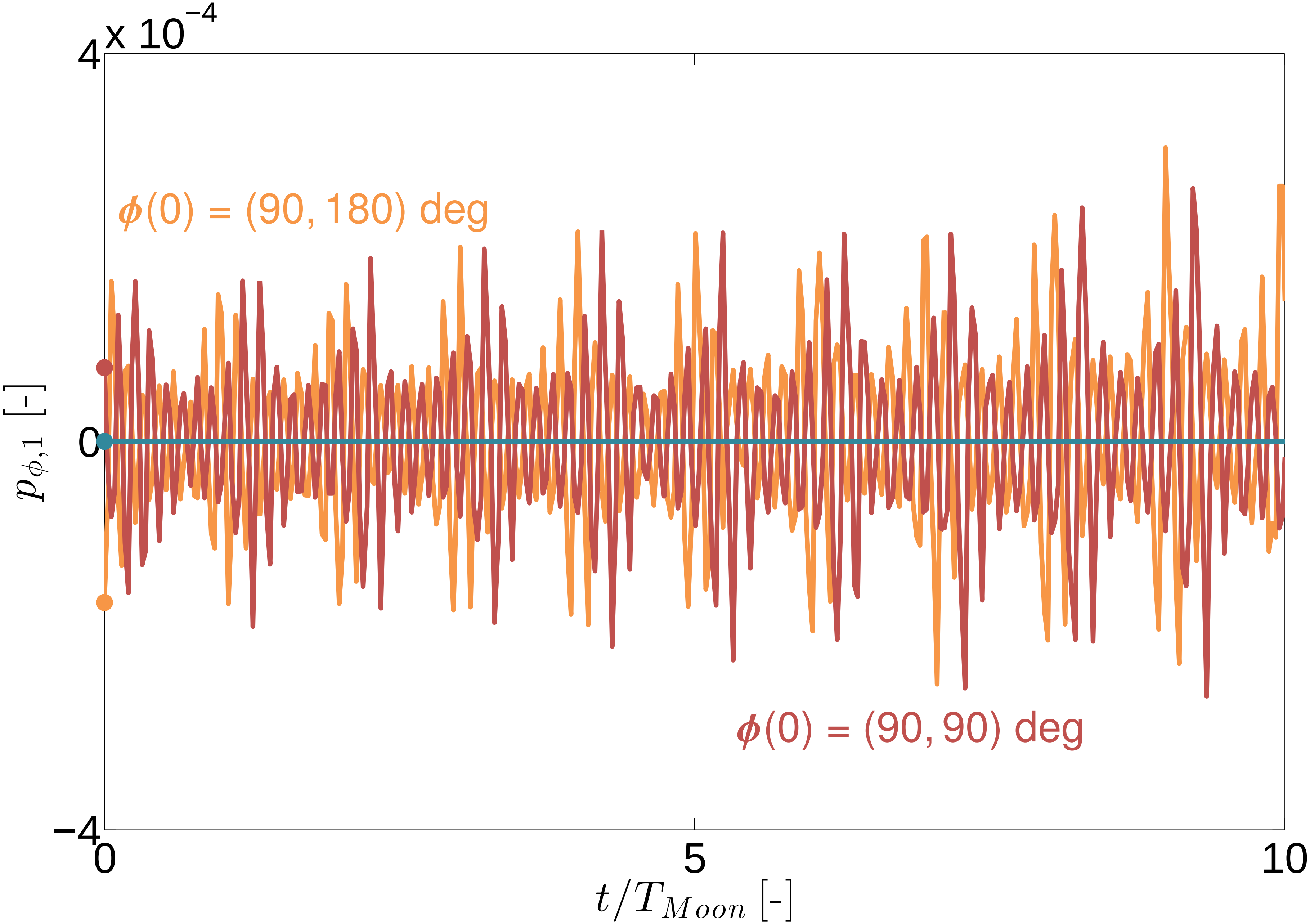


Near identity transformation:

$$\mathcal{T}(\boldsymbol{J}, \boldsymbol{\phi}) = -i \sum_{0<|\boldsymbol{k}|\leq N} \frac{\boldsymbol{c_k}}{\boldsymbol{k} \cdot \omega(\boldsymbol{J})} \exp(i\boldsymbol{k} \cdot \boldsymbol{\phi})$$

Where $\boldsymbol{c_k}$ are Fourier coefficients of $\frac{\mathrm{d}\,\boldsymbol{I}}{\mathrm{d}\,t}$

2. Transforming $\boldsymbol{p_\phi}(0)$ is the key



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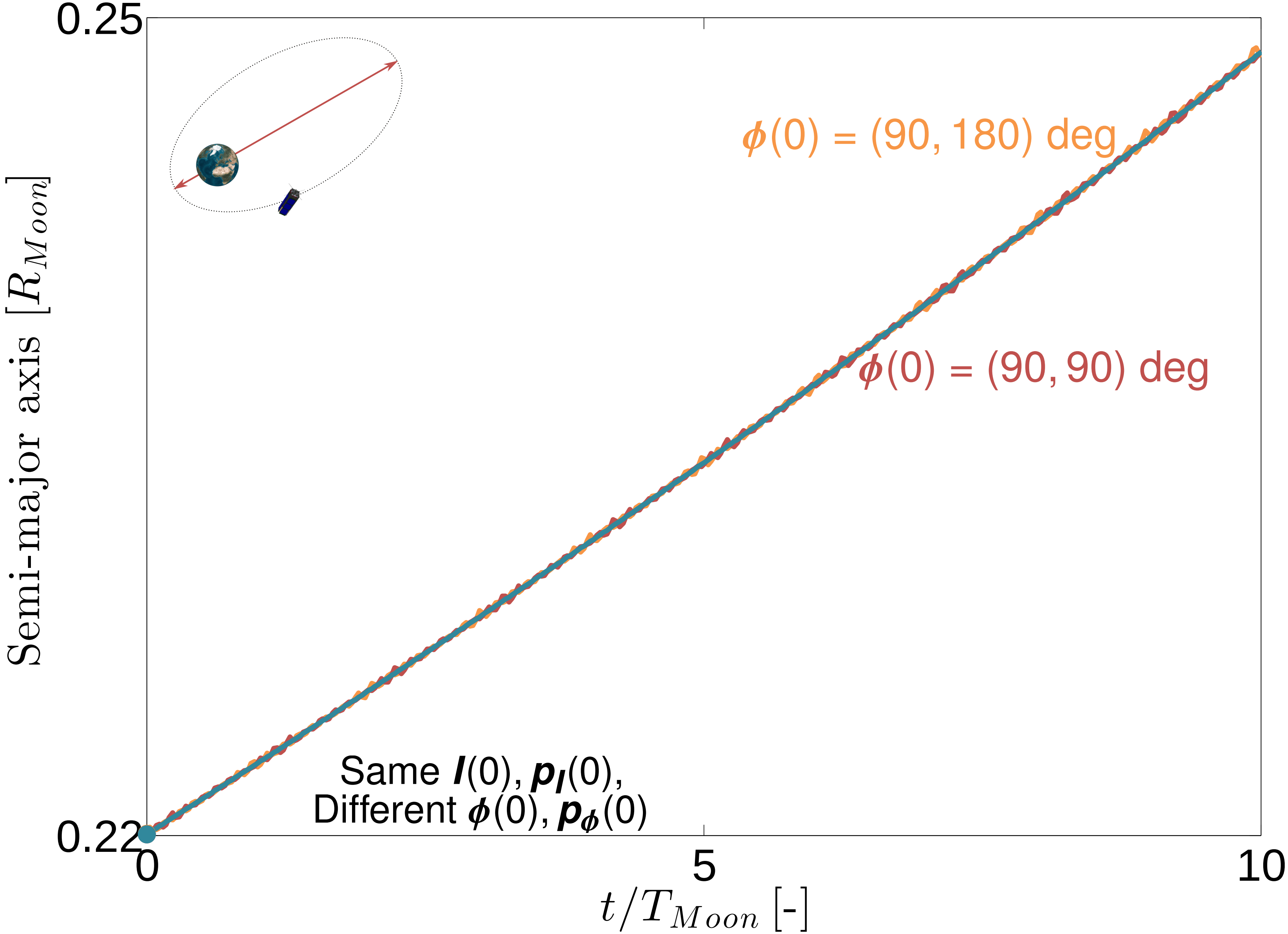
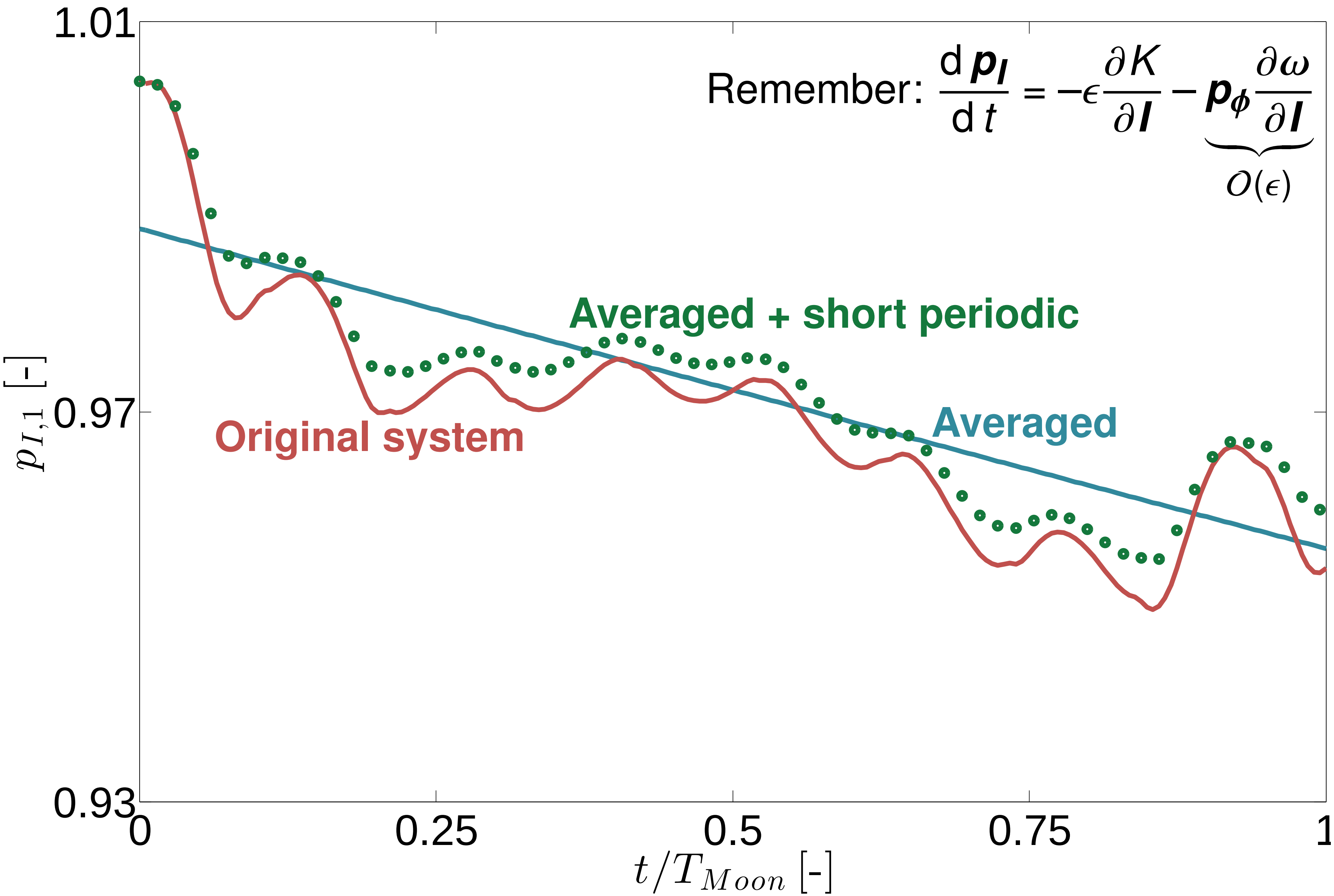


2. The classical transformation is not adequate for $\boldsymbol{p_I}$



2. Nested transform for the short-periodic variations of $\boldsymbol{p_I}$

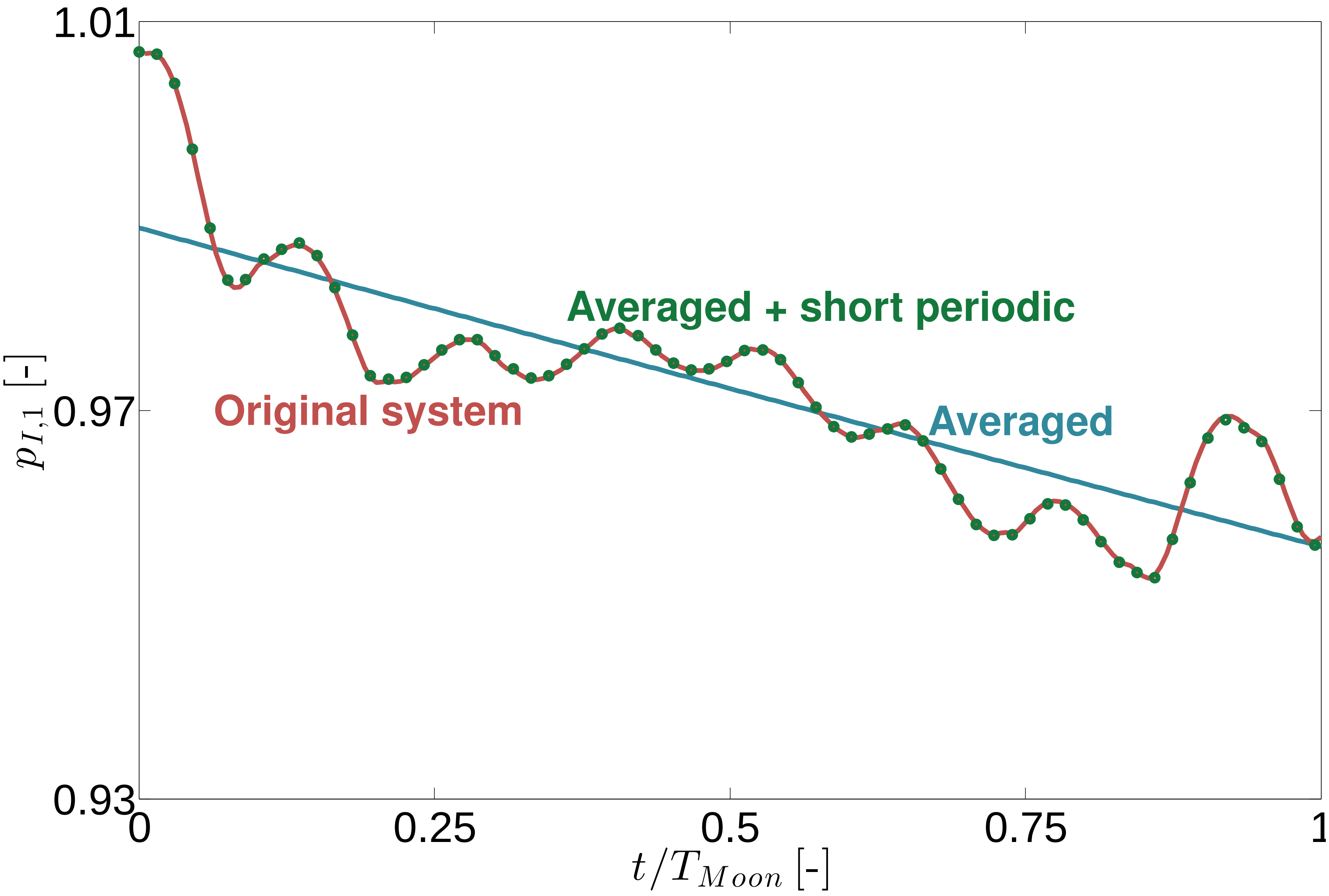
First, build the transformation of $\boldsymbol{p_\phi}$:

$$\hat{\boldsymbol{p}}_{\boldsymbol{\phi}}(\boldsymbol{\phi}) = \boldsymbol{p_\phi} + \mathcal{T}_{\boldsymbol{p_\phi}}(\boldsymbol{J}, \boldsymbol{\phi}, \boldsymbol{p_J}, \boldsymbol{p_\phi})$$

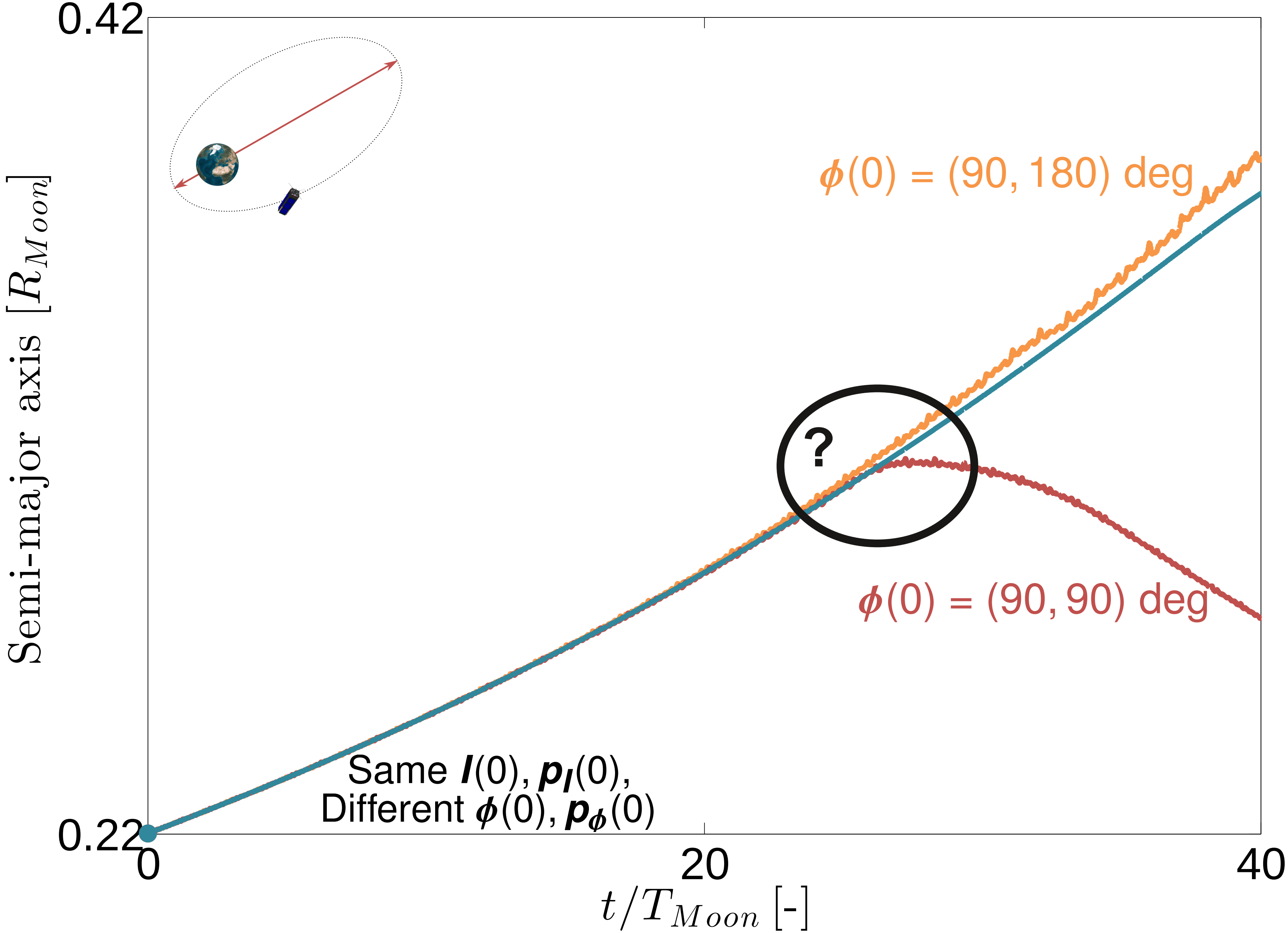
Then, use this information to evaluate the Fourier coefficients of:

$$\frac{\mathrm{d}\,\boldsymbol{p_I}}{\mathrm{d}\,t} = -\epsilon\left[\frac{\partial K}{\partial \boldsymbol{I}} - \hat{\boldsymbol{p}}_{\boldsymbol{\phi}}(\boldsymbol{\phi})\frac{\partial \omega}{\partial \boldsymbol{I}}\right]$$

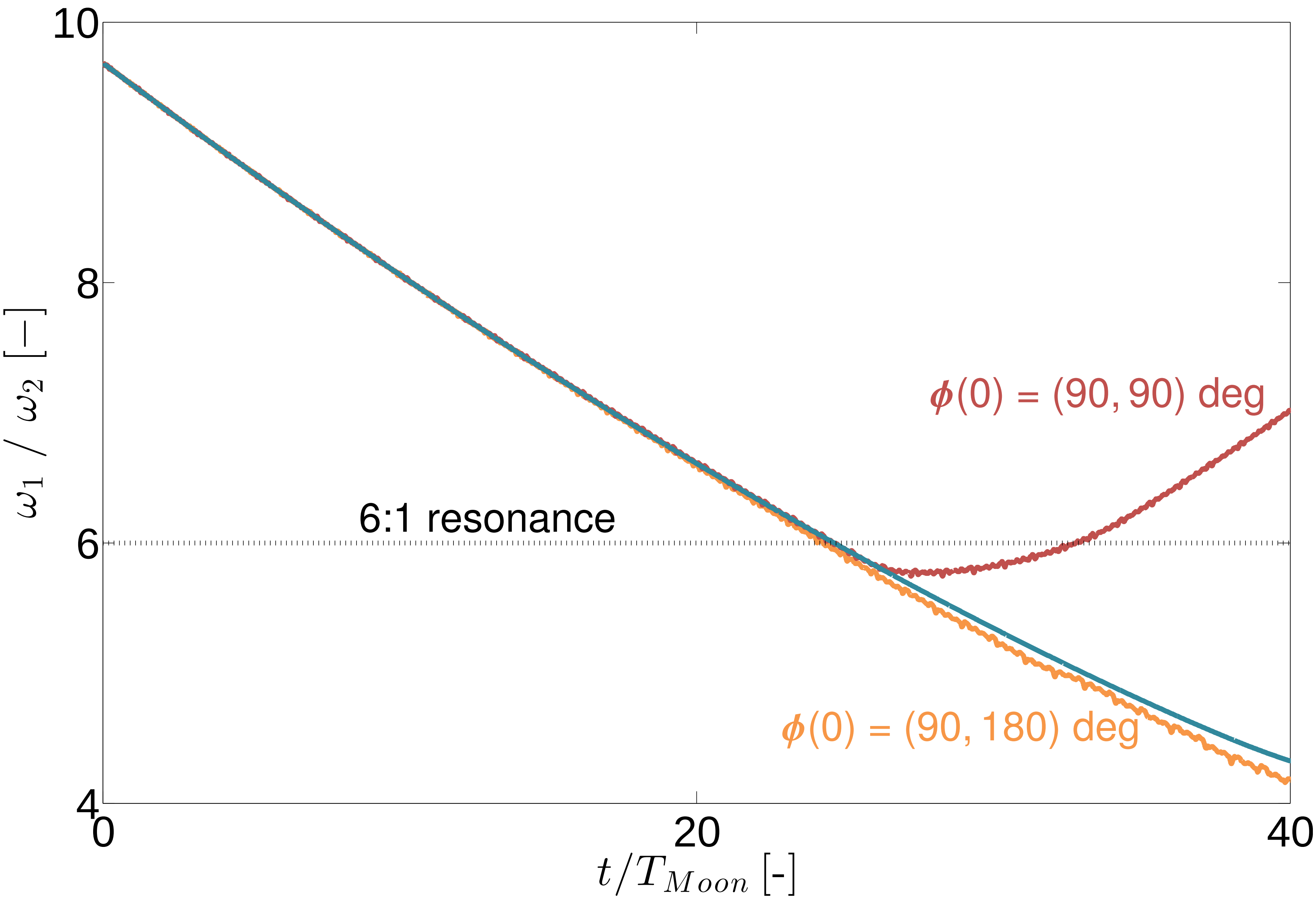
2. Short-periodic variations of p_I are accurately evaluated



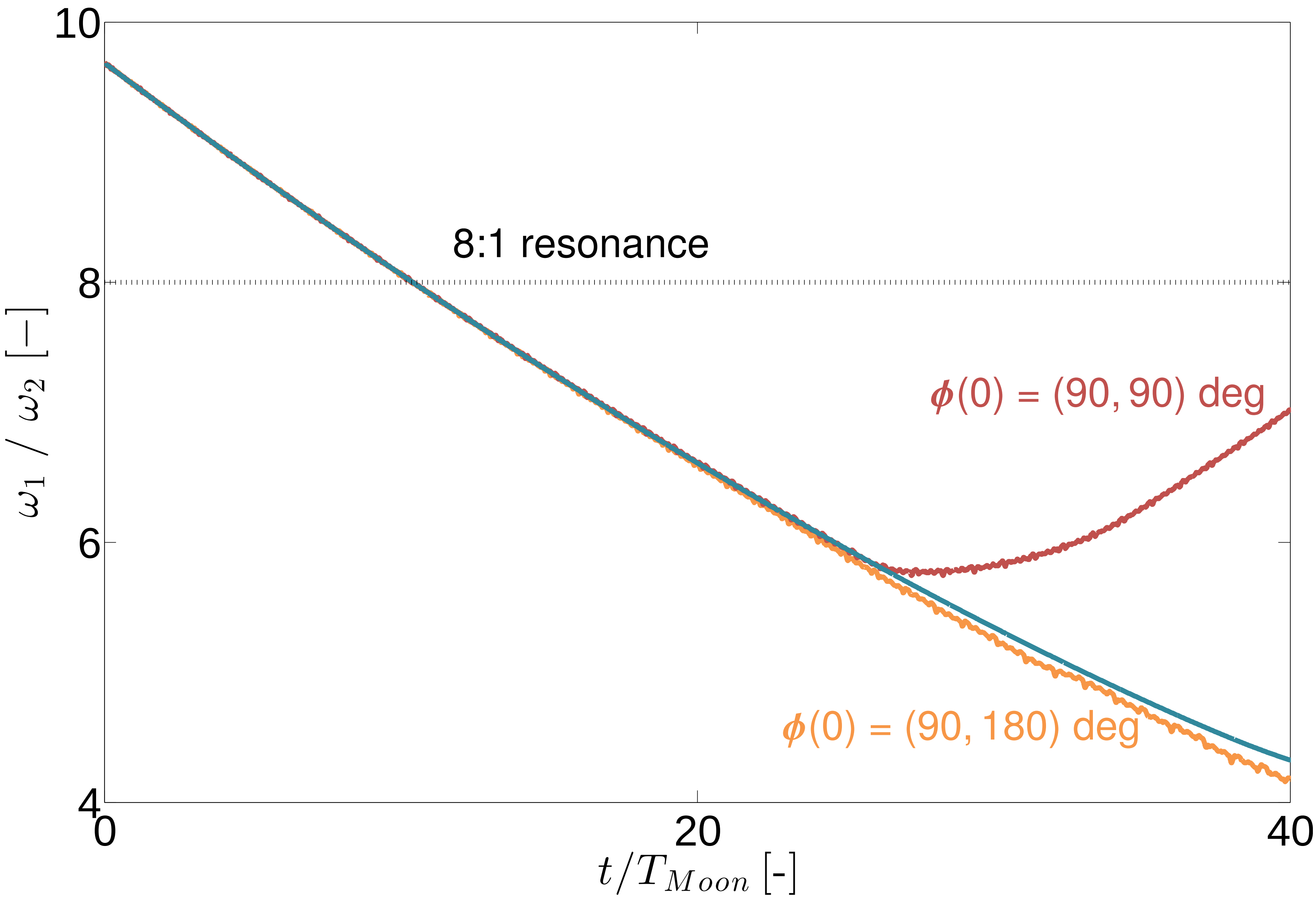
2. Transforming initial conditions is not yet enough



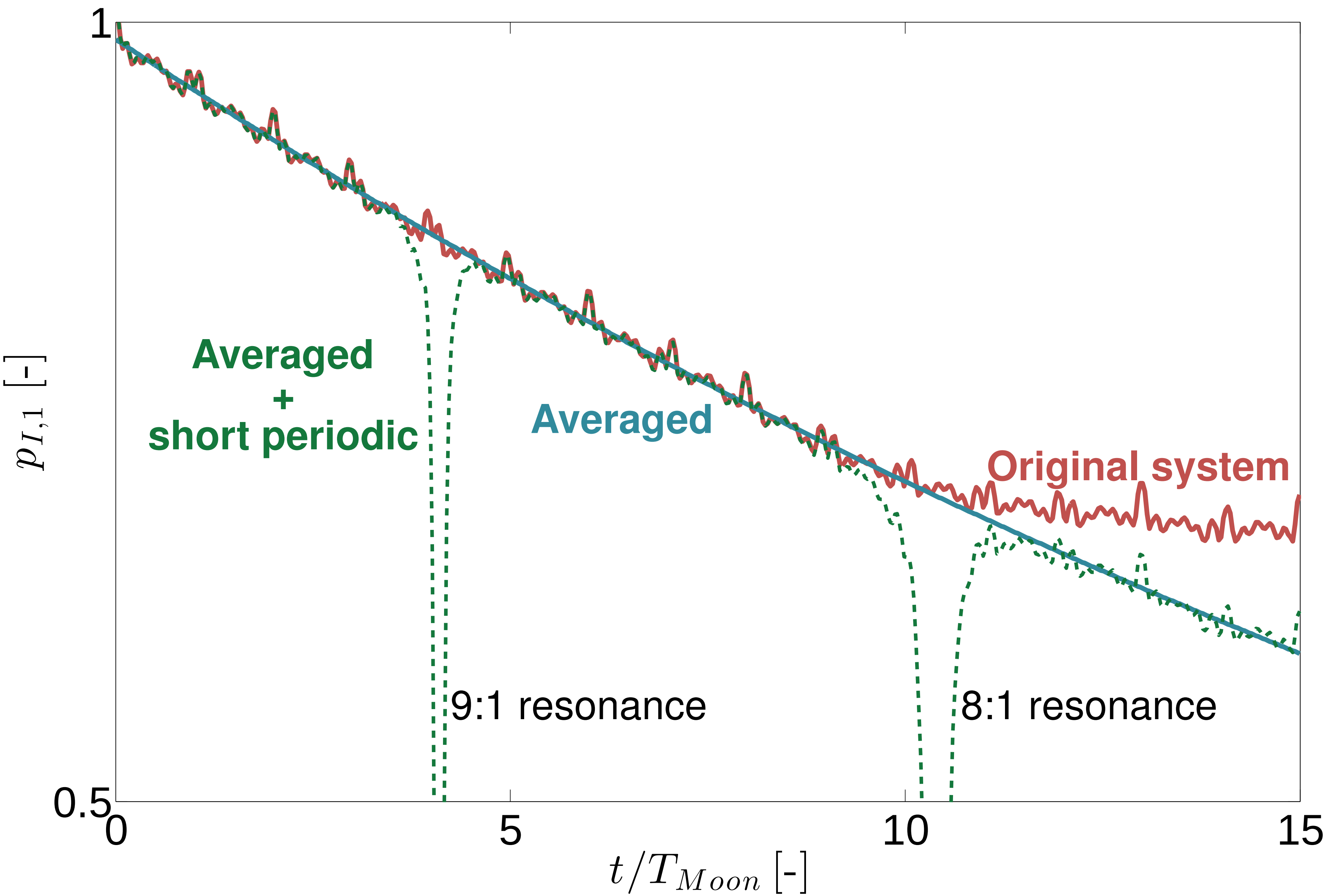
2. Is it a resonance effect?



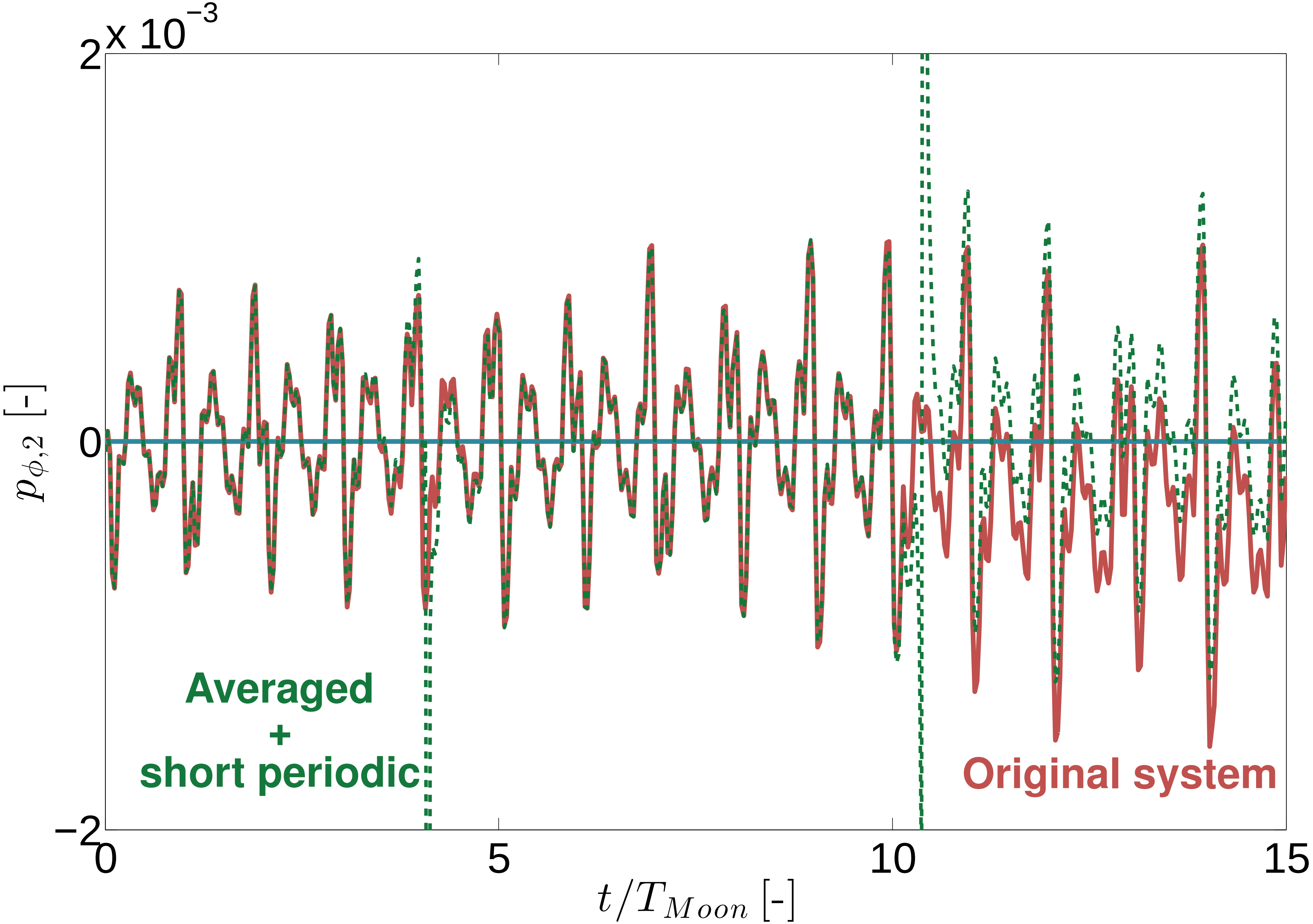
2. Yes, but divergence happened much earlier!



2. What happens when resonances are crossed?



2. Resonance crossing induces small jumps of p_ϕ



2. Resonant averaged form

Assume that there is $\boldsymbol{k}$ such that:

$$|\omega(\boldsymbol{J}) \cdot \boldsymbol{k}| \leq c\sqrt{\epsilon}$$

Perform the change of variables:

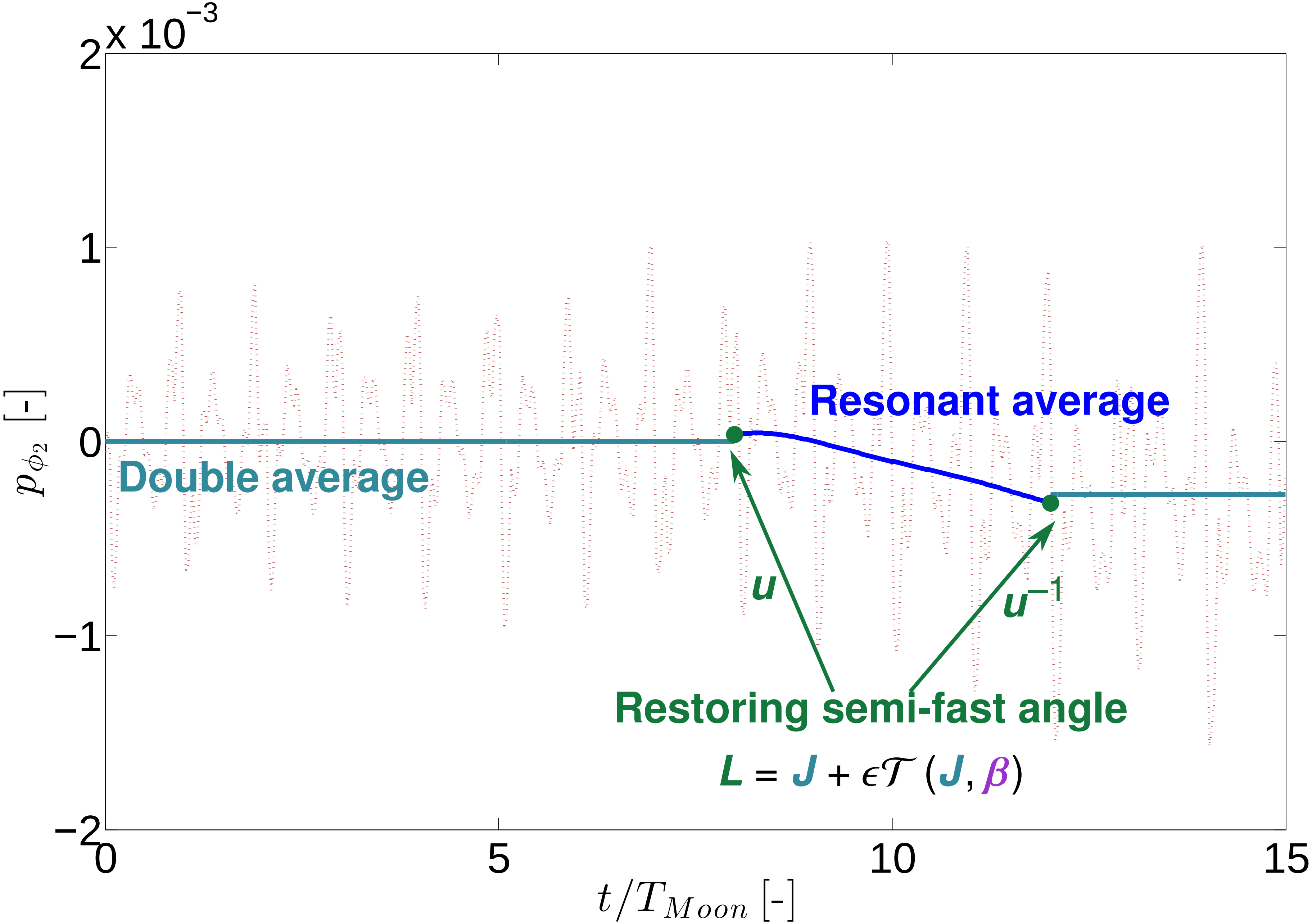
$$\boldsymbol{L} = \boldsymbol{J}, \qquad \beta = \frac{\boldsymbol{k} \cdot \boldsymbol{\phi}}{||\boldsymbol{k}||^2}, \qquad \alpha = \frac{\boldsymbol{k}^\perp \cdot \boldsymbol{\phi}}{||\boldsymbol{k}||^2}$$

$$\boldsymbol{p_L} = \boldsymbol{p_J}, \qquad p_\beta = \frac{\boldsymbol{k} \cdot \boldsymbol{p_\phi}}{||\boldsymbol{k}||^2}, \qquad p_\alpha = \frac{\boldsymbol{k}^\perp \cdot \boldsymbol{p_\phi}}{||\boldsymbol{k}||^2}$$

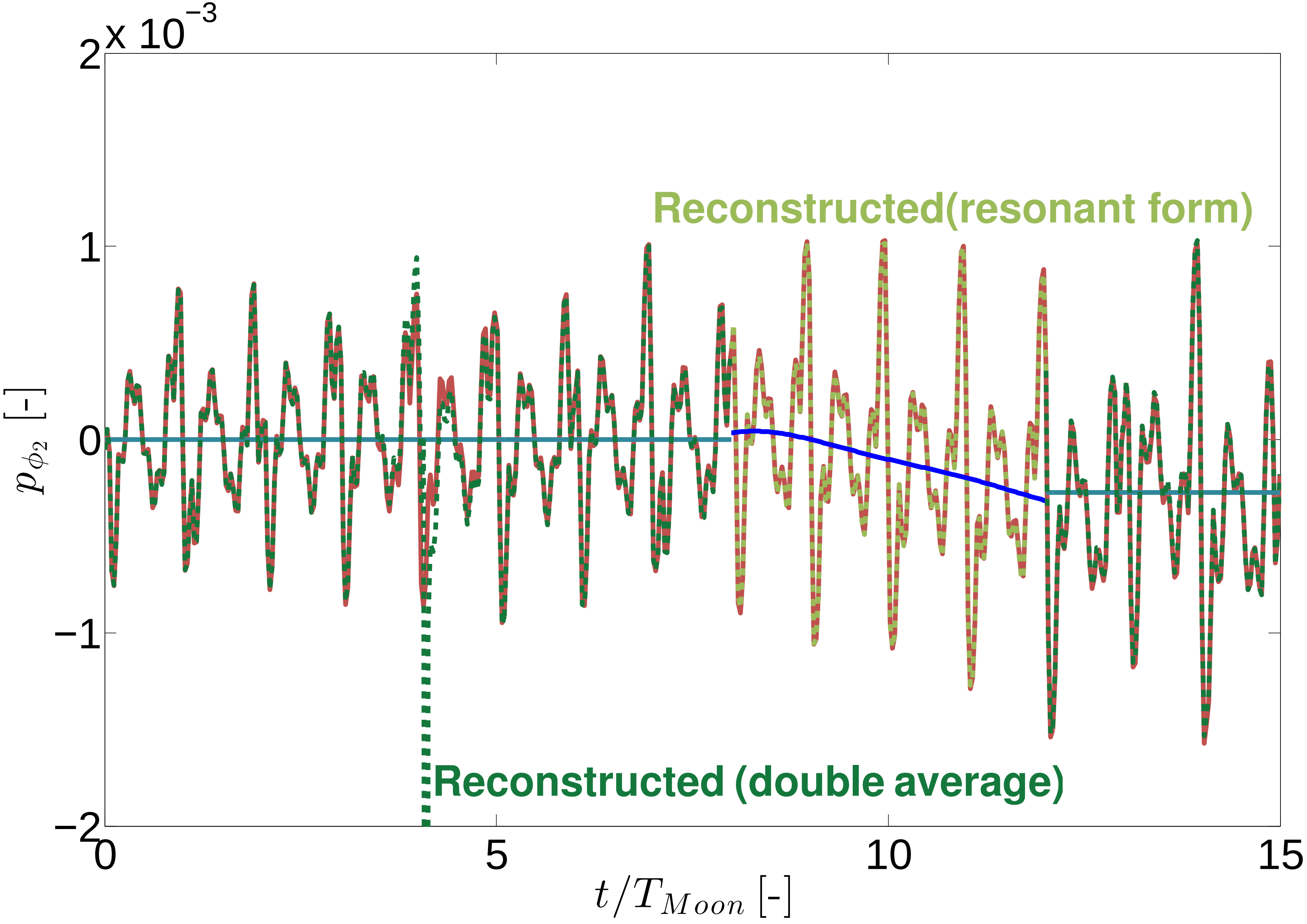
Average with respect to α

$$\overline{\mathcal{H}}_{\boldsymbol{k}} = \int_0^{2\pi} \mathcal{H}\left(\boldsymbol{L},\ \boldsymbol{p_L},\ \left[\frac{\boldsymbol{k}}{||\boldsymbol{k}||^2}\ \frac{\boldsymbol{k}^\perp}{||\boldsymbol{k}||^2}\right]\begin{Bmatrix}\beta \\ \alpha\end{Bmatrix},\ \left[\frac{\boldsymbol{k}}{||\boldsymbol{k}||^2}\ \frac{\boldsymbol{k}^\perp}{||\boldsymbol{k}||^2}\right]\begin{Bmatrix}p_\beta \\ p_\alpha\end{Bmatrix},\ 0\right) \mathrm{d}\alpha$$

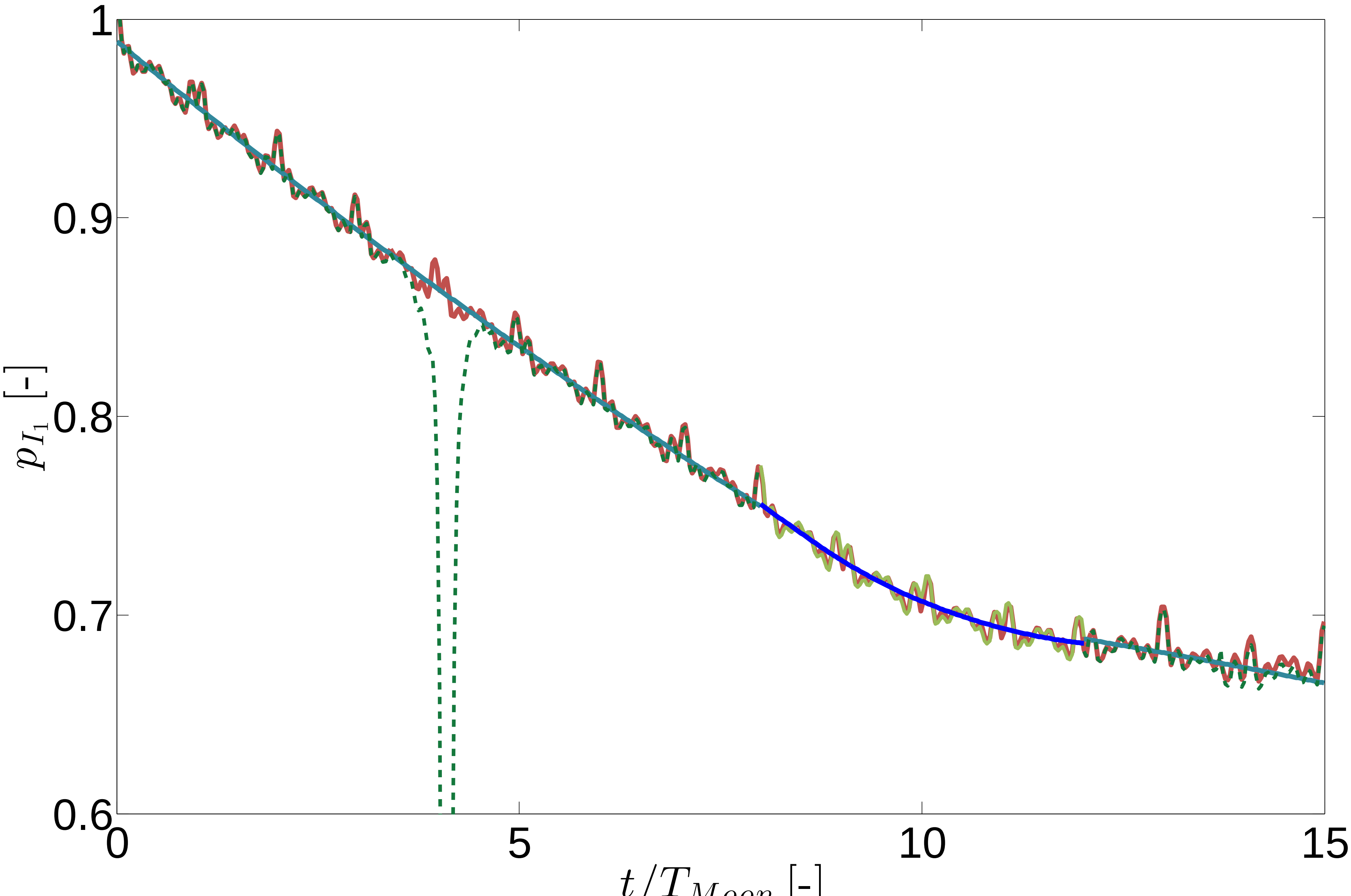
2. Transformation to interface averaged forms



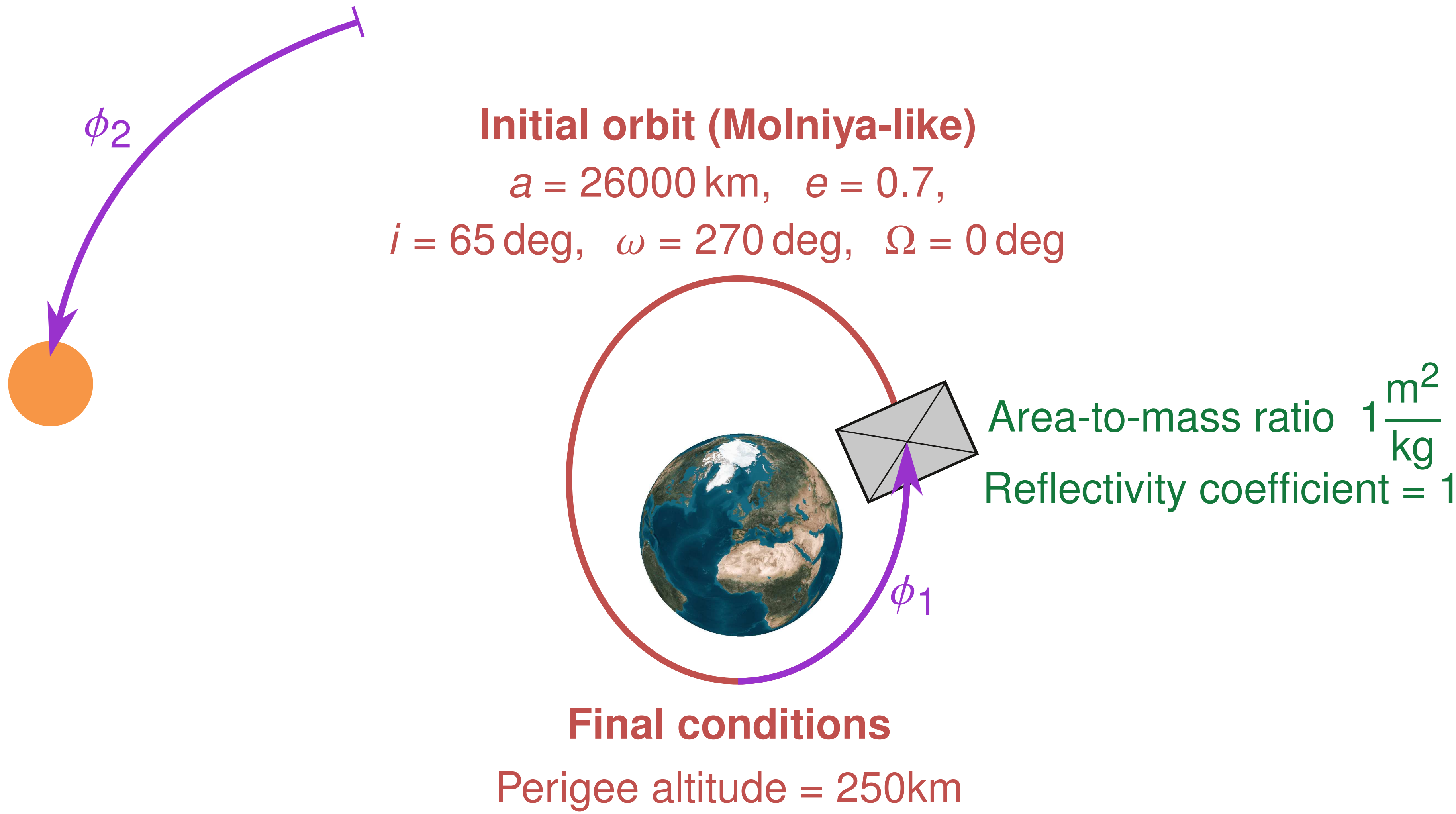
2. Jumps of adjoints to fast variables are properly



2. The transform enable 'gluing' of different forms



3. De-orbiting leveraging on solar radiation pressure



3. Mathematical modeling

Assumptions

- SRP is the only perturbation
- "Cannonball" model (SRP toward Sun direction)
- "Perfect sail" (SRP is negligible when u = 0)
- Attitude dynamics is neglected

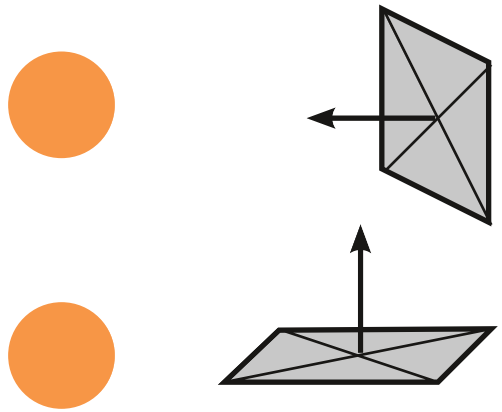
Optimal control

- Switching function

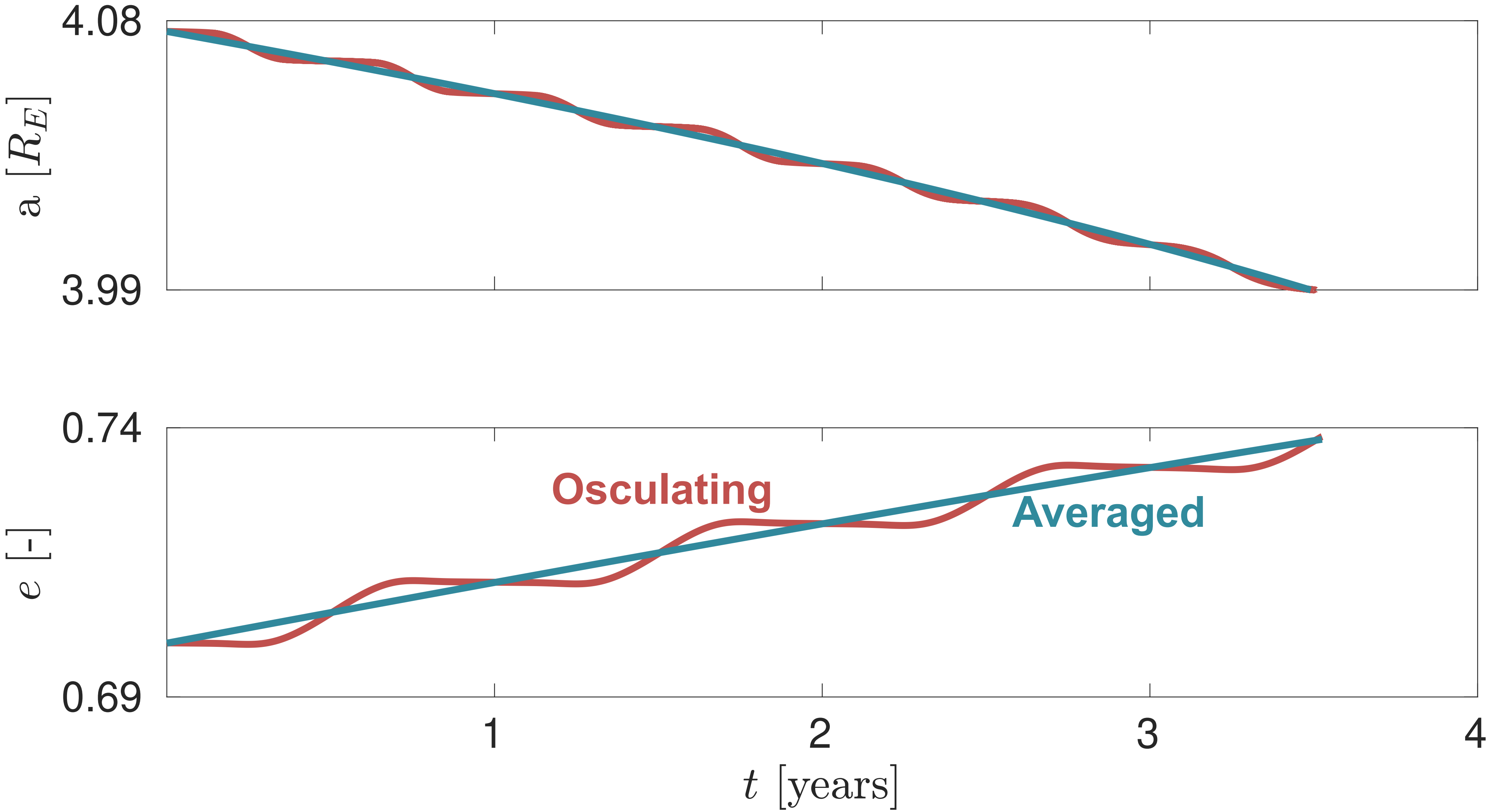
$$s = \boldsymbol{f_1}(\boldsymbol{l}, \phi) \cdot \boldsymbol{p_l} + \boldsymbol{g_1}(\boldsymbol{l}, \phi) \cdot \boldsymbol{p_\phi}$$

- Control

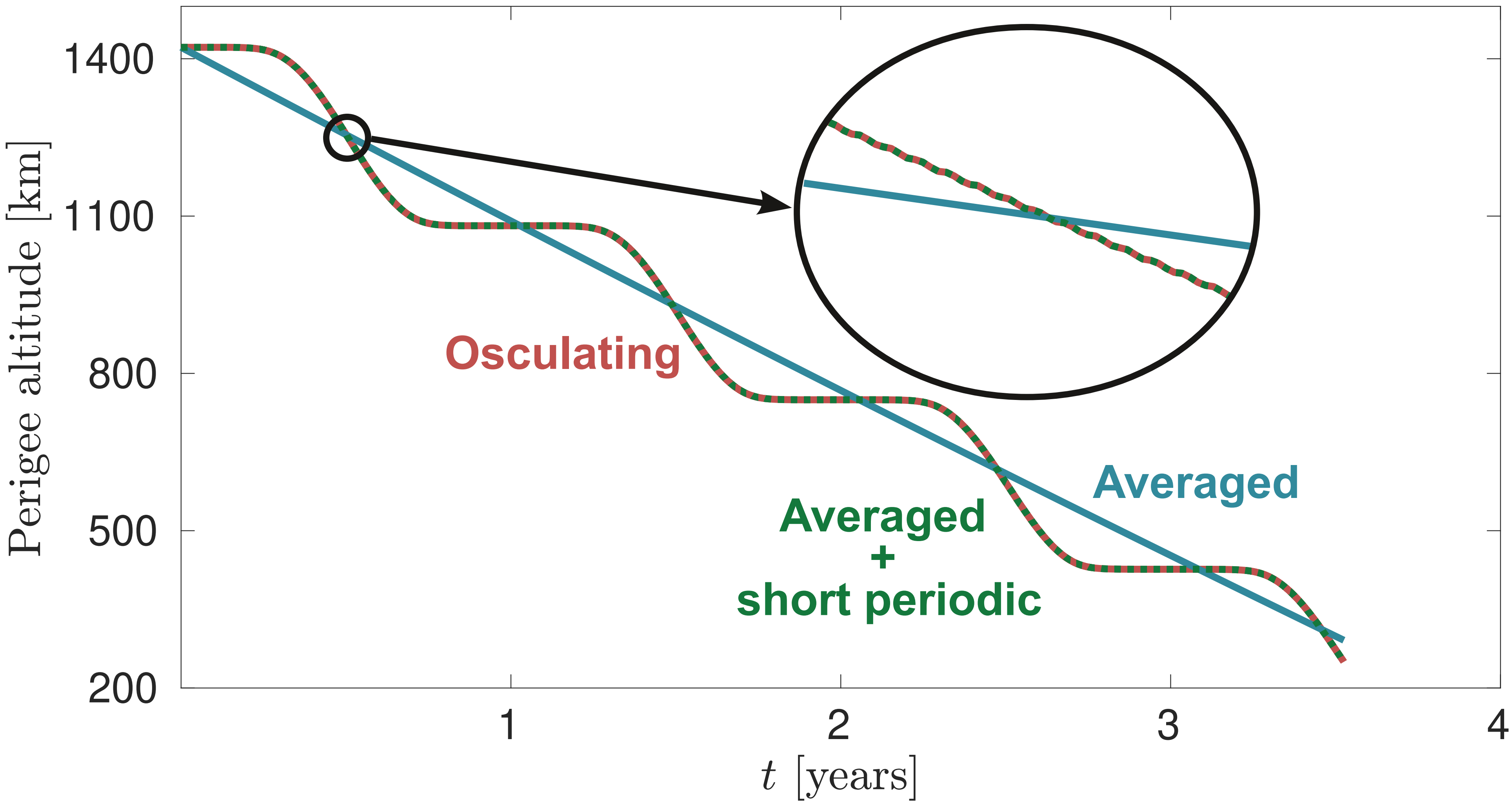
$$u = \begin{cases} 1 & \text{if} \quad s(\boldsymbol{l}, \phi, \boldsymbol{p_l}, \boldsymbol{p_\phi}) > 0 \\ 0 & \text{otherwise} \end{cases}$$



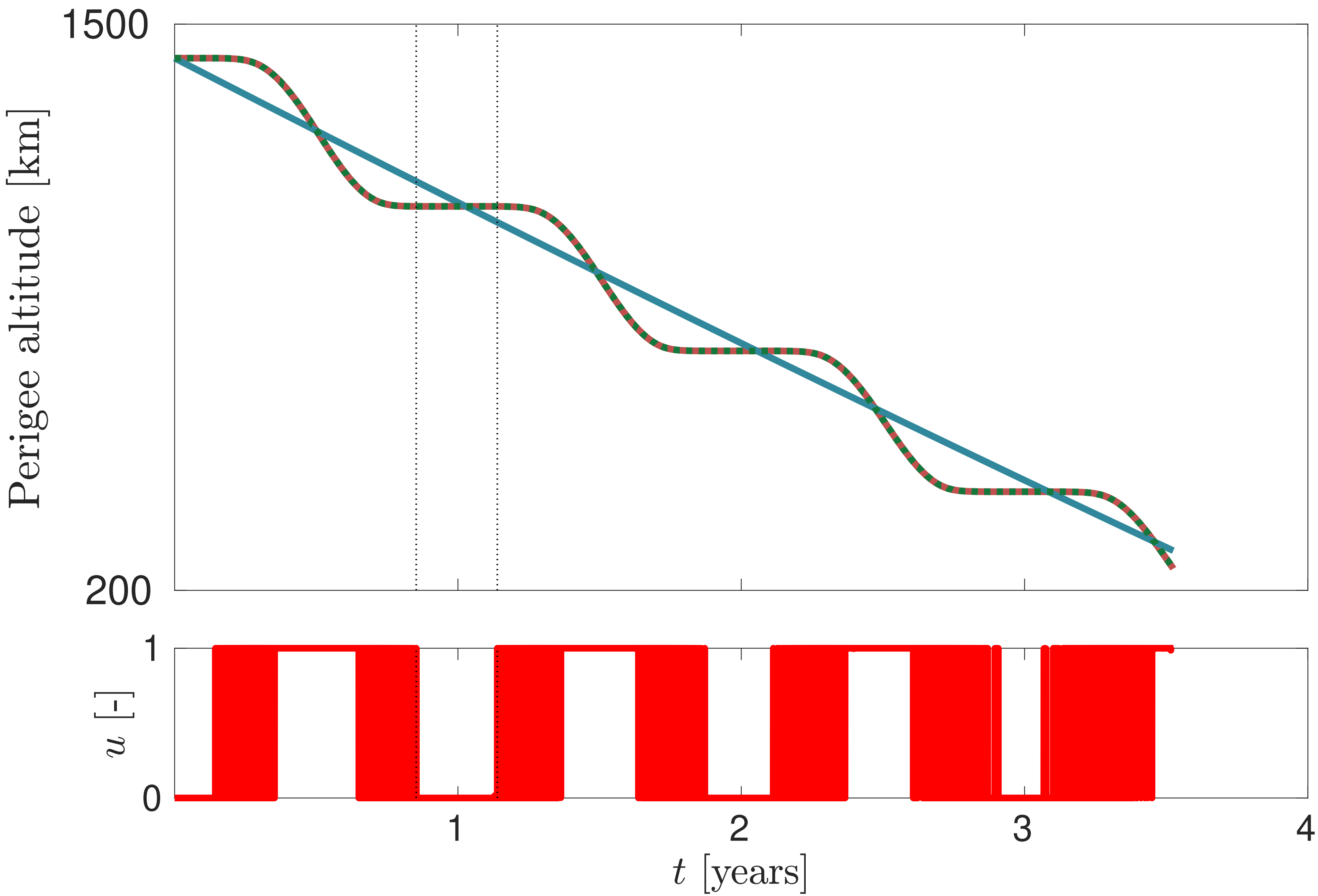
3. Semi-major axis and eccentricity



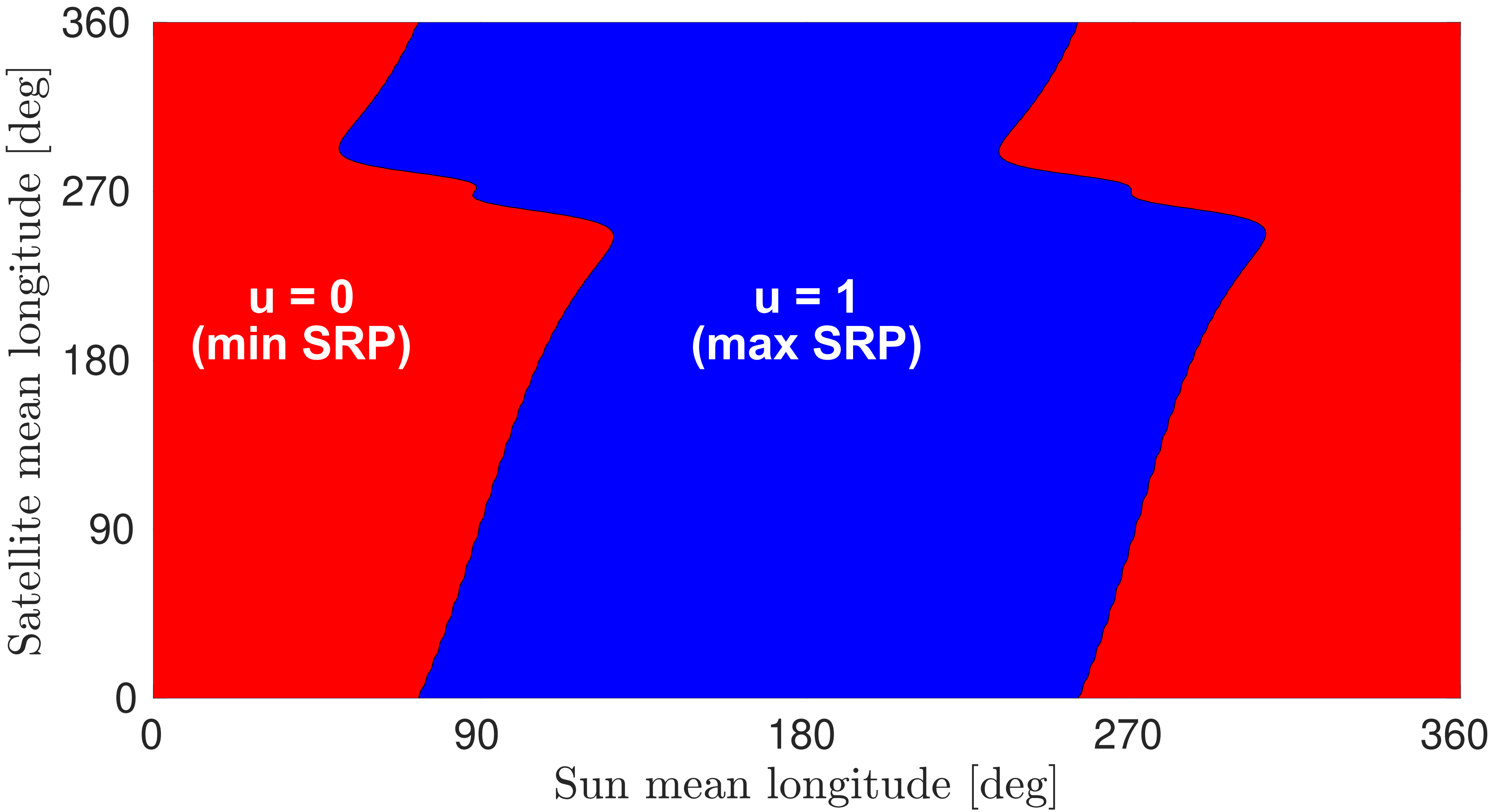
3. Trajectory of the perigee altitude



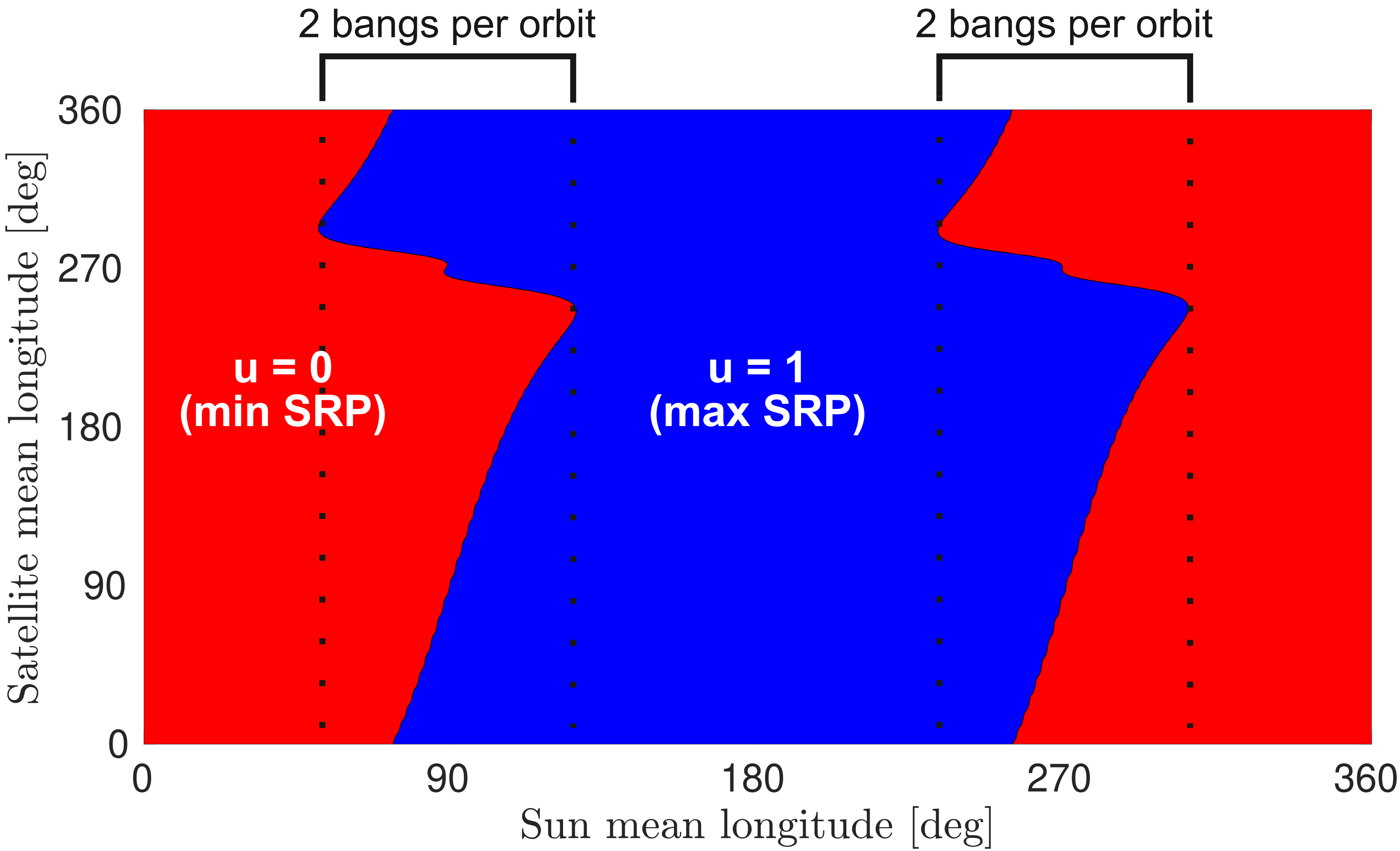
3. Short-periodic oscillations include the control structure



3. Control as a function of the phases at initial time



3. "Four-seasons" control structure



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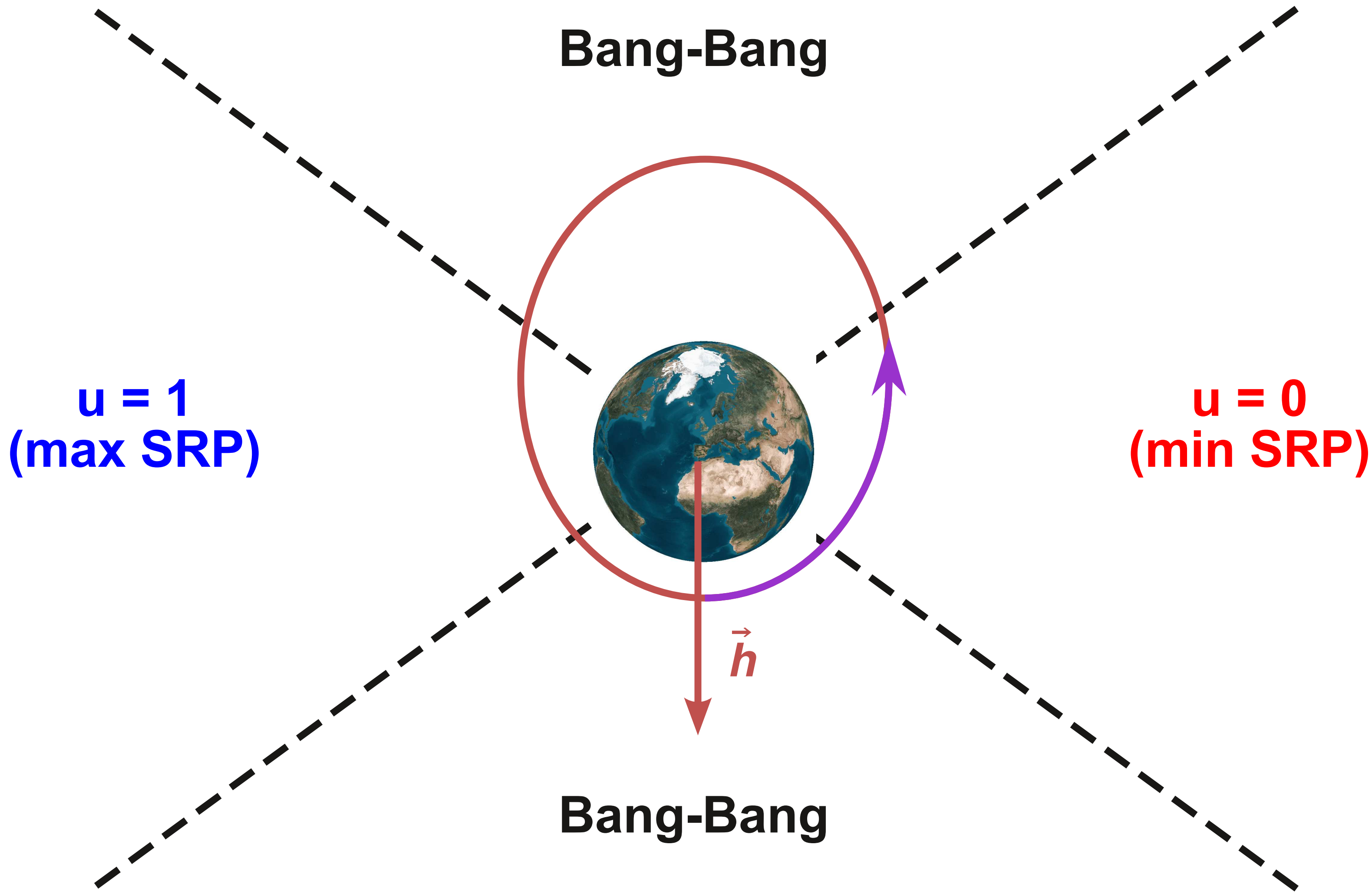

Ecliptic plane

3. Way forward

Complexity of the model

Orbital perturbations ► The second fast angle is: $l_{Sun} - \Omega$

Eclipses ► Similar treatment of bang-bang (regularization)

Singular arcs

Conclusion

Non-conventional fast-oscillating dynamical problem

Analogies with other problems in space mechanics (e.g., **quasi-satellite orbits**)

Key role of the transformation of the **adjoints to fast variables**

Benefits of averaged control system:

- Reduced set of unknown
- Smoothed trajectories
- Control structure is not required

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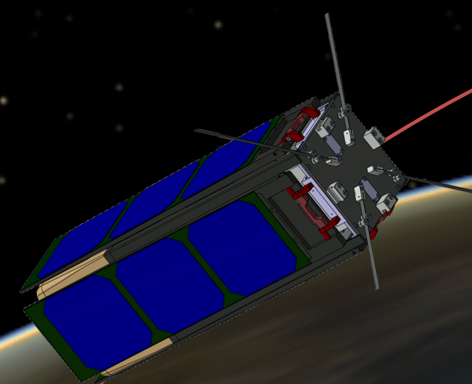

Logrono, 25/04/2019

INVENTEURS DU MONDE NUMÉRIQUE