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THE FUNDAMENTAL MATRIX OF SINGULARLY PERTURBED MARKOV CHAINS

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Abstract

We consider a singularly perturbed (finite state) Markov chain and provide a complete characterization of the fundamental matrix. In particular, we obtain a formula for the regular part simpler than a previous Schweitzer's formula, and the singular part is obtained via a reduction process similar to Delebecque's reduction for the stationary distribution. In contrast to previous approaches, one works with aggregate Markov chains of much smaller dimension than the original chain, an essential feature for practical computation. An application to mean first-passage times is also presented.

MARKOV CHAINS; FUNDAMENTAL MATRIX; SINGULAR PERTURBATION

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1. Introduction

Singularly perturbed Markov chains have been studied in the pioneering works of Simon and Ando [31], Schweitzer [27], Courtois [6], Pervozvanskii et al [10, 22, 23], Korolyuk and Turbin [17], Delebecque and Quadrat [7, 8], Phillips and Kokotovic [24] and later by Coderch et al [5], Rohlicek and Willsky [25], Filar, Bielecki and Abbad [1, 3], Latouche and Louchard [20, 21], Hassin and Haviv [12], Haviv and Ritov [11], Lasserre [19] and many others. For various applications of singularly perturbed chains the interested reader is referred to the above references and the references therein.

In the above papers, the original Markov chain has m ergodic classes which, after perturbation, reduce to a single one (with possibly transient states in some of the above works), i.e. the

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perturbation breaks down the class structure of the original chain. Actually, Delebecque considers the general case when the perturbed chain has also several (but less) recurrent classes [8].

To compute the Taylor's series expansion of the perturbed stationary distribution $\pi(\epsilon)$, Delebecque introduced a nice reduction process [8] that handles successive aggregated Markov chains with small dimension matrices (in the same spirit, see also the works by Coderch et al [5], Rohlicek and Willsky [25]). On the other hand, for linear analytic perturbations, Schweitzer [28, 29] first proved that $\pi(\epsilon)$ has a *power series* expansion in ϵ similar in form to the power series expansion in the *regular* case. As this nice result requires knowledge of the fundamental matrix $Z(\epsilon)$, he also provided [28] a formula for $Z(\epsilon)$ which consists of a regular part $Z^R(\epsilon) \stackrel{def}{=} \sum_{k=0}^{\infty} \epsilon^k Z_k$ and a singular part $Z^S(\epsilon) \stackrel{def}{=} \sum_{k=1}^{s} \epsilon^{-k} Z_{-k}$ (cf. eq. (9)). However, Schweitzer's numerical procedure [28] for computing $Z^S(\epsilon)$ and Z_0 (needed for the power series expansion of $\pi(\epsilon)$) is rather involved and does not use a reduction process, therefore leading to computations with large size matrices. In addition, the formula given in Schweitzer for $Z(\epsilon)$ is rather complicated, and, in contrast to the formula for $\pi(\epsilon)$, does not appear as a simple extension of the regular perturbation case (cf. Schweitzer's report [28]).

As already mentioned, the asymptotic expansion of $Z(\epsilon)$ is needed to get the power series expansion of $\pi(\epsilon)$. However, another useful application is the computation of first-passage times. In particular, for singularly perturbed chains, the singular part $Z^S(\epsilon)$ provides the time-scale analysis of first-passage times to states in different ergodic classes (of the unperturbed chain) (see also a direct approach in Latouche and Louchard [20] for series expansions of the first and second moments). Finally, let us mention an interesting application to the Hamilton Cycle Problem in Filar [4, 9], a famous combinatorial optimization problem.

The purpose and contribution of the present paper is twofold:

(a) for linear perturbations, it provides a <u>simple</u> formula for the regular part $Z^R(\epsilon)$, with very much the same flavour as the regular perturbation case. Indeed, it readily simplifies to Schweitzer's regular perturbation formula (cf. Schweitzer [28, 29]) whenever the perturbation is regular.

(b) The singular terms of $Z(\epsilon)$ (there are finitely many) are computed via a simple reduction process, similar in flavour to Delebecque's reduction process for the stationary distribution (cf. also [5, 25]). In contrast to the numerical procedure described in Schweitzer's report, a nice and important feature of this reduction process is that at each stage, one works with **aggregated** Markov chains on much smaller state spaces. For instance, as in Delebecque [8], after the first stage, there are no more than m states, where m is the number of ergodic classes.

In addition, and again as in Delebecque [8], this reduction process permits to handle general analytic perturbations with the same complexity as for linear perturbations, thus permitting to handle different time scales in the perturbed matrix.

Finally, the computational complexity of the reduction-process algorithm is $O(s^2N^3)$ where s is the order of the pole in the Laurent series expansion of $Z(\epsilon)$, N the number of states of the Markov chain and assuming s is already known (or has been determined by, e.g., Hassin and Haviv's $O(N^4)$ algorithm [12]). If s is not known, then the reduction-process algorithm is run with $s = 1, 2, \ldots$ until some rank-condition is satisfied for some value s. When $s \ll N$ (the usual case in practical applications), then not only s, the order of the pole, but also the whole singular part of $Z(\epsilon)$ is obtained in $O(s^3N^3)$ in contrast to $O(N^4)$ just for the pole in Hassin and Haviv [12].

2. Notation and definitions

Consider a perturbed finite Markov chain with transition matrix $P(\varepsilon)$ depending on a small parameter ε. Two types of perturbation are discussed. The first type is a linear perturbation

$$P(\varepsilon) = P_0 + \varepsilon B, \tag{1}$$

where $P_0 \in \mathbb{R}^{N\times N}$ (N – the number of states) is the transition matrix of the unperturbed chain and εB is the perturbation term. The second type is an analytic perturbation

$$P(\varepsilon) = P_0 + \varepsilon G_1 + \varepsilon^2 G_2 + \ldots$$

or

$$G(\varepsilon) = G_0 + \varepsilon G_1 + \varepsilon^2 G_2 + \ldots, \tag{2}$$

where $G(\varepsilon) = P(\varepsilon) - I \in \mathbb{R}^{N\times N}$, $G_0 = P_0 - I \in \mathbb{R}^{N\times N}$ are the generators of the perturbed and unperturbed chains respectively.

Let ξ_n be the vector $[1\cdots 1]^T \in \mathbb{R}^{n\times 1}$. Since each row of a transition matrix is a probability distribution, we immediately have

$$P(\varepsilon)\xi_N = P_0\xi_N = \xi_N \quad \text{and} \quad G(\varepsilon)\xi_N = G_0\xi_N = 0. \tag{3}$$

For the same reasons, $B\xi_N = 0$ and $G_k\xi_N = 0, k \geq 1$.

For any Markov chain, there exists a unique *ergodic projection* [14, 16] which describes the long run behaviour of the chain. By definition, the ergodic projections for the unperturbed and perturbed chains are

$$P^* \stackrel{def}{=} \lim_{T\to\infty} \frac{1}{T+1}\sum_{t=0}^{T} P_0^t \quad \text{and} \quad P^*(\varepsilon) \stackrel{def}{=} \lim_{T\to\infty} \frac{1}{T+1}\sum_{t=0}^{T} P^t(\varepsilon),$$

respectively. There are a number of algorithms for computing P^* [14, 16] and $P^*(\varepsilon)$ [8, 5, 6, 25, 29]. Furthermore it is not difficult to show [8, 29] that the perturbed ergodic projection $P^*(\varepsilon)$ admits the Taylor series expansion at the punctured neighbourhood of $\varepsilon = 0$.

$$P^*(\varepsilon) = P_0^* + \varepsilon P_1^* + \varepsilon^2 P_2^* + ..., \quad \varepsilon \neq 0 \tag{4}$$

The constant term P_0^* will be called the *limiting ergodic projection.* Note that in general $P_0^* \neq P^*$ [27]. The perturbation is said to be **singular** if the ergodic projection $P^*(\varepsilon)$ has such a kind of discontinuity at $\epsilon = 0$. It turns out that this situation occurs when the perturbation unites several ergodic classes into one [27, 29]. We are mainly interested in this type of perturbations since the case of regular perturbations is fairly well understood. Therefore, in the sequel, we make the following standard assumption about the character of the singularity.

ASSUMPTION A. *For any $\varepsilon > 0$ sufficiently small, the perturbed matrix $P(\varepsilon)$ is a unichain transition matrix, i.e, the perturbed chain has a unique ergodic class and possibly, a set of transient states.*

The above assumption implies that $P^*(\varepsilon) = \xi_N \pi(\varepsilon)$ for $\varepsilon \neq 0$, where the *stationary probability distribution* $\pi(\varepsilon)$ of the perturbed chain is the unique solution of the system

$$\begin{cases} \pi(\varepsilon)P(\varepsilon) = \pi(\varepsilon), \\ \pi(\varepsilon)\xi_N = 1 \end{cases}$$

and $\pi(\epsilon)$ has a Taylor series expansion

$$\pi(\varepsilon) = \pi_0 + \varepsilon\pi_1 + \varepsilon^2\pi_2 + ... , \quad \varepsilon \neq 0. \tag{5}$$

In case of a linear perturbation (cf. (1)), Schweitzer [28, 29] proved that $\pi(\epsilon)$ can be expressed by the *updating* formula

$$\pi(\varepsilon) = \pi_0[I - \varepsilon B Z_0]^{-1}, \quad \varepsilon \neq 0, \tag{6}$$

where Z_0 is the constant term in the fundamental matrix Laurent series expansion (cf. (9) below). Hence, the coefficients $\pi_k, k = 0, 1, ...$ in (5) form a *power* sequence

$$\pi_k = \pi_0 (B Z_0)^k, \quad k = 0, 1, ...$$

The *fundamental matrix* of the original chain is defined by the formula [14, 16]

$$Z \stackrel{def}{=} (I - P_0 + P^*)^{-1} = (P^* - G_0)^{-1}. \tag{7}$$

Of course, the same formula is true for the perturbed chain

$$Z(\varepsilon) \stackrel{def}{=} (I - P(\varepsilon) + P^*(\varepsilon))^{-1} = (P^*(\varepsilon) - G(\varepsilon))^{-1}. \tag{8}$$

In this paper, we are mainly concerned with the characterization of the fundamental matrix $Z(\varepsilon)$ of the perturbed chain.

In the case of a singular perturbation, $Z(\epsilon)$ also has a discontinuity at $\epsilon = 0$. Moreover, in this case, $||Z(\varepsilon)|| \to \infty$ as $\varepsilon \to 0$ and $Z(\varepsilon)$ admits a Laurent series expansion [29, 11]

$$Z(\varepsilon) = \frac{1}{\varepsilon^s} Z_{-s} + ... + \frac{1}{\varepsilon} Z_{-1} + Z_0 + \varepsilon Z_1 + ..., \varepsilon \neq 0. \tag{9}$$

Note that s, the order of the pole, is finite and $s \leq N$. We denote the singular and the regular parts of the fundamental matrix expansion (9) by $Z^S(\varepsilon)$ and $Z^R(\varepsilon)$, respectively. Schweitzer has also obtained formulas for the matrices Z_k. However, they are rather complicated and their computation requires to handle large size matrices (cf. [28]).

In the sequel, we propose a different approach that permits to compute the terms of the singular part $Z^S(\epsilon)$, as well as Z_0, efficiently. We apply a reduction process similar in flavour to Delebecque's reduction [8] for the Taylor series of $\pi(\epsilon)$. Right after the first stage of this reduction process, one handles *aggregated* Markov chains with no more than m states (m being the number of ergodic classes), a drastic reduction of the dimension. In addition, we also provide a simple formula for the regular part $Z^R(\epsilon)$, which readily simplifies to the usual formula in the case of regular perturbations.

We introduce the *deviation matrices* (or *reduced resolvents*) H and $H(\varepsilon)$ for the original and perturbed chains:

$$H \stackrel{def}{=} Z - P^* \quad \text{and} \quad H(\varepsilon) \stackrel{def}{=} Z(\varepsilon) - P^*(\varepsilon).$$

We immediately have

$$H^S(\varepsilon) = Z^S(\varepsilon) \quad \text{and} \quad H^R(\varepsilon) = Z^R(\varepsilon) - P^*(\varepsilon). \tag{10}$$

The fundamental matrix and the reduced resolvent also satisfy [14, 16], i.e.,

$$HG_0 = G_0 H = ZG_0 = G_0 Z = P^* - I, \tag{11}$$

$$HP^* = P^* H = 0, \quad ZP^* = P^* Z = P^*, \tag{12}$$

(with G_0 as in (2)) and similarly, for the perturbed chain

$$H(\varepsilon)G(\varepsilon) = G(\varepsilon)H(\varepsilon) = Z(\varepsilon)G(\varepsilon) = G(\varepsilon)Z(\varepsilon) = P^*(\varepsilon) - I, \tag{13}$$

$$H(\varepsilon)P^*(\varepsilon) = P^*(\varepsilon)H(\varepsilon) = 0, \quad Z(\varepsilon)P^*(\varepsilon) = P^*(\varepsilon)Z(\varepsilon) = P^*(\varepsilon). \tag{14}$$

The following identities will be useful as well,

$$Z\xi_N = Z(\varepsilon)\xi_N = P^*\xi_N = P^*(\varepsilon)\xi_N = \xi_N, \tag{15}$$

$$H\xi_N = H(\varepsilon)\xi_N = 0. \tag{16}$$

3. The regular part of the fundamental matrix

In this section, we obtain a closed form analytic formula for the regular part $Z^R(\varepsilon)$ of the fundamental matrix expansion (9). We use arguments similar to those in Schweitzer's elegant technique introduced in [29]. We also show that Schweitzer's formula for regular perturbations is in fact a particular case, i.e., our formula readily simplifies to Schweitzer's one in case of regular perturbations.

Theorem 1 Let $P(\varepsilon) = P_0 + \varepsilon B$ be the transition matrix of a linearly perturbed Markov chain and let Assumption A hold. Then the regular part $Z^R(\epsilon)$ of the fundamental matrix $Z(\epsilon)$ is given by

$$Z^R(\varepsilon) = \{[I - P^*(\varepsilon)]Z_0 + P_0^*\}[I - \varepsilon B Z_0]^{-1} - P^*(\varepsilon)Z^S(\varepsilon) \tag{17}$$

Proof. For arbitrary $0 < \epsilon_1,\ \epsilon_2$, we have the following identity (see e.g. [19])

$$Z(\varepsilon_1) - Z(\varepsilon_2) = (\varepsilon_1 - \varepsilon_2)Z(\varepsilon_1)BZ(\varepsilon_2) + Z(\varepsilon_1)P^*(\varepsilon_2) - P^*(\varepsilon_1)Z(\varepsilon_2)$$

Under Assumption A, and using(15), we have $Z(\varepsilon_1)P^*(\varepsilon_2) = Z(\varepsilon_1)\xi_N\pi(\varepsilon_2) = \xi_N\pi(\varepsilon_2) = P^*(\varepsilon_2)$. Hence,

$$Z(\varepsilon_1) - Z(\varepsilon_2) = (\varepsilon_1 - \varepsilon_2)Z(\varepsilon_1)BZ(\varepsilon_2) + P^*(\varepsilon_2) - P^*(\varepsilon_1)Z(\varepsilon_2) \tag{18}$$

With ε_2 fixed in (18), the regular parts with respect to ε_1 satisfy:

$$Z^R(\varepsilon_1) - Z(\varepsilon_2) = (\varepsilon_1 - \varepsilon_2)Z^R(\varepsilon_1)BZ(\varepsilon_2) + Z_{-1}BZ(\varepsilon_2) + P^*(\varepsilon_2) - P^*(\varepsilon_1)Z(\varepsilon_2) \tag{19}$$

Letting $\varepsilon_1 = \varepsilon_2$ in (19) yields

$$Z^R(\varepsilon_2) - Z(\varepsilon_2) = Z_{-1}BZ(\varepsilon_2) + P^*(\varepsilon_2) - P^*(\varepsilon_2)Z(\varepsilon_2) \tag{20}$$

and since $P^*(\varepsilon) = P^*(\varepsilon)Z(\varepsilon)$, we have

$$Z^R(\varepsilon_2) - Z(\varepsilon_2) = Z_{-1}BZ(\varepsilon_2)$$

so that

$$Z^S(\varepsilon) = -Z_{-1}BZ^S(\varepsilon) \tag{21}$$

and

$$0 = Z_{-1}BZ^R(\varepsilon). \tag{22}$$

If instead, we fix ε_1 in (18) and consider the regular parts with respect to ϵ_2, we obtain

$$Z^S(\varepsilon) = -Z^S(\varepsilon)BZ^{-1} \tag{23}$$

and

$$Z^R(\varepsilon)BZ_{-1} = P^*(\varepsilon) - P^*(\varepsilon)Z^R(\varepsilon) = P^*(\varepsilon)Z^S(\varepsilon). \tag{24}$$

Taking the regular parts in (19) with respect to ε_2 (with of course, ε_1 fixed) yields

$$\begin{aligned} Z^R(\varepsilon_1) - Z^R(\varepsilon_2) &= (\varepsilon_1 - \varepsilon_2)Z^R(\varepsilon_1)BZ^R(\varepsilon_2) - Z^R(\varepsilon_1)BZ_{-1} + Z_{-1}BZ^R(\varepsilon_2) + P^*(\varepsilon_2) \\ &\quad -P^*(\varepsilon_1)Z^R(\varepsilon_2). \end{aligned}$$

The term $Z_{-1}BZ^R(\varepsilon_2)$ vanishes in view of (22). Then with $\varepsilon_1 := \varepsilon$ and letting $\varepsilon_2 \to 0$, one gets

$$Z^R(\varepsilon) - Z_0 = \varepsilon Z^R(\varepsilon)BZ_0 - Z^R(\varepsilon)BZ_{-1} + P_0^* - P^*(\varepsilon)Z_0.$$

$$Z^R(\varepsilon)[I - \varepsilon BZ_0] = [I - P^*(\varepsilon)]Z_0 + P_0^* - Z^R(\varepsilon)BZ_{-1}.$$

Using (24), we get

$$Z^R(\varepsilon)[I - \varepsilon BZ_0] = [I - P^*(\varepsilon)]Z_0 + P_0^* - P^*(\varepsilon)Z^S(\varepsilon). \tag{25}$$

We shall now show that

$$P^*(\varepsilon)Z^S(\varepsilon) = P^*(\varepsilon)Z^S(\varepsilon)[I - \varepsilon BZ_0]. \tag{26}$$

It suffices to prove that $Z^S(\varepsilon)BZ_0 = 0$. Indeed, from (22) and (23),

$$Z^S(\varepsilon)BZ_0 = -Z^S(\varepsilon)BZ_{-1}BZ_0 = 0.$$

Finally, substituting (26) into (25) and multiplying (25) by $[I - \varepsilon BZ_0]^{-1}$ from the right-hand-side, we get the required formula (17).

QED

Two useful corollaries follow directly from Theorem 1. First, consider the term $P^*(\varepsilon)Z^S(\varepsilon)$. This term is regular, despite it is the product of the perturbed ergodic projection and the *singular* part of the fundamental matrix. The next corollary shows the explicit regular structure of this product.

Corollary 1 The formula (17) is equivalent to

$$Z^R(\varepsilon) = \{[I - P^*(\varepsilon)]Z_0 + P_0^*\}[I - \varepsilon BZ_0]^{-1} - P^*(\varepsilon)\sum_{i=1}^{s}(BZ_0)^i Z_{-i}. \tag{27}$$

Proof. In view of Assumption (A) and (6) we have $P^*(\varepsilon) = \xi_N \pi(\varepsilon) = P_0^*[I - \varepsilon B Z_0]^{-1}$ and hence $P_k^* = P_0^*(BZ_0)^k, k = 0, 1, ...$ in the power series (4).

Note that *only* the regular part of $P^*(\varepsilon)Z^S(\varepsilon)$ contributes to equation (17), since $Z^R(\varepsilon)$ and $\{[I - P^*(\varepsilon)]Z_0 + P_0^*\}[I - \varepsilon BZ_0]^{-1}$ are both regular. Then,

$$[P^*(\varepsilon)Z^S(\varepsilon)]^R = [\sum_{k=1}^{\infty} P_0^*(\varepsilon BZ_0)^k \sum_{i=1}^{s} \varepsilon^{-i} Z_{-i}]^R = \sum_{i=1}^{s}\sum_{k=i}^{\infty} P_0^*(BZ_0)^k Z_{-i}\varepsilon^{k-i} =$$

$$= \sum_{i=1}^{s}\sum_{k=0}^{\infty} P_0^*(\varepsilon BZ_0)^k (BZ_0)^i Z_{-i} = \sum_{k=0}^{\infty} P_0^*(\varepsilon BZ_0)^k \sum_{i=1}^{s}(BZ_0)^i Z_{-i} = P^*(\varepsilon)\sum_{i=1}^{s}(BZ_0)^i Z_{-i}$$

This yields formula (27).

QED

By using (10) and (6), one easily obtains the counterpart of (17) for the deviation matrix $H(\varepsilon)$.

Corollary 2 The regular part of the deviation matrix $H(\varepsilon)$ is given by

$$H^R(\varepsilon) = [I - P^*(\varepsilon)]H_0[I - \varepsilon BZ_0]^{-1} - P^*(\varepsilon)H^S(\varepsilon).$$

Remark 1 Schweitzer's formula for regular perturbations [29]

$$Z(\varepsilon) = \{[I - P^*(\varepsilon)]Z(0) + P^*\}[I - \varepsilon BZ(0)]^{-1}$$

is a particular case of (17) (since then, $Z^S(\epsilon) = 0$).

Remark 2 The matrices $Z_k,\ k \geq 1$ are easily obtained from (27), with

$$Z_k = Z_0(BZ_0)^k - P_0^*\sum_{j=0}^{k}(BZ_0)^j Z_0 (BZ_0)^{k-j} + P_0^*(BZ_0)^k\{I - \sum_{i=1}^{s}(BZ_0)^i Z_{-i}\}. \tag{28}$$

Of course, in practical applications, a more efficient (and numerically stable) computational scheme for the above matrices $Z_k, k \geq 1$ is via recurrent formulas, in the same spirit as in [27, 28]. For instance, if we define the two k-dependent expressions

$$U_k = (BZ_0)^k \quad \text{and} \quad W_k = \sum_{j=0}^{k}(BZ_0)^j Z_0(BZ_0)^{k-j},\ \ k = 1, \ldots,$$

then (28)) is equivalent to

$$Z_k = Z_0 U_k - P_0^*[W_k - U_k\{I - \sum_{i=1}^{s}(BZ_0)^i Z_{-i}\}],\ \ k = 1, \ldots, \tag{29}$$

with U_k and W_k computed recursively in (30) below.

$$U_{k+1} = U_k(BZ_0), \quad W_{k+1} = W_k(BZ_0) + U_{k+1}Z_0, \quad k = 1, \ldots\ . \tag{30}$$

4. The singular part and the reduction process

From Remark 2, once the matrices $Z_{-k}, k = \overline{1,s}$ and Z_0 have been computed, then the matrices $Z_k, k \geq 1$ are obtained directly via the recurrent formulas (29), (30). Note also that Z_0 is a prerequisite in Schweitzer's updating formula (6).

Therefore, our aim in this section is to provide an efficient algorithm for computing $Z_{-k}, k = \overline{0,s}$ (including Z_0). The basic technique is the so-called *reduction process* for analytically perturbed linear operators, introduced in Kato [2, 15]. Delebecque [8] first applied the reduction process for computing the stationary distribution $\pi(\epsilon)$ of singularly perturbed Markov chains. We shall use the reduction process to compute the first regular term Z_0 and the singular part of the fundamental matrix expansion (9). It turns out that the reduction process is easier for the deviation matrix $H(\varepsilon)$ than for the fundamental matrix $Z(\varepsilon)$. Hence, we first obtain the matrices $H_{-k}, k = \overline{1,s}$ and H_0 of the deviation matrix expansion. Of course, we immediately restore the marices $Z_{-k}, k = \overline{0,s}$ from (10), i.e.

$$Z_{-k} = H_{-k},\ k = \overline{1,s}; \quad Z_0 = H_0 + P_0^* \tag{31}$$

The reduction process for analytic perturbations has practically the same level of difficulty as for linear perturbations. Therefore, and as in Delebecque [8], we consider the general case of analytic perturbations (2).

Under Assumption A, the deviation matrix $H(\varepsilon)$ of the perturbed Markov chain is uniquely defined by the following equations (cf. [14, 11])

$$H(\varepsilon)G(\varepsilon) = P^*(\varepsilon) - I \tag{32}$$

$$H(\varepsilon)\xi_N = 0 \tag{33}$$

To simplify further summations, we introduce a new notation for the coefficients H_k. Let us denote $X_k \stackrel{def}{=} H_{k-s}, k = 0, 1, \ldots$, so that $H(\varepsilon) = (X_0 + \varepsilon X_1 + \ldots)/\varepsilon^s = X(\varepsilon)/\varepsilon^s$.

$$X(\varepsilon)G(\varepsilon) = \varepsilon^s[P^*(\varepsilon) - I] \tag{34}$$

$$X(\varepsilon)\xi_N = 0 \tag{35}$$

Then, substitute (2) and (4) into (34) and equate the terms with the same power of ε to get

$$
\begin{array}{ll}
(36.0) & X_0 G_0 = 0 \\
(36.1) & X_1 G_0 + X_0 G_1 = 0 \\
\dots & \\
(36.s-1) & X_{s-1} G_0 + \dots + X_0 G_{s-1} = 0 \\
(36.s) & X_s G_0 + \dots + X_0 G_s = P_0^* - I \\
(36.s+1) & X_{s+1} G_0 + \dots + X_0 G_{s+1} = P_1^* \\
\dots &
\end{array}
$$

The above system is called the system of *determining equations*. It follows from (35) that each equation (36.k) is coupled with the normalization condition

$$X_k \xi_N = 0, \;\; k \geq 0. \tag{37}$$

Because of null right-hand-sides in (36.0)-(36.s-1), one cannot solve this system of determining equations by straightforward recurrent methods. In principle, the *infinite* system (36)-(37) can be solved by several (general) approaches in Perturbation Theory [2, 17, 18, 30]. Also similar equations were considered by Sain and Massey [26] and Howlett [13] in the study of linear control systems. However, we describe below a *specific* (and more efficient) reduction process which takes into account the particular properties of the fundamental matrix.

In this specific reduction process, let the n-th step reduced determining system be:

$$
\begin{array}{ll}
(38.0) & X_0^{(n)} G_0^{(n)} = 0 \\
(38.1) & X_1^{(n)} G_0^{(n)} + X_0^{(n)} G_1^{(n)} = 0 \\
\dots & \\
(38.s-n-1) & X_{s-n-1}^{(n)} G_0^{(n)} + \dots + X_0^{(n)} G_{s-n-1}^{(n)} = 0 \\
(38.s-n) & X_{s-n}^{(n)} G_0^{(n)} + \dots + X_0^{(n)} G_{s-n}^{(n)} = A_0^{(n)} \\
(38.s-n+1) & X_{s-n+1}^{(n)} G_0^{(n)} + \dots + X_0^{(n)} G_{s-n+1}^{(n)} = A_1^{(n)} \\
\dots &
\end{array}
$$

With $n = 0$, we retrieve the original determining system (36). Namely, $X_k^{(0)} = X_k, G_k^{(0)} = G_k$ and $A^{(0)} = P_0^* - I, A_k^{(0)} = P_k^*, k \geq 1$. The n-th step reduced equations (38) are also coupled with the normalization conditions

$$X_k^{(n)} \xi_{m_n} = 0, \quad k = 0, 1, \dots \tag{39}$$

The matrix $G_0^{(n)} \in \mathbb{R}^{m_n \times m_n}$ can be interpreted as the generator of the n-th step aggregated Markov chain [8, 5]. m_n is the number of ergodic classes of the $(n-1)$-th step aggregated Markov chain and in particular, m_1 is the number of ergodic classes in the original chain. The corresponding aggregated ergodic projection $P^{*(n)}$ satisfies

$$P^{*(n)}G_0^{(n)} = G_0^{(n)}P^{*(n)} = 0. \tag{40}$$

The aggregated ergodic projection is constructed by the following decomposition [3, 8]

$$P^{*(n)} = Q^{(n)}M^{(n)}, \tag{41}$$

where

$$Q^{(n)} \in \mathbb{R}^{m_n \times m_{n+1}} : G_0^{(n)}Q^{(n)} = 0, Q^{(n)}\xi_{m_{n+1}} = \xi_{m_n}; \tag{42}$$

$$M^{(n)} \in \mathbb{R}^{m_{n+1} \times m_n} : M^{(n)}G_0^{(n)} = 0, M^{(n)}\xi_{m_n} = \xi_{m_{n+1}}. \tag{43}$$

In addition,

$$M^{(n)}Q^{(n)} = I_{m_{n+1}}.$$

The n-th step reduced resolvent $H^{(n)}$ is computed via the formula

$$H^{(n)} = (P^{*(n)} - G_0^{(n)})^{-1} - P^{*(n)}. \tag{44}$$

Of course, this matrix satisfy (11)-(12), i.e.,

$$H^{(n)}G_0^{(n)} = G^{(n)}H^{(n)} = P^{*(n)} - I, \tag{45}$$

$$H^{(n)}P^{*(n)} = P^{*(n)}H^{(n)} = 0. \tag{46}$$

We now may formulate the main result of this section, which allows us to solve the system (36), step-by-step.

Theorem 2 (A reduction step) Let the n-th step $(n < s)$ reduced determining system (38) with the normalization conditions (39) be given. Then the unknown $X_k^{(n)}$ satisfy:

$$X_k^{(n)} = X_k^{(n+1)}M^{(n)} + \begin{cases} \sum_{i=0}^{k-1} X_i^{(n)}G_{k-i}^{(n)}H^{(n)}, & k < s-n, \\ \sum_{i=0}^{k-1} X_i^{(n)}G_{k-i}^{(n)}H^{(n)} - A_{k+n-s}^{(n)}H^{(n)}, & k \geq s-n, \end{cases} \tag{47}$$

where $X_k^{(n+1)} \in \mathbb{R}^{m_n \times m_{n+1}}, k = 0, 1, \ldots$ *are solutions of the next* $(n+1)$*-th step reduced determining equations*

$$\begin{array}{ll}(48.0) & X_0^{(n+1)} G_0^{(n+1)} = 0 \\ (48.1) & X_1^{(n+1)} G_0^{(n+1)} + X_0^{(n+1)} G_1^{(n+1)} = 0 \\ \cdots & \\ (48.s-n-2) & X_{s-n-2}^{(n+1)} G_0^{(n+1)} + \ldots + X_0^{(n+1)} G_{s-n-2}^{(n+1)} = 0 \\ (48.s-n-1) & X_{s-n-1}^{(n+1)} G_0^{(n+1)} + \ldots + X_0^{(n+1)} G_{s-n-1}^{(n+1)} = A_0^{(n+1)} \\ (48.s-n) & X_{s-n}^{(n+1)} G_0^{(n+1)} + \ldots + X_0^{(n+1)} G_{s-n}^{(n+1)} = A_1^{(n+1)} \\ \cdots & \end{array}$$

The matrices $G_k^{(n+1)}$ *and the right-hand-sides* $A_k^{(n+1)}$ *are given by*

$$G_k^{(n+1)} = M^{(n)} \sum_{p=1}^{k+1} \sum_{\nu_1+\ldots+\nu_p=k+1} G_{\nu_1}^{(n)} H^{(n)} G_{\nu_2}^{(n)} \cdots H^{(n)} G_{\nu_p}^{(n)} Q^{(n)}, \tag{49}$$

$$A_k^{(n+1)} = \sum_{i=0}^{k} A_i^{(n)} \sum_{p=1}^{k-i} \sum_{\nu_1+\ldots+\nu_p=k-i} G_{\nu_1}^{(n)} H^{(n)} G_{\nu_2}^{(n)} \cdots H^{(n)} G_{\nu_p}^{(n)} Q^{(n)}, \quad k = 0, 1, \ldots . \tag{50}$$

The new reduced equations (48) are also coupled with the normalization conditions

$$X_k^{(n+1)} \xi_{m_{n+1}} = 0, \quad k = 0, 1, \ldots . \tag{51}$$

Proof. Define the new unknowns

$$X_k^{(n+1)} \stackrel{def}{=} X_k^{(n)} Q^{(n)}. \tag{52}$$

The first determining equation (38.0) implies

$$X_0^{(n)} = X_0^{(n)} P^{*(n)}. \tag{53}$$

Substituting the decomposition of the ergodic projection (41) into (53), we get

$$X_0^{(n)} = X_0^{(n)} Q^{(n)} M^{(n)} = X_0^{(n+1)} M^{(n)}. \tag{54}$$

Multiply (38.1) by $Q^{(n)}$ from the right and use (42) to obtain

$$X_0^{(n)} G_1^{(n)} Q^{(n)} = 0. \tag{55}$$

Then, substitute (54) into (55) to get

$$X_0^{(n+1)} M^{(n)} G_1^{(n)} Q^{(n)} = 0$$

or

$$X_0^{(n+1)} G_0^{(n+1)} = 0,$$

where $G_0^{(n+1)} \stackrel{def}{=} M^{(n)} G_1^{(n)} Q^{(n)}$. The above equation is the first required equation (48.0). The n-th step deviation matrix $H^{(n)}$ plays a crucial role in obtaining the next reduced equations. Indeed, consider the following decomposition of $X_1^{(n)}$

$$\begin{aligned} X_1^{(n)} &= X_1^{(n)} P^{*(n)} + X_1^{(n)} (I - P^{*(n)}) \\ &= X_1^{(n)} Q^{(n)} M^{(n)} - X_1^{(n)} G_0^{(n)} H^{(n)} \\ &= X_1^{(n+1)} M^{(n)} - X_1^{(n)} G_0^{(n)} H^{(n)}. \end{aligned} \tag{56}$$

Here, the definition of $X_1^{(n+1)}$ (52) and the property (45) of the deviation matrix $H^{(n)}$ have been used. Now, using (38.1) for $X_1^{(n)} G_0^{(n)}$ and substituting into (56) yields

$$\begin{aligned} X_1^{(n)} &= X_1^{(n+1)} M^{(n)} + X_0^{(n)} G_1^{(n)} H^{(n)} \\ &= X_1^{(n+1)} M^{(n)} + X_0^{(n+1)} M^{(n)} G_1^{(n)} H^{(n)}. \end{aligned} \tag{57}$$

Note that we have expressed the n-th step unknown $X_1^{(n)}$ in terms of new unknowns $X_0^{(n+1)}, X_1^{(n+1)}$. Similar expressions are obtained for $X_k^{(n)}, k \geq 2$.

Substitute now (54) and (57) into (38.2) to get

$$[X_1^{(n+1)} M^{(n)} + X_0^{(n+1)} M^{(n)} G_1^{(n)} H^{(n)}] G_1^{(n)} + X_0^{(n+1)} M^{(n)} G_2^{(n)} Q^{(n)} = 0.$$

Multiplying (38.2) by $Q^{(n)}$ from the right and using (42) yields

$$X_1^{(n+1)} M^{(n)} G_1^{(n)} Q^{(n)} + X_0^{(n+1)} M^{(n)} (G_1^{(n)} H^{(n)} G_1^{(n)} + G_2^{(n)}) Q^{(n)} = 0,$$

or, equivalently,

$$X_1^{(n+1)} G_0^{(n+1)} + X_0^{(n+1)} G_1^{(n+1)} = 0,$$

with $G_1^{(n+1)} \stackrel{def}{=} M^{(n)} (G_1^{(n)} H^{(n)} G_1^{(n)} + G_2^{(n)}) Q^{(n)}$. Thus, we have got the second next step reduced equation (48.1) and an expression for $G_1^{(n+1)}$. The other next step reduced equations are obtained with similar arguments. The general formulas (49) and (50) are proved by induction.

Note that if $X_k^{(n+1)}, k \geq 0$ are known, then the coefficients $X_k^{(n)}, k \geq 0$ are easily calculated by the recurrent formula (47). Indeed,

$$\begin{aligned} X_k^{(n)} &= X_k^{(n)} P^{*(n)} + X_k^{(n)} (I - P^{*(n)}) \\ &= X_k^{(n+1)} M^{(n)} - X_k^{(n)} G_0^{(n)} H^{(n)} \\ &= X_k^{(n+1)} M^{(n)} + \left\{ \begin{array}{ll} \sum_{i=0}^{k-1} X_i^{(n)} G_{k-i}^{(n)} H^{(n)}, & k < s - n, \\ \sum_{i=0}^{k-1} X_i^{(n)} G_{k-i}^{(n)} H^{(n)} - A_{k+n-s}^{(n)} H^{(n)}, & k \geq s - n, \end{array} \right. . \end{aligned}$$

Finally, we show that the normalization condition (51) holds. To prove this, we need the second identity in (43) and the property of the ergodic projection $P^{*(n)}\xi_{m_n} = \xi_{m_n}$. For example, consider the case $k < s-n$.

$$
\begin{aligned}
0 = X_k^{(n)}\xi_{m_n} &= X_k^{(n+1)}M^{(n)}\xi_{m_n} + \sum_{i=0}^{k-1} X_i^{(n)}G_{k-i}^{(n)}H^{(n)}\xi_{m_n} \\
&= X_k^{(n+1)}\xi_{m_{n+1}} + \sum_{i=0}^{k-1} X_i^{(n)}G_{k-i}^{(n)}H^{(n)}P^{*(n)}\xi_{m_n} \\
&= X_k^{(n+1)}\xi_{m_{n+1}},
\end{aligned}
$$

since $H^{(n)}P^{*(n)} = 0$.

QED

Note that the $(n+1)$-th step reduced system (48) has exactly the same structure as the n-th step reduced system (38). The only one (but important) difference between the structures of these two systems is that the system (48) has **less** equations with null right-hand-sides. Thus, after s reduced steps, the system (below) of reduced equations has non-zero right-hand-sides.

(58.0) $\quad X_0^{(s)}G_0^{(s)} = A_0^{(s)}$

(58.1) $\quad X_1^{(s)}G_0^{(s)} + X_0^{(s)}G_1^{(s)} = A_1^{(s)}$

...

(58.k) $\quad X_k^{(s)}G_0^{(s)} + ... + X_0^{(s)}G_k^{(s)} = A_k^{(s)}$

...

The next proposition gives the simple recurrent formulas for the solution of the (final step) reduced system (58).

Proposition 1 The solutions $X_k^{(s)}, k = 0, 1, ...$ of the system (58) are given by

(59)
$$
X_0^{(s)} = -A_0^{(s)}H^{(s)}; \; X_k^{(s)} = [\sum_{i=0}^{k-1} X_i^{(s)}G_{k-i}^{(s)} - A_k^{(s)}]H^{(s)}, \quad k = 1, \dots .
$$

Proof. The final-step aggregated generator $G_0^{(s)}$ has the same number of ergodic classes as the perturbed chain [8]. Hence, in view of Assumption A, the aggregated generator $G_0^{(s)}$ is a unichain generator and the corresponding ergodic projection is just $P^{*(s)} = \xi_{m_s}\pi_0^{(s)}$, where $\pi_0^{(s)} \in \mathbb{R}^{1\times m_s}$ is a unique stationary distribution of the aggregated generator $G_0^{(s)}$.

Of course, the final-step reduced system (58) is coupled with the normalization conditions $X_k^{(s)}\xi_{m_s} = 0, k = 0, 1, \dots$. Multiplying by $\pi_0^{(s)}$, we get

(60)
$$
X_k^{(s)}\xi_{m_s}\pi_0^{(s)} = X_k^{(s)}P^{*(s)} = 0, \quad k = 0, 1, \dots .
$$

Now, using the modified normalization conditions (60) and the decomposition of $X_k^{(s)}$ into subspaces $\mathcal{R}(P^{*(s)})$ and $\mathcal{R}(I-P^{*(s)})$, we obtain the recurrent formulas (59)

$$\begin{aligned} X_k^{(s)} &= X_k^{(s)}P^{*(s)} + X_k^{(s)}(I-P^{*(s)}) \\ &= X_k^{(s)}(I-P^{*(s)}) = -X_k^{(s)}G_0^{(s)}H^{(s)} \\ &= [\sum_{i=0}^{k-1} X_i^{(s)}G_{k-i}^{(s)} - A_k^{(s)}]H^{(s)}, \end{aligned}$$

QED

Using Theorem 2 and Proposition 1, we are able to outline a practical algorithm for the computation of matrices $Z_{-k}, k=\overline{0,s}$.

Computational algorithm:

1. Determine the order of singularity s, using for instance, the method proposed in [12] (knowledge of the order of singularity makes the algorithm more straightforward and economical).
2. By using the formulas in Theorem 2, make s reduction steps (i.e. successively calculate the matrices $G_k^{(n)}, k=\overline{0,2s-n}$ and the right-hand-sides $A_k^{(n)}, k=\overline{0,s}$ for $n=\overline{1,s}$). As a result, one gets the final system of reduced determining equations (58).
3. Compute $X_k^{(s)}, k=\overline{0,s}$ in (58) using the recurrent formulas (59).
4. Using (47), restore successively all the $X_k^{(n)}, k=\overline{0,s}$ from $n=s-1$ down to $n=0$. In particular,
$$H_k = X_{k+s} = X_{k+s}^{(0)}, \quad k=\overline{-s,0}.$$
5. Finally, via (31), compute the matrices of the fundamental matrix expansion
$$Z_{-k}=H_{-k}, k=\overline{1,s}; \quad Z_0 = H_0 + P_0^* \ .$$

Now, let us make several remarks about the reduction process.

Remark 3 To compute the matrices $G_k^{(n+1)}$ and $A_k^{(n+1)}$, instead of using (49) and (50), one may also use recurrent formulas that are more efficient. In the spirit of Remark 2 and [17, 27, 28], define

$$V_k^{(n)} = \sum_{p=1}^{k+1} \sum_{\nu_1+\ldots+\nu_p=k+1} G_{\nu_1}^{(n)}H^{(n)}G_{\nu_2}^{(n)}\cdots H^{(n)}G_{\nu_p}^{(n)},$$

which can be calculated by the simple recurrent formula

(61) $$V_k^{(n)} = G_{k+1}^{(n)} + \sum_{i=1}^{k} G_i^{(n)}H^{(n)}V_{k-i}^{(n)}, \quad k=0,1,\ldots\ .$$

We then have

$$G_k^{(n+1)} = M^{(n)} V_k^{(n)} Q^{(n)} \quad \textit{and} \quad A_k^{(n+1)} = \sum_{i=0}^{k} A_i^{(n)} V_{k-i-1}^{(n)} Q^{(n)}. \tag{62}$$

Remark 4 One may obtain every *matrix* $Z_k,\ k = \overline{-s,\infty}$ *by the reduction process. However, in the case of linear perturbation the matrices* Z_k *with indicies* $k > 0$ *can be computed more efficiently via the recurrent formulas (29),(30) (see Remark 2).*

Finally, we discuss the computational complexity of the above algorithm. Obviously, Steps 2 and 4 have the highest computational burden. In fact, Step 2 even prevails. Therefore, it suffices to estimate the number of arithmetic operations in Step 2 to obtain the computational complexity of the reduction process.

Step 2 consists of s reduction steps. Note that the first reduction step is the most demanding from a computational point of view, since it reduces the determining system from the full state space into the aggregated chain subspace with dimension m_1 equal to the number of ergodic classes in the original unperturbed chain. It is not difficult to see that the number of operations in this procedure is $O(s^2N^3)$. Indeed, multiplying two $N \times N$-matrices requires $O(N^3)$ operations and the recurrent formulas (61),(62) for $k = \overline{0, 2s-1}$ require approximately $O(s^2)$ such multiplications. After this crucial first step, we only deal with matrices whose dimension does not exceed m_1. The complexity of the other reduction steps can be estimated as $O(s^3m_1^3)$. Thus, Step 2 requires $O(s^2N^3 + s^3m_1^3)$ operations.

Let us now discuss this evaluation. In most practical applications $m_1 << N$ and $s << N$, i.e. the number of ergodic classes and the order of singularity are much less than N, the number of states of the original chain. Therefore, the complexity of the algorithm is in fact $O(N^3)$ (or $O(N^4)$, see remark below about the determination of s). However, if $m_1 \sim N$, then the complexity of the algorithm is $O(s^3N^3)$. The latter may increase significantly (even up to $O(N^6)$), if s is of the same order of magnitude as N. However, the authors are convinced that the cases of large s are quite rare.

Finally, note that the determination of s by the method proposed in [12] requires $O(N^4)$ operations. However, if s has not been determined, then the above reduction-process algorithm can be run with $s := 1, 2, \ldots$ until $G_0^{(s)}$ has rank $m_s - 1$, in which case s is the order of the pole. Therefore, assuming that $m_1 \ll N$, the computational complexity to determine <u>both</u> s and the singular part Z^s is just $O(s^3N^3)$. When $s \ll N$ (as one may expect in practice), compare with $O(N^4)$ just for getting s in [12].

5. Computing mean first-passage times

The fundamental matrix expansion (9) is very useful to get a close approximation of mean first-passage times of the singularly perturbed Markov chain. In particular, it gives the time-scales to link states in different ergodic classes (of the unperturbed chain), an crucial information (e.g. for simulation).

For a regular Markov chain, the *mean first-passage time* between states i and j are given by the formula [14, 16]

$$m^{ij} = \frac{Z^{jj} - Z^{ij}}{\pi^j}, \tag{63}$$

where $Z := \{Z^{ij}\},\ i,j \in \overline{1,N}$ and $\pi := \{\pi^j\},\ j \in \overline{1,N}$ are the fundamental matrix and the stationary distribution of the Markov chain.

Since by Assumption (A), the singularly perturbed Markov chain is regular, (63) is valid, so that

$$m^{ij}(\varepsilon) = \frac{Z^{jj}(\varepsilon) - Z^{ij}(\varepsilon)}{\pi^j(\varepsilon)}, \quad \varepsilon \neq 0. \tag{64}$$

Substituting the expansions (5) and (9) into (64), we immediately obtain the Laurent series expansion

$$m^{ij}(\varepsilon) = \frac{1}{\varepsilon^{q_{ij}}} m^{ij}_{-q_{ij}} + \ldots + \frac{1}{\varepsilon} m^{ij}_{-1} + m^{ij}_0 + \varepsilon m^{ij}_1 + \ldots, \quad \varepsilon \neq 0.$$

Note that in general, the order of singularity q_{ij} of $m^{ij}(\varepsilon)$, is different from the order of singularity s of the fundamental matrix, and depends on the particular states i, j considered.

Hassin and Haviv [12], Haviv and Ritov [11] have proposed a nice elementary example of a singularly perturbed Markov chain with second-order singularity. Our reduction process and the Laurent series expansion for mean first passage times are illustrated on this elementary example.

Consider the singularly perturbed Markov chain with following transition matrix

$$P(\varepsilon) = P_0 + \varepsilon G_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \varepsilon \begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

It is a second-order perturbation (cf. [11, 12]). Hence, the reduction process consists of just two steps. Using standard definitions [14, 16] we get the matrices of normalized left and right eigenvectors M, Q, the ergodic projection P^* and the reduced resolvent H of the unperturbed chain, and using [8], the Taylor's series expansion of the perturbed ergodic projection $P^*(\varepsilon) =$

$P_0^* + \varepsilon P_1^* + ...$

$$M^{(0)} = M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; Q^{(0)} = Q = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix};$$

$$P^{*(0)} = P^* = QM$$

$$H^{(0)} = H = (I - P_0 + P^*)^{-1} - P^* = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_0^{(0)} = P_0^* - I = \begin{bmatrix} -1 & 0.5 & 0 & 0.5 \\ 0 & -0.5 & 0 & 0.5 \\ 0 & 0.5 & -1 & 0.5 \\ 0 & 0.5 & 0 & -0.5 \end{bmatrix}; A_1^{(0)} = P_1^* = \begin{bmatrix} 0.5 & -0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & 0.5 & -0.5 \end{bmatrix};$$

$$A_2^{(0)} = P_2^* = \begin{bmatrix} -0.5 & 0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & -0.5 & 0.5 \end{bmatrix};$$

Now, after the first step of the reduction process, we have

$$G_0^{(1)} = MG_1Q = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}; G_1^{(1)} = MG_1HG_1Q = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix};$$

$$G_2^{(1)} = MG_1HG_1HG_1Q = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}; G_3^{(1)} = MG_1HG_1HG_1HG_1Q = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix};$$

$$A_0^{(1)} = A_0^{(0)}Q = \begin{bmatrix} -0.5 & 0.5 \\ -0.5 & 0.5 \\ 0.5 & -0.5 \\ 0.5 & -0.5 \end{bmatrix}; A_1^{(1)} = (A_1^{(0)} + A_0^{(0)}HG_1)Q = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ -1 & 1 \\ 0 & 0 \end{bmatrix};$$

$$A_2^{(1)} = (A_2^{(0)} + A_1^{(0)}HG_1 + A_0^{(0)}HG_1HG_1)Q = \begin{bmatrix} -1 & 1 \\ 0 & 0 \\ 1 & -1 \\ 0 & 0 \end{bmatrix}.$$

In this particular example, $M^{(1)} = Q^{(1)} = I_2, P^{*(1)} = I_2$ and $H^{(1)} = 0$, so that the next reduction step is straightforward. Namely,

$$G_0^{(2)} = G_1^{(1)} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}; G_1^{(2)} = G_2^{(1)} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}; G_2^{(2)} = G_3^{(1)} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix};$$

and $A_k^{(2)} = A_k^{(1)}, k = 0, 1, 2$.

The above matrices are used in the (final step) reduced system (58) that can be solved easily via the recurrent formulas (59).

$$P^{*(2)} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}, H^{(2)} = H = (P^{*(2)} - G_0^{(2)})^{-1} - P^{*(2)} = \begin{bmatrix} 0.25 & -0.25 \\ -0.25 & 0.25 \end{bmatrix};$$

$$X_0^{(2)} = -A_0^{(2)} H^{(2)} = \begin{bmatrix} 0.25 & -0.25 \\ 0.25 & -0.25 \\ -0.25 & 0.25 \\ -0.25 & 0.25 \end{bmatrix}; X_1^{(2)} = (X_0^{(2)} G_1^{(2)} - A_1^{(2)}) H^{(2)} = \begin{bmatrix} -0.25 & 0.25 \\ 0.25 & -0.25 \\ 0.25 & -0.25 \\ -0.25 & 0.25 \end{bmatrix};$$

$$X_2^{(2)} = (X_0^{(2)} G_2^{(2)} + X_1^{(2)} G_1^{(2)} - A_2^{(2)}) H^{(2)} = 0$$

Then, we use the recurrent formulas (47) to successively restore $X_k^{(1)}, k = 0, 1, 2$ and $X_k^{(0)}, k = 0, 1, 2$. Again, due to $M^{(1)} = I_2$ and $H^{(1)} = 0$, we have $X_k^{(1)} = X_k^{(2)}, k = 0, 1, 2$.

$$X_0^{(0)} = X_0^{(1)} M^{(0)} = \begin{bmatrix} 0 & 0.25 & 0 & -0.25 \\ 0 & 0.25 & 0 & -0.25 \\ 0 & -0.25 & 0 & 0.25 \\ 0 & -0.25 & 0 & 0.25 \end{bmatrix};$$

$$X_1^{(0)} = X_1^{(1)} M^{(0)} + X_0^{(0)} G_1 H^{(0)} = \begin{bmatrix} 0.25 & -0.5 & -0.25 & 0.5 \\ 0.25 & 0 & -0.25 & 0 \\ -0.25 & 0.5 & 0.25 & -0.5 \\ -0.25 & 0 & 0.25 & 0 \end{bmatrix};$$

$$X_2^{(0)} = X_2^{(1)} M^{(0)} + (X_1^{(0)} G_1 - A^{(0)}) H^{(0)} = \begin{bmatrix} 0.5 & -0.5 & 0.5 & -0.5 \\ 0 & 0 & 0 & 0 \\ 0.5 & -0.5 & 0.5 & -0.5 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Finally, from (31), we get

$$Z_{-2} = X_0^{(0)},\ Z_{-1} = X_1^{(0)},\ Z_0 = X_2^{(0)} + P_0^* = \begin{bmatrix} 0.5 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0.5 \end{bmatrix}.$$

Using MATLAB symbolic toolbox for a direct expansion of $(I-P(\varepsilon)+P^*(\varepsilon))^{-1}$, one may check that both results coincide. Schweitzer's formula (6) also permits to check the validity of the Taylor's series expansion of $\pi(\epsilon)$. Indeed,

$$\pi_k = \pi_0 (G_1 Z_0)^k, \quad k = 0, 1, \ldots .$$

Substituting the expression obtained for Z_0 into the above equation, we get

$$\pi_k = (-1)^k [\ -0.5 \ \ 0.5 \ \ -0.5 \ \ 0.5\ ], \quad k = 1, 2, \ldots ,$$

which coincides with the expression for $\pi_k, k \geq 1$ given in [11].

Now, let us apply the coefficients of the fundamental matrix expansion to compute mean first-passage times via formula (64). Due to the symmetry of this particular example, one only needs to compute the mean first-passage times to states 1 and 2.

$$m^{21}(\varepsilon) = \frac{Z^{11}(\varepsilon) - Z^{21}(\varepsilon)}{\pi^1(\varepsilon)} = [(\frac{0.25}{\varepsilon} + 0.5 + \ldots) - (\frac{0.25}{\varepsilon} + 0 + \ldots)]/[0.5\varepsilon - 0.5\varepsilon^2 + \ldots] = \frac{1}{\varepsilon} + \ldots,$$

$$m^{31}(\varepsilon) = \frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} + \ldots, \quad m^{41}(\varepsilon) = \frac{1}{\varepsilon^2} + \frac{2}{\varepsilon} + \ldots,$$

$$m^{12}(\varepsilon) = \frac{1}{\varepsilon} + \ldots, \quad m^{32}(\varepsilon) = \frac{1}{\varepsilon^2} + \frac{0}{\varepsilon} + \ldots, \quad m^{42}(\varepsilon) = \frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} + \ldots .$$

The mean first-passage times exhibit different time scales. Namely, the mean first-passage times $m^{21}(\varepsilon), m^{12}(\varepsilon)$ between states in the same ergodic class are of order ε^{-1} whereas the mean first-passage times $m^{31}(\varepsilon), m^{41}(\varepsilon), m^{32}(\varepsilon), m^{42}(\varepsilon)$, i.e. between states in different ergodic classes, are of order ϵ^{-2}. It confirms the intuition that a transition between states in different ergodic classes takes much more time to occur than a transition between states in the same class. Finally, we would like to emphasize that the present approach permits to handle perturbations with <u>any</u> order of singularity, whereas Latouche and Louchard [20] restrict to the case of first-order singularity.

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