Laurent series for the inversion of perturbation operators on Hilbert space

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Abstract

The paper studies the perturbation operators on Hilbert spaces which depend on complex parameter. Linear, polynomial and analytical perturbations are considered. It is demonstrated that polynomial and analytical perturbations can be transformed to the case of linear perturbed operators in augmented space. Then, the generalization of an efficient algorithm of Schweitzer and Stewart can be applied to obtain a Laurent expansion for the inverse of perturbed operators.

1 Introduction

In this paper we consider the perturbation of linear operators on Hilbert space. The perturbation is characterized by a complex parameter z. We study three types of perturbations: linear, polynomial and analytic. The linear perturbation is represented by

$$A(z) = A + zB, \tag{1}$$

where $A, B \in \mathcal{L}(H,K)$ and H, K are Hilbert spaces. The polynomial perturbation is given by

$$A(z) = A_0 + zA_1 + ... + z^m A_m, \tag{2}$$

where $A_0, A_1, ..., A_m \in \mathcal{L}(H,K)$. We refer to $A(z)$ as a perturbed operator and to A, B as unperturbed operators. In the case of analytical perturbation the perturbed operator can be represented by power series

$$A(z) = A_0 + zA_1 + z^2 A_2 + ... \tag{3}$$

which is convergent in some neighbourhood of $z = 0$. Moreover, we destinct between *regular* and *singular* perturbations [1]. Namely, the perturbation is said to be regular if the unperturbed operator A (or A_0) is invertible, i.e. it is a one-to-one mapping from H onto K. If the unperturbed operator is not invertible but the perturbed operator $A(z)$ has an inverse for some domain of z around the origin, then the perturbation is said to be singular.

A lot of work has been done in the perturbation analysis of eigenvalue problem. An interested reader can find excellent expositions in the books of Kato [6] and Baumgartel [4]. However, there are only few papers on the inversion of perturbed operators [1, 5, 8, 10, 14, 16]. Moreover, practically all papers on this subject concern with finite dimensional spaces.

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The present work studies the operator perturbation in general Hilbert spaces. The infinite dimensional case has been also analyzed in [7] and [15]. However, we introduce less restrictive conditions. For example, in [7] the authors demand that in the case of singular perturbations the zero eigenvalue of the unperturbed operator is to be semisimple and in [15] the authors ask for finite dimensionality of the null space of the unperturbed operator. The approach of this paper does not require these assumptions. Furthermore, here we study not only the case of linear perturbations but also the case of polynomial perturbations and more general case of analytical perturbations. Finally, we would like to note that the iterative method of Section 3 is a generalization of the finite dimensional method proposed in the paper of Schweitzer and Stewart [14] which seems to be most computationally efficient.

The paper is organized as follows: In Section 2 we apply the classical Neumann series expansion to the regular perturbation case. This makes the paper self-contained. In Section 3 we explore the properties of inverses for the perturbed operators in general Hilbert spaces. There we also provide an efficient iterative procedure for the inversion of linearly perturbed operators. Then in Section 4 we show how the results of the previous section can be applied to polynomial and analytical perturbations. At the end of the paper we place two illustrative examples.

2 Regular perturbation

Firstly, in order to make the paper self-contained, let us consider the application of classical Neumann series expansion to the case of regular perturbations. The results can be summarized as follows:

Theorem 1 *Let $A, B \in \mathcal{L}(H, K)$ and let A be a one-to-one mapping of H onto K. Then the perturbation operator can be expanded as Maclaurin series*

$$(A + Bz)^{-1} = \sum_{n=0}^{\infty} z^n (-A^{-1}B)^n A^{-1} \tag{4}$$

for all $|z| < 1/\|A^{-1}B\|$.

PROOF: Note that

$$(A + zB)^{-1} = (A(I + zA^{-1}B))^{-1} = (I + zA^{-1}B)^{-1}A^{-1}$$

Then, by applying the Neumann series to $(I + zA^{-1}B)^{-1}$, we obtain the expansion (4) with the radius of convergence $1/\|A^{-1}B\|$.

□

Note that the case of polynomial or analytical perturbations which are regular can be treated in a similar way.

3 Singular perturbation

In this section we consider the case of singular perturbation, that is when the operator A does not have an inverse but the perturbed operator $A + zB$ does for sufficiently small z but distinct from zero. It turns out that the inverse of the perturbed operator can be expanded as a Laurent series under certain conditions. Here we propose the conditions for the existence of Laurent series with a finite order pole and provide an algorithm for the calculation of its coefficients. The next lemma provides an auxilirary result.

Lemma 1 *Let $A, B \in \mathcal{L}(H,K)$ be bounded linear maps. Let $M = Null(A)$ and define $N = B(M) \subseteq K$. If there exists some fixed value $z_0 \in \mathbb{C}$ with $z_0 \neq 0$ for which $(A + Bz_0)^{-1}$ exists then B is bounded below on M and N is a closed subspace of K.*

PROOF: By the Banach Inverse Theorem the map $(A + Bz_0)$ is bounded below on H. Therefore we can find $\epsilon > 0$ such that

$$\|(A + Bz_0)h\| \geq \epsilon \|h\|$$

for all $h \in H$. Since $Am = 0$ for all $m \in M$ it follows that

$$\|Bm\| \geq \frac{\epsilon}{|z_0|} m$$

for all $m \in M$. Now if $\{n_r\}$ is a Cauchy sequence in $N = B(M)$ then $n_r = Bm_r$ where $\{m_r\}$ is a corresponding sequence in M. Because B is bounded below on M the sequence $\{m_r\}$ must also be a Cauchy sequence. And since the null space is always closed, the sequence $\{m_r\}$ has a limit in $M = N(A)$. If $m_r \to m$ and $n_r \to n$ then $Bm = n$ as B is a continuous operator. The later implies that $n \in B(M) = N$ and hence N is a closed subspace of K.

□

Since $M = Null(A)$ and $N = B(M)$ are closed subspaces and since the respective orthogonal complements are also closed we can decompose the Hilbert spaces H and K as follows: $H = M \oplus M^{\perp}$ and $K = N \oplus N^{\perp}$. Let $P : H \mapsto M$ and $Q : K \mapsto N$ denote the corresponding projection operators. The following properties will be used in the sequel.

$$AP = 0 \tag{5}$$

$$(I - Q)BP = 0 \tag{6}$$

Now we are able to formulate the main result of this section. This is a generalization of the results given in Schweitzer and Stewart [14].

Theorem 2 *Suppose that $A \in \mathcal{L}(H,K)$ is singular. Then the mapping $(A + Bz) \in \mathcal{L}(H,K)$ is a one-to-one mapping of H onto K for some $z \neq 0$ if and only if $(A|_{M^{\perp},N^{\perp}} + zB|_{M^{\perp},N^{\perp}}) \in \mathcal{L}(M^{\perp}, N^{\perp})$ is a one-to-one mapping of $M^{\perp}$ onto $N^{\perp}$ for that z and $B|_{M,N} \in \mathcal{L}(M,N)$ is a one-to-one mapping of M onto N. In this case we have*

$$\begin{aligned}(A + zB)^{-1} &= \frac{1}{z} B|_{M,N}^{-1} Q \\ &\quad + [I - \frac{1}{z} B|_{M,N}^{-1}(A|_{M^{\perp},N} + zB|_{M^{\perp},N})](A|_{M^{\perp},N^{\perp}} + zB|_{M^{\perp},N^{\perp}})^{-1}(I - Q)\end{aligned} \tag{7}$$

In addition, if $A|_{M^{\perp},N^{\perp}} \in \mathcal{L}(M^{\perp}, N^{\perp})$ is a one-to-one mapping of $M^{\perp}$ onto $N^{\perp}$ then $(A + zB)^{-1}$ exists in the region $|z| < 1/\|A|_{M^{\perp},N^{\perp}}^{-1} B|_{M^{\perp},N^{\perp}}\|$ and can be expanded in this region as a Laurent series with a pole of order 1 at $z = 0$.

PROOF: Since by Lemma 1 underlying Hilbert spaces can be decomposed in the direct sums $H = M \oplus M^{\perp}, K = N \oplus N^{\perp}$, for any $x \in K$ and $y \in H$ we can write $x = x_M + x_M^{\perp}$, where $x_M = Px \in M, x_M^{\perp} = (I - P)x \in M^{\perp}$ and $y = y_N + y_N^{\perp}$, where $y_N = Py \in N, y_N^{\perp} = (I - Q)y \in M^{\perp}$. Then, using properties (5) and (6), one can write the equation $(A + zB)x = y$ in the form of the following equivalent system.

$$\begin{aligned} zB|_{M,N} x_M + (A|_{M^{\perp},N} + zB|_{M^{\perp},N}) x_M^{\perp} &= y_N \\ (A|_{M^{\perp},N^{\perp}} + zB|_{M^{\perp},N^{\perp}}) x_M^{\perp} &= y_N^{\perp} \end{aligned}$$

or, equivalently,

$$\begin{aligned} (A|_{M^\perp,N^\perp} + zB|_{M^\perp,N^\perp})x_M^\perp &= y_N^\perp \\ zB|_{M,N}x_M &= y_N - (A|_{M^\perp,N} + zB|_{M^\perp,N})x_M^\perp \end{aligned}$$

It is clear that the above system has a unique solution for $z \neq 0$ if and only if $B|_{M,N}$ is a one-to-one mapping of M onto N and $(A|_{M^\perp,N^\perp} + B|_{M^\perp,N^\perp}z)$ is a one-to-one mapping of $M^\perp$ onto $N^\perp$. Solving this system of equations gives.

$$\begin{aligned} x_M &= \frac{1}{z}B|_{M,N}^{-1} - \frac{1}{z}B|_{M,N}^{-1}(A|_{M^\perp,N} + zB|_{M^\perp,N})(A|_{M^\perp,N^\perp} + zB|_{M^\perp,N^\perp})^{-1}y_N^\perp \\ x_M^\perp &= (A|_{M^\perp,N^\perp} + zB|_{M^\perp,N^\perp})^{-1}y_N^\perp \end{aligned}$$

Note that we can write $B|_{M,N}^{-1}$, $(A|_{M^\perp,N^\perp} + zB|_{M^\perp,N^\perp})^{-1}$, since $B|_{M,N}$ and $(A|_{M^\perp,N^\perp} + zB|_{M^\perp,N^\perp})$ are one-to-one and onto mappings. Finally, we obtain

$$x = (A + zB)^{-1}y = x_M + x_M^\perp$$

$$= \frac{1}{z}B|_{M,N}^{-1}Qy - \frac{1}{z}B|_{M,N}^{-1}(A|_{M^\perp,N} + zB|_{M^\perp,N})(A|_{M^\perp,N^\perp} + zB|_{M^\perp,N^\perp})^{-1}(I-Q)y$$
$$+(A|_{M^\perp,N^\perp} + zB|_{M^\perp,N^\perp})^{-1}(I-Q)y$$
$$= \frac{1}{z}B|_{M,N}^{-1}Qy + [I - \frac{1}{z}B|_{M,N}^{-1}(A|_{M^\perp,N} + zB|_{M^\perp,N})](A|_{M^\perp,N^\perp} + zB|_{M^\perp,N^\perp})^{-1}(I-Q)y$$

This proves formula (7). Now suppose that $A|_{M^\perp,N^\perp}$ has an inverse. Then, according to Theorem 1, the perturbed operator $(A|_{M^\perp,N^\perp} + zB|_{M^\perp,N^\perp})$ is invertible in the region $|z| < 1/\|A|_{M^\perp,N^\perp}^{-1}B|_{M^\perp,N^\perp}\|$ and hence the operator $(A+zB)$ is invertible in the punctured disc $0 < |z| < 1/\|A|_{M^\perp,N^\perp}^{-1}B|_{M^\perp,N^\perp}\|$. Upon substituting the Maclaurin series (4) for $(A|_{M^\perp,N^\perp} + zB|_{M^\perp,N^\perp})$ into (7), one immediately obtains a Laurent series for $(A+zB)$ with a pole of order one. This completes the proof.

□

Remark 1 *The utility of decompositions $H = M \oplus M^\perp, K = N \oplus N^\perp$ can be clearly seen with the help of the next block-operator representation*

$$A + zB = \begin{bmatrix} zB|_{M,N} & A|_{M^\perp,N} + zB|_{M^\perp,N} \\ 0 & A|_{M^\perp,N^\perp} + zB|_{M^\perp,N^\perp} \end{bmatrix}.$$

Now let us note that Theorem 2 points out to the iterative procedure for the computation of the Laurent series for $(A+zB)^{-1}$. Indeed, if the operator $(A+zB)$ is invertible in some domain around $z=0$, then the *reduced* operator $(A|_{M^\perp,N^\perp} + zB|_{M^\perp,N^\perp})$ is invertible as well. Note that the operator $(A|_{M^\perp,N^\perp} + zB|_{M^\perp,N^\perp})$ satisfies all conditions of Theorem 2 and hence formula (7) can be applied agian. If a new unperturbed reduced operator is invertible, then the orginal perturbed operator $(A+zB)$ can be expanded in a Laurent series with the pole of order two and we can terminate this *reduction* process. Otherwise, we can make the next iteration. Note that the authors also proposed a different reduction process in [1]. However, these two reduction processes have the folowing common fiture: the reduced operators act on "smaller" underlying spaces. In the case of finite dimensional Hilbert spaces it implies that the reduction process has a finite number of steps and the inverse of the linearly perturbed operator can be always expanded as a Laurent series with a finite order pole [1, 13, 14]. However, in the case of infinite dimensional Hilbert spaces the inverse operator $(A+zB)^{-1}$ can have an essensial singularity at $z=0$ and therefore the described above iterative process on principle can go on indefenitely.

4 Extention to polynomial and analytic perturbation

Now let us show that the method described in the previous section can be easily aplied to polynomial or even analytical perturbations. First, let us show that the polynomial perturbation (2) can be transformed to an equivalent linear perturbation problem.

Note that the problem of inversion of the perturbation operator is equivalent to the solution of the infinite system of *fundamental* equations. We formulate the next result for the polynomial perturbation, however it is also true for the general case of analytic perturbations.

Lemma 2 *Let the inverse of perturbed operator $A(\varepsilon)$ can be expanded as a Laurent series*

$$A^{-1}(z) = \frac{1}{z^s}(X_0 + zX_1 + z^2X_2 + ...). \tag{8}$$

Then the coefficients $\{X_k\}_{k=0}^{\infty}$ form a unique solution to the system of fundamental equations

$$\sum_{i=0}^{m} A_i X_{l-i} = \begin{cases} I, & l = s, \\ 0, & otherwise \end{cases} \tag{9}$$

PROOF:

□

The above lemma points out to an alternative way of computing Laurent series expansion for the inverse of the perturbed operator. This approach was investigated in [1]. Now we want to demonstrate that the method of Schweitzer and Stewart can be also applied to polynomial and analytical perturbations. It can be done by transforming the original perturbation problem to some *linear* perturbation problem in *augmented* space.

Theorem 3 *Let the inverse of the polynomial perturbation operator (2) can be expanded as a Laurent series (8) in some punctured neibourhood of the origin. And let us introduce the following augmented operators from H^m to K^m.*

$$\mathcal{A} := \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{m-1} & A_{m-2} & \cdots & A_0 \end{bmatrix} \quad \mathcal{B} := \begin{bmatrix} A_m & A_{m-1} & \cdots & A_1 \\ 0 & A_m & \cdots & A_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_m \end{bmatrix}$$

Then a solution of the linear perturbation problem

$$(\mathcal{A} + z\mathcal{B})\mathcal{X}(z) = z^{s_*}\mathcal{I}, \quad \mathcal{I} = [I \cdots 0]^T \tag{10}$$

can be expanded as a Laurent series

$$\mathcal{X}(z) = \frac{1}{z^{s_*}}\left(\begin{bmatrix} 0 \\ \vdots \\ 0 \\ X_0 \\ \vdots \\ X_{mod(s,m)} \end{bmatrix} + \begin{bmatrix} X_{mod(s,m)+1} \\ \vdots \\ \\ \\ \vdots \\ X_{mod(s,m)+m} \end{bmatrix} + ... \right) \tag{11}$$

where $s_* = \lfloor s/m \rfloor + 1$ and $\{X_k\}_{k=0}^{\infty}$ are coefficients of the Laurent series (8).

PROOF: The proof is based on the fact that the system of fundamental equations for (10) precisely coincides with the system of fundamental equations (9).

□

Thus, we can apply the reduction process of Section 3 first to the augmented problem and then we can easily retrieve the coefficients of the original Laurent series (8).

It is interesting that the augmentation scheme is not unique. For example, the next scheme is proposed in the book of Baumgartel [4]. Even though that method was used for the perturbation analysis of generalized eigenvalue problem, it is also applicable for the inversion of perturbation operators.

Theorem 4 *Let the augmented operators be defined as follows:*

$$\mathcal{A} := \begin{bmatrix} A_0 & A_1 & \cdots & A_{m-1} \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \end{bmatrix} \qquad \mathcal{B} := \begin{bmatrix} 0 & 0 & \cdots & A_m \\ -I & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & -I & 0 \end{bmatrix}$$

Then the most upper block of the solution $\mathcal{X}(z)$ for equation (10) with $s_* = s$ coincides with the solution of the original equation $A(z)X(z) = z^s I$.

Each of the presented augmentation schemes has its own merits. Using the second method to obtain first k coefficients of Laurent series (8) one needs to calculate k augmented coefficients $\mathcal{X}_i, i = 0, ..., k-1$, whereas if one utilizes the first augmentation scheme, he needs to compute approximately m times less augmented coefficients. However, both augmented matrices $\mathcal{A}$ and $\mathcal{B}$ are close to upper triangular form and have a lot of zero elements in the second method. Since by using Schweitzer and Stewart method we try to put simultaneously matrices $\mathcal{A}$ and $\mathcal{B}$ in triangular form, each iteration of the reduction process for augmented matrices of the second method could be more computationally efficient.

Now we show that in turn the results on polynomial perturbations can be applied to the general case of analytical perturbations (3). Suppose again that the Laurent series for the inverse $A^{-1}(z)$ has a pole of finite order s. Then the number of terms which uniquely determine the inversion procedure is finite [1]. Namely, there exists m such that the inverse $(A_0 + ... + z^m A_m)^{-1}$ exists for sufficiently small z. Moreover, any $m \geq s$ can be taken. Therefore we may write

$$\begin{aligned} A^{-1}(z) &= [(A_0 + ... + z^s A_s) + z^{s+1} A_{s+1} + ...]^{-1} \\ &= [(A_0 + ... + z^s A_s)(I + (A_0 + ... + z^s A_s)^{-1}(z^{s+1} A_{s+1} + ...))]^{-1} \\ &= [I + (A_0 + ... + z^s A_s)^{-1}(z^{s+1} A_{s+1} + ...)]^{-1}(A_0 + ... + z^s A_s)^{-1} \end{aligned} \tag{12}$$

By using the method of augmented operators, we obtain

$$(A_0 + ... + z^s A_s)^{-1} = \frac{1}{z^s} \sum_{k=0}^{\infty} z^k \tilde{X}_k.$$

Now substitute the above expansion into (12).

$$\begin{aligned} A^{-1}(z) &= \left[I + \frac{1}{z^s} \sum_{k=0}^{\infty} z^k \tilde{X}_k (z^{s+1} A_{s+1} + z^{s+2} A_{s+2} + ...) \right]^{-1} \frac{1}{z^s} \sum_{k=0}^{\infty} z^k \tilde{X}_k \\ &= \left[I + \sum_{k=1}^{\infty} z^k \sum_{i=0}^{k-1} \tilde{X}_i A_{s+k-i} \right]^{-1} \frac{1}{z^s} \sum_{k=0}^{\infty} z^k \tilde{X}_k \end{aligned}$$

By expanding the expression

$$\left[I + \sum_{k=1}^{\infty} z^k \sum_{i=0}^{k-1} \tilde{X}_i A_{s+k-i} \right]^{-1}$$

as Neumann power series, we obtain the simple recursive formulae for coefficients $X_k, k = 0, 1, ...$ of the Laurent series for $A^{-1}(z)$. For example,

$$X_{-s} = \tilde{X}_{-s},$$

$$X_{-s+1} = \tilde{X}_{-s+1} - \tilde{X}_{-s} A_{s+1} \tilde{X}_{-s},$$

$$X_{-s+2} = \tilde{X}_{-s+2} - \tilde{X}_{-s} A_{s+1} \tilde{X}_{-s+1} - \tilde{X}_{-s+1} A_{s+1} \tilde{X}_{-s} - \tilde{X}_{-s} A_{s+2} \tilde{X}_{-s} + \tilde{X}_{-s} A_{s+1} \tilde{X}_{-s} A_{s+1} \tilde{X}_{-s}.$$

5 Example 1

Let $H = H^2((0,1)) \cap H_0^1((0,1))$ be the Hilbert space of measurable functions $x : (0,1) \mapsto \mathbb{R}$ with

$$\int_{(0,1)} \left[|x(t)|^2 + |x'(t)|^2 + |x''(t)|^2 \right] dt < \infty,$$

$$\langle x_1, x_2 \rangle_H = \int_{(0,1)} \left[x_1(t) x_2(t) + x_1'(t) x_2'(t) + x_1''(t) x_2''(t) \right] dt,$$

and

$$x(0) = x(1) = 0.$$

Let $K = L^2((0,1))$ be the Hilbert space of measurable functions $y : (0,1) \mapsto \mathbb{R}$ with

$$\int_{(0,1)} |y(t)|^2 dt < \infty,$$

and

$$\langle y_1, y_2 \rangle_K = \int_{(0,1)} y_1(t) y_2(t) dt.$$

Define $A \in \mathcal{L}(H, K)$ and $B \in \mathcal{L}(H, K)$ by setting

$$A(x) = x'' + \pi^2 x$$

and

$$B(x) = x$$

for all $x \in H$. Note that $\|x''\|_K^2 \leq \|x\|_H^2$. For each $y \in K$ and $z \in \mathbb{C}$ we wish to find $x \in H$ to solve the differential equation

$$[x''(t) + \pi^2 x(t)] + z x(t) = y(t).$$

With the above notation this equation can be rewritten in the form

$$(A + zB)(x) = y$$

and hence the solution is given by

$$x = (A + zB)^{-1}(y)$$

provided the inverse exists. If we set

$$e_k(t) = \sqrt{2} \sin k\pi t$$

for each $j = 1, 2, \ldots$ and each $t \in (0, 1)$ then it is well known that each $x \in H$ can be written as $x = \sum_{k=1}^{\infty} x_k e_k$ where $\sum_{k=1}^{\infty} [1 + k^2\pi^2]^2 {x_k}^2 < \infty$ and each $y \in K$ can be written as $y = \sum_{k=1}^{\infty} y_k e_k$ where $\sum_{k=1}^{\infty} {y_k}^2 < \infty$. It is easily seen that A is singular because $A(e_1) = 0$. Nevertheless it is easy to see that $(A + zB)$ is non-singular and by equating coefficients in the respective Fourier series that the solution to the differential equation is given by

$$x_1 = y_1/z \ \text{ and } \ x_k = y_k/[z + \pi^2(1 - k^2)] \text{ for } k > 1.$$

By writing the solution in the form

$$\begin{aligned} x &= \frac{y_1 e_1}{z} + \sum_{k=2}^{\infty} \frac{y_k e_k}{\pi^2(1 - k^2)} \left[1 - \frac{z}{\pi^2(1 - k^2)} + \cdots\right] \\ &= \frac{y_1 e_1}{z} + \sum_{k=2}^{\infty} \frac{y_k e_k}{\pi^2(1 - k^2)} - \sum_{k=2}^{\infty} \frac{y_k e_k}{[\pi^2(1 - k^2)]^2} z + \cdots \end{aligned}$$

we can see that the expansion is a Laurent series with a pole of order 1 at $z = 0$.

6 Example 2

Consider a Markov chain with continuous state space. Let $J = [0, 1]$. We consider the set $\mathcal{M}$ of non-negative measurable functions $f : J \mapsto \mathbb{R}$ with

$$\bar{f} = \int_J f(s) ds = 1$$

and the transformation $S : \mathcal{M} \mapsto \mathcal{M}$ defined by

$$Sf(t) = \bar{f}$$

for all $t \in J$. Clearly $S^n = S$ for all $n \in \mathbb{N}$. We consider the transformation $S_\epsilon : \mathcal{M} \mapsto \mathcal{M}$ defined by

$$S_\epsilon = (1 - \epsilon)I + \epsilon S$$

where $I : \mathcal{M} \mapsto \mathcal{M}$ is the identity transformation. The transformation S_ϵ is a perturbation of the identity transformation that allows a small probability of transition between states. Transition times are determined by the operator

$$[I - S_\epsilon + S_\epsilon^\star]^{-1}$$

where $S_\epsilon^\star = \lim_{n\to\infty} S_\epsilon$. Now it is easy to see that

$${S_\epsilon}^n = (1 - \epsilon)^n I + [1 - (1 - \epsilon)^n]S$$

and so $S_\epsilon^\star = S$. The equation

$$[I - S_\epsilon + S_\epsilon^\star]f = g$$

is easily solved to give

$$f = \frac{1}{\epsilon}[g - \bar{g}] + \bar{g}.$$

In this case the operator inversion has the form

$$[I - S_\epsilon + S_\epsilon^\star]^{-1} = [S + \epsilon(I - S)]^{-1}$$

where S is singular. Thus we expect a pole at $\epsilon = 0$ in our Laurent series solution.

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