Stochastic Coalitional Better-response Dynamics for Finite Games with Application to Network Formation Games*

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Abstract

We consider a coalition formation among players, in an n-player strategic game, over infinite horizon. At each time a randomly selected coalition makes a joint deviation, from a current action profile to a new action profile, which is strictly beneficial for all the players belonging to the coalition. Such deviations define a stochastic coalitional better-response (CBR) dynamics. The stochastic CBR dynamics either converges to a $\mathcal{K}$-stable equilibrium or becomes stuck in a closed cycle. We also assume that at each time a selected coalition makes mistake in deviation with small probability. We prove that all $\mathcal{K}$-stable equilibria and all action profiles from closed cycles, having minimum stochastic potential, are stochastically stable. Similar statement holds for strict $\mathcal{K}$-stable equilibrium. We apply the stochastic CBR dynamics to the network formation games. We show that all strongly stable networks and closed cycles of networks are stochastically stable.

Keywords— Strong Nash equilibrium, Coalitional better-response, Stochastic stability, Network formation games, Strongly stable networks.

1 Introduction

In a repeated play of a strategic game over infinite horizon, a Nash equilibrium that is played in the long run depends on an initial action profile as well as the way all the players choose their actions at each time. Young [25] considered an n-player strategic game where at each time all the players make a simultaneous move and each player chooses an action that is the best response to k previous games among the m, $k \leq m$, most recent games in the past. In general, this dynamics need not converge to a Nash equilibrium, it may stuck into a closed cycle. Young [25] also considered the case where at each time with small probability each player makes mistake and chooses some non-optimal action. These mistakes add mutations into the dynamics. In general the mutations can be sufficiently small. This leads to the definition of a stochastically stable Nash equilibrium which is selected by the stochastic dynamics as mutations vanish. Young proposed an algorithm to compute the stochastically stable Nash equilibria. He showed that the risk dominant Nash equilibrium of a 2×2 coordination game is stochastically stable. Kandori et al. [16] considered a different dynamic model where at each time each player plays with every other player in a pairwise contest. The pairwise contest is given by 2×2 symmetric matrix game and each player chooses an action which has higher expected average payoff. The mutations are present into dynamics due to wrong actions taken by the players. They showed

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that the risk dominant Nash equilibrium of a 2×2 coordination game is stochastically stable. That is, for 2×2 coordination games the dynamics given by [25] and [16] select the same Nash equilibrium. Fudenberg et al. [10] proposed a dynamics where at each time only one player is selected to choose actions. The mutations with small probability also occur at each time. The risk dominant Nash equilibrium of a 2×2 coordination game need not be stochastically stable under this dynamics.

The equilibrium concept which are stable against the coalitional deviations are more suitable for the situations where players can a priori communicate, being in a position to form a coalition and jointly deviate in a coordinated way. Sawa [24] introduced such an equilibrium concept which is called (strict) $\mathcal{K}$-stable equilibrium, where $\mathcal{K}$ is a set of all feasible coalitions. A (strict) $\mathcal{K}$-stable equilibrium corresponds to a (strict) strong Nash equilibrium [1] if there is no restriction on coalition formation. As motivated from the application of $\mathcal{K}$-stable equilibria in network formation games considered in [7, 14, 13], we restrict ourselves to only pure actions. A $\mathcal{K}$-stable equilibrium need not always exist and in such case there exist some set of action profiles forming a closed cycle. Recently, some stochastic dynamics due to coalitional deviations have been proposed [4, 19, 24]. Sawa [24] studied the stochastic stability in general finite games where the mutations are present through a logit choice rule. Newton [19] considered the situation where profitable coalitional deviations are given greater importance than unprofitable single player deviations. Avrachenkov et al. [4] studied the stochastic stability for network formation games with teams. In general, the stochastic stability results depend on the way actions being chosen during the infinite play. Some other famous works on stochastic stability in different settings include [8, 9, 10, 20, 21, 11, 23, 18, 22].

In this paper, we consider the coalition formation in a strategic game where at each time players are allowed to form a coalition and make a joint deviation from the current action profile if it is strictly beneficial for all the members of the coalition. Such deviations define a coalitional better-response (CBR) dynamics. We assume that the coalition formation is random and at each time only one coalition can be formed among all feasible coalitions. We also consider the possibility of making wrong decision at each time by the selected coalition. These wrong decisions are made with small probability. These mistakes work as mutations and add perturbations into CBR dynamics. We prove that the perturbed CBR dynamics selects $\mathcal{K}$-stable equilibria or closed cycles, that have minimum stochastic potential among all action profiles, in the long run as mutations vanish. If there is no restriction on coalition formation, CBR dynamics selects all the strong Nash equilibria and closed cycles, i.e., all the strong Nash equilibria and closed cycles are stochastically stable. The similar CBR dynamics can be given for the case where each time a coalition deviate from a current action profile such that all the players from the coalition are at least as well off at new action profile and at least one player is strictly better off. In this case the similar results hold for strict $\mathcal{K}$-stable equilibrium and strict strong Nash equilibrium (SSNE). We apply CBR dynamics corresponding to SSNE to network formation games where nodes (players) of a network form a coalition and make a move to a new network if it offers each player at least as much as it is in the current network and at least one player gets strictly better payoff. We prove that all strongly stable networks and closed cycles are stochastically stable. The CBR dynamics generalizes the stochastic dynamics considered in [4] by considering the general finite games. The stochastic dynamics for pairwise stable networks considered in [15] can also be viewed as a special case of stochastic CBR dynamics.

The paper is organized as follows. Section 2 contains the model and few definitions. We describe the CBR dynamics in Section 3. Section 4 contains the application of CBR dynamics to network formation games. We conclude our paper in Section 5.

2 The Model

We consider an n-player strategic game. Let $N = \{1, 2, \cdots, n\}$ be a finite set of players. For each $i \in N$, let A_i be a finite action set of player i whose generic element is denoted by a_i. Then, $A = \prod_{i=1}^{n} A_i$ is defined as a set of all action profiles whose generic element is denoted by $a = (a_1, a_2, \cdots, a_n)$. The payoff function of player i is defined as $u_i : A \to \mathbb{R}$. We consider the situation where players are allowed to communicate with each other. As a consequence they can form a coalition and revise their strategies jointly. In many situations it may not be feasible to form all types of coalitions. Let $\mathcal{K} \subseteq \mathcal{P}(N) \setminus \phi$ be the set of all feasible coalitions, where $\mathcal{P}(N)$ denotes the power set of N and ϕ an empty set. For a coalition $S \in \mathcal{K}$, define $A_S = \prod_{i \in S} A_i$ whose element is denoted by a_S and a_{-S} denotes an action profile of the players outside S. A coalition of players jointly deviate from a current action profile if new action profile is strictly beneficial for all the players from the coalition. Such deviations are called improving deviations and it leads to the definition of a $\mathcal{K}$-stable equilibrium. In some cases, a coalition of players jointly deviate from a current action profile if at new action profile each player is at least as well off and one player is strictly better off. Such deviations leads to the definition of a strict $\mathcal{K}$-stable equilibrium.

Definition 2.1. *An action profile a^* is said to be a $\mathcal{K}$-stable equilibrium if there is no $S \in \mathcal{K}$ and $a \in A$ such that*

1. $a_i = a_i^*, \ \forall\, i \notin S.$
2. $u_i(a) > u_i(a^*), \ \forall\, i \in S.$

If $\mathcal{K} = \mathcal{P}(N) \setminus \phi$, a $\mathcal{K}$-stable equilibrium is a strong Nash equilibrium (SNE) [1]. Let $A(S, a)$ be the set of all action profiles reachable from a via deviation of coalition S. It is defined as

$$A(S, a) = \{a' | a'_i = a_i, \ \forall\, i \notin S \text{ and } a'_i \in A_i, \ \forall\, i \in S\}.$$

A coalition always has option to do nothing, so $a \in A(S, a)$. Let $\mathcal{I}_1(S, a)$ be a set of improved action profiles reachable from an action profile a via improving deviations of coalition S, i.e.,

$$\mathcal{I}_1(S, a) = \{a' | a'_i = a_i, \ \forall\, i \notin S \text{ and } u_i(a') > u_i(a), \ \forall\, i \in S\}. \tag{1}$$

For an improved action profile $a' \in \mathcal{I}_1(S, a)$, an action profile a'_S of all the players from S is called a *better-response* of coalition S against a fixed action profile a_{-S} of the players outside S. Define, $\overline{\mathcal{I}}_1(S, a) = A(S, a) \setminus \mathcal{I}_1(S, a)$ as a set of all action profiles due to the erroneous decisions of coalition S. The set $\overline{\mathcal{I}}_1(S, a)$ is always nonempty for all S and a because $a \in \overline{\mathcal{I}}_1(S, a)$. A $\mathcal{K}$-stable equilibrium need not always exist. In such a case there exist a set of action profiles lying on a closed cycle and all such action profiles can be reached from each other via an improving path. The definitions of closed cycle and improving path are as follows:

Definition 2.2 (Improving Path). *An improving path from a to a' is a sequence of action profiles and coalitions $a^1, S_1, a^2, \cdots, a^{m-1}, S_{m-1}, a^m$ such that $a^1 = a$, $a^m = a'$ and $a^{k+1} \in \mathcal{I}_1(S_k, a^k)$ for all $k = 1, 2, \cdots, m-1$.*

Definition 2.3 (Cycles). *A set of action profiles C form a cycle if for any $a \in C$ and $a' \in C$ there exists an improving path connecting a and a'. A cycle is said to be a closed cycle if no action profile in C lies on an improving path leading to an action profile that is not in C.*

Theorem 2.4. *There always exists a $\mathcal{K}$-stable equilibrium or a closed cycle of action profiles, or both $\mathcal{K}$-stable equilibrium and closed cycle.*

Proof. An action profile is a $\mathcal{K}$-stable equilibrium if and only if it is not possible for any feasible coalition from set $\mathcal{K}$ to make an improving deviation from it to another action profile. So, start at an action profile. Either it is $\mathcal{K}$-stable equilibrium or there exists a coalition that can make an improving deviation to another action profile. In the first case result is established. For the second case the same thing holds, i.e., either this new action profile is a $\mathcal{K}$-stable equilibrium or there exists a coalition that can make an improving deviation to another action profile. Given the finite number of action profiles, the above process either finds an action profile which is a $\mathcal{K}$-stable equilibrium or it reaches to one of previous profiles, i.e., there exists a cycle. Thus, we have proved that there always exists either a $\mathcal{K}$-stable equilibrium or a cycle. Suppose there are no $\mathcal{K}$-stable equilibria. Given the finite number of action profiles and non-existence of $\mathcal{K}$-stable equilibria there must exists a maximal set C of action profiles such that for any $a \in C$ and $a' \in C$ there exists an improving path connecting a and a', and no action profile in C lies on an improving path leading to an action profile that is not in C. Such a set C is a closed cycle. □

An strict $\mathcal{K}$-stable equilibrium can be defined similarly. An action profile a^* in Definition 2.1 is said to be strict $\mathcal{K}$-stable equilibrium if the condition 1 is same and the condition 2 is $u_i(a) \geq u_i(a^*)$ for all $i \in S$ with at least one strict inequality. Further if $\mathcal{K} = \mathcal{P}(N) \setminus \phi$, a^* is a strict strong Nash equilibrium (SSNE). In this case, for a given action profile a and a coalition $S \in \mathcal{K}$ the set of improved action profiles $\mathcal{I}_2(S, a)$ is defined as,

$$\mathcal{I}_2(S,a) = \{a' | a'_i = a_i, \ \forall\, i \notin S, \text{ and } u_i(a') \geq u_i(a), \ \forall\, i \in S, u_j(a') > u_j(a), \text{ for some } j \in S\}. \tag{2}$$

and $\overline{\mathcal{I}}_2(S,a) = A(S,a) \setminus \mathcal{I}_2(S,a)$. The definitions of improving path and cycles can be defined analogously to previous case. A result similar to Theorem 2.4 holds, i.e., there always exists at least a strict $\mathcal{K}$-stable equilibrium or a closed cycle of action profiles or both. A strict $\mathcal{K}$-stable equilibrium is always a $\mathcal{K}$-stable equilibrium, i.e., the set of strict $\mathcal{K}$-stable equilibrium is a subset of the set of $\mathcal{K}$-stable equilibrium.

Now, we give few examples of two player game illustrating the presence of $\mathcal{K}$-stable equilibrium and closed cycle. In particular, we allow only the coalitions of size 1, and hence a $\mathcal{K}$-stable equilibrium is a Nash equilibrium.

Example 2.5. *Consider a two player game*

$$\begin{array}{c} \\ a_1 \\ a_2 \end{array} \begin{array}{c} \begin{array}{cc} b_1 & \quad b_2 \end{array} \\ \begin{pmatrix} (4,3) & (2,5) \\ (6,1) & (3,2) \end{pmatrix} \end{array}.$$

Here (a_2, b_2) is the only Nash equilibrium that can be reached from all other action profiles via improving deviations. The situation is described in Figure 1.

A directed edge $(a_1, b_1) \xrightarrow{\{1\}} (a_2, b_1)$ of Figure 1 represents a deviation by player 1. The other directed edges are similarly defined. The self loop at (a_2, b_2) shows that a unilateral deviation from (a_2, b_2) is not possible.

Example 2.6. *Consider a two player game*

$$\begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \end{array} \begin{array}{c} \begin{array}{ccc} b_1 & \quad b_2 & \quad b_3 \end{array} \\ \begin{pmatrix} (4,4) & (0,0) & (0,0) \\ (0,0) & (4,5) & (1,6) \\ (0,0) & (2,5) & (6,1) \end{pmatrix} \end{array}.$$

The Example 2.6 has both Nash equilibrium and closed cycle. The action profile (a_1, b_1) is a Nash equilibrium and the closed cycle is given by Figure 2.

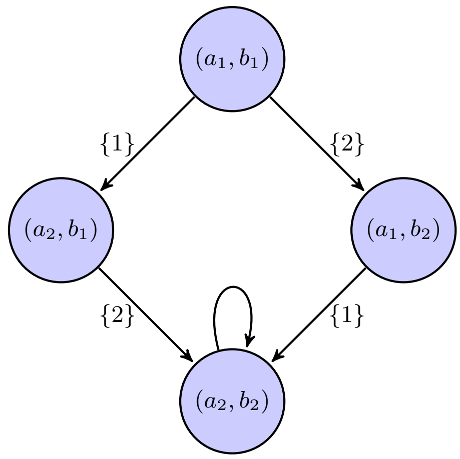

Figure 1: Nash equilibrium and Improving deviations

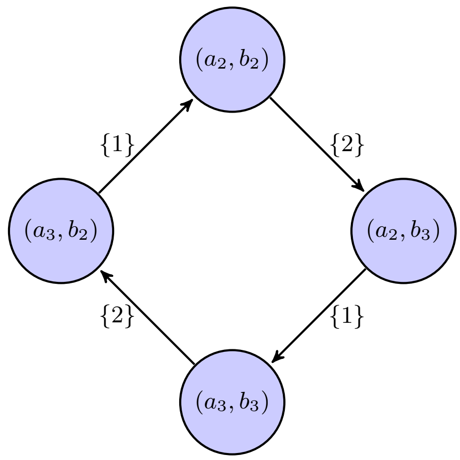

Figure 2: Closed Cycle

3 Dynamic play

We consider the n player strategic game defined in Section 2 where players can a priori communicate with each other and form a coalition. They jointly deviate from the current action profile to a new action profile if new action profile is strictly beneficial for all the members of the coalition. We consider the coalition formation over infinite horizon. At each time a coalition is randomly formed and it makes an improving deviation from a current action profile to a new action profile according to the improved action profile sets defined by (1). That is, at new action profile the actions of the players outside the coalition remain same as before and each player of the coalition is strictly benefited. If there are no such improved action profiles for a coalition then it does not deviate. The same thing repeats at next stage and it continues for infinite horizon. Such deviations define a CBR dynamics. We assume that the coalition formation is random and at each time only one coalition can be formed. If there are more than one improved action profiles for a coalition then each improved action profile can be chosen with positive probability. That is, the CBR dynamics is stochastic in nature. The CBR dynamics defines a Markov chain over a finite set of action profiles A. We also assume that at each time a selected coalition makes mistake and jointly deviate to an action profile where all the members of selected coalition are not strictly benefited. This happens with very small probability. Such mistakes work as mutations and it adds another level of stochasticity in the CBR dynamics. As a consequence we have perturbed Markov chain, see e.g., [2, 3]. We are interested in the action profiles which are going to be selected by the CBR dynamics as mutations vanish. We next describe the stochastic CBR dynamics as discussed above.

3.1 A stochastic CBR dynamics without mistakes

At each time $t = 0, 1, 2, \cdots$ a coalition S_t is selected randomly with probability $p_{S_t} > 0$. We assume that at each time selected coalition makes an improving deviation from current action profile a^t, i.e., at time $t+1$, the new action profile is $a^{t+1} \in \mathcal{I}_1(S_t, a^t)$ with probability $p_{\mathcal{I}_1}(a^{t+1}|S_t, a^t)$ where $p_{\mathcal{I}_1}(\cdot|S_t, a^t)$ is a probability distribution over finite set $\mathcal{I}_1(S_t, a^t)$. If there is no improving deviation for coalition S_t, $a^{t+1} = a^t$. Let X_t^0 denotes the action profile at time t, then $\{X_t^0\}_{t=0}^{\infty}$ is a finite Markov chain on set A. The transition law P^0 of the Markov chain is defined as follows:

$$P^0(X_{t+1}^0 = a'|X_t^0 = a) = \sum_{S\in\mathcal{K};\mathcal{I}_1(S,a)\neq\phi} p_S\, p_{\mathcal{I}_1}(a'|S,a)1_{\mathcal{I}_1(S,a)}(a') + \sum_{S\in\mathcal{K};\mathcal{I}_1(S,a)=\phi} p_S 1_{\{a'=a\}}(a'), \tag{3}$$

where 1_B is an indicator function for a given set B. It is clear that the $\mathcal{K}$-stable equilibria and closed cycles are the recurrent classes of P^0. A $\mathcal{K}$-stable equilibrium corresponds to an absorbing state of P^0 and a closed cycle corresponds to a recurrent class of P^0 having more than one action profiles.

From Example 2.6 it is clear that in general the closed cycles together with $\mathcal{K}$-stable equilibria can be present in a game. In that case, the CBR dynamics need not converge. In Example 2.6 the CBR dynamics need not converge to Nash equilibrium (a_1, b_1) because once CBR dynamics enter into closed cycle given in Figure 2 then it will never come out of it. The closed cycle $C = \{(a_2, b_2), (a_2, b_3), (a_3, b_3), (a_3, b_2)\}$ is a recurrent class and (a_1, b_1) is an absorbing state of Markov chain P^0 corresponding to the game given in Example 2.6.

We call a game acyclic if it has no closed cycles. The acyclic games include coordination games. There exists at least one $\mathcal{K}$-stable equilibrium for acyclic games from Theorem 2.4. For acyclic games the Markov chain defined by (3) is absorbing. Hence, from the theory of Markov chain the CBR dynamics given in Section 3.1 will be at $\mathcal{K}$-stable equilibrium in the long run no matter from where it starts (see [17]).

3.2 A stochastic CBR dynamics with mistakes

We assume that at each time t a selected coalition S_t makes error and deviate from a^t to an action profile where at least one player from the coalition S_t is not strictly better off. We assume that at action profile a^t, coalition S_t makes error with probability $\varepsilon f(S_t, a^t) \in (0, 1)$, where $f(S_t, a^t)$ takes into account the fact that some coalitions can be more prone to make errors than others and that some action profiles may lead to wrong choices more often than others. The parameter ε allows us to tune the frequency of errors. Therefore, at time $t+1$ the coalition S_t selects an improving deviation with probability $(1 - \varepsilon f(S_t, a^t))$. It selects $a^{t+1} \in \mathcal{I}_1(S_t, a^t)$ according to distribution $p_{\mathcal{I}_1}(\cdot)$ defined in Section 3.1. By combining the probabilities we obtain that a coalition S_t selects $a^{t+1} \in \mathcal{I}_1(S_t, a^t)$ with probability $(1-\varepsilon f(S_t, a^t))p_{\mathcal{I}_1}(a^{t+1}|S_t, a^t)$. The coalition S_t selects a non-improving deviation with probability $\varepsilon f(S_t, a^t)$. Let $p_{\overline{\mathcal{I}}_1}(\cdot|S_t, a^t)$ be a probability distribution over finite set $\overline{\mathcal{I}}_1(S_t, a^t)$. Then, the coalition chooses $a^{t+1} \in \overline{\mathcal{I}}_1(S_t, a^t)$

with probability $\varepsilon f(S_t, a^t) p_{\overline{\mathcal{I}}_1}(\cdot|S_t, a^t)$. If there is no improving deviation, then with probability $\Big(1 - \varepsilon f(S_t, a^t)\Big)$ the coalition does not modify the action profile, i.e., $a^{t+1} = a^t$, and with the complementary probability selects an action profile in $\overline{\mathcal{I}}_1(S_t, a^t)$ according to the distribution $p_{\overline{\mathcal{I}}_1}(\cdot|S_t, a^t)$. The transition law P^ε of perturbed Markov chain $\{X_t^\varepsilon\}_{t=0}^\infty$ is defined as below:

$$\begin{aligned} P^\varepsilon(X_{t+1}^\varepsilon = a' | X_t^\varepsilon = a) = & \sum_{S \in \mathcal{K}; \mathcal{I}_1(S,a) \neq \phi} p_S\big((1 - \varepsilon f(S, a)) p_{\mathcal{I}_1}(a'|S, a) 1_{\mathcal{I}_1(S,a)}(a') \\ & \qquad + \varepsilon f(S, a) p_{\overline{\mathcal{I}}_1}(a'|S, a) 1_{\overline{\mathcal{I}}_1(S,a)}(a')\big) \\ & + \sum_{S \in \mathcal{K}; \mathcal{I}_1(S,a) = \phi} p_S\big((1 - \varepsilon f(S, a)) 1_{\{a' = a\}}(a') \\ & \qquad + \varepsilon f(S, a) p_{\overline{\mathcal{I}}_1}(a'|S, a) 1_{\overline{\mathcal{I}}_1(S,a)}(a')\big), \end{aligned} \tag{4}$$

for all $a, a' \in A$.

The perturbed Markov chain $\{X_t^\varepsilon\}_{t=0}^\infty$ is irreducible because given nonzero errors it is possible to reach all the action profiles starting from any action profile in a finite number of steps. It is also aperiodic because with positive probability the state does not change. Hence, there exists a unique stationary distribution μ^ε for perturbed Markov chain. However, when $\varepsilon = 0$, there can be several stationary distributions corresponding to different $\mathcal{K}$-stable equilibria or closed cycles. Such Markov chains are called singularly perturbed Markov chains [2, 3]. We are interested in the action profiles to which stationary distribution μ^ε assigns positive probability as $\varepsilon \to 0$. This leads to the definition of a stochastically stable action profile.

Definition 3.1. *An action profile a is stochastically stable relative to process P^ε if $\lim_{\varepsilon \to 0} \mu_a^\varepsilon > 0$.*

We recall few definitions from Young [25]. If $P^\varepsilon(a'|a) > 0$, $a, a' \in A$, the *one step resistance* from an action profile a to an action profile $a' \neq a$ is defined as the minimum number of mistakes (mutations) that are required for the transition from a to $a' \neq a$ and it is denoted by $r(a, a')$. From (4) it is clear that the transition from a to a' has the probability of order ε if $a' \notin \mathcal{I}_1(S, a)$ for all S and thus has resistance 1 and is of order 1 otherwise, so has resistance 0. So, in our setting $r(a, a') \in \{0, 1\}$ for all $a, a' \in A$. A zero resistance between two action profiles corresponds to a transition with positive probability under P^0. One can view the action profiles as the nodes of a directed graph that has no self loops and the weight of a directed edge between two different nodes is represented by one step resistance between them. Since P^ε is an irreducible Markov chain then there must exist at least one directed path between any two recurrent classes H_i and H_j of P^0 which starts from H_i and ends at H_j. The resistance of any path is defined as the sum of the weights of the corresponding edges. The resistance of a path which is minimum among all paths from H_i to H_j is called as *resistance* from H_i to H_j and it is denoted by r_{ij}. The resistance from any action profile $a^i \in H_i$ to any action profile $a^j \in H_j$ is r_{ij} because inside H_i and H_j action profiles are connected with a path of zero resistance. Now we recall the definition of stochastic potential of a recurrent class H_i of P^0 from [25]. It can be computed by restricting to a reduced graph. Construct a graph $\mathcal{G}$ where total number of nodes are the number of recurrent classes of P^0(one action profile from each recurrent class) and a directed edge from a^i to a^j is weighted by r_{ij}. Take a node $a^i \in \mathcal{G}$ and consider all the spanning trees such that from every node $a^j \in \mathcal{G}$, $a^j \neq a^i$, there is a unique path directed from a^j to a^i. Such spanning trees are called a^i-trees. The resistance of an a^i-tree is the sum of the resistances of its edges. The stochastic potential of a^i is the resistance of an a^i-tree having minimum resistance among all a^i-trees. The stochastic potential of each node in H_i is same and it is a stochastic potential of H_i [25].

Theorem 3.2. *For the stochastic CBR dynamics defined in Section 3.2, all $\mathcal{K}$-stable equilibria and all the action profiles from closed cycles, that have minimum stochastic potential, are stochastically stable. Furthermore, if one action profile in a closed cycle is stochastically stable then all the action profiles in the closed cycle are stochastically stable.*

Proof. We know that the Markov chain P^ε is aperiodic and irreducible. From (3) and (4) it is easy to see that

$$\lim_{\varepsilon \to 0} P^\varepsilon(a'|a) = P^0(a'|a), \ \forall \ a, a' \in A.$$

From (4) it is clear that, if $P^\varepsilon(a'|a) > 0$ for some $\varepsilon \in (0, \varepsilon_0]$, then we have

$$0 < \varepsilon^{-r(a,a')} P^\varepsilon(a'|a) < \infty.$$

Markov chain P^ε satisfies all three required conditions of Theorem 4 in [25] from which it follows that as $\varepsilon \to 0$, μ^ε converges to a stationary distribution μ^0 of P^0 and an action profile a is stochastically stable, i.e., $\mu_a^0 > 0$, if and only if a is contained in a recurrent class of P^0 having minimum stochastic potential. We know that the recurrent classes of Markov chain P^0 are $\mathcal{K}$-stable equilibria or closed cycles. Therefore, all the $\mathcal{K}$-stable equilibria and the action profiles from closed cycles having minimum stochastic potential are stochastically stable. The proof of last part follows from the fact that the stochastic potential of each action profile in a closed cycle is the same. □

Remark 3.3. *The stochastic stability results do not depend on the function $f(\cdot)$ or the distributions of $p_{\mathcal{I}_1}(\cdot)$, $p_{\overline{\mathcal{I}}_1}(\cdot)$ and $p = (p_S)_{S \in \mathcal{K}}$.*

Corollary 3.4. *If $\mathcal{K} = \mathcal{P}(N) \setminus \phi$, all strong Nash equilibria and all action profiles from closed cycles are stochastically stable for the stochastic CBR dynamics defined in Section 3.2.*

Proof. If $\mathcal{K} = \mathcal{P}(N) \setminus \phi$, all the strong Nash equilibria and closed cycles are the recurrent classes of P^0. Due to the formation of grand coalition it is always possible to reach one action profile from another action profile by at most one error. Then, the resistance r_{ij} between any two distinct recurrent classes H_i and H_j is always 1. Hence, the stochastic potential of each recurrent class of P^0 is $J - 1$, where J is the number of recurrent classes of P^0. In fact, a spanning tree in graph $\mathcal{G}$ includes only $J - 1$ links and each of them has resistance 1. The proof then follows from Theorem 3.2.

□

We can have a similar CBR dynamics without mistakes and with mistakes as given in Sections 3.1 and 3.2 respectively, if for all $S \in \mathcal{K}$ and $a \in A$ the set of improved action profiles is $\mathcal{I}_2(S, a)$ given by (2). We have the following results.

Theorem 3.5. *For a stochastic CBR dynamics corresponding to improved action profile sets defined by* (2)*, all strict $\mathcal{K}$-stable equilibria and all the action profiles from closed cycles, that have minimum stochastic potential, are stochastically stable. Furthermore, if one action profile in a closed cycle is stochastically stable then all the action profiles in the closed cycle are stochastically stable.*

Proof. The proof follows from the similar arguments given in Theorem 3.2. □

Corollary 3.6. *If $\mathcal{K} = \mathcal{P}(N) \setminus \phi$, all strict strong Nash equilibria and all action profiles from closed cycles are stochastically stable for the stochastic CBR dynamics corresponding to improved action profile sets defined by* (2)*.*

Proof. The proof follows from the similar arguments given in Corollary 3.4. □

3.2.1 Equilibrium selection in coordination games

We discuss the Nash equilibrium selection, in a 2×2 coordination game, by stochastic CBR dynamics defined in Section 3.2. Our equilibrium selection results are different from the results given in [16],[25]. We first consider the case where only the coalitions of size 1 are formed. In this case, a $\mathcal{K}$-stable equilibrium is a Nash equilibrium. Consider a 2×2 coordination game,

$$\begin{array}{cc} & \begin{array}{cc} s_1 & s_2 \end{array} \\ \begin{array}{c} s_1 \\ s_2 \end{array} & \begin{pmatrix} (a_{11}, b_{11}) & (a_{12}, b_{12}) \\ (a_{21}, b_{21}) & (a_{22}, b_{22}) \end{pmatrix} \end{array},$$

where $a_{jk}, b_{jk} \in \mathbb{R}$, $j, k \in \{1, 2\}$ and $a_{11} > a_{21}$, $b_{11} > b_{12}$, $a_{22} > a_{12}$, $b_{22} > b_{21}$. The action set of player i, $i = 1, 2$, is $A_i = \{s_1, s_2\}$. Here (s_1, s_1) and (s_2, s_2) are two Nash equilibria. In this game there are two types of Nash equilibria. One is payoff dominant and another is risk dominant. If $a_{11} > a_{22}$, $b_{11} > b_{22}$, then (s_1, s_1) is payoff dominant and if $a_{11} < a_{22}$, $b_{11} < b_{22}$, then (s_2, s_2) is payoff dominant. In other cases payoff dominant Nash equilibrium does not exist. From [25], define,

$$R_1 = \min\left\{\frac{a_{11} - a_{21}}{a_{11} - a_{12} - a_{21} + a_{22}}, \frac{b_{11} - b_{12}}{b_{11} - b_{12} - b_{21} + b_{22}}\right\},$$

$$R_2 = \min\left\{\frac{a_{22} - a_{12}}{a_{11} - a_{12} - a_{21} + a_{22}}, \frac{b_{22} - b_{21}}{b_{11} - b_{12} - b_{21} + b_{22}}\right\}.$$

If $R_1 > R_2$, then (s_1, s_1) is a risk dominant Nash equilibrium and if $R_2 > R_1$, then (s_2, s_2) is a risk dominant Nash equilibrium.

The state space of Markov chain is $\{(s_1, s_1), (s_1, s_2), (s_2, s_1), (s_2, s_2)\}$, where (s_1, s_1) and (s_2, s_2) are the absorbing states of Markov chain P^0. From Remark 3.3, the stochastic stability results do not depend on the distributions of $p_{\mathcal{I}_1}(\cdot)$, $p_{\overline{\mathcal{I}}_1}(\cdot)$ and $p = (p_S)_{S \in \mathcal{K}}$ and function $f(\cdot)$. Therefore, we assume that all the distributions are uniform and function $f(\cdot)$ has constant value 1. Under this assumption, the transition probability matrix of perturbed Markov chain is

$$P^{\varepsilon} = \begin{pmatrix} 1 - \frac{\varepsilon}{2} & \frac{\varepsilon}{4} & \frac{\varepsilon}{4} & 0 \\ \frac{1-\varepsilon}{2} & \varepsilon & 0 & \frac{1-\varepsilon}{2} \\ \frac{1-\varepsilon}{2} & 0 & \varepsilon & \frac{1-\varepsilon}{2} \\ 0 & \frac{\varepsilon}{4} & \frac{\varepsilon}{4} & 1 - \frac{\varepsilon}{2} \end{pmatrix}.$$

The unique stationary distribution of P^{ε} is $\mu^{\varepsilon} = \left(\frac{1-\varepsilon}{2-\varepsilon}, \frac{\varepsilon}{2(2-\varepsilon)}, \frac{\varepsilon}{2(2-\varepsilon)}, \frac{1-\varepsilon}{2-\varepsilon}\right)$. As $\varepsilon \to 0$, $\mu^{\varepsilon} \to (\frac{1}{2}, 0, 0, \frac{1}{2})$. That is, both the Nash equilibria are stochastically stable. This happens because we require only 1 mutation to reach from one Nash equilibrium to another Nash equilibrium. Therefore, the resistance from one Nash equilibrium to another Nash equilibrium is 1. Then, the stochastic potential of both the Nash equilibria will be 1. For the case when all types of coalitions can be formed the CBR dynamics always selects a payoff dominant Nash equilibrium whenever it exists because it is an SNE. If a payoff dominant Nash equilibrium does not exist, both the Nash equilibria are strong Nash equilibria and in that case CBR dynamics selects both the Nash equilibria. The stochastic dynamics by Young [25] and Kandori et al. [16] always select a risk dominant Nash equilibrium.

Among symmetric coordination games if we go beyond 2×2 matrix games the result by Young [25] cannot be generalized, i.e., it need not select a risk dominant Nash equilibrium. Consider an example of 3×3 matrix game from [25],

$$\begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \end{array} \begin{array}{c} \begin{array}{ccc} s_1 & s_2 & s_3 \end{array} \\ \begin{pmatrix} (6,6) & (0,5) & (0,0) \\ (5,0) & (7,7) & (5,5) \\ (0,0) & (5,5) & (8,8) \end{pmatrix} \end{array}.$$

In this game, (s_1, s_1), (s_2, s_2) and (s_3, s_3) are three Nash equilibria. The stochastic dynamics by Young [25] selects (s_2, s_2) that is not a risk dominant Nash equilibrium. A Nash equilibrium of an $m \times m$ symmetric coordination game is risk dominant if it is risk dominant in all pairwise contest (see [12]). We now discuss the equilibrium selection by CBR dynamics in above 3×3 coordination game. We first consider the case where only the coalitions of size 1 are formed. The state space of Markov chain is $\{(s_1, s_1), (s_1, s_2), (s_1, s_3), (s_2, s_1), (s_2, s_2), (s_2, s_3), (s_3, s_1), (s_3, s_2), (s_3, s_3)\}$, where (s_1, s_1), (s_2, s_2), and (s_3, s_3) are the absorbing states of Markov chain P^0. We label the states (s_1, s_1) as 1, (s_2, s_2) as 2 and (s_3, s_3) as 3. Then, the resistance from (s_1, s_1) to (s_2, s_2) is denoted by r_{12}, where $r_{12} = 1$. Similarly, $r_{13} = 2$, $r_{31} = 2$, $r_{21} = 1$, $r_{23} = 1$, $r_{32} = 1$. There are three 1-trees as given below.

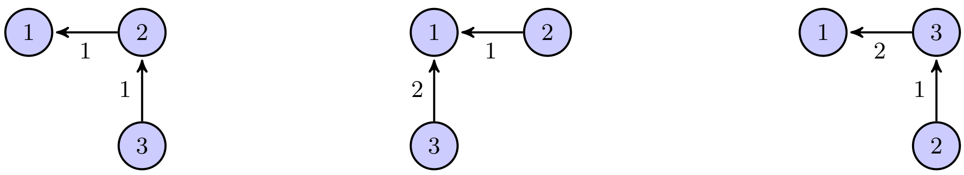

Figure 3: 1-trees

The minimum resistance of a 1-tree among all 1-trees is 2. Hence, the stochastic potential of (s_1, s_1) is 2. Similarly, by constructing 2-trees and 3-trees we can calculate the stochastic potential of (s_2, s_2) and (s_3, s_3). The stochastic potential of (s_2, s_2) and (s_3, s_3) is also 2. Hence, all the Nash equilibria are stochastically stable from Theorem 3.2. For the case where there is no restriction in coalition formation, CBR dynamics selects (s_3, s_3) because it is a strong Nash equilibrium.

4 Application to network formation games

In this section we consider the network formation games, see e.g., some recent books [13], [6], [5]. In general, the networks which are stable against the deviation of all the coalitions are called strongly stable networks. In the literature, there are two definitions of strongly stable networks. The first definition is, corresponding to SNE, due to [7]. The second definition is, corresponding to SSNE, due to [14]. A strongly stable network according to the definition of [14] is also strongly stable network according to the definition of [7]. The definition of a strongly stable network according to [14] are more often considered in the literature. We also consider the strong stability of networks according to [14]. We discuss the dynamic formation of networks over infinite horizon. We apply the stochastic CBR dynamics corresponding to SSNE to network formation games to discuss the stochastic stability of networks.

4.1 The model

Let $N = \{1, 2, \cdots, n\}$ be a finite set of players also called as nodes. The players are connected through undirected edges. An edge can be defined as a subset of N of size 2, e.g., $\{ij\} \subset N$ defines an edge between player i and player j. The collection of edges define a network. Let G denotes a set of all networks on N. For each $i \in N$, let $u_i : G \to \mathbb{R}$ be a payoff function of player i, where $u_i(g)$ is a payoff which player i receives at network g.

To reach from one network to another requires the addition of new links or the destruction of existing links. It is always assumed in the literature that forming a new link requires the consent of both the players while a player can delete a link unilaterally. The coalition formation in network formation games has also been considered in the literature. Some players in a network can form a coalition and make a joint move to another network by adding or severing some links, if new network is at least as beneficial as the previous network for all the players of coalition and at least one player is strictly benefited (see [14]). We recall few definitions from [14] describing the coalitional moves in network formation games and the stability of networks against all possible coalitional deviations.

Definition 4.1. *A network g' is reachable from g via deviation by a coalition S as denoted by $g \to_S g'$, if*

1. *$ij \in g'$ and $ij \notin g$ then $\{i, j\} \subset S$.*
2. *$ij \in g$ and $ij \notin g'$ then $\{i, j\} \cap S \neq \phi$.*

The first condition of the above definition requires that a new link can be added only between the nodes which are the part of a coalition S and the second condition requires that at least one node of any deleted link has to be a part of a coalition S. We denote $G(S, g)$ as a set of all networks which are reachable from g via deviation by S, i.e., $G(S, g) = \{g' | g \to_S g'\}$.

Definition 4.2. *A deviation by a coalition S from a network g to a network g' is said to be improving if*

1. *$g \to_S g'$,*
2. *$u_i(g') \geq u_i(g), \ \forall\, i \in S$ (with at least one strict inequality).*

We denote $\mathcal{I}_2(S, g)$ as a set of all networks g' which are reachable from g by an improving deviation of S, i.e.,

$$\mathcal{I}_2(S, g) = \big\{g' | g \to_S g', u_i(g') \geq u_i(g),\ \forall\, i \in S, u_j(g') > u_j(g) \text{ for some } j \in S\big\}.$$

It is clear that $g \notin \mathcal{I}_2(S, g)$ for all S. We denote $\overline{\mathcal{I}}_2(S, g) = G(S, g) \setminus \mathcal{I}_2(S, g)$ as a set of all networks which are reachable from g due to erroneous decisions of S. This set is always nonempty as $g \in \overline{\mathcal{I}}_2(S, g)$ for all S.

Definition 4.3. *A network g is said to be strongly stable if it is not possible for any coalition S to make an improving deviation from network g to some other network g'.*

A strongly stable network need not always exist and in that case there exists some set of networks lying on a closed cycle and all the networks in a closed cycle can be reached from each other via an improving path. An improving path and a closed cycle in network formation games can be defined similarly to Definitions 2.2 and 2.3, respectively.

Theorem 4.4. *There exists at least a strongly stable network or a closed cycle of networks.*

Proof. The proof follows from the similar arguments used in Theorem 2.4. □

4.2 Dynamic network formation

The paper by Jackson and Watts [15] is the first one to consider the dynamic formation of networks. They considered the case where at each time only a pair of players form a coalition and only a link between them can be altered. We consider the situation where at each time a subset of players form a coalition and deviate from a current network to a new network if at new network the payoff of each player of the coalition is at least as much as at current network and at least one player has strictly better payoff. This process continues over infinite horizon. A coalition can make all possible changes in the network and as a result more than one link can be created or severed at each time. So, we consider the following network formation rules by [14] given below:

- The creation of a link between two nodes requires the agreement of the whole coalition.
- A coalition can create/severe simultaneously multiple links among its members.

The CBR dynamics corresponding to SSNE can be applied to dynamic network formation. That is, at time t a network is g_t and a coalition S_t is selected with probability $p_{S_t} > 0$ and it makes an improving deviation to a new network that is at least as beneficial as g_t for all players of coalition S_t and at least one player of S_t is strictly benefited. So, at time $t+1$ network is $g_{t+1} \in \mathcal{I}_2(S_t, g_t)$ with probability $p_{\mathcal{I}_2}(g_{t+1}|S_t, g_t)$. If an improving deviation is not possible for selected coalition S_t, then $g_{t+1} = g_t$. The above process defines a Markov chain over state space G and its transition probabilities can be defined similarly to (3). In general this Markov chain is multichain. We can also assume that at each time selected coalition S_t makes error with small probability $f(S_t, g_t)\varepsilon$. That is, $g_{t+1} \in \mathcal{I}_2(S_t, g_t)$ with probability $(1 - f(S_t, g_t)\varepsilon)p_{\mathcal{I}_2}(g_{t+1}|S_t, g_t)$ and $g_{t+1} \in \overline{\mathcal{I}}_2(S_t, g_t)$ with probability $f(S_t, g_t)\varepsilon p_{\overline{\mathcal{I}}_2}(g_{t+1}|S_t, g_t)$. The transition probabilities of the perturbed Markov chain can be defined similar to (4). The presence of mutations makes the Markov chain ergodic for which there exists a unique stationary distribution. We are interested in the stochastically stable networks, i.e., the networks to which positive probabilities are assigned by the stationary distribution as $\varepsilon \to 0$. The stochastic stability analysis similar to the one given in Section 3.2 holds here. Thus, we have the following result.

Theorem 4.5. *All the strongly stable networks and closed cycles of a network formation game are stochastically stable.*

Proof. The proof follows directly from Corollary 3.6. □

5 Conclusions

We introduce coalition formation among players in an n-player strategic game over infinite horizon and propose a stochastic CBR dynamics. The mutations are present in the dynamics due to erroneous decisions taken by the coalitions. We show that all $\mathcal{K}$-stable equilibria and all action profiles from closed cycles, that have minimum stochastic potential, are stochastically stable. Similar development holds for strict $\mathcal{K}$-stable equilibrium. When there is no restriction on coalition formation, all SNE and closed cycles are stochastically stable. Similar development holds for SSNE. We apply CBR dynamics to network formation games and prove that all strongly stable networks and closed cycles of networks are stochastically stable.

Acknowledgements

This research was sponsored by the European Commission within the framework of the CONGAS project FP7-ICT-2011-8-317672.

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