STABILITY OF CONSTANT RETRIAL RATE SYSTEMS WITH NBU INPUT*

K. Avrachenkov[1], E. V. Morozov[2], R. S. Nekrasova[3] and B. Steyaert[4]

We study the stability of a single-server retrial queueing system with constant retrial rate, general input and service processes. First, we present a review of some relevant recent results related to the stability criteria of similar systems. Sufficient stability conditions were obtained by (Avrachenkov and Morozov, 2014), which hold for a rather general retrial system. However, only in case of Poisson input an explicit expression is provided; otherwise one has to rely on simulation. On the other hand, the stability criteria derived by (Lillo, 1996) can be easily computed, but only hold for the case of exponential service times.

We present new sufficient stability conditions, which are less tight than the ones obtained by (Avrachenkov and Morozov, 2010), but have an analytical expression under rather general assumptions. A key assumption is that interarrival times belongs to the class of *new better than used* (NBU) distributions. We illustrate the accuracy of the condition based on this assumption (in comparison with known conditions when possible) for a number of non-exponential distributions.

1. Introduction

We consider a general finite-capacity retrial queueing system with a constant retrial rate; this system will be denoted by Σ in the remainder of this paper. The external (primary) arrivals follow a renewal input with arrival epochs $\{t_k\}$ and rate λ. The system has m identical servers, and customers have i.i.d. service times $\{S_i\}$, with a generic element S and rate $\mu := 1/\mathsf{E}S$. If a new customer finds all m servers busy and the buffer (of size $n < \infty$) full, it joins an infinite-capacity virtual buffer (or *orbit*). If the orbit is non-empty, then an orbital (secondary) customer attempts to rejoin the primary queue after an exponentially distributed time with rate μ_0. Thus, unlike most classical retrial models, the orbit rate in Σ does not depend on the orbit size (i.e., the number of orbit customers). Such a model is referred to as a retrial model with *constant retrial rate*. It then follows that the orbit can be interpreted as a single-server $\cdot/M/1$-type queue with service rate μ_0, and where the jobs rejected from the primary queue provide the input. The merged stream to the orbit is in general not a GI-type arrival stream, since it is a combination of the rejected part of the primary customers and the secondary customers returning to the orbit after unsuccessful attempts to enter the primary queue.

Retrial systems with constant retrial rate have been investigated in a considerable number of contributions that mainly focused on Markovian systems. Let us list the most relevant papers that consider such a model. In [11] Fayolle introduced a retrial system with constant retrial rate, and derived stability conditions for the bufferless $M/G/1/0$ primary queue. In [1] Artalejo has obtained stability conditions for the Markovian $M/M/2/0$ case. In [18] Ramalhoto and Gómez-Corral have deduced stability conditions for the $M/M/1/1$ case. For the general Markovian $M/M/m/n$ case, the authors of [18] have obtained decomposition results, assuming ergodicity (stability). The ergodicity conditions for the multiserver Markovian $M/M/n/0$ case with a recovery probability have been derived by Artalejo, Gómez-Corral and Neuts in [2]. Retrial systems with constant retrial rate can be adopted to model a range of telecommunication systems, such as a telephone exchange system [11], multiple access systems [9], [10], short TCP transfers [4], [5], as well as logistic systems [13]. Such a system can be successfully applied to model the multi-access protocol ALOHA, with restrictions for the individual retrial rates.

It is useful to note that the only source of instability of the system is an unlimited growth of the orbit size. In this paper we briefly discuss some known stability results. These conditions are defined by the system parameters λ, μ_0, μ, and the condition discussed in [6] also includes the loss probability P_{loss} in an auxiliary loss system. In

[1] Inria Sophia Antipolis, France, e-mail: K.Avrachenkov@sophia.inria.fr
[2] Institute of Applied mathematical research, KarRC RAS, Russia, emorozov@karelia.ru
[3] Institute of Applied mathematical research, KarRC RAS, Russia, ruslana.nekrasova@mail.ru
[4] Ghent University, TELIN Department, Belgium bs@telin.UGent.be
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some cases this probability can be found explicitly, otherwise, simulation is required to obtain an estimation for this quantity.

The paper is organized as follows. In Section 2 we give a short review of the known stability conditions of the described system. In particular, the stability criteria for the $GI/M/1/0$-type retrial system presented in [13] are discussed in detail. Then, in Section 3, we present a sufficient stability condition, which will be verified analytically (with no simulation). This is useful for those cases where the loss probability P_{loss}, as indicated above, can be found merely by estimation. Finally, in Section 4, we present simulation results. We would like to stress that these results demonstrate a remarkable consistency with the theoretical formulas.

2. Stability conditions

In this section, the stability conditions of the retrial system Σ are discussed. We consider a general $GI/G/m/n$-type retrial system, and construct an associated auxiliary *loss system* (denoted as $\Sigma^{(0)}$), which has the same configuration as the original system Σ, and an extra independent Poisson input with rate μ_0. We assume that the overflow stream from the system $\Sigma^{(0)}$ is directed to a $\cdot/M/1/\infty$ system (with a service rate μ_0), which is treated as a *virtual orbit*. The server in the original system Σ is less loaded than in the system $\Sigma^{(0)}$ (because in the former system some gaps in the stream from orbit to server occur, as opposed to the latter system). In [6], the following sufficient stability condition of Σ has been obtained:

$$(\lambda + \mu_0)\mathsf{P}_{loss} < \mu_0, \tag{1}$$

where P_{loss} is the stationary loss probability in the system $\Sigma^{(0)}$. Indeed, condition (1) is strictly proved for a limited class of service time distributions. This proof is based on a delicate coupling construction and sample-path monotonicity of the stream of the rejected customers in system $\Sigma^{(0)}$. Condition (1) has a clear probabilistic interpretation: the left-hand side is the input rate to the virtual orbit (system $\cdot/M/1/\infty$), while the right-hand side represents its service rate. From standard queueing theory we know that this condition will imply the stability of the virtual orbit, and hence, the stability of the real orbit in the (less loaded) original system Σ. For the $M/G/1/0$ system $\Sigma^{(0)}$, by the Erlang formula, we may write

$$\mathsf{P}_{loss} = \frac{\lambda + \mu_0}{\mu + \lambda + \mu_0}, \tag{2}$$

where $\mu := 1/\mathsf{E}S$. Setting $\lambda = 1$ we can rewrite (1) in the form

$$\frac{1}{\mu_0} + 1 < \mu, \tag{3}$$

which allows numerical investigation of the stability region, by varying the two parameters μ_0 and μ (Figure 1 shows this region for an $M/M/1$-type system).

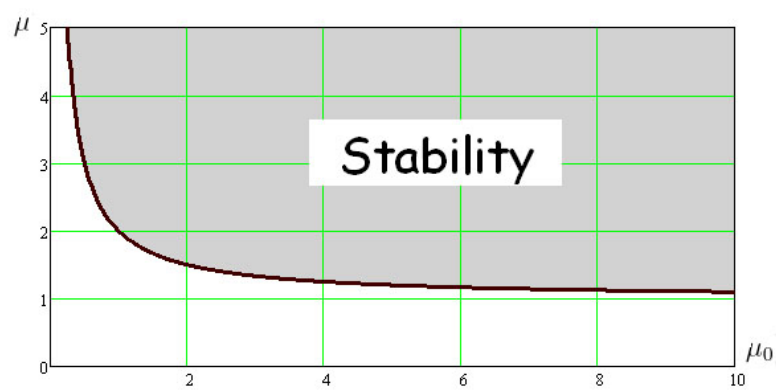

Fig. 1. Stability/instability region of the $M/M/1/0$-type system with $\lambda = 1$.

Thus, the stability region of the $M/G/1/0$-type retrial system can be determined analytically by means of (3). Now we focus on the stability criterion of a $GI/M/1/0$-type retrial system obtained in [13]:

$$\frac{\lambda(\mu + \mu_0)^2}{\mu\Big[\lambda\mu[1 - C(\mu + \mu_0)] + \mu_0(\mu + \mu_0)\Big]} < 1, \tag{4}$$

where

$$C(s) = \int_0^\infty e^{-xs} dF(x),\ s>0, \tag{5}$$

is the Laplace-Stieltjes transform of the distribution function F of the interarrival times between primary arrivals. For the $M/M/1/0$-type retrial system, $C(s) = \lambda/(s+\lambda)$, and in this case it is easy to check that conditions (1) and (4) are indeed equivalent. Note that in a recent paper [12], the stability criterion (4) is given in another equivalent form

$$\frac{\mu+\mu_0}{\mu}\Big[\frac{\mu}{\mu+\mu_0}\big[1-C(\mu+\mu_0)\big]+\frac{\mu_0}{\lambda}\Big]^{-1} < 1. \tag{6}$$

Returning to the $GI/M/1/0$-type system, we note that for a non-Poisson input, inequality (1) is only a sufficient (but not a necessary) condition, and thus the stability region found by (1) (that is, the domains of parameters λ, μ, μ_0 where the orbit is stable) is expected to be a subspace of the exact stability region satisfying (4). Note that, for now, P_{loss} is an unknown quantity, and we can use simulation results to obtain an estimate $\mathsf{P}_{loss}(t)$ instead, to check condition

$$\mathsf{P}_{loss}(t)(\lambda+\mu_0) < \mu_0, \tag{7}$$

where $\mathsf{P}_{loss}(t) := R(t)/A(t)$ and $R(t)$ and $A(t)$ are the number of losses and arrivals, respectively, in system $\Sigma^{(0)}$ during interval $(0, t]$. The accuracy of condition (7) is discussed in more detail in Section 4.

3. Stability condition based on NBU property

As it was mentioned above, the explicit expression for P_{loss} is available only for Poisson arrivals (with rate λ). Hence, in general, to verify sufficient stability condition (1), we need to simulate system $\Sigma^{(0)}$ and estimate the probability P_{loss}.

In this section, we present a new sufficient stability condition of Σ, which can be computed analytically and thus does not require simulations, but is rougher, in a sense, than condition (1). More precisely, we assume that P_{loss} is upper bounded by an analytically available value $\mathcal{L}$. Then $\mathsf{P}_{loss}(\lambda+\mu_0) \leqslant \mathcal{L}(\lambda+\mu_0)$, and if condition

$$\mathcal{L}(\lambda+\mu_0) < \mu_0, \tag{8}$$

holds, then condition (1) holds as well. We stress that there is a trade-off: condition (8) is less tight than (1), but because the value of $\mathcal{L}$ is available analytically in a closed-form expression, then there is no need for simulation experiments. Hence, the advantage/disadvantage of condition (8) depends on the specific model that is considered. As simulation results in Section 4 show, in some cases the new condition is easy to check and is quite accurate.

Let us show how the quantity $\mathcal{L}$ can be computed in some particular cases. For a classical bufferless *loss* system $GI/G/1/0$ with general interarrival time τ and general service time S, the following relation connecting the probability P_{loss} and stationary busy probability $\mathsf{P}_b^{(0)}$ has been obtained (by mean of regenerative arguments) [17]:

$$\mathsf{P}_{loss} = 1-\frac{1}{\rho}\mathsf{P}_b^{(0)}, \tag{9}$$

where $\rho := \mathsf{E}S/\mathsf{E}\tau$ is the traffic intensity,

$$\mathsf{P}_{loss} := \lim_{t\to\infty}\mathsf{P}_{loss}(t),\ \ \mathsf{P}_b^{(0)} := \lim_{t\to\infty}\frac{B(t)}{t},$$

and $B(t)$ is the busy time of the server in interval $[0, t]$. Thus, if we can determine a lower bound $\mathcal{B}$ of the probability $\mathsf{P}_b^{(0)}$, then the required upper bound $\mathcal{L}$ of the probability P_{loss} immediately follows:

$$\mathsf{P}_{loss} = 1-\frac{\mu}{\lambda+\mu_0}\mathsf{P}_b^{(0)} \leqslant 1-\frac{\mu}{\lambda+\mu_0}\mathcal{B} := \mathcal{L}. \tag{10}$$

In the loss system $GI/G/1/0$, each arrival to an empty server induces a regeneration. The length of a regeneration cycle is the sum of the busy period B (which stochastically equals a service time S) and the idle period I, which equals a remaining interarrival time $\hat{\tau}$. Consider for a moment a loss system $M/G/1/0$, with Poisson input with rate λ and service rate $\mu = 1/\mathsf{E}S$. In such a case $I =_{st} \tau$ and we have the well-known expression

$$\mathsf{P}_b^{(0)} = \frac{\mathsf{E}S}{\mathsf{E}S+\mathsf{E}\tau} = \frac{\rho}{1+\rho}. \tag{11}$$

However, expression (11) is not applicable to the loss system $\Sigma^{(0)}$, where the input is a superposition of two independent streams, Poisson μ_0-input, and in general non-Poisson λ-input. In this case a regeneration occurs if and only if a λ-customer enters an empty system, and hence the busy period is a random sum of service times. To obtain a lower bound for $\mathsf{P}_b^{(0)}$ in system $\Sigma^{(0)}$, we construct a modified loss system $\Sigma^{(1)}$ as follows. Recall that both $\lambda-$ and μ_0-customer has the same service time S. Then, at each service completion epoch, the next interarrival time in the λ-input is generated anew. Thus, in system $\Sigma^{(1)}$, the remaining time to the next λ-arrival is replaced by an independent variable distributed as τ. As a result, each arrival to an empty server generates a regeneration instant, and the busy period is stochastically equivalent to service time S. At the same time, in the system $\Sigma^{(1)}$, the idle period I is stochastically equal to a minimum of τ and an exponential variable η with rate μ_0 (describing the interarrival time between μ_0-arrivals). Denote $\gamma = \min(\tau, \eta)$ and note that $\mathsf{E}\gamma < \infty$. Then by standard regenerative arguments, the busy probability $\mathsf{P}_b^{(1)}$ in the modified system $\Sigma^{(1)}$ satisfies

$$\mathsf{P}_b^{(1)} = \frac{\mathsf{E}S}{\mathsf{E}S + \mathsf{E}\gamma}. \tag{12}$$

If F_τ is the cumulative distribution function of τ, then, denoting its tail as by $\bar{F}_\tau = 1 - F_\tau$, we obtain

$$\mathsf{E}\gamma = \int_0^\infty \mathsf{P}(\gamma \geqslant x)dx = \int_0^\infty \bar{F}_\tau(x)e^{-\mu_0 x}dx. \tag{13}$$

Constructing system $\Sigma^{(1)}$, we replace the tails of some interarrival times between λ-customers in system $\Sigma^{(0)}$ by the entire intervals τ. Thus, if $\hat{\tau} \leqslant_{st} \tau$, one can expect, that system $\Sigma^{(0)}$ is more heavily loaded than system $\Sigma^{(1)}$ and our main assumption is $\mathsf{P}_b^{(0)} \geqslant \mathsf{P}_b^{(1)}$, or

$$\mathsf{P}_b^{(0)} \geqslant \mathsf{P}_b^{(1)} = \frac{\mathsf{E}S}{\mathsf{E}S + \int_0^\infty \bar{F}_\tau(x)e^{-\mu_0 x}dx}. \tag{14}$$

Note, that the tail distribution is denoted by $\bar{F} = 1 - F$. Inequality $\hat{\tau} \leqslant_{st} \tau$ holds if and only if

$$\bar{F}_\tau(x+y) \leqslant \bar{F}_\tau(y)\bar{F}_\tau(x), \quad x \geqslant 0,\, y \geqslant 0, \tag{15}$$

in which case F_τ belongs to the class of *new-better-than-used* (NBU) distributions. Equivalently, (15) can be written, for any $x \geqslant 0$, $y \geqslant 0$, as $\mathsf{P}(\tau - y > x|\tau > y) \leqslant \mathsf{P}(\tau > x)$. In other words, an independently resampled variable τ is stochastically larger than or equal to the remaining value $\hat{\tau} = \tau - y > x$ conditioned on the event $\{\tau > y\}$. If a distribution satisfies the opposite inequality (to (15)), then it is known as a *new-worse-than-used* (NWU) distribution.

In case (14) holds, then

$$\mathcal{L} := 1 - \frac{\mu}{\lambda + \mu_0} \cdot \frac{\mathsf{E}S}{\mathsf{E}S + \mathsf{E}\gamma}, \tag{16}$$

is an upper bound of P_{loss} in system $\Sigma^{(0)}$. So, condition (8) implies stability of the original retrial system Σ.

Remark. Note that the proof of statement (14) can not be based on the classical monotonicity properties of queues, as the inputs in $\Sigma^{(0)}$ and $\Sigma^{(1)}$ are a superposition of two streams [21]. Therefore, we have to rely on simulation to validate this result.

Consider the $Weibull/M/1/0$-type retrial system with a Weibull distribution describing the primary input. In this case

$$F_\tau(x) = 1 - e^{-x^w}, \quad x > 0. \tag{17}$$

The property (15) holds, for instance, for a light-tailed Weibull distribution with $w > 1$. Note that for the system under consideration

$$\lambda = \frac{1}{\mathsf{E}\tau} = \frac{1}{\int_0^\infty u^{1/w}\, e^{-u}du}, \tag{18}$$

and

$$\mathsf{E}\gamma = \mathsf{E}\min(\tau, \eta) = \int_0^\infty e^{-x^w - \mu_0 x}dx. \tag{19}$$

Now we compare two stability regions: i) based on criterion (4); and ii) based on condition (8).

To illustrate criterion (4), for each μ_0 we need to calculate the root $\mu^L := \mu^L(\mu_0)$ of equation

$$\frac{\lambda(\mu+\mu_0)^2}{\mu\Big[\lambda\mu[1-C(\mu+\mu_0)]+\mu_0(\mu+\mu_0)\Big]} = 1. \tag{20}$$

Thus, function $\mu^L(\mu_0)$ strictly delimits the stability zone. Similarly, to illustrate condition (8), we find $\mu^C := \mu^C(\mu_0)$ as a solution of equation, see (16),

$$\mathcal{L}(\lambda+\mu_0) = \Big(1 - \frac{\mu}{\lambda+\mu_0}\cdot\frac{1}{1+\mu\cdot \mathsf{E}\gamma}\Big)\cdot(\lambda+\mu_0) = \mu_0. \tag{21}$$

Figure 2 shows a comparison between the two stability regions, for the $Weibull/M/1/0$-type retrial queueing system, with $w = 4$. The first region is based on stability criterion (4). The second region is based on condition (8). The area above the curves is the stability zone for the respective condition. It is obvious that the stability region, based on the function $\mu^C(\mu_0)$, is a subspace of the region delimited by $\mu^L(\mu_0)$, because condition (8) is a sufficient condition (but not a necessary one, in general). And thus we lose the stability area between the two curves when we rely on condition (8) only. As we obtained, the stability region obtained by condition (1) is closer to the exact stability zone than a region obtained by condition (8).

However, because P_{loss} is typically unknown, it is often preferable to use condition (8) instead of relying on simulation.

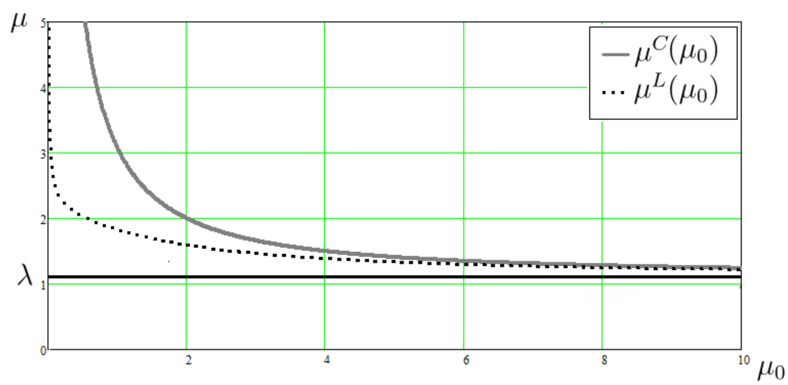

Fig. 2. Stability region for the $Weibull/M/1/0$-type retrial system, $w = 4$, $\lambda = 1.103$.

Note that if μ_0 increases, then the stability region delimited by condition (8) approaches the actual stability region, delimited by requirement (4). It can be deduced from [15] that as μ_0 increases, the original retrial system approaches the classical system with an infinite buffer, for which the stability criterion is $\rho := \lambda/\mu < 1$. This explains why the curve $\mu^L(\mu_0)$ depicted on Figure 2 approaches the constant $\mu = \lambda = 1.103$ as $\mu_0 \to \infty$.

Figure 3 gives a comparison of the two stability regions for the $Weibull/M/1/0$-type system with parameter $w = 2$. It is seen that the region obtained by condition (8) on Figure 3 is closer to the real stability region than the one on Figure 2. Thus, decreasing $w \downarrow 1$ makes the predicted stability region more accurate. Note that for $w = 1$ Weibull distribution becomes exponential with rate 1. Then inequality (15) becomes an equality, and in this case $\mathsf{P}_{loss} = \mathcal{L}$. Hence, conditions (8) and (1) are equivalent and coincide with the criterion (4). This property is illustrated by Figure 4.

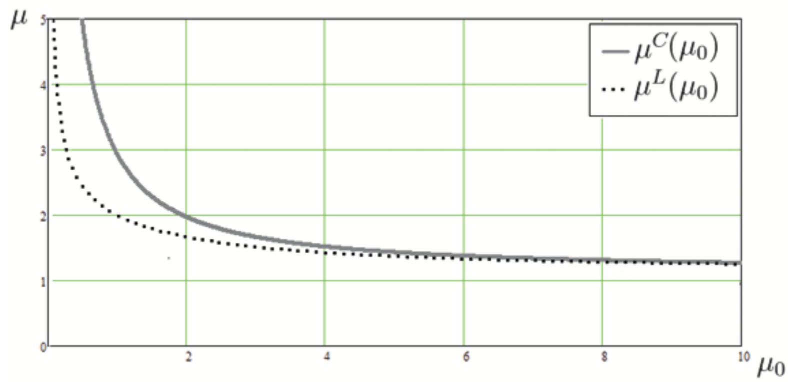

Fig. 3. Stability region for the $Weibull/M/1/0$-type retrial system, $w = 2,\ \lambda = 1.128$.

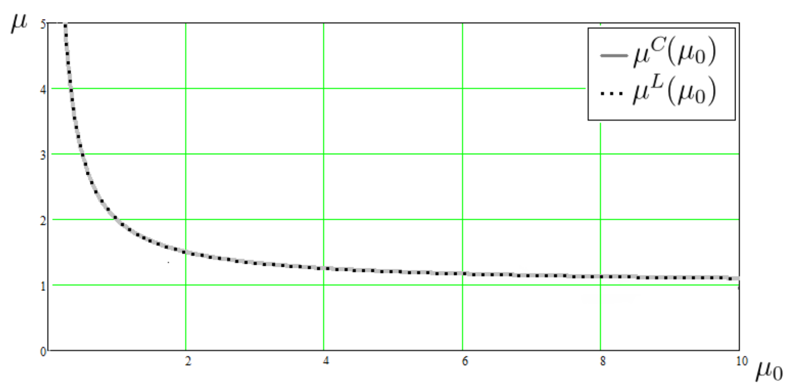

Fig. 4. Stability region for the $Weibull/M/1/0$-type retrial system, $w = 1,\ \lambda = 1$.

The Weibull distribution with parameter $w < 1$ is *heavy-tailed* (15) and thus belongs to the class of NWU distributions. As it is seen, in this case, $\mathsf{P}_b^{(0)} < \mathsf{P}_b^{(1)}$. Thus, the fulfilment of (8) does not guarantee the fulfilment of the sufficient condition (1) in such a case. Figure 5 shows a comparison of the two regions delimited by $\mu^C(\mu_0)$ and $\mu^L(\mu_0)$ for the $Weibull/M/1/0$-type system with $w = 0.8$. In this case, the values which provide an equality in condition (8) are outside the actual stability zone.

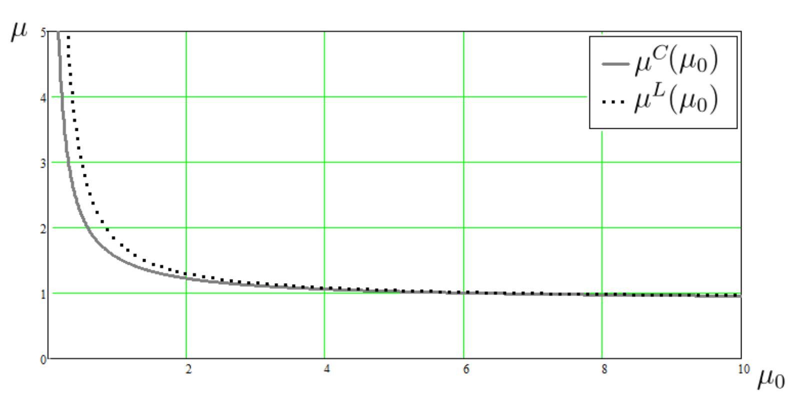

Fig. 5. Stability region for the $Weibull/M/1/0$-type retrial system, $w = 0.8,\ \lambda = 0.883$.

Next, we use simulation to verify (14). Obviously, it is preferable to use condition (8) to designate the stability region, rather than to rely on estimation based on (7). Verifying condition (7) demands, in general, a considerable simulation effort, and the results are naturally less accurate. At the same time, the method based on the (rougher) sufficient condition is applicable to a general $GI/G/1/0$-type retrial system with NBU λ-input.

4. Simulation results

In this section we present and discuss some simulation results, which indeed confirm the theoretical conclusions obtained in previous sections.

4.1. Accuracy of the sufficient condition

Consider the loss system $\Sigma^{(0)}$, where the input stream is a superposition of two independent streams: a primary (in general, non-Poisson) renewal stream with rate λ, and a Poisson stream with rate μ_0. First, we illustrate how to apply condition (7) when only simulation is available to estimate the unknown probability P_{loss} by means of the estimate $\mathsf{P}_{loss}(t)$. The goal is to find, for fixed λ, μ_0, $\varepsilon > 0$ and simulation time t, the proper value of μ such that

$$\Big|\mathsf{P}_{loss}(t)(\lambda+\mu_0)-\mu_0\Big| \leqslant \varepsilon. \tag{22}$$

Let us define

$$\Gamma(t) := \mu_0 - \mathsf{P}_{loss}(t)(\lambda+\mu_0). \tag{23}$$

Then it is expected that if $\Gamma(t) > 0$, the system is stable, and the system is unstable otherwise.

Simulation results for Pareto λ-input with parameters

$$t = 20000,\ \ \varepsilon = 0.01,\ \ \mu_0 \geqslant 0.02,\ \ k = 1,\dots,500,$$

are given on Figure 6, where the stability region obtained by criterion (4) is depicted as well. Note that in this case

$$\mathsf{P}(\tau \leqslant x) = 1 - x^{-\alpha},\ \ x \geqslant 1,\ \ \alpha > 1, \tag{24}$$

and $\lambda = (\alpha-1)/\alpha$. The values of μ that delimit the stability area can be read from the figure.

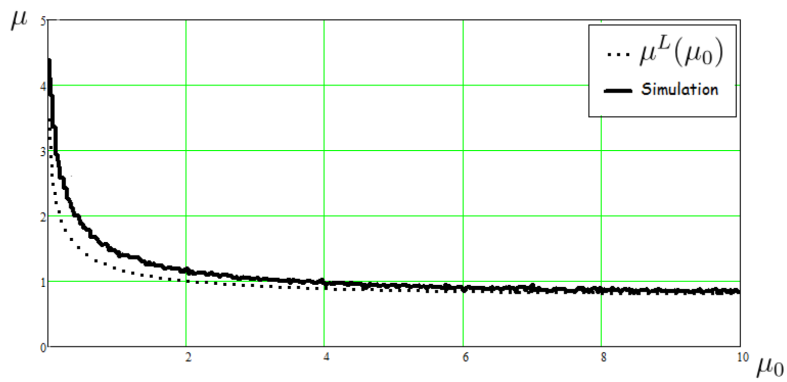

Fig. 6. Stability region for the $Pareto/M/1/0$-type retrial system, $\alpha = 4$, $\lambda = 0.75$.

Note that the values of μ_0 and μ that fall within the area above the graph assure the stability of the retrial system with the same configuration. Figure 6 shows that the stability region based on the sufficient condition is quite close to the real stability region, calculated from the proposed criterion (the two curves almost coincide). The tests for other values of the input rate ($\alpha = 2$, $\alpha = 3$, $\alpha = 5$) had shown similar results as for $\alpha = 4$. Thus, the sufficient condition for the $Pareto/M/1/0$-type retrial system is actually quite accurate. Note that a Pareto distribution is an NWU one, and condition (8) is not applicable in this case.

4.2. Accuracy of the new stability condition

Now we present some simulation results of systems $\Sigma^{(0)}$ and $\Sigma^{(1)}$ with an NBU distribution for the λ-input. Recall that the stability analysis based on condition (8) is applicable only if

$$\mathsf{P}_b^{(0)} \geqslant \mathsf{P}_b^{(1)}. \tag{25}$$

The results of the estimation of $\mathsf{P}_b^{(0)}$, P_{loss}, $\mathcal{L}$, $\mathsf{P}_b^{(1)}$ for Weibull λ-input and exponential service times are presented in Table 1 (where the corresponding estimators are equipped with a “hat”).

The experiments presented in Table 1 confirm that for Weibull distribution with parameter $w > 1$ (in which case Weibull distribution is NBU) the statement (25) holds, implying that parameter $\mathcal{L}$ (defined in (16)) is an

Table 1. Simulation results for $\Sigma^{(0)}$ and $\Sigma^{(1)}$ with Weibull λ-input and exponential service times.

w	λ	μ_0	μ	$\hat{\mathsf{P}}_b^{(0)}$	$\mathsf{P}_b^{(1)}$	$\hat{\mathsf{P}}_{loss}$	$\hat{\mathcal{L}}$
4.0	1.103	1	5	0.317	0.286	0.243	0.321
4.0	1.103	6	6	0.547	0.512	0.542	0.567
2.0	1.128	3	4	0.520	0.489	0.495	0.527
2.0	1.128	3	2	0.684	0.667	0.670	0.677
1.0	1.000	2	4	0.429	0.428	0.430	0.429
1.0	1.000	1	6	0.248	0.248	0.250	0.257
0.8	0.883	1	2	0.470	0.482	0.498	0.488
0.8	0.883	2	4	0.416	0.425	0.427	0.410

upper bound for P_{loss}, and condition (8) is a sufficient condition for stability. For $w = 1$, the λ-input becomes Poisson, and the (theoretical) equality $\mathsf{P}_b^{(0)} = \mathsf{P}_b^{(1)}$ is reflected in Table 1. For NWU Weibull distribution (with parameter $w < 1$), inequality (25) is violated (see the two bottom lines in Table 1), and thus condition (8) is not applicable.

A comparison of the three stability zones based on sufficient condition (22), criterion (4), and rough sufficient condition (8) for a $Weibull/M/1/0$-type system is presented in Figure 7.

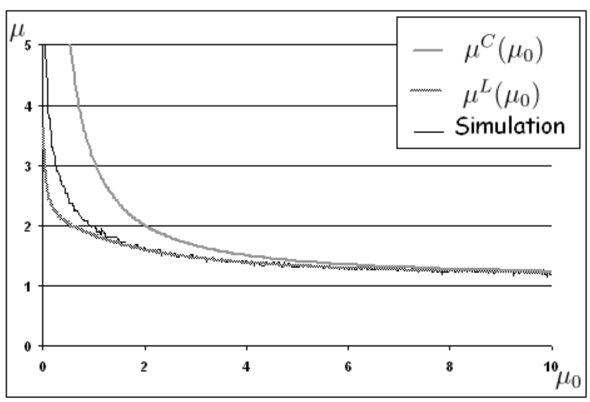

Fig. 7. Stability region for the $Weibull/M/1/0$-type retrial system, $w = 4$.

Similar results for a $D/M/1/0$-type system with deterministic λ-input with rate $\lambda := 1/D = 3$ and exponential service times are presented in Figure 8. (Note that deterministic input is NBU.)

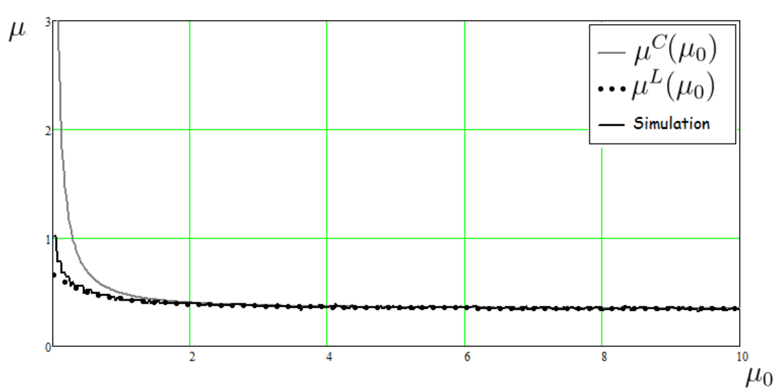

Fig. 8. Stability region for the $D/M/1/0$-type retrial system, $D = 3$.

We would like to emphasize that the method based on the rough condition is applicable for the general retrial queuing system. This is illustrated for the $Weibull/D/1/0$-type retrial system with Weibull parameter $w = 4$ in Figure 9. Note that in this case $\mu = 1/D$, where D is the deterministic service time, and criterion (4) is not applicable.

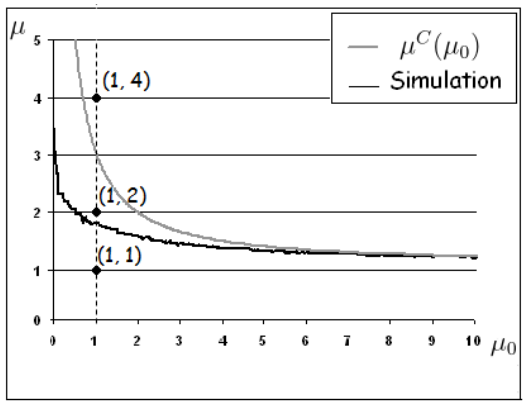

Fig. 9. Stability region for the $Weibull/D/1/0$-type retrial system, $w = 3$.

Let $N(t)$ be the orbit size at instant t. To conclude, we demonstrate the dynamics of the process $\{N(t), t \geqslant 0\}$ for the set of parameters $\{(\mu_0, \mu)\} = \{(1, 4), (1, 2), (1, 1)\}$. The points (1, 4), (1, 2) belong to the stability region, and this is confirmed by Figure 10 where we observe that $N(t)$ is a stable process. However, the pair (1, 1) belongs to the instability region, and Figure 11 shows the expected growth of the orbit size.

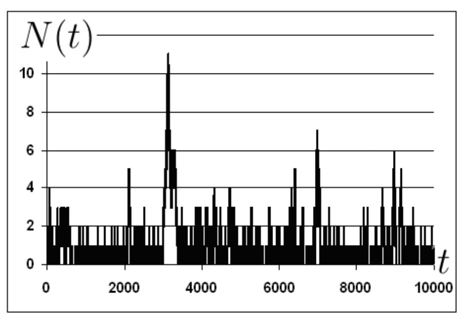

Fig. 10. Dynamics of the orbit for the $Weibull/D/1/0$-type retrial system, $w = 3$, $\mu_0 = 1$, $\mu = 2$, $D = 0.5$.

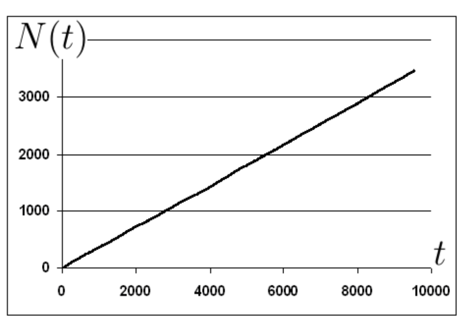

Fig. 11. Dynamics of the orbit for the $Weibull/D/1/0$-type retrial system, $w = 3$, $\mu_0 = 1$, $\mu = 1$, $D = 1$.

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