Inversion of perturbed linear operators that are singular at the origin

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***Abstract*— We consider the inversion of perturbed linear operators on Hilbert space. Namely, we study linear operators that depend on a small parameter and are singular when the parameter is equal to zero. First we consider the af£ne dependence on the parameter. We treat subsequently the cases of bounded operators with closed range, bounded operators with non closed range, and densely de£ned and closed unbounded operators. Then, we extend the results to the cases of polynomial and analytic perturbations.**

I. Introduction

Let H and K be Hilbert spaces and consider bounded linear operators $C \in \mathcal{L}(H,K)$ and $D \in \mathcal{L}(H,K)$. Let $A(z) = C + Dz$ be a linear perturbation of C that depends on a single complex parameter z.

If the mapping C is onto with range $\mathcal{R}(C) = K$ but is not $1-1$ but there is some $z_0 \neq 0$ for which the inverse operator $A(z_0)^{-1} \in \mathcal{L}(K,H)$ exists we show that the inverse operator $A(z)^{-1} \in \mathcal{L}(K,H)$ is well de£ned in some deleted neighbourhood $\{z \mid 0 < |z| < s\}$ and can be represented in this region by a convergent Laurent series with an isolated singularity at $z = 0$.

If the mapping C is $1-1$ but not onto and if the range $\mathcal{R}(C) \subseteq K$ is closed we use the adjoint operator to show that a similar result holds.

If $\mathcal{R}(C)$ is not closed we show that C is not bounded below. In other words there is no constant $\epsilon > 0$ such that $\|Cx\| \geq \epsilon\|x\|$ for all $x \in H$. By changing the topology in $\mathcal{R}(C)$ we will show that C can be extended to a bounded linear operator with a closed range in which case the previous results can be applied.

In the case where $C : \mathcal{D}(C) \subset H \mapsto K$ is a densely de£ned and closed unbounded linear operator we show that by changing the topology on H we can replace C by a bounded operator and once again apply the previous results. It will follow that case where C is densely de£ned and closed but unbounded is essentially the inverse of the case where C is bounded but $\mathcal{R}(C)$ is not closed.

The general results of the present paper can be applied to the singularly perturbed linear systems (see e.g., Pervozvanskii and Gaitsgory [7]) and descriptor type systems (see e.g., Campbell [3]) in Banach spaces as in Krein [5].

II. Previous work

There is an extensive body of work on the inversion of operator valued functions. We shall not attempt a complete review here but instead refer to a previous work by Avrachenkov *et al* [1] where a comprehensive review of the literature is given and where there is an extensive bibliography. We have previously provided a substantial extension to Hilbert space [4] of a method used by Schwietzer and Stewart [8] for inversion of perturbed matrix functions that are singular at the origin. In our previous paper [4] we showed that the methods can also be used in Banach space in certain special cases. In this paper we provide a complete description of the new method.

III. Prerequisite theory

We follow Luenberger [6] and Yosida [9]. Although it is essentially a corollary to the open mapping theorem the Banach inverse theorem is an equally important and celebrated result.

Theorem 3.1: (Banach) Let X and Y be Banach spaces and let $C \in \mathcal{L}(X,Y)$. If C is a $1-1$ mapping of X onto Y then the inverse mapping $C^{-1} \in \mathcal{L}(Y,X)$.

Corollary 3.2: Let X and Y be Banach spaces and let $C \in \mathcal{L}(X,Y)$. Assume that $\mathcal{R}(C)$ is closed. Then there is a constant $\epsilon > 0$ such that $\|Cx\| \geq \epsilon\|x\|$ for all $x \in X$.

IV. Linear perturbation of a bounded linear operator with closed range on Hilbert space

A. The case in which $\mathcal{N}(C) = \{0\}$ and $\mathcal{R}(C) = K$

In this case C is non-singular and we have a standard Maclaurin series expansion for $(C + Dz)^{-1}$.

Theorem 4.1: (Neumann) Let $C \in \mathcal{L}(H,K)$ be a $1-1$ mapping of H onto K. If $D \in \mathcal{L}(H,K)$ then there exists a real number $s > 0$ such that $(C+Dz)^{-1} \in \mathcal{L}(K,H)$ is well de£ned with

$$(C + Dz)^{-1} = \sum_{n=0}^{\infty} (-1)^n (C^{-1}D)^n C^{-1} z^n$$

for all $|z| < s$.

B. The case in which $\mathcal{N}(C) \neq \{0\}$ and $\mathcal{R}(C) = K$

In this case C is singular.

Lemma 4.2: (Howlett & Avrachenkov) Let $C, D \in \mathcal{L}(H,K)$ be bounded linear maps. Let $M = \mathcal{N}(C)$ and de£ne $N = D(M) \subseteq K$. If $(C + Dz_0)^{-1}$ is well de£ned for some $z_0 \in \mathbb{C}$ with $z_0 \neq 0$ then D is bounded below on M and N is a closed subspace of K.

1) Orthogonal decomposition of H and K: Since $M = \mathcal{N}(C)$ is a closed subspace and the orthogonal complement $M^\perp$ is also closed it follows that $H_1 = M$ and $H_2 = M^\perp$ are each Hilbert spaces. In the same way we note that $N = D(M)$ is a closed subspace and since the orthogonal complement $N^\perp$ is also closed it follows that $K_1 = N$ and $K_2 = N^\perp$ are each Hilbert spaces. Let $P_i \in \mathcal{L}(H, H_i)$ for each $i = 1, 2$ and $Q_j \in \mathcal{L}(K, K_j)$ for each $j = 1, 2$ be the natural projections. We can de£ne unitary operators $R \in \mathcal{L}(H, H_1 \times H_2)$ and $S \in \mathcal{L}(K, K_1 \times K_2)$ by setting

$$R = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}$$

such that

$$SCR^* = \begin{bmatrix} 0 & C_{12} \\ 0 & C_{22} \end{bmatrix} \quad SDR^* = \begin{bmatrix} D_{11} & D_{12} \\ 0 & D_{22} \end{bmatrix}.$$

2) The general inversion formula:

Theorem 4.3: Suppose that $C \in \mathcal{L}(H, K)$ is singular with $\mathcal{N}(C) \neq \{0\}$ and $\mathcal{R}(C) = K$ and that for some $z_0 \neq 0$ the inverse operator $(C + Dz_0)^{-1}$ is well-de£ned. Then D_{11}^{-1} is well-de£ned and $\mathcal{R}(C_{22}) = K_2$. The mapping $(C + Dz) \in \mathcal{L}(H, K)$ is a $1-1$ mapping of H onto K if and only if $z \neq 0$ and $(C_{22} + D_{22}z) \in \mathcal{L}(H_2, K_2)$ is a $1-1$ mapping of H_2 onto K_2. In this case

$$\begin{aligned}(C + Dz)^{-1} &= P_1^* D_{11}^{-1} Q_1 / z \\ &\quad + \left[P_2^* - P_1^* D_{11}^{-1}(C_{12} + D_{12}z)/z\right](C_{22} + D_{22}z)^{-1} Q_2.\end{aligned}$$

If $\mathcal{N}(C_{22}) = \{0\}$ then $C_{22} \in \mathcal{L}(H_2, K_2)$ is a $1-1$ mapping of H_2 onto K_2 and for some real number $\epsilon > 0$ the operator $(C_{22} + D_{22}z) \in \mathcal{L}(H_2, K_2)$ is a $1-1$ mapping of H_2 onto K_2 in the region $|z| < \epsilon$ and the operator $(C + Dz)^{-1}$ can be represented in the region $0 < |z| < \epsilon$ by a convergent Laurent series with a pole of order 1 at $z = 0$.

Remark 4.4: Since $(C + Dz_0) \in \mathcal{L}(H, K)$ is non-singular it follows that $(C_{22} + D_{22}z_0) \in \mathcal{L}(H_2, K_2)$ is also non-singular. If $C_{22} \in \mathcal{L}(H_2, K_2)$ is singular with $\mathcal{N}(C_{22}) \neq \{0\}$ and $\mathcal{R}(C_{22}) = K_2$ then we can apply Theorem 4.3 to the operator $(C_{22} + D_{22}z)$. Thus the procedure is essentially recursive and will be continued until the non-trivial null space of C is eliminated.

Example 4.5: The procedure does not necessarily terminate. Consider the mappings on l^2 de£ned by the in£nite matrices

$$C = \begin{bmatrix} 0 & C_{12} \\ 0 & C_{22} \end{bmatrix} = \left[\begin{array}{c|cccc} 0 & 1 & 0 & 0 & \cdots \\ \hline 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right]$$

and

$$D = \begin{bmatrix} D_{11} & D_{12} \\ 0 & D_{22} \end{bmatrix} = \left[\begin{array}{c|cccc} 1 & 0 & 0 & 0 & \cdots \\ \hline 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right]$$

and the linearly perturbed in£nite matrix

$$C + Dz = \begin{bmatrix} D_{11}z & C_{12} + D_{12}z \\ 0 & C_{22} + D_{22}z \end{bmatrix}$$

$$= \left[\begin{array}{c|cccc} z & 1 & 0 & 0 & \cdots \\ \hline 0 & z & 1 & 0 & \cdots \\ 0 & 0 & z & 1 & \cdots \\ 0 & 0 & 0 & z & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right] = (C + Iz).$$

It is easy to see that the reduced problem to calculate $(C_{22} + D_{22}z)^{-1}$ is the same as the original problem to calculate $(C + Dz)^{-1}$. By applying a recursive calculation it is easy to guess that the answer will be

$$(C + Iz)^{-1} = \begin{bmatrix} z^{-1} & -z^{-2} & z^{-3} & -z^{-4} & \cdots \\ 0 & z^{-1} & -z^{-2} & z^{-3} & \cdots \\ 0 & 0 & z^{-1} & -z^{-2} & \cdots \\ 0 & 0 & 0 & z^{-1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$= I \cdot \frac{1}{z} - C \cdot \frac{1}{z^2} + C^2 \cdot \frac{1}{z^3} + \cdots.$$

In this case we have an essential singularity at the origin.

C. *The case in which $\mathcal{N}(C) = \{0\}$ and $\mathcal{R}(C) \neq K$*

In this case C is singular.

Theorem 4.6: Suppose that $C \in \mathcal{L}(H, K)$ is singular with $\mathcal{N}(C) = \{0\}$ and $\mathcal{R}(C) \neq K$ and that for some $z_0 \neq 0$ the inverse operator $(C + Dz_0)^{-1}$ is well-de£ned. Then $C^* \in \mathcal{L}(K, H)$ is singular with $\mathcal{N}(C^*) \neq \{0\}$ and $\mathcal{R}(C^*) = H$ and $(C^* + D^*\zeta_0)^{-1}$ is well-de£ned for some $\zeta_0 \neq 0$. If $(C^* + D^*\zeta)^{-1}$ is well-de£ned for $0 < |\zeta| < \epsilon$ then $(C + Dz)^{-1}$ is also well-de£ned and

$$(C + Dz)^{-1} = [(C^* + D^*\bar{z})^{-1}]^*$$

for $0 < |z| < \epsilon$.

D. *The case in which $\mathcal{N}(C) \neq \{0\}$ and $\mathcal{R}(C) \neq K$*

The inversion procedure must incorporate the removal of non-trivial null spaces for both C and C^*.

V. Bounded linear operators with non-closed range on Hilbert space

In this section we show that by modifying the topology of the range space we can ensure that the range is closed. The previous inversion procedures can then be used.

Lemma 5.1: Let H and K be Hilbert spaces and let $C \in \mathcal{L}(H, K)$. If $\mathcal{R}(C) \subseteq K$ is not closed then for each $\epsilon > 0$ we can £nd $x = x(\epsilon) \in H$ such that $\|Cx\| < \epsilon\|x\|$.

By de£ning a new inner product on $\mathcal{R}(C)$ called the energy inner product we can overcome this dif£culty. Let $M = \mathcal{N}(C)$.

De£nition 5.2: Let the new inner product $\langle \cdot, \cdot \rangle_E : \mathcal{R}(C) \times \mathcal{R}(C) \mapsto \mathbb{C}$ be de£ned by the formula

$$\langle y, v \rangle_E = \langle y, v \rangle_K + \langle x_M^\perp, u_M^\perp \rangle_H$$

for each $y, v \in \mathcal{R}(C)$ where $x_M^\perp, u_M^\perp \in M^\perp$ are the uniquely de£ned elements with $Cx_M^\perp = y$ and $Cu_M^\perp = v$.

With the new inner product and the associated new topology on $\mathcal{R}(C)$ we have the following result.

Lemma 5.3: The space $\mathcal{R}(C)$ is closed.

Remark 5.4: The new inner product is simply a more appropriate measurement tool on the space $\mathcal{R}(C)$ in relation to the operator C. One could argue that the elements of the space $\mathcal{R}(C)$ remain unchanged.

Lemma 5.5: The space $\mathcal{R}(C) = K_E$ is a Hilbert space with the inner product $\langle \cdot, \cdot \rangle_E$. We call K_E the energy space.

Without loss of generality we can therefore suppose that $C \in \mathcal{L}(H, K)$ where $\mathcal{R}(C) \subseteq K$ is a closed subspace.

VI. Unbounded linear operators on Hilbert space

In this section we will show that by changing the topology in the domain space we can ensure that the operator is bounded. Previous inversion techniques can then be used.

A. Fundamental results

We follow Balakrishnan [2] and Yosida [9]. Let H and K be Hilbert spaces and let $C : \mathcal{D}(C) \subseteq H \mapsto K$ be a linear operator de£ned on a linear subspace $\mathcal{D}(C) \subseteq H$.

Lemma 6.1: If C is densely de£ned then C^* is a closed linear operator. If C is closed then C^* is densely de£ned.

Theorem 6.2: (J. von Neumann) Let C be densely de£ned and closed. Then the operators C^*C and CC^* are self-adjoint with $(I + C^*C)^{-1} \in \mathcal{L}(H)$ and $(I + CC^*)^{-1} \in \mathcal{L}(K)$.

B. The energy domain

Let H and K be Hilbert spaces and let $C : \mathcal{D}(C) \subseteq H \mapsto K$ be a densely de£ned and closed linear operator. For each $x, u \in \mathcal{D}(C)$ we de£ne a new inner product

$$\langle x, u \rangle_E = \langle x, u \rangle_H + \langle Cx, Cu \rangle_K$$

and a corresponding norm $\|x\|_E = (\langle x, x \rangle_E)^{1/2}$. The set $H_E \subseteq H$ is de£ned as the completion of $\mathcal{D}(C)$ in the new norm. The point $x \in H$ will belong to H_E if there exists a sequence $\{x_n\}_{n \in \mathbb{N}} \in \mathcal{D}(C)$ with $\|x_n - x\|_E \to 0$ as $n \to \infty$. In such cases we also have $y = Cx \in K$ with $\|Cx_n - y\|_K \to 0$. The set H_E becomes a Hilbert space under the new inner product.

C. Reformulation as a bounded linear operator

De£nition 6.3: For each $x \in H_E$ there is a sequence $x_n \in \mathcal{D}(C)$ such that $\|x_n - x\|_H \to 0$ and a point $y \in K$ such that $\|Cx_n - y\|_K \to 0$ as $n \to \infty$. We the de£ne the extended mapping $C : H_E \mapsto K$ by setting $Cx = y$.

Lemma 6.4: The extended mapping $C : H_E \mapsto K$ is a bounded linear mapping. That is $C \in \mathcal{L}(H_E, K)$.

Lemma 6.5: The new adjoint mapping $C_E^* : K \mapsto H_E$ is de£ned in terms of the original adjoint mapping $C^* : \mathcal{D}(C^*) \subseteq K \mapsto H$ by the formulae

$$C_E^* = C^*(I + CC^*)^{-1} = (I + C^*C)^{-1}C^*.$$

VII. Extension to polynomial and analytic perturbations

Let us show that inversion of a polynomial perturbation is equivalent to inversion of a corresponding linear perturbation in augmented space. There might be a number of methods to implement this idea. Here we discuss two schemes.

Theorem 7.1: Suppose the inverse of the polynomial perturbation operator

$$A(z) = A_0 + zA_1 + ... + z^m A_m$$

can be expanded as a Laurent series

$$A^{-1}(z) = \frac{1}{z^s}(X_0 + zX_1 + ...) \tag{VII.1}$$

in some punctured neighbourhood of the origin. De£ne augmented operators from H^m to K^m by setting

$$\mathcal{C} := \begin{bmatrix} A_0 & 0 & \cdots & 0 \\ A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{m-1} & A_{m-2} & \cdots & A_0 \end{bmatrix},$$

$$\mathcal{D} := \begin{bmatrix} A_m & A_{m-1} & \cdots & A_1 \\ 0 & A_m & \cdots & A_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_m \end{bmatrix}, \quad \mathcal{I} := \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Then the solution of the linear perturbation problem

$$(\mathcal{C} + z\mathcal{D})\mathcal{X}(z) = z^{s_*}\mathcal{I} \quad \text{(VII.2)}$$

can be expanded as a Laurent series

$$\mathcal{X}(z) = \frac{1}{z^{s_*}} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ X_0 \\ \vdots \\ X_{\mathrm{mod}(s,m)} \end{bmatrix} + \frac{1}{z^{s_*-1}} \begin{bmatrix} X_{\mathrm{mod}(s,m)+1} \\ \vdots \\ \\ \\ \vdots \\ X_{\mathrm{mod}(s,m)+m} \end{bmatrix} + ... \quad \text{(VII.3)}$$

where $s_* = \lfloor s/m \rfloor + 1$ and $\{X_k\}_{k=0}^{\infty}$ are coef£cients of the Laurent series $(VII.1)$, and where $\lfloor s/m \rfloor$ is the integer part and $\mathrm{mod}(s,m)$ is the remainder of the division of s by m.

Thus, we can apply the procedure of Theorem 4.3 £rst to the augmented problem and then we can easily retrieve the coef£cients of the Laurent series (VII.1).

The second scheme is given in the next theorem.

Theorem 7.2: Let the augmented operators be de£ned as follows:

$$\mathcal{C} := \begin{bmatrix} A_0 & A_1 & \cdots & A_{m-1} \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \end{bmatrix}$$

$$\mathcal{D} := \begin{bmatrix} 0 & 0 & \cdots & A_m \\ -I & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & -I & 0 \end{bmatrix}$$

Then the uppermost block of the solution $\mathcal{X}(z)$ for equation (VII.2) with $s_* = s$ coincides with the solution of the original equation $A(z)X(z) = z^s I$.

Each of the presented augmentation schemes has its own merits. Using the second method to obtain the £rst k coef£cients of the Laurent series one needs to calculate k augmented coef£cients $\mathcal{X}_i, i = 0, ..., k-1$, whereas if one utilizes the £rst augmentation scheme, one needs to compute about m times less augmented coef£cients. However, in the second method both augmented matrices $\mathcal{C}$ and $\mathcal{D}$ are close to upper triangular form and have a lot of zero elements. Since the procedure of Theorem 4.3 is based on simultaneous reduction of matrices $\mathcal{C}$ and $\mathcal{D}$ to upper block triangular form, each iteration of the second method could be more computationally ef£cient.

Now we show that in turn the results on polynomial perturbation can be applied to the general case of analytic perturbation represented by a series

$$A(z) = A_0 + zA_1 + z^2 A_2 + ...$$

that converges in some neighbourhood of the origin. Suppose again that the Laurent series for the inverse $A^{-1}(z)$ has a pole of £nite order s, as in (VII.1). Then the number of terms which uniquely determine the inversion procedure is £nite. Namely, there exists m such that the inverse $(A_0 + ... + z^m A_m)^{-1}$ exists for suf£ciently small z. Moreover, any $m \geq s$ can be taken. Therefore we may write

$$A^{-1}(z) = [(A_0 + ... + z^s A_s) + z^{s+1}A_{s+1} + ...]^{-1}$$

$$= [(A_0 + ... + z^s A_s)(I + (A_0 + ... + z^s A_s)^{-1}(z^{s+1}A_{s+1} + ...))]^{-1}$$

$$= [I + (A_0 + ... + z^s A_s)^{-1}(z^{s+1}A_{s+1} + ...)]^{-1}(A_0 + ... + z^s A_s)^{-1}.$$

Now we can apply any of the methods for polynomial perturbations outlined in Theorems 7.1 and 7.2.

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